1/ GSP تمرین سری 3 52

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دانشگاه صنعتی شریف

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شماره سوال: سوال 1

(I) We know that G is k-regular graph which means $\sum_{i=1}^{n} A(i,j) = d(j) = k \quad \forall n$ and also $A \cdot \mathbb{1}_{n\times 1} = k\,\mathbb{1}_{n\times 1}$, hence $(k, \mathbb{1}_{n\times 1})$ is an eigen-pair of A. Furthermore we know that if $i \sim j$, then $A^2(i,j) = r$ (if $i \sim j$, they have r common neighbors). Otherwise ($i \nsim j$), then $A^2(i,j)$. So we can reconstruct A^2 with respect to A:

$$A^2 = \underbrace{rA}_{i\sim j} + \underbrace{s(\mathbb{1}_{n\times n} - I - A)}_{i \nsim j} + \underbrace{kI}_{i=j} \rightarrow \text{let } P(A) = A^2 + (s-r)A + (s-k)I = s\,\mathbb{1}_{n\times n}.$$

Now it's clear that the matrix $P(A) = s\,\mathbb{1}_{n\times n}$ is of rank 1 and the only nonzero eigenvalue is ns with eigenvector $\mathbb{1}_{n\times 1}$. So let $\{v_i\}_{i=2}^{n}$ form a basis perpendicular to the subspace with basis $v_1 = \mathbb{1}_{n\times 1}$. Then we know that $\forall i \geq 2: P(A)v_i = 0$; since at least one of v_i's are not zero vector, we'd get that:

$$\left.\begin{array}{l} A^2 v_i = \lambda^2 v_i \\ A v_i = \lambda v_i \end{array}\right\} \quad \left(\lambda^2 + (s-r)\lambda + (s-k)\right) v_i = 0 \rightarrow \lambda_A \in \begin{cases} k \\ \dfrac{-(s-r) \pm \sqrt{(s-r)^2 + 4(k-s)}}{2} \end{cases}$$

Since $L = D - A = kI - A \rightarrow \Lambda_L = -\Lambda_A + k$ which suggests that

$$\Lambda_L \in \left\{ 0, \frac{2k - r + s \pm \sqrt{(r-s)^2 + 4(k-s)}}{2} \right\} \quad \blacksquare \text{ Q.E.D}$$

(II) Since $x^T \mathbb{1} = 0 \rightarrow \vec{x} \perp \mathbb{1} \rightarrow x = \sum_{v_i \in \mathrm{eig}(L) \perp \mathbb{1}} \gamma_i v_i = \gamma_2 v_2 + \gamma_3 v_3 \quad (v_2 \perp v_3)$

$\langle x, x \rangle = \gamma_2^2 + \gamma_3^2 = 1 \;(*)$. If $y = Hx = f(L)x = \left(\sum_{i=0}^{M-1} \alpha_i L^i\right) x = \left(\sum_{i=0}^{M-1} \alpha_i L^i (\gamma_2 v_2 + \gamma_3 v_3)\right)$

$$\rightarrow y = \sum_{i=0}^{M-1} \alpha_i (\theta_2^i v_2 + \theta_3^i v_3) = \gamma_2 f(\theta_2) v_2 + \gamma_3 f(\theta_3) v_3$$

$$\langle x, y \rangle = \langle \gamma_2 f(\theta_2) v_2 + \gamma_3 f(\theta_3) v_3, \gamma_2 v_2 + \gamma_3 v_3 \rangle = \gamma_2^2 f(\theta_2) + \gamma_3^2 f(\theta_3) = 0 \quad (**)$$

$$(*), (**) \rightarrow \begin{cases} \gamma_2^2 = \dfrac{+f(\theta_3)}{f(\theta_3) - f(\theta_2)} \\ \gamma_3^2 = \dfrac{+f(\theta_2)}{f(\theta_2) - f(\theta_3)} \end{cases}$$

$$\rightarrow \langle y, y \rangle = \gamma_2^2 f(\theta_2)^2 + \gamma_3^2 f(\theta_3)^2 = \frac{-f(\theta_2) f(\theta_3)^2 + f(\theta_3) f(\theta_2)^2}{f(\theta_2) - f(\theta_3)} = f(\theta_2) f(\theta_3)$$

$$\begin{cases} y^H y = \|y\|_2^2 = f(\theta_2) f(\theta_3) \\ \|y\|_2 = \sqrt{f(\theta_2) f(\theta_3)} \end{cases}$$

Q.E.D $\blacksquare$

2/ with the assumption that the vector $\frac{1}{\sqrt{n}}\mathbb{1}$ is an eigenvector of both

$\forall i \in \{1, \dots, n\}$

$y_i = \sqrt{n} \cdot y_{i-1} \circledast_G \delta_i$ $\qquad U^H = \begin{bmatrix} \bar{u}_{11} & \cdots & \bar{u}_{n1} \\ \vdots & & \vdots \\ \bar{u}_{1n} & \cdots & \bar{u}_{nn} \end{bmatrix}$

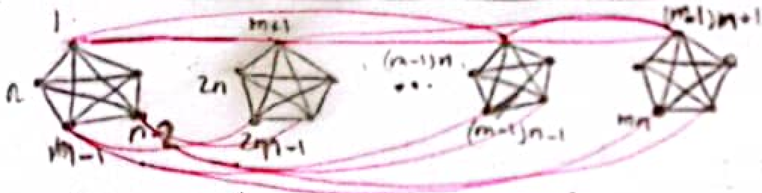
$\hat{y}_i(k) = \sqrt{n} \cdot \hat{y}_{i-1}(k) \cdot \mathcal{F}\{\delta_i\}(k) = n^{1/2} \cdot \hat{y}_{i-1}(k) \cdot \bar{u}_{ik} = n^{n/2}\, \hat{y}_0(k) \left(\prod_{i=1}^{n} \bar{u}_{ik} \right)$

$\langle y_n, \mathbb{1} \rangle \underset{\text{Parseval theorem}}{=} \langle \hat{y}_n, \hat{\mathbb{1}} \rangle = \langle \hat{y}_n, \underbrace{U^H \mathbb{1}}_{\delta_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}} \rangle = \hat{y}_n(0) = \cancel{n^{n/2}}\, \hat{y}_0(0) \Big(\underbrace{\prod_{i=1}^{n} \bar{u}_{i1}}_{\cancel{\left(\frac{1}{\sqrt{n}}\right)}} \Big) = \hat{y}_0(0) \quad (**)$

$\langle y_0, \mathbb{1} \rangle = \langle \hat{y}_0, \hat{\mathbb{1}} \rangle = \langle \hat{y}_0, U^H \mathbb{1} \rangle = \hat{y}_0(0) \quad (*)$

thus $(*)$, $(**)$ will result in $\boxed{\langle y_n, \mathbb{1}_{n\times 1} \rangle = \langle y_0, \mathbb{1}_{n\times 1} \rangle}$ Q.E.D. ■

:3 سؤال



$H = \underbrace{K_n}_{G_1} \otimes \underbrace{K_m}_{G_2}$

shifting the signal δ_k with respect to the l-th vertex in graph H would be $y = \delta_l \circledast \delta_k$. However, since the laplacian of graphs K_n, K_m are circular, we know that their Basis will be the DFT matrix of sizes $n \times n$, $m \times m$, respectively. Furthermore, we know for graphs H, G_1, G_2 such that $H = G_1 \otimes G_2$, we have:

$L_H = L_{G_1} \otimes I_m + I_n \otimes L_{G_2}$ and also $\beta_{ij} = \vec{\psi}_i \otimes \vec{\Phi}_j$ will be the eigenvector of H with eigenvalue $\gamma_i + \lambda_j$. $\left((\gamma_i, \vec{\psi}_i) \in \text{Eigen}(G_1),\ (\lambda_j, \vec{\Phi}_j) \in \text{Eigen}(G_2) \right)$

Thus $\underline{U} = \Psi \otimes \Phi$, where $\Psi = DFT(n)$, $DFT(m) = \Phi$

$\bullet\ \Lambda_H = \{0, \overbrace{n, n, \dots, n}^{\# n-1}, \overbrace{m, m, \dots, m}^{\# m-1}, \overbrace{m+n, m+n, \dots, m+n}^{\#(m-1)(n-1)}\}$

$\Lambda_{G_1} = \{0, \overbrace{n, n, \dots, n}^{\# n-1}\}$, $\Lambda_{G_2} = \{0, \overbrace{m, m, \dots, m}^{m-1}\}$

$\begin{cases} DFT^{(n)}(l,k) = \frac{1}{\sqrt{n}} e^{-j\frac{2\pi}{n}(l-1)(k-1)} \\ DFT^{(m)}(l,k) = \frac{1}{\sqrt{m}} e^{-j\frac{2\pi}{m}(l-1)(k-1)} \end{cases}$

Now since $\mathcal{F}\{\delta_k \circledast_G \delta_l\}(i) = (U^H\delta_k)(i)\,(U^H\delta_l)(i)$, we can say that:

$\underset{mn\times 1}{\vec{\eta}} = \mathcal{F}\{\delta_k \circledast_G \delta_l\} = (\underbrace{U^H \delta_k}_{u_k}) \odot (\underbrace{U^H \delta_l}_{u_l})$; $\vec{\eta}(a) = U_{ka} \cdot U_{la}$

$u_k(a) = U_{ka} = DFT^{(n)}(k_1, a_1)\, DFT^{(m)}(k_2, a_2) = \frac{1}{\sqrt{mn}} \exp\Big(\underbrace{-\frac{2\pi j}{n} k_1 a_1 - \frac{2\pi j}{m} k_2 a_2}_{\theta_{ak}} \Big)$

$\left\{ \begin{array}{ll} k_1 = \lfloor \frac{k}{n} \rfloor - 1 & k_2 = (k \bmod n) - 1 \\ a_1 = \lfloor \frac{a}{n} \rfloor - 1 & a_2 = (a \bmod n) - 1 \end{array} \right\} \rightarrow$ change of coordinates

$$\vec{\eta}(a) = U_{ka} U_{la} = u_k(a) u_l(a) = \frac{1}{mn} \exp(\theta_{ak} + \theta_{al})$$

$$\theta_{ak} + \theta_{al} = -\frac{2\pi j}{n} a_1 k_1 - \frac{2\pi j}{m} a_2 k_2 - \frac{2\pi j}{n} a_1 l_1 - \frac{2\pi j}{m} a_2 l_2$$

$$= -\frac{2\pi j}{n} a_1 (\underbrace{k_1 + l_1}_{t_1}) - \frac{2\pi j}{m} a_2 (\underbrace{k_2 + l_2}_{t_2}) = \theta_{at} + 2k\pi$$

therefore we can find $\forall k, l, \exists t$ such that $\theta_{ak} + \theta_{al} = \theta_{at} + 2k\pi$, we just have to put $0 \leq t_1 = k_1 + l_1 = \lfloor \frac{k}{n} \rfloor + \lfloor \frac{l}{n} \rfloor$ & $t_2 = k_2 + l_2 = (k + l \mod n) - 2 \geq 0$.

& we can find t from $t_1, t_2 \longrightarrow t = n t_1 + t_2$.

thus we got $\forall a \in \{1, \dots, mn\}, \exists t: \vec{\eta}(a) = \frac{1}{mn} \exp(\theta_{ak} + \theta_{al}) = \frac{1}{mn} \exp(\theta_{at})$.

So we can see that $\vec{\eta}$ as a vector is the t-th column of U, thus:

$$u_k \odot u_l = (U^H \delta_k) \odot (U^H \delta_l) = \frac{1}{\sqrt{mn}} U_t = \frac{1}{\sqrt{mn}} (U^H \delta_t).$$

$$\Rightarrow \delta_k \circledast_G \delta_l = \mathcal{F}^{-1}\{\mathcal{F}\{\delta_k \circledast_G \delta_l\}\} = \mathcal{F}^{-1}\{u_k \odot u_l\} = \mathcal{F}^{-1}\left\{\frac{1}{\sqrt{mn}} U^H \delta_t\right\}$$

$$= \frac{U U^H}{\sqrt{mn}} \delta_t = \boxed{\frac{1}{\sqrt{mn}} \delta_t}$$

Q.E.D ■

where $t = n\left(\lfloor \frac{k-1}{n} \rfloor + \lfloor \frac{l-1}{n} \rfloor \mod n\right) + \left[(l + k - 2) \mod n \mod mn\right]$

سوال ۴:

① $D = \begin{bmatrix} I_{N/M} & 0_{N/M} & \cdots & 0_{N/M} \end{bmatrix} \in \mathbb{C}^{\frac{N}{M} \times M}$ $\qquad (A^M)_{1,1} \in \mathbb{C}^{\frac{N}{M} \times \frac{N}{M}}$ first $\frac{N}{M}$ rows and cols

$$B^M = \begin{bmatrix} A^M_{1,1} & \Phi \\ A^M_{2,1} & A^M_{2,2} \end{bmatrix}$$

$$\bar{A} = D B^M D^T = \begin{bmatrix} I_{N/M} & \Phi \end{bmatrix} \begin{bmatrix} A^M_{1,1} & \Phi \\ A^M_{2,1} & A^M_{2,2} \end{bmatrix} \begin{bmatrix} I_{N/M} \\ \Phi \end{bmatrix} = A^M_{1,1}$$

$$D B^M = \begin{bmatrix} I_{N/M} & \Phi \end{bmatrix} \begin{bmatrix} A^M_{1,1} & \Phi \\ A^M_{2,1} & A^M_{2,2} \end{bmatrix} = \begin{bmatrix} A^M_{1,1} & 0 \end{bmatrix} = \bar{A} D$$

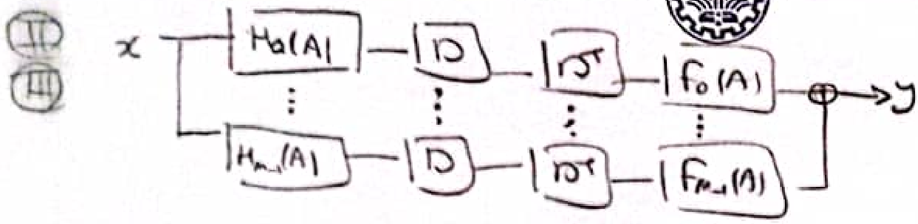
$D B^M = \bar{A} D \Rightarrow$
$D (B^M)^k = (D B^M) B^{M^{k-1}} = \bar{A} D (B^M)^{k-1}$

finally $\Rightarrow D (B^M)^k = \bar{A}^k D \longrightarrow D H(B^M) = D\left(\sum_{k=0}^{L} h_k (B^M)^k\right) = \sum_{k=0}^{L} h_k D (B^M)^k$

$$= \sum_{k=0}^{L} h_k \bar{A}^k D = \boxed{\left(\sum_{k=0}^{L} h_k \bar{A}^k\right) D = H(\bar{A}) D = D H(B^M)}$$

Q.E.D. ■

(II) (III)



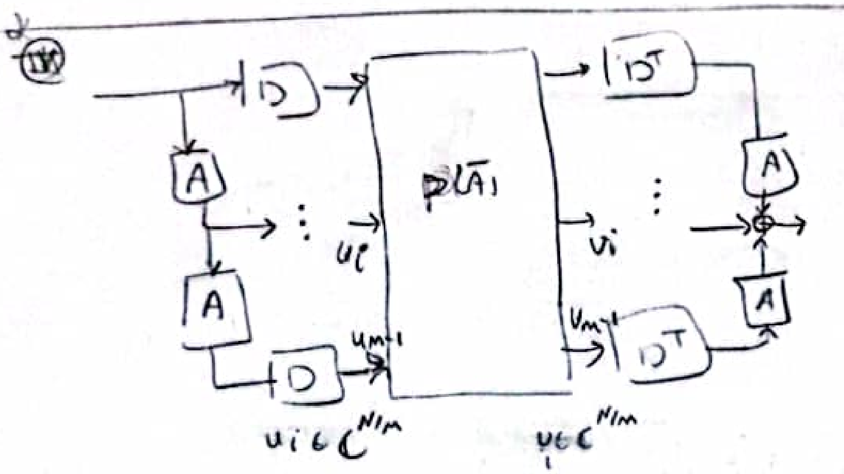
$$D^TDA^M = \begin{bmatrix} I_{N/M} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} I_{N/M} & 0 & \cdots & 0 \end{bmatrix} \begin{pmatrix} A^M_{1,1} & A^M_{1,2} \\ A^M_{2,1} & A^M_{2,2} \end{pmatrix} = \begin{bmatrix} A^M_{1,1} & A^M_{1,2} \\ 0 & 0 \end{bmatrix}$$

$$A^M D^T D = \begin{bmatrix} A^M_{1,1} & A^M_{1,2} \\ A^M_{2,1} & A^M_{2,2} \end{bmatrix} \begin{bmatrix} I_{N/M} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} I_{N/M} & 0 & \cdots & 0 \end{bmatrix} = \begin{pmatrix} A^M_{1,1} & 0 \\ A^M_{2,1} & 0 \end{pmatrix}$$

So $D^TDA^M = A^M D^T D \iff A^M_{2,1} = A^M_{1,2} = 0$

In general the described filter-Banks is <u>not GSI</u> and depends on whether $D^TDA^M = A^M D^T D$ is true or not, which would be true if and only if $A^M_{1,2} = A^M_{2,1} = 0$. Here we checked $SA^M = A^M S$ in general form, for standard GSI we can simply use $M=1$.

(IV)



$u_i = DA^i x$

$v_i = \sum_{j=0}^{M-1} P_{ij}(\bar{A})\, u_j$

$u = \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{M-1} \end{bmatrix}$

$\underbrace{P(\bar{A})}_{N\times N} u = v = \begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ \vdots \\ v_{M-1} \end{bmatrix}$

$P(A) = \begin{bmatrix} P_{ij}(A) \end{bmatrix} \in \mathbb{C}^{N\times N}$, $P_{ij}(A) \in \mathbb{C}^{N/M \times N/M}$

$$y = \sum_{i=0}^{M-1} A^{M-i-1} D^T v_i = \left[A^{M-1}D^T \,|\, A^{M-2}D^T \,|\, \cdots \,|\, A^1 D^T \,|\, A^0 D^T\right] \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{M-1} \end{bmatrix}$$

$$y = \left[A^{M-1} \cdots A^1 A^0\right] \operatorname{diag}(D^T)\, v = \left[A^{M-1} | \cdots | A^1 | A^0\right] \operatorname{diag}(D)^T v$$

$$\longrightarrow y = \left[A^{M-1} | \cdots | A^1 | A^0\right] \operatorname{diag}(D^T)\, P(\bar{A})\, \operatorname{diag}(D) \begin{bmatrix} A^0 \\ A^1 \\ \vdots \\ A^{M-1} \end{bmatrix} x$$

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جمع نمرات	۵	۴
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دانشگاه صنعتی شریف

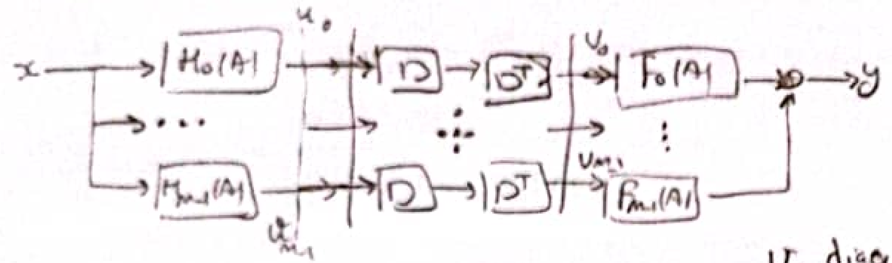
شماره دانشجویی:

نام و نام خانوادگی:

شماره صندلی: نام استاد:

شماره سوال:

the filter_Bank of Previous Section



$$\begin{cases} H_i(A) = \sum_{r=0}^{M-1} E_{ir}(A^M) \\ F_i(A) = \sum_{r=0}^{M-1} R_{ir}(A^M) \end{cases}$$

$V = \operatorname{diag}(D^TD)$; $y = [F_0(A) | F_1(A) | \cdots | F_{M-1}(y)]$

$$u = \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{M-1} \end{bmatrix} = \begin{bmatrix} H_0(A) \\ H_1(A) \\ \vdots \\ H_{M-1}(A) \end{bmatrix} x \longrightarrow y = [F_0(A) | \cdots | F_{M-1}(A)] \operatorname{diag}(D^TD) \begin{bmatrix} H_0(A) \\ \vdots \\ H_{M-1}(A) \end{bmatrix} x$$

$$⊗ \quad F_i(A) = [A^M | \cdots | A^0] \underbrace{\begin{bmatrix} R_{i,0}(A^M) \\ \vdots \\ R_{i,M-1}(A^M) \end{bmatrix}}_{R_i(A^M)} = [A^{M-1} | \cdots | A^0] \, R_i(A^M)$$

Likewise

$$** \quad H_i(A) = [E_{i,0}(A^M) | \cdots | E_{i,M-1}(A^M)] \begin{bmatrix} A^0 \\ \vdots \\ A^{M-1} \end{bmatrix} = E_i(A^M) \begin{bmatrix} A^0 \\ A^1 \\ \vdots \\ A^{M-1} \end{bmatrix}$$

$*$ Putting all together for all i

$$\begin{bmatrix} H_0(A) \\ H_1(A) \\ \vdots \\ H_{M-1}(A) \end{bmatrix} = \underbrace{\begin{bmatrix} E_0(A^M) \\ E_1(A^M) \\ \vdots \\ E_{M-1}(A^M) \end{bmatrix}}_{E(A^M)} \begin{bmatrix} A^0 \\ A^1 \\ \vdots \\ A^{M-1} \end{bmatrix} = \mathbb{E}(A^M) \begin{bmatrix} A^0 \\ A^1 \\ \vdots \\ A^{M-1} \end{bmatrix} \; ; \; E(A^M) = \begin{bmatrix} E_{0,0}(A^M) & \cdots & E_{0,M-1}(A^M) \\ \vdots & \ddots & \\ E_{M-1,0}(A^M) & \cdots & E_{M-1,M-1}(A^M) \end{bmatrix}$$

$**$ the same here

$$[F_0(A) | F_1(A) | \cdots | F_{M-1}(A)] = [A^{M-1} | \cdots | A^1 | A^0] \overbrace{[R_0(A^M) | \cdots | R_{M-1}(A^M)]}^{R^T(A^M)}$$

$A = A^T \longrightarrow A^M = (A^M)^T$ is the needed constraints in Noble

$$y=[A^{m-1}|\dots|A^1|A^0]\,R^T(A^m)\,\mathrm{diag}(D)^T\mathrm{diag}(D)\,E(A^m)\begin{bmatrix}A^0\\A^1\\\vdots\\A^{m-1}\end{bmatrix}x$$

if $A^m_{2,1}=\phi$ in

$$B^m=\begin{bmatrix}A^m_{1,1} & A^m_{1,2}\\ \cancel{A^m_{2,1}} & A^m_{2,2}\end{bmatrix}=A^m \longrightarrow DH(A^m)=H(\bar{A})D$$

$$DE_{in}(A^m)=E_{ir}(\bar{A})D \longrightarrow \mathrm{diag}(D)\,E(A^m)=E(\bar{A})\,\mathrm{diag}(D)$$

$$DR_{ir}(A^m)=R_{ir}(\bar{A})D \longrightarrow \mathrm{diag}(D)\,R(A^m)=R(\bar{A})\,\mathrm{diag}(D)$$

$$\longrightarrow R^T(A^m)\,\mathrm{diag}(D^T)=\mathrm{diag}(D)^T R^T(\bar{A})$$

$$\Rightarrow y=[A^{m-1}|\dots|A^1|A^0]\,\mathrm{diag}(D^T)\,\underbrace{R^T(\bar{A})E(\bar{A})}_{P(\bar{A})}\mathrm{diag}(D)\begin{bmatrix}A^0\\A^1\\\vdots\\A^{m-1}\end{bmatrix}x$$

$$\Rightarrow\begin{cases}A=A^T,\ P(A)=R^T(\bar{A})E(\bar{A})\\ P_{ij}(A)=\sum_{r=0}^{N-1}R_{ri}(\bar{A})E_{rj}(\bar{A})\end{cases}$$

Q.E.D. ■

سوال ۵:

(I) $$\min_{W\in W_n}\Big\{\underbrace{\|Z\odot W\|_{1,1}+\gamma\sigma\sum_{i,j}w_{ij}(\log w_{ij}-1)}_{L(Z,W)}\Big\}$$

$$\frac{\partial L}{\partial w_{ij}}=\frac{\partial}{\partial w_{ij}}\Big(\sum_{i,j}z_{ij}w_{ij}+\gamma\sigma\sum_{i,j}w_{ij}(\log w_{ij}-1)\Big)=z_{ij}+\gamma\sigma\log(w_{ij})=0$$

$$\rightarrow\ \boxed{w_{ij}=\exp\Big(\frac{-z_{ij}}{\gamma\sigma}\Big)=\exp\Big(-\frac{\|x_i-x_j\|^2}{\gamma\sigma}\Big)}$$

Q.E.D. ■

(II) let $W=F(Z,\alpha,\beta)$ be the solution to $\min_W\{\|Z\odot W\|_{1,1}-\alpha\mathbb{1}^T\log(W\mathbb{1})+\beta\|W\|_F^2\}$.

then we can say that $\gamma F(Z,\alpha,\beta)=\gamma W=\arg\min_W\{\gamma\sum_{i,j}w_{ij}z_{ij}-\alpha\mathbb{1}^T\log(W\mathbb{1}\gamma)+\sum_{i,j}2\beta\gamma^2w_{ij}^2\}$

$$=\arg\min_W\{\gamma\sum_{i,j}w_{ij}z_{ij}-\alpha\mathbb{1}^T\log(W\mathbb{1})-n\gamma\alpha+\sum_{i,j}2\beta^2\gamma^2w_{ij}^2\}\quad(\text{assumption } \gamma>0,\ \alpha>0)$$

$$=\arg\min_W\{\sum_{i,j}w_{ij}z_{ij}-\frac{\alpha}{\gamma}\mathbb{1}^T\log(W\mathbb{1})+\sum_{i,j}2\beta\gamma w_{ij}\}=F(Z,\frac{\alpha}{\gamma},\beta\gamma)$$

thus: $W=F(Z,\alpha,\beta)=F(Z,\frac{\alpha}{\gamma},\beta\gamma)$

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using the same technique we get:

$$W = F(Z,\alpha,\beta) = \arg\min_W \left\{ \|W \circ Z\|_{1,1} - \alpha \mathbb{1}^T \log(W\mathbb{1}) + \beta \|W\|_F^2 \right\}$$

$$= \alpha \arg\min_W \left\{ \alpha \sum_{i,j} w_{ij} z_{ij} - \alpha \mathbb{1}^T \log(W\mathbb{1}) - \underbrace{n\alpha \log\alpha}_{cte} + \beta \sum_{ij} \alpha^2 w_{ij}^2 \right\} \quad \Big\} \times 1/\alpha$$

$$= \alpha \cdot \arg\min_W \left\{ \|W \circ Z\|_{1,1} - \mathbb{1}^T \log(W\mathbb{1}) + \beta\alpha \underbrace{\sum_{ij} w_{ij}^2}_{\|W\|_F^2} \right\} = \alpha F(Z,1,\alpha\beta)$$

at last $\Rightarrow$ $\boxed{W = F(Z,\alpha,\beta) = \gamma F(Z, \frac{\alpha}{\gamma}, \gamma\beta) = \alpha F(Z, 1, \beta\alpha)}$ Q.E.D. ■

(III)

$$J(w) = \sum_{ij} z_{ij} w_{ij} - \alpha \sum_i \log\left(\sum_j w_{ij}\right) + \beta \sum_{ij} w_{ij}^2$$

$$\rightarrow \frac{\partial J}{\partial w_{ij}} = 2z_{ij} - \alpha \underbrace{\left(\frac{1}{\sum_k w_{ik}} + \frac{1}{\sum_k w_{kj}} \right)}_{A} + 4\beta w_{ij} = 0$$

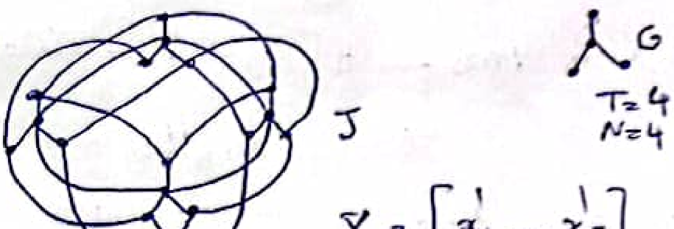
in order to get maximum w_{ij}, i,j must have

let $A \le \frac{2\alpha}{w_{ij}} \Rightarrow 4\beta w_{ij} = A \le \frac{2\alpha}{w_{ij}} \rightarrow w_{ij} \le \sqrt{\frac{\alpha}{\beta}}$ Q.E.D. ■

: 6 سؤال

(I) It was obvious that Graph J was constructed by $G \times T$.

since in graph J we put vec(X) on graph J instead of x_i on graph G. the mean condition does not matter since $\forall t \; \exists c \; E[x_t] = c\mathbb{1}_N$.



$$X = \begin{bmatrix} | & & | \\ x_1 & \cdots & x_T \\ | & & | \end{bmatrix}$$

So its the moment condition, therefore there must be a ~~vector~~ Diagonal matrix D such that $\Sigma = U_J D U_J^H$ so that Σ and $\mathcal{L} = U_J \Lambda_J U_J^H$ can be commute: $\Sigma \mathcal{L} = \mathcal{L} \Sigma$.

since T is a cyclic graph and has a toeplitz Laplacian matrix, the $DFT(N)$ matrix will be its basis $\rightarrow U_T = DFT(N)$, let $\omega_\tau = \frac{2\pi(\tau-1)}{T} = \Omega(\tau,\tau)$, $\Lambda_G(n,n) = \lambda_n$

III) $U_J = \underbrace{U_G}_{\in \mathbb{C}^{N\times N}} \otimes \underbrace{U_T}_{\in \mathbb{C}^{T\times T}}$, $\Sigma = h(L_G, L_T) = U_J \underbrace{h(\Lambda_G, \Omega_T)}_{\in \mathbb{R}^{NT\times NT}} U_J^H = \text{cov}(\text{vec}(X))$

$$\underbrace{H_{t_1,t_2}}_{\in \mathbb{R}^{N\times N}} = \Sigma_{N(t_1-1):Nt_1,\, N(t_2-1):Nt_2} = \sum_{\tau=1}^{T} U_J(t_1,\tau)\, \underbrace{h(\Lambda_G, \Omega_T)_{\tau\tau}}_{\in \mathbb{R}^{N\times N}}\, U_J^H(\tau, t_2)$$

is block matrix

$$= \sum_{\tau=1}^{T} U_G\, U_T(t_1,\tau)\, h_{\tau\tau}(\cdot,\cdot)\, \overline{U_T(\tau,t_2)}\, U_G^H$$

$$= \frac{1}{\sqrt{T}}\frac{1}{\sqrt{T}} \sum_{\tau=1}^{T} e^{-j\omega_\tau t_2}\, U_G\, h_{\tau\tau}(\cdot,\cdot)\, U_G^H\, e^{j\omega_\tau t_1}$$

Q.E.D. ■

$$= \frac{1}{T}\sum_{\tau=1}^{T} e^{+j\omega_\tau(t_1-t_2)} \underbrace{U_G\, h_{\tau\tau}(\Lambda_G, \Omega_T)\, U_G^H}_{h_{\omega_\tau}(L_G)} = \boxed{\frac{1}{T}\sum_{\tau=1}^{T} e^{j\omega_\tau(t_1-t_2)}\, h_{\omega_\tau}(L_G)}$$

so we can see that $H_{t_1,t_2} = f(t_1-t_2)$ and doesn't rely on t_1 and t_2.

III) (i) X is JWSS $\rightarrow \mathbb{E}[X] = c\,\mathbb{1}_{N\times T}$ (*)

$\hookrightarrow H_{t_1,t_2} = f(t_1-t_2)$ (**)

$\Rightarrow \boxed{\text{TWSS} \subseteq \text{JWSS}}$

X is TWSS $\rightarrow \mathbb{E}[X] = c\,\mathbb{1}_{N\times T}$ $\rightarrow$ already is true since (*) due to JWSS.

$\hookrightarrow \begin{cases} H_{t_1,t_2} = f(t_1-t_2+1-1) = f(t_1-t_2) = f(t_1-t_2+\cancel{1}-\cancel{1}) \\ t_2 = 1 \\ t_1 = t_1+1 \end{cases}$ $\rightarrow$ already is true since (**) due to JWSS

(ii) X is VWSS $\rightarrow \mathbb{E}[X] = c\,\mathbb{1}_{N\times T}$ $\rightarrow$ already being satisfied due to (*), JWSS.

$\hookrightarrow \Sigma_{t_1 t_2} = \gamma_{t_1,t_2}(L_G)$ $\Longrightarrow$ $\boxed{\text{VWSS} \subseteq \text{JWSS}}$

It follows from the definition that for a JWSS process, each block of Σ has to be a linear graph filter $\Sigma_{t_1,t_2} = \gamma_{t_1,t_2}(L_G)$. Hence the Covariance matrix can be written as:

$$\Sigma = \begin{bmatrix} \gamma_{1,1}(L_G) & \gamma_{1,2}(L_G) & \cdots & \gamma_{1,T}(L_G) \\ \gamma_{2,1}(L_G) & \gamma_{2,2}(L_G) & \cdots & \vdots \\ \vdots & & \ddots & \vdots \\ \gamma_{T,1}(L_G) & \cdots & & \gamma_{T,T}(L_G) \end{bmatrix}$$

Thus if JWSS holds, automatically VWSS and TWSS will hold as well since $(\text{JWSS} \supseteq \text{TWSS})$ & $(\text{JWSS} \supseteq \text{VWSS})$. Q.E.D. ■

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ادامه سوال ۶:

(IV) JWSS ✓, $\forall n, t: \lambda_n \neq 0, \omega_t \neq 0$.

By construction of the JFT basis $\hat{X}(0,0)$ captures the DC-offset of a signal, and condition $\left(\mathbb{E}[\hat{X}(n,\tau)] = 0 \text{ if } \begin{smallmatrix} \lambda_n \neq 0 \\ \omega_t \neq 0 \end{smallmatrix} \right)$ is equivalent to $\mathbb{E}[X] = c\mathbb{1}_{NT}$.

Moreover, if the graph is connected and (a) holds, at least one of $\mathbb{E}[\hat{X}(n_1,\tau_1)]$ and $\mathbb{E}[\hat{X}(n_2,\tau_2)]$ must be zero when $n_1 \neq n_2$ or $\tau_1 \neq \tau_2$ and

hence:

$$\mathbb{E}\left[\hat{X}(n_1,\tau_1)\hat{X}(n_2,\tau_2)\right] = \mathbb{E}\left[\hat{X}(n_1,\tau_1)\hat{X}(n_2,\tau_2)\right] - \mathbb{E}\left[\hat{X}(n_1,\tau_1)\right]\mathbb{E}\left[\hat{X}(n_2,\tau_2)\right]$$

$$= \left[U_J^H \Sigma U_J\right]\left(N(\tau_1-1)+n_1, N(\tau_2-1)+n_2\right)$$

thus, the second condition is equivalent to stating that $\Sigma = U_J D U_J^H$ for some diagonal matrix. Now if such D exists we will have

$$\Sigma = U_J D U_J^H = U_J h(\Lambda_G, \Omega_T) U_J^H = h(L_G, L_T),$$ which is the

second moment condition of JWSS Process. so we're done!

Q.E.D. ■