سوال 1:

(I) Since $M = M^T \in \mathbb{R}^{n\times n}$, M can be written as $M = Q\Lambda Q^T$ where $\Lambda = \operatorname{diag}(\mu_1, \dots, \mu_n)$ and $Q = \begin{bmatrix} | & & | \\ \psi_1 & \cdots & \psi_n \\ | & & | \end{bmatrix}$. For the sake of showing that a subspace S of dimension k exists such that $\min\limits_{\substack{x\neq 0\\ x\in S}} \left\{ \frac{x^T M x}{x^T x} \right\} = \mu_k$, we can simply put $\{\psi_1, \dots, \psi_k\}$ as the orthogonal bases of the subspace S. Therefore $S = \operatorname{span}\{\psi_1, \dots, \psi_k\}$. Now we'd get:

$$\Rightarrow \frac{x^T M x}{x^T x} = \frac{x^T\left(\sum\limits_{i=1}^{n} \mu_i \psi_i \psi_i^T\right) x}{\sum\limits_{i=1}^{n} x_i^2} = \frac{\sum\limits_{i=1}^{n} \mu_i \langle x, \psi_i\rangle^2}{\sum\limits_{i=1}^{n} x_i^2} = \frac{\sum\limits_{i=1}^{n} \mu_i \sum\limits_{j=1}^{k} \langle x_j \psi_j, \psi_i\rangle^2}{\sum\limits_{i=1}^{n} x_i^2}$$

Now since $x \in S$ with orthogonal vectors $\{\psi_1, \dots, \psi_k\}$; then we can write x as $x = \sum\limits_{i=1}^{k} x_i \psi_i$. Since $\forall i\neq j: \langle \psi_i, \psi_j\rangle = 0$, we'll obtain that:

$$\Rightarrow \frac{x^T M x}{x^T x} = \frac{\sum\limits_{i=1}^{k} \mu_i x_i^2}{\sum\limits_{i=1}^{k} x_i^2} \longrightarrow \operatorname*{arg\,min}_{(x_1, \dots, x_n)} \left\{ \frac{x^T M x}{x^T x} \right\} = \hat{e}_k = (0, \dots, 0, \underset{k\text{th}-\text{element}}{1}, \dots, 0)$$

therefore the minimum value for the Rayleigh quotient will be μ_k.

$$\min_{\substack{x\in S\\ x\neq 0}} \left\{ \frac{x^T M x}{x^T x} \right\} = \frac{\mu_k x_k^2}{x_k^2} = \mu_k \Rightarrow$$

$\boxed{\text{So there exist a subspace } S \text{ with } \dim(S) = k, \text{ that } \mu_k = \frac{x^{*T} M x^*}{x^{*T} x^*} \text{ is acheivable for some } x^* \in S.}$ ■ Q.E.D.

(II) Let $T = \operatorname{span}\{\psi_k, \dots, \psi_n\}$ & S an arbitrary subspace of $\mathbb{R}^n$ with $\dim(S) = k$. Then we can say that $T \cap S$ has at least one element; therefore:

$$\max_{x\in T} \left\{ \frac{x^T M x}{x^T x} \right\} \underset{(i)}{\geq} \max_{x\in T\cap S} \left\{ \frac{x^T M x}{x^T x} \right\} \underset{(ii)}{\geq} \min_{x\in T\cap S} \left\{ \frac{x^T M x}{x^T x} \right\} \underset{(iii)}{\geq} \min_{x\in S} \left\{ \frac{x^T M x}{x^T x} \right\}$$

(i) since $T\cap S \subseteq T$; (ii) $\max\{\dots\} \geq \min\{\dots\}$; (iii) since $T\cap S \subseteq S$

Thus, we've proven that: $\boxed{\min\limits_{\substack{x\in S\\ x\neq 0}} \left\{ \frac{x^T M x}{x^T x} \right\} \leq \max\limits_{\substack{x\in T\\ x\neq 0}} \left\{ \frac{x^T M x}{x^T x} \right\}}$ ■ Q.E.D

(III) Simple as before we can see that $\operatorname*{arg\,max}\limits_{(x_k, \dots, x_n)\in T} \left\{ \frac{\sum\limits_{i=k}^{n} \mu_i x_i^2}{\sum\limits_{i=k}^{n} x_i^2} \right\} = \hat{e}_k$

Thus: $\max\limits_{x\in T} \left\{ \frac{x^T M x}{x^T x} \right\} = \mu_k$. Therefore:

$$\boxed{\forall x\in T: \frac{x^T M x}{x^T x} \leq \mu_k \qquad T = \operatorname{span}\{\psi_k, \dots, \psi_n\}}$$

So far we know $\min\limits_{x\in S}\left\{\frac{x^TMx}{x^TX}\right\} \le \max\limits_{x\in T}\left\{\frac{x^TMX}{x^TX}\right\} = \mu_k$ سؤال 1:

However, In Part (I) we saw: $\max\limits_{\substack{S\subseteq\mathbb{R}^n\\ \dim(S)\ge k}}\ \min\limits_{\substack{x\in S\\ x\ne 0}}\left\{\frac{x^TMx}{x^Tx}\right\} \le \mu_k$

which suggests that $\hookrightarrow S = \mathrm{Span}\{\varphi_1,\dots,\varphi_k\}$

$$\boxed{\max_{\substack{S\subseteq\mathbb{R}\\ \dim(S)\ge k}}\ \min_{\substack{x\in S\\ x\ne 0}}\left\{\frac{x^TMx}{x^Tx}\right\} = \mu_k}$$

Q.E.D. ∎

the other side of the equation can be done using the same technique!

We define $B^k_{n\times n}$ such that except the k-th row and k-th column which are zero, all others are identical to A. so we'd get: سؤال 2:

$$B^k_{n\times n} = \begin{cases} A(i,j) & i,j\ne k\\ 0 & \text{o.w.}\end{cases}$$

we can see that for an $x\in\mathbb{R}^n$:

$x^TAx = x^TB^kx$ if $x_k=0$. thus one'd obtain that:

$$\max_{\substack{\dim(S)=k\\ S\subseteq\mathbb{R}^n}}\ \min_{\substack{x\ne 0\\ x_k=0\\ x\in S}}\left\{\frac{x^TAx}{x^Tx}\right\} = \max_{\substack{\dim(S)=k\\ S\subseteq\mathbb{R}^n}}\ \min_{\substack{x\ne 0\\ x_k=0\\ x\in S}}\left\{\frac{x^TB^kx}{x^Tx}\right\} = \max_{\substack{\dim(S)=k\\ S\subseteq\mathbb{R}^{n-1}}}\ \min_{\substack{x\ne 0\\ x\in S}}\left\{\frac{x^TBx}{x^Tx}\right\} \quad \boxed{(*)}$$

Now removing the $x_k=0$ condition will imply that:

$$\max_{\substack{\dim(S)=k\\ S\subseteq\mathbb{R}^n}}\ \min_{\substack{x\ne 0\\ x\in S}}\left\{\frac{x^TAx}{x^Tx}\right\} \ge \max_{\substack{\dim(S)\ge k\\ S\subseteq\mathbb{R}^{n-1}}}\ \min_{\substack{x\in S\\ x\ne 0\\ x_k=0}}\left\{\frac{x^TAx}{x^Tx}\right\} \quad \boxed{(**)}$$

Now by aggregating the results from (*), (**), we'd obtain that:

$$\forall k\in\{1,2,\dots,n\}:\ \lambda_k = \max_{\substack{\dim(S)=k\\ S\subseteq\mathbb{R}^n}}\ \min_{\substack{x\ne 0\\ x\in S}}\left\{\frac{x^TAx}{x^Tx}\right\} \ge \max_{\substack{\dim(S)=k\\ S\subseteq\mathbb{R}^{n-1}}}\ \min_{\substack{x\in S\\ x\ne 0}}\left\{\frac{x^TBx}{x^Tx}\right\} = \mu_k$$

which shows $\forall k:\ \lambda_k \ge \mu_k$.

We follow the same approach for the other direction ($\cancel{\forall}\ \forall k \geq 2$: $\gamma_{k-1} \geq \lambda_k$)

Since $x^T A x = x^T \tilde{B}^k x$ if $x_k = 0$, we have:

$$\min_{\substack{T \subseteq \mathbb{R}^{n-1} \\ \dim(T) = n-k}} \max_{\substack{x \neq 0 \\ x \in T}} \left\{ \frac{x^T \tilde{B}^k x}{x^T x} \right\} \geq \min_{\substack{\dim(T) \\ = n-k+1 \\ T \subseteq \mathbb{R}^n}} \max_{\substack{x \neq 0 \\ x_k = 0 \\ x \in T}} \left\{ \frac{x^T \tilde{B}^k x}{x^T x} \right\} = \min_{\substack{\dim(T) \\ = n-k+1 \\ T \subseteq \mathbb{R}^n}} \max_{\substack{x \neq 0 \\ x \in T \\ x_k = 0}} \left\{ \frac{x^T A x}{x^T x} \right\}$$

Now we know that:

$$\min_{\substack{\dim(T) \\ = n-k+1 \\ T \subseteq \mathbb{R}^n}} \max_{\substack{x \neq 0 \\ x_k = 0 \\ x \in T}} \left\{ \frac{x^T A x}{x^T x} \right\} \geq \min_{\substack{\dim(T) \\ = n-k+1 \\ T \subseteq \mathbb{R}^n}} \max_{\substack{x \neq 0 \\ x \in T}} \left\{ \frac{x^T A x}{x^T x} \right\} = \lambda_k$$

which implies that:

$$\min_{\substack{T \subseteq \mathbb{R}^{n-1} \\ \dim(T) = n-(k-1)+1}} \max_{\substack{x \neq 0 \\ x \in T}} \left\{ \frac{x^T B x}{x^T x} \right\} = \gamma_{k-1} \geq \lambda_k = \min_{\substack{\dim(T) \\ = n-k+1 \\ T \subseteq \mathbb{R}^n}} \max_{\substack{x \neq 0 \\ x \in T}} \left\{ \frac{x^T A x}{x^T x} \right\}$$

Eventually, we proved that: $\forall k$: $\lambda_k \geq \gamma_k \geq \lambda_{k+1}$ which will end the proof since we'd obtain that: $\boxed{\lambda_1 \geq \gamma_1 \geq \lambda_2 \geq \dots \geq \lambda_{n-1} \geq \gamma_{n-1} \geq \lambda_n}$ ∎

where $\begin{cases} \lambda_1 \geq \dots \geq \lambda_n \to \text{eigenvalues of } A \\ \gamma_1 \geq \dots \geq \gamma_{n-1} \to \text{eigenvalues of } B \end{cases}$

the interlacting theorem

Q.E.D

let $P(A) = (A-\lambda_1 I)(A-\lambda_2 I) = (A-\lambda_2 I)(A-\lambda_1 I) = 0$ be the characteristic polynomial of A. let also $\{(\lambda_1, v_1), (\lambda_2, v_2)\}$ be the (eigenvalue, eigenvector) pairs. then we have $\begin{cases} Av_1 = \lambda_1 v_1 \\ Av_2 = \lambda_2 v_2 \end{cases}$ $\boxed{(*)}$

from (*) & $\underline{A = AB - BA}$, we can deduce that:

$$\begin{cases} Av_1 = \lambda_1 v_1 = (A-\lambda_1 I) Bv_1 \\ Av_2 = \lambda_2 v_2 = (A-\lambda_2 I) Bv_2 \end{cases} \quad \boxed{(**)}$$

by (***)

$$\overset{(*)}{\Longrightarrow} \begin{cases} (A-\lambda_2 I) Av_1 = (A-\lambda_2 I)\lambda_1 v_1 = \lambda_1(\lambda_1-\lambda_2) v_1 = 0 \\ (A-\lambda_1 I) Av_2 = (A-\lambda_1 I)\lambda_2 v_2 = \lambda_2(\lambda_2-\lambda_1) v_2 = 0 \end{cases}$$

$$\overset{(**)}{\Longrightarrow} \begin{cases} (A-\lambda_2 I) Av_1 = \overbrace{(A-\lambda_2 I)(A-\lambda_1 I)}^{0} Bv_1 = 0 \\ (A-\lambda_1 I) Av_2 = \underbrace{(A-\lambda_1 I)(A-\lambda_2 I)}_{0} Bv_2 = 0 \end{cases} \quad \boxed{(***)}$$

So we obtained that $\begin{cases} \lambda_1(\lambda_1-\lambda_2) = 0 \\ \lambda_2(\lambda_2-\lambda_1) = 0 \end{cases}$ which has two solutions:

(i) either $\lambda_1 = \lambda_2 = 0$ which means $P(A) = A^2 = 0$ ✓

(ii) either $\lambda_1 = \lambda_2 = \lambda \neq 0$ which means $P(A) = (A-\lambda I)^2 = A^2 - 2\lambda A + \lambda^2 I = 0$

In this case we know that:

$$tr\{A\} = tr\{AB - BA\} = tr\{AB\} - tr\{BA\} = 0 = \sum_{k=1}^{2} \lambda_k = \lambda_1 + \lambda_2 = 2\lambda$$

$\hookrightarrow \lambda = 0 \longrightarrow P(A) = A^2 = \underline{0}$ ✓ $\qquad$ Q.E.D ■

Firstly, we know that, the maximum & minimum eigenvalue of : ٤ سؤال
a matrix S has the forms: $\lambda_{max}(S) = \sup_{\|x\|=1} \{x^T S x\}$ & $\lambda_{min}(S) = \inf_{\|x\|=1} \{x^T S x\}$.

(I) so now let A be a symmetric block matrix $A = \begin{bmatrix} B & C \\ C^T & D \end{bmatrix}$ $B \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{n\times m}$, $D \in \mathbb{R}^{m\times m}$.

furthermore let $[x_1^T \; x_2^T]^T = x$ be the eigenvector associated with the largest eigenvalue of A.

– First we assume that $(x_1 = 0) \vee (x_2 = 0)$

- if $x_1 = 0 \rightarrow x^T A x = x_2^T D x_2 = \hat{x}_2^T D \hat{x}_2 = \lambda_{max}(D)$
- if $x_2 = 0 \rightarrow x^T A x = x_1^T B x_1 = \hat{x}_1^T B \hat{x}_1 = \lambda_{max}(A)$

with the assumption $(x_1=0) \vee (x_2=0)$ we have:

$$\lambda_{min}(A) + \lambda_{max}(A) \le 2\lambda_{max}(A) = \lambda_{max}(B) + \lambda_{max}(D)$$ ✓ Q.E.D. ■

Now we consider the general case $x = [x_1^T, x_2^T]^T$ where $\|x\|^2 = \|x_1\|^2 + \|x_2\|^2 = 1$.

$$\rightarrow \lambda_{max}(A) = x^T A x = x_1^T B x_1 + x_2^T D x_2 + 2x_1^T C x_2$$
$$= \|x_1\|^2 \cdot \hat{x}_1 B \hat{x}_1 + \hat{x}_2^T D \hat{x}_2 \cdot \|x_2\|^2 + 2\|x_1\| \cdot \|x_2\| \, \hat{x}_1^T C \hat{x}_2 \quad (*)$$

Now, let $y = \left[\frac{-\|x_2\|}{\|x_1\|} x_1^T, \frac{-\|x_1\|}{\|x_2\|} x_2^T\right]^T \rightarrow \|y\|^2 = \|x_1\|^2 + \|x_2\|^2 = 1$.

since $\lambda_{min}(A) = \inf_{\|\tilde{y}\|=1} \{\tilde{y}^T A \tilde{y}\} \le y^T A y = \|x_2\|^2 \cdot \hat{x}_1^T B \hat{x}_1 + \|x_1\|^2 \hat{x}_2^T D \hat{x}_2$ $\forall \|y\|=1$ $\quad (**)$
$+ 2\|x_1\| \cdot \|x_2\| \hat{x}_1^T C \hat{x}_2$

Now adding the results $(*)$ & $(**)$ leads to:

$$\lambda_{max}(A) + \lambda_{min}(A) \le \underbrace{(\|x_1\|^2 + \|x_2\|^2)}_{1} \hat{x}_1^T B \hat{x}_1 + \underbrace{(\|x_2\|^2 + \|x_1\|^2)}_{1} \hat{x}_2^T D \hat{x}_2$$

$$\le \lambda_{max}(B) + \lambda_{max}(D)$$ ■ Q.E.D.

the last line of Proof was due to the fact that

$$\left.\begin{array}{l}\forall \|\hat{x}_1\|=1 \\ \|\hat{x}_2\|=1\end{array}\right\} \rightarrow \left\{\begin{array}{l}\hat{x}_1^T B \hat{x}_1 \le \lambda_{max}(B) \\ \hat{x}_2^T D \hat{x}_2 \le \lambda_{max}(D)\end{array}\right.$$

(II) Now that we've proven the formula for the base of induction of $k=2$.
which is true since for $A = \begin{bmatrix} M_{1,1} & M_{1,2} \\ M_{1,2}^T & M_{2,2} \end{bmatrix} \rightarrow (2-1)\lambda_{min}^{(A)} + \lambda_{max}^{(A)} \leq \sum_{i=1}^{2} \lambda_{max}(M_{i,i})$

So we assume that the assumption is true for all $i < k$. & we will prove it for $i = k$. we have

$$A = \left[\begin{array}{cccc|c} M_{1,1} & M_{1,2} & \cdots & M_{1,k-1} & M_{1,k} \\ M_{1,2}^T & \ddots & \tilde{M}_{1,1} & \vdots & \vdots \\ \vdots & & \ddots & \vdots & \vdots \\ M_{1,k-1}^T & \cdots & \cdots & M_{k-1,k-1} & M_{k-1,k} \\ \hline M_{1,k}^T & \cdots & \cdots & M_{k-1,k}^T & M_{k,k} \end{array}\right]$$

$\tilde{M}_{1,2}$, $\tilde{M}_{2,1}$, $\tilde{M}_{2,2}$

So let

$$\tilde{M}_{1,1} = \begin{bmatrix} M_{1,1} & \cdots & M_{1,k-1} \\ \vdots & \ddots & \vdots \\ M_{1,k-1}^T & \cdots & M_{k-1,k-1} \end{bmatrix}$$

$$\tilde{M}_{1,2} = \begin{bmatrix} M_{1,k} & \cdots & M_{k-1,k} \end{bmatrix}^T \qquad \tilde{M}_{2,1} = \tilde{M}_{1,2}^T \qquad \tilde{M}_{2,2} = M_{k,k}$$

Since each $\tilde{M}_{i,i}$ is symmetric, We know that

$$\begin{cases} \lambda_{min}(A) + \lambda_{max}(A) \leq \lambda_{max}(\tilde{M}_{1,1}) + \lambda_{max}(\tilde{M}_{2,2}) & (*) \\ \text{by induction assumption: } (k-2)\lambda_{min}(\tilde{M}_{1,1}) + \lambda_{max}(\tilde{M}_{1,1}) \leq \sum_{i=1}^{k-1} \lambda_{max}(M_{i,i}) & (**) \end{cases}$$

furthermore, since $\tilde{M}_{1,1}$ is the matrix A with the n-th row & column, By Interlacing theorem that we proved earlier in Problem 1, we know that: $\lambda_{min}(\tilde{M}_{11}) \geq \lambda_{min}(A)$ $(***)$. thus from $(*)$, $(**)$ & $(***)$, we can obtain that:

$$\lambda_{min}(A) + \lambda_{max}(A) \leq \lambda_{max}(M_{k,k}) + \sum_{i=1}^{k-1} \lambda_{max}(M_{i,i}) - (k-2)\lambda_{min}(\tilde{M}_{1,1})$$

$$\overset{(***)}{\leq} \sum_{i=1}^{k} \lambda_{max}(M_{i,i}) - (k-2)\lambda_{min}(A)$$

which leads to $\boxed{\lambda_{min}^{(A)}(1 + \cancel{k-2}) + \lambda_{max}^{(A)} \leq \sum_{i=1}^{k} \lambda_{max}(M_{i,i})}$ with $k-1$ written beneath

So Now that we've proven that case k is also true, By induction we can prove that for any k, the statement is true (Base of induction was true as well)

$$\boxed{\forall k: \quad (k-1)\lambda_{min}^{(A)} + \lambda_{max}^{(A)} \leq \sum_{i=1}^{k} \lambda_{max}^{(M_{i,i})}}$$

Q.E.D ∎

$$P_A(x) = \det(xI - A) = \det\left(\begin{bmatrix} x-a_{11} & -a_{12} & \cdots & -a_{1n} \\ \vdots & \ddots & & \vdots \\ -a_{n1} & \cdots & & x-a_{nn} \end{bmatrix}\right) = \sum_{k=0}^{n} (-1)^k \sigma_k(A)\, x^{n-k}$$

Since λ_i's are the roots to the polynomial $P_A(x)$ then we can write $P_A(x)$ as $P_A(x) = \prod_{i=1}^{n} (x - \lambda_i) = \sum_{k=0}^{n} b_k x^{n-k}$ where

$$b_k = (-1)^k \sum_{\substack{S \subseteq \{1,\dots,n\} \\ |S| = k}} \prod_{i \in S} \lambda_i = (-1)^k \left[\lambda_1 \lambda_2 \cdots \lambda_k + \lambda_1 \lambda_2 \cdots \lambda_{k+1} + \cdots + \lambda_{n-k+1} \cdots \lambda_{n-1} \lambda_n \right]$$

So we can deduce that $b_k = (-1)^k \sum_{\substack{S \subseteq [n] \\ |S| = k}} \prod_{i \in S} \lambda_i = (-1)^k \sigma_k(A)$

which results in $\boxed{\sigma_k(A) = \sum_{\substack{S \subseteq [n] \\ |S| = k}} \prod_{i \in S} \lambda_i}$ Q.E.D. ∎

On the other hand, we can find $\sigma_k(A) \cdot (-1)^k$ from $\det(xI - A)$ since for having x^{n-k}, we must choose $n-k$ element on the main diameter, & the rest from other remaining rows and columns. So for getting the coefficient of x^{n-k}, we must sum over all possible multiplica of all choosings of k distinct elements from $\{1, 2, \dots, n\}$.

Since in these multiplications, we don't choose from the $n-k$ already chosen elements lying on the main diameter, it would be the determinant of the matrix M_π such that k rows & k columns have been deleted.

thus:

$$\{\text{coeff of } x^{n-k}\} = \left\{ \sum_{\pi \in S_n} \left(\operatorname{sgn}(\pi) \prod_{i=1}^{n-k} (x - a_{\pi(i), \pi(i)}) \prod_{j \in \{1,\dots,n\} / \{\pi(1), \dots, \pi(n-k)\}} (-a_{j, \pi(j)}) \right) \right\}_{\text{coeff of } x^{n-k}}$$

$$\rightarrow \{\text{coeff of } x^{n-k}\} = \sum_{\pi \in S_n} (-1)^k \operatorname{sgn}(\pi) \prod_{j \in \{\cdots\}/\{\cdots\}} a_{j, \pi(j)} = (-1)^k \sum_{\substack{S \subseteq n \\ |S| = k}} \det(A(S,S))$$

which follows $\sigma_k(A) = \sum_{\substack{S \subseteq n \\ |S| = k}} \det(A(S,S))$ Q.E.D ∎

(I) $A = A^T$, $\det(B) \neq 0$, $C = BAB^T$: 6 ...

if $v \in Null(A) \to Av = 0 \to BAB^T v = 0 \longrightarrow$ since $\det(B) \neq 0$, there exists B^{-1} such that $BB^{-1} = I$. $B^{-1}BAB^T v = A(\underbrace{B^T v}_{u}) = 0$, Thus we obtained that

$$\boxed{\forall v \in Null(BAB^T) \to \exists u = B^T v : u \in Null(A) \quad (*)}$$

on the other hand if $u \in Null(A) \to Au = 0 \xrightarrow[B^{-1} \text{ exists}]{} v = B^{-T} u \in Null(BAB^T)$

since $BAB^T v = BAB^T B^{-T} u = B\underline{Au} = 0$

$$\boxed{\forall u \in Null(A) \to \exists v = B^{-T} u : v \in Null(BAB^T)} \quad (**)$$

$(*)$, $(**)$ will result in that the number of zero eigenvalues in matrices A, BAB^T (with constraints $\det(B) \neq 0$, $A = A^T$) is equal. Q.E.D. ■

(II) $\{\gamma_1, \dots, \gamma_k\}$: positive eigenvalues of C ; $\Gamma_k = span(v_1, \dots, v_k)$

$\Lambda_k = \{B^T y ; y \in \Gamma_k\}$

$$\lambda_k = \max_{\substack{S \subseteq \mathbb{R}^n \\ \dim(S) \geq k}} \min_{\substack{x \in S \\ x \neq 0}} \left\{ \frac{x^T A x}{x^T x} \right\} \geq \min_{\substack{x \in \Lambda_k \\ x \neq 0}} \left\{ \frac{x^T A x}{x^T x} \right\} = \min_{\substack{y \in \Gamma_k \\ y \neq 0}} \left\{ \frac{y^T BAB^T y}{\|B^T y\|^2} \right\} = \min_{i \in [k]} \gamma$$

Now since $\lambda_k \geq \min\{\gamma_1, \dots, \gamma_k\} \geq 0 \longrightarrow$ implies $\lambda_k \geq 0$. Q.E.D. ■

this shows that A has at least k eigenvalues which are positive.

(III) $\{\gamma_n, \dots, \gamma_{n-k+1}\}$: negative eigenvalues of C ; $\Gamma_{n-k+1} = span\{v_n, \dots, v_{n-k+1}\}$

$\Lambda_{n-k+1} = \{B^T y, y \in \Gamma_{n-k+1}\}$ let $S = \{n-k+1, \dots, n\}$

$$\lambda_{n-k+1} = \min_{\substack{T \subseteq \mathbb{R}^n \\ \dim(S) \geq k}} \max_{\substack{x \neq 0 \\ x \in T}} \left\{ \frac{x^T A x}{x^T x} \right\} \leq \max_{\substack{x \neq 0 \\ x \in \Lambda_{n-k+1}}} \left\{ \frac{x^T A x}{x^T x} \right\} = \max_{y \in \Gamma_{n-k+1}} \left\{ \frac{y^T BAB^T y}{\|B^T y\|^2} \right\} = \max_{i \in S} \gamma_i$$

Now since $\lambda_{n-k+1} \leq \max\{\gamma_n, \dots, \gamma_{n-k+1}\} < 0 \longrightarrow$ implies $\lambda_{n-k+1} < 0$ Q.E.D

Now let us assume that A has N_0 zero eigenvalues, N_1 positive eigenvalues & N_2 negative eigenvalues. where $n = N_0 + N_1 + N_2$

Since $C = BAB^T$ has exactly N_0 zero eigenvalue, N_1' positive eigenvalues & N_2' negative eigenvalues, where $n = N_0 + N_1' + N_2'$;

ادامه سوال 6:

we can deduce that $N_1' + N_2' = N_1 + N_2$.

furthermore; In prior to this, we derived that

$$\left.\begin{array}{ll} (*) & N_1 \geq N_1' \\ (**) & N_2 \geq N_2' \\ (***) & N_1 + N_2 = N_1' + N_2' \end{array}\right\} \longrightarrow \text{which implies } \begin{cases} N_1' = N_1 \\ N_2' = N_2 \end{cases}$$

$\rightarrow$ So we know that the matrices A, BAB^T have exactly the same number of positive, negative, zero eigenvalues.

سوال 7:

Since $A \leq B$, then $\forall x \in \mathbb{R}^n \;\; x^T B x \geq x^T A x$. $(*)$

thus from Courant-Fischer theorem we obtain:

$$\lambda_k^{(A)} = \max_{\substack{\dim(S)=k \\ S \subseteq \mathbb{R}^n}} \; \min_{\substack{x \in S \\ x \neq 0}} \left\{ \frac{x^T A x}{x^T x} \right\} \overset{*}{\leq} \max_{\substack{\dim(S)=k \\ S \subseteq \mathbb{R}^n}} \; \min_{\substack{x \in S \\ x \neq 0}} \left\{ \frac{x^T B x}{x^T x} \right\} = \lambda_k^{(B)}$$

So we easily showed that $\boxed{\forall k \in \{1, \dots, n\}: \lambda_k^{(A)} \leq \lambda_k^{(B)}}$.

which is the essential condition for $B \succeq A$.

Q.E.D. ■