Graph Signal Processing
Computer Homework 1 (Report)

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Problem 1

A. Firstly, we have forked and installed the repository from this link. We needed to install a mingw-w64 compatible with Matlab.

B. Using `gsp_comet` and `gsp_ring` we defined the desired graphs G_1 and G_2 as below. Then using `gsp_graph_product` We got Kronicker product $G_t = G_1 \bigotimes G_2$ and Cartesian product $G_s = G_1 \times G_2$ of these two graphs which are depicted below:

(a) Graph 1 (b) Graph 2

Figure 1: Graphs G_1, G_2 with their Kronicker product G_t(left) and their Cartesian product G_s (right)

```
    w1 = [0, 0.7, 1.1, 2.3];     W1 = zeros(4);
    W1(1, :) = w1;               W1 = W1 + W1';
    G1 = gsp_comet(4, 3);   % Defining G1 ----------
    G1.W = W1;
    W2 = [0, 1.6, 2.4; 0, 0, 0.8; 0, 0, 0];     W2 = W2 + W2';
    G2 = gsp_ring(3);       % Defining G2 ----------
    G2.W = W2;
    pt.verbose = 1;
    pt.rule = 'kronecker';
    Gs = gsp_graph_product(G1, G2, pt);  % Defining Kronicker product
```

```
    pt.rule = 'cartesian';
    Gt = gsp_graph_product(G1, G2, pt);  % Defining Cartesian product
```

We know that for G_t and G_s the adjacency matrix are $A_t = A_1 \otimes A_2$ and $A_s = A_1 \otimes I_4 + I_3 \otimes A_2$, respectively. Where A_1, A_2 are the adjacency matrices for G_1, G_2, respectively. You can further investigate on the weight matrices for G_t, G_s at `Gt_matrices.txt` and `Gs_matrices.txt` on the results directory.

C. In this section we chose $H = G_t$ and generated a random graph signal on H and plotted the result using `gsp_plot_signal` on our graph which is depicted down below:

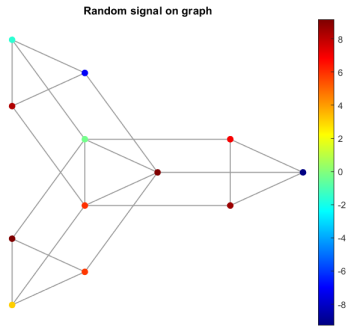


Figure 2: A random signal x_H plotted with its associated graph H

```
    H = Gt;
    x = rand(size(H.A, 1), 1) * 20 - 10;
    % Plotting the signal
    figure('Position', [100, 100, 600, 500]);
    title('Random signal on graph')
    gsp_plot_signal(H, x)
```

D. Now we will get the eigenvalues and eigenvectors of the Laplacian matrix of graph H, and show its spectrum using `gsp_compute_fourier_basis` and `gsp_gft`.

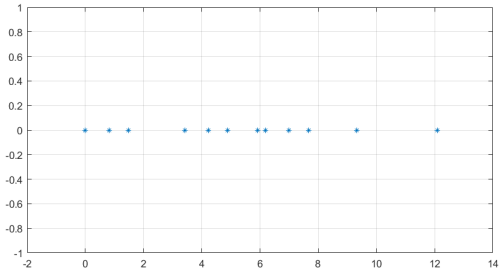


Figure 3: The spectrum of the graph $\{\lambda_1, ..., \lambda_n\}$

Note that since the Laplacian matrix L_H is symmetric, the eigenvalues of L_H are real-valued.
We can also show the random signal x_H in part C, to show the spectrum of X_H over the eigenvalues that we got:

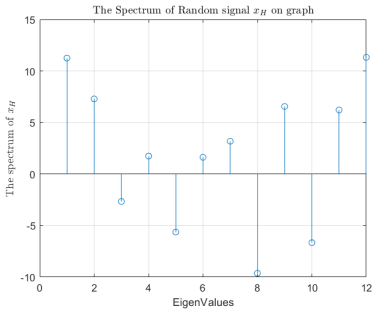


Figure 4: The spectrum of the random graph signal x_H over $\{\lambda_1, ..., \lambda_n\}$

```
H  = gsp_compute_fourier_basis(H);% computing the fourier basis of graph H
spectrum = H.e;                   % getting the eigenvalues(spectrum basis)
x_hat = gsp_gft(H, x);            % computing the spectrum of x
```

E. Next, we will obtain the four eigenvectors corresponding to the two largest and two smallest eigenvalues. These eigenvalues will then be plotted as signals on the graph. The desired plot is shown below:

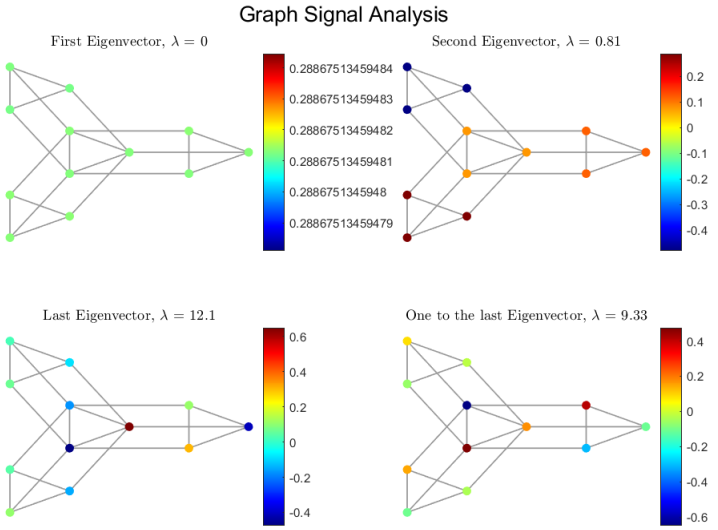


Figure 5: The spectrum of the random graph signal x_H over $\{\lambda_1, ..., \lambda_n\}$

As you can see, The signal with the least eigenvalue(which is zero) is smooth, and as the eigenvalues increases, the graph gets more and more non-smooth. As the eigenvector correspondent to eigenvalue $12.1 = \lambda_{max}$ is of the most non-smoothness.

Problem 2

A. In this context we have a Stochastic Block Matrix $SBM(n, p, q)$ where we have two communities where each person is friend with its own community with the probability p and is friend with the other community with the probability q. Thus, the expected value of the adjacency matrix over the distributions would be:

$$\mathbb{E}[A] + pI_{n\times n} = \left[\begin{array}{ccc|ccc} p & \cdots & p & q & \cdots & q \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ p & \cdots & p & q & \cdots & q \\ \hline q & \cdots & q & p & \cdots & p \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ q & \cdots & q & p & \cdots & p \end{array}\right]$$

Here is the implementation code of defining our SBM graph:

```
% SBM params
N = 1000;      p = 0.5;      q = 0.2;
% Generate SBM graph
dim = 2;      seed = 99;      rng(seed)
sig = randi([0, dim - 1], [N, 1]);
sig = 2 * rescale(sig) - 1;          % sig \in {+1,-1}
P = (sig == sig') * p + (sig ~= sig') * q;
A = ones(N) - (rand(N) > P);
A = triu(A);
A = abs(A - A');     % --- the Adjancency Matrix
% Create graph and compute Fourier basis
G = gsp_graph(A);
G = gsp_create_laplacian(G, 'normalized');
G.type = 'normalized'; % or we could use unnormalized version
G = gsp_compute_fourier_basis(G);
```

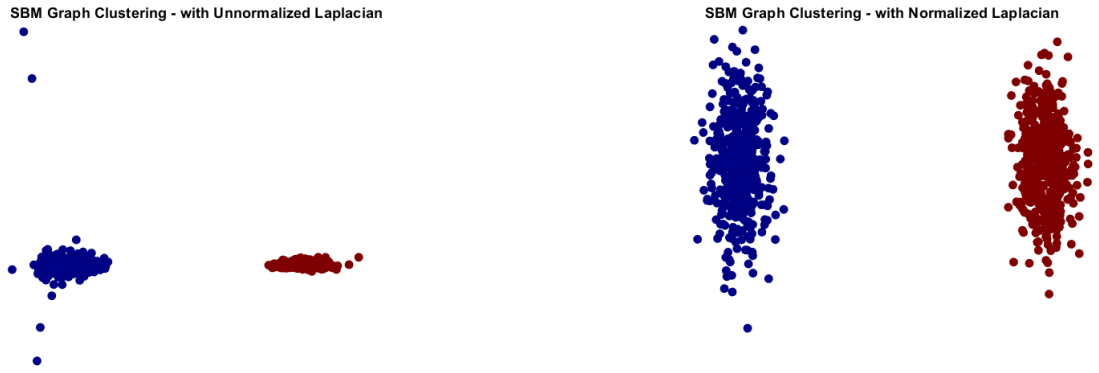
We know that the $(u_1, \lambda_1) = (\not\Vdash_n, 0)$ is a pair of eigenvector, eigenvalues of the Laplacian matrix. So we would get the eigenvalues of the second least eigenvalues(if they are nonzero, otherwise we would switch to bigger ones) and place each node j with a point in 2 dimensional grid as $(u_2(j), u_3(j))$. And the key point is that the sign of the first coordinate of points(nodes) which is $(u_2(j)$ will tell us to which community each node belongs. It is as if that our rule would be:

$$\hat{\sigma_j} = \begin{cases} +1, & \text{if } u_2(j) \geq 0 \\ -1, & \text{if } u_2(j) < 0 \end{cases}$$

Here, we used both Normalized Laplacian(the right image) and Unnormalized Laplacian(the left image) matrices. And this is the code and the final result of community detection:

```
G = gsp_graph(A);
% Use 'combinatorial' for unnormalized Laplacian
G = gsp_create_laplacian(G, 'combinatorial');
G.type = 'combinatorial';
G = gsp_compute_fourier_basis(G);

% Getting Fourier basis for visualization
U23 = G.U;
G.coords = U23(:, 2:max(3, dim));
% Plotting parameters
param = struct;
param.colorbar = 0;
G.plotting.edge_width = 0.1;
figure;
gsp_plot_signal(G, sig, param)
title('SBM Graph Clustering - with Unnormalized Laplacian');
```



(a) Graph 1 (b) Graph 2

Figure 6: $SBM(n = 1000, p = 0.5, q = 0.2)$ classification with the eigenvalues of Unnormalized and Normalized Laplacian Matrix of the graph

Furthermore, the task of classification with Normalized Laplacian Matrix seems better than the Unnormalized Laplacian Matrix.

B. First of all, the reason behind $p, q = \Theta(\frac{\log n}{n})$ is that if p or q is of $o(\frac{\log n}{n})$, then it would be a sparse graph and we wouldn't allow that.
On the other hand, as the distance between p and q rises, the classification gets harder, therefore we plot the ratio of true labeling(accuracy) versus a measure of $|\sqrt{\alpha} - \sqrt{\beta}|$. Here is the code and the result for plotting the accuracy.

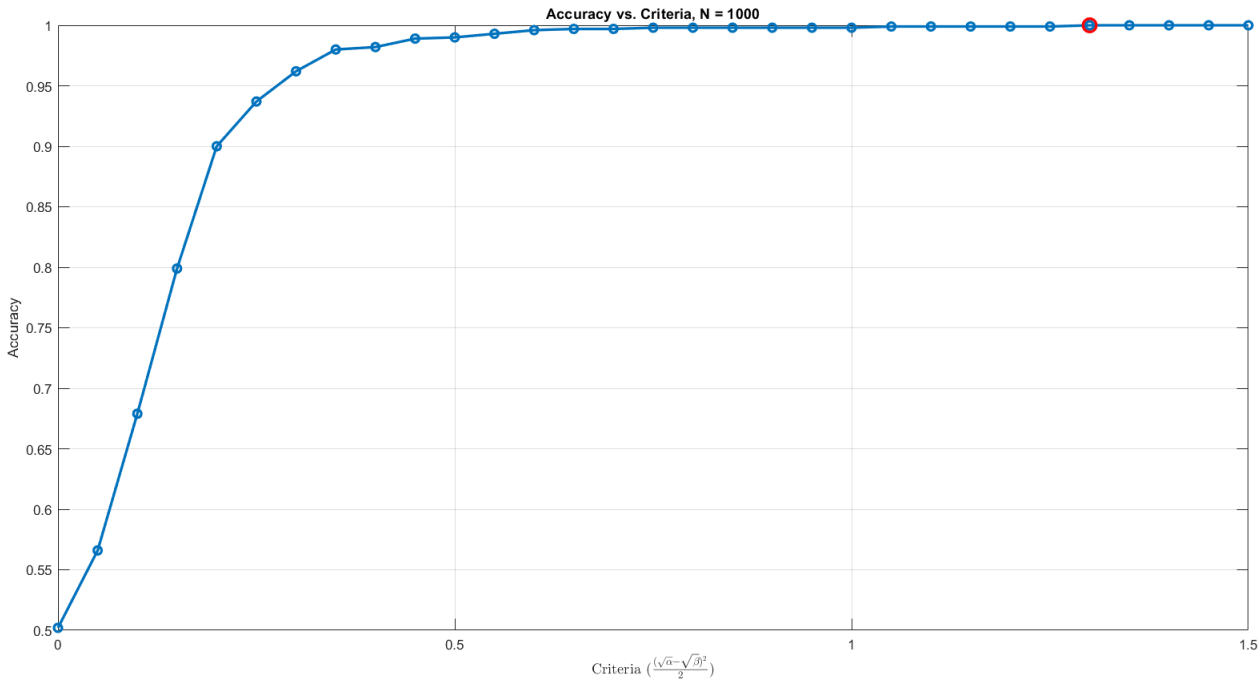


Figure 7: The accuracy versus criteria $|\sqrt{\alpha} - \sqrt{\beta}|^2$

```
% Parameters of SBM
N = 1000;
seed = 99;
dim = 2;

% Sets of alpha and beta
criteria = 0:0.05:1.5;
alphas = 4 * ones(1, numel(criteria));
betas = (sqrt(alphas) - sqrt(2*criteria)).^2;

% Initialize arrays to store results
acc = zeros(1, numel(criteria));

disp('Simulation start')

% Loop through sets
for i = 1:numel(alphas)
    alpha = alphas(i);
    beta = betas(i);

    % Generate SBM graph
    p = alpha * log(N) / N;
    q = beta * log(N) / N;

    % Run clustering and store accuracy
    acc(i) = run_clustering(N, p, q, dim, seed);
end

disp('Simulation finished')

% Find the point where accuracy becomes 1 after a certain criterion
criterion_threshold = 1;
idx_threshold = find(acc == 1, 1, 'first');

% Plot accuracy vs. criteria
figure;
plot(criteria, acc, '-o', 'LineWidth', 2);
hold on;
plot(criteria(idx_threshold), 1, 'ro', 'MarkerSize', 10, 'LineWidth', 2);
hold off;
xlabel('Criteria (\(\frac{(\sqrt{\alpha} - \sqrt{\beta})^2}{2}\))',
    'Interpreter', 'latex');
ylabel('Accuracy');
title(['Accuracy vs. Criteria , N = ' num2str(N)]);
grid on;


% Display threshold information
disp(['Criterion threshold for N = ' num2str(N) ': '
    num2str(criteria(idx_threshold))]);

function acc = run_clustering(N, p, q, dim, seed)
    % Generate SBM graph
    rng(seed)
    sig = randi([0, dim - 1], [N, 1]);
    sig = 2 * rescale(sig) - 1;
    P = (sig == sig') * p + (sig ~= sig') * q;
    A = ones(N) - (rand(N) > P);
```

```
    A = triu(A);
    A = abs(A - A');

    % Create graph and compute Fourier basis
    G = gsp_graph(A);
    G = gsp_create_laplacian(G, 'normalized');
    G.type = 'normalized';
    G = gsp_compute_fourier_basis(G);

    % Choose subset of Fourier basis for visualization
    U23 = G.U;
    dim = 2;
    G.coords = U23(:, 2:max(dim, 3));

    % Perform k-means clustering
    k = 2;
    [clusters, ~] = kmeans(G.coords, k);
    % clusters = G.coords(:,1) > 0;

    % True labels based on the sign of sig
    true_labels = (sig + 1) / 2 + 1;

    % Calculate accuracy
    accuracy = sum(clusters == true_labels) / numel(true_labels);
    accuracy = max(accuracy, 1 - accuracy);

    % Display results
    alpha = p * N / log(N);
    beta = q * N / log(N);
    c = (sqrt(alpha) - sqrt(beta)) ^ 2 / 2;
    disp(['alpha = ' num2str(alpha) ', beta = ' num2str(beta) ...
        ', criteria = ' num2str(c) ', Accuracy: ' num2str(accuracy)]);

    acc = accuracy;

end
```