محمدپارسا دینی 400101204 GSP-HW2 تمرین سری 2 پردازش سیگنال های گرافی

:10

(I) We know that the variational form of $\lambda_{max}^{(M)} = \sup_{x:\|x\|=1} \{x^T M x\}$.

If $\forall i \in (n): u(i) \geq 0$. Let $\exists_{?} j: u(j) \neq 0, u(i) = 0$: $(M_{ik} \geq 0, u(k) \geq 0)$

$$(Mu)_{(i)} = \mu\, u(i) \rightarrow \sum_{k=1}^{n} M_{ik}\, u(k) = \mu \cdot u(i) = 0 \quad \dot{\times}$$

Since $\forall k: M_{ik} \geq 0, u(k) \geq 0$ & $\exists j: u(j) \neq 0$ the running-sum would not have the chance to get zero which is a paradox. thus:

if $\forall i \in (n): u(i) \geq 0 \rightarrow \nexists j: u(j) \neq 0$ Q.E.D. ■

which suggests that: $\boxed{\forall i \in [n]: u(i) > 0 \leftarrow \forall i \in [n]: u(i) \geq 0}$

(II) Since $\mu_1 = u_1^T M u_1 \geq y^T M y$. Let $y_{(j)} = |u_1^{(j)}|\ \forall j$: Thus:

$$y^T M y = \sum_{i,j} y_i y_j M_{ij} = \sum_{i,j} |u_1^{(j)}| \cdot |u_1^{(i)}| M_{ij} \geq \sum_{i,j} u_1^{(j)} u_1^{(i)} M_{ij} = \overset{=\mu_1}{u_1^T M u_1}$$

which suggests $(y^T M y \leq \mu_1) \wedge (y^T M y \geq \mu_1) \longrightarrow y^T M y = \mu_1$

thus: $y^T M y = u_1^T M u_1 = \mu_1$ & since $\mu_1 = \sup_{\|z\|=1} \{z^T M z\} = u_1^T M u_1$

therefore $y = |u_1| \underset{+}{=} u_1$, which proves that: $\forall j: u_1(j) = |u_1(j)|$

$\longrightarrow \boxed{\forall j: u_1(j) \geq 0}$ Q.E.D. ■ $\longrightarrow$ from (I) $\rightarrow \boxed{\forall j: u_1(j) > 0}$

(III) let $y(i) = |u_n(i)|\ \forall i \in (n)$. Now we have:

$$\mu_1 = \sup_{\|z\|=1} \{z^T M z\} \geq y^T M y = \sum_{i,j} |u_n^{(i)}|\, |u_n^{(j)}|\, M_{ij} \overset{\text{Triangle Inequality}}{\geq} \left| \sum_{i,j} u_n^{(i)} u_n^{(j)} M_{ij} \right| = |\mu_n|$$

Thus: $\boxed{\mu_1 \geq |\mu_n| \geq -\mu_n}$ Q.E.D. ■

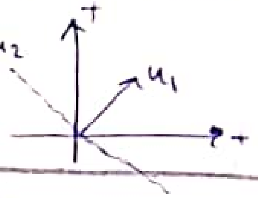
(IV) We know that for symmetric matrices $M = M^T$, if $\mu_i \neq \mu_j \longrightarrow u_i \perp u_j$.

Even if $\mu_i = \mu_j \longrightarrow u_i \perp u_j$.

$u_1^T u_2 = 0 \longrightarrow \sum_{i=1}^{n} u_1^{(i)} u_2^{(i)} = 0$. Now, since $\forall i: u_1(i) > 0$. Therefore, u_2 must have positive & negative elements:

$$\boxed{\exists\, i, j \in [n]: (u_2(i) > 0) \wedge (u_2(j) < 0)}$$ Q.E.D. ■

we can see this in other words as well. since all elements of the first eigenvector are positive, and the fact that $u_2 \perp u_1$; u_2 must have negative elements as it's impossible for u_2 to have positive elements and be perpendicular to u_1.



Ⅴ Again let $\forall i \in [n]: y(i) = |u_2(i)|$, then $\{M_{ij} \geq 0\}$:

$$\mu_2 = u_2^T M u_2 = \sum_{i,j} u_2^{(i)} u_2^{(j)} M_{ij} \leq \sum_{i,j} \underbrace{|u_2^{(i)}|}_{y(i)}\underbrace{|u_2^{(j)}|}_{y(j)} M_{ij} = y^T M y \leq u_1^T M u_1 = \mu_1 .$$

Now since u_2 has negative elements as well, we can deduce that:

$u_2^T M u_2 < y^T M y$ which suggests $\mu_2 = u_2^T M u_2 < y^T M y \leq u_1^T M u_1 = \mu_1$.

$\boxed{\text{Thus: } \mu_2 < \mu_1}$ Q.E.D. ■

Ⅵ G is connected and $\mu_1 = -\mu_n$, then we have: (let $\forall i : |u_n(i)| = y(i)$)

$$\mu_1 = -\mu_n = u_1^T M u_1 = -u_n^T M u_n \xrightarrow{\text{Claim:}} \sum_{i,j} M_{ij}\left(u_n^{(i)} u_n^{(j)} + |u_n^{(i)} u_n^{(j)}|\right) = 0 \quad (*)$$

Proof:

$\rightarrow \mu_1 = u_1^T M u_1 \geq y^T M y \geq -u_n^T M u_n = -\mu_n$. So all the inequalities must be the equal case.

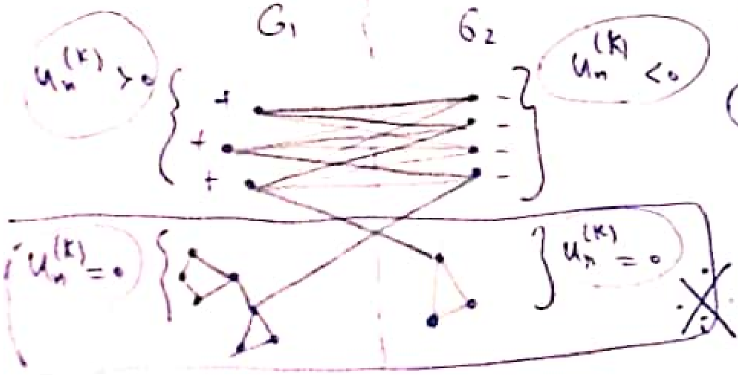
(*) → Since $M_{ij} \geq 0$, there are two cases:

if $M_{ij} \neq 0$, then $\underbrace{u_n^{(i)} u_n^{(j)} + |u_n^{(i)} u_n^{(j)}| = 0}_{\hookrightarrow\ u_n^{(i)} u_n^{(j)} < 0}$, otherwise; we don't know anything about $u_n^{(i)}, u_n^{(j)}$.

thus, if $\forall i,j: u_n^{(i)} u_n^{(j)} < 0$, then (i,j) are $\overset{(M_{ij}=0)}{\text{disconnected}}$. otherwise, they could be connected or not. so the graph G, is bipartite. Since we can label each node k with $u_n^{(k)} > 0$ to subgraph G_1 and with $u_n^{(k)} < 0$ to the other subgraph G_2.

$$\boxed{(M_{ij} = 1) \Leftrightarrow u_n^{(i)} u_n^{(j)} < 0}$$

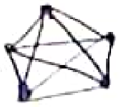
Q.E.D. ■



(*) $u_n(k) = 0$ does not happen since for $\mu_1 = -\mu_n$ we must have: $\forall i\ u_1(i) = |u_n(i)|$ which proves $u_n(k) = 0$ doesn't happen.

سوال 2:

Ⓘ K_n



$$\rightarrow \mathcal{L} = \begin{bmatrix} n-1 & -1 & -1 & \cdots & -1 \\ -1 & n-1 & -1 & \cdots & -1 \\ -1 & -1 & n-1 & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & n-1 \end{bmatrix} = nI - \mathbb{1}_{n\times n}$$

$\det(\lambda I - \mathcal{L}) = \det(-nI + \mathbb{1}_{n\times n} + \lambda I) = \det((\lambda - n)I + \mathbb{1}_{n\times n})$. Let $\Lambda_{\mathcal{L}}, \Lambda_H$ be the set of eigenvalues of $\mathcal{L}$, $-\mathbb{1}_{n\times n}$, respectively.

$$\Lambda_L = \{\lambda : \det(-\mathcal{L} + \lambda I) = 0\} = \{\lambda : \det((\lambda-n)I - (-\mathbb{1}_{n\times n}))\} = \{\lambda + n : \det(\lambda I - (-\mathbb{1}_{n\times n})) = 0\}$$

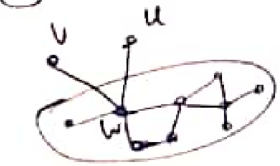
$$\rightarrow \Lambda_L = n + \{\lambda : \det(\lambda I - \{-\mathbb{1}_{n\times n}\}) = 0\} = n + \Lambda_H.$$

However, the matrix $-\mathbb{1}_{n\times n}$ has $(n-1)$ zero eigenvalues since $\operatorname{rank}(-\mathbb{1}_{n\times n}) = 1$.

Let $v = \begin{bmatrix} +1 \\ +1 \\ \vdots \\ +1 \end{bmatrix} \rightarrow (-\mathbb{1}_{n\times n})v = \begin{bmatrix} -n \\ -n \\ \vdots \\ -n \end{bmatrix} = -nv \rightarrow \lambda = n \rightarrow \Lambda_H = \{\underbrace{0, 0, \cdots, 0}_{n-1}, n\}$

which suggest $\Lambda_L = \{\underbrace{n, n, \cdots, n}_{n-1}, 0\}$.

Ⓘⓘ we have nodes u, v with degree 1, that have a common neighbor w.



$$\mathcal{L} = \begin{array}{c} u \\ v \\ w \\ \\ \\ \end{array} \left[\begin{array}{cccccccccccc} 0&0&0&1&0&0&0&0&0&-1&0&0 \\ 0&0&0&0&0&0&1&0&0&-1&0&0 \\ &&&-1&&&-1&&&m&& \\ &&&0&&&0&&&\vdots&& \\ &&&\vdots&&&\vdots&&&&& \end{array}\right] \\ \qquad\qquad\quad u \qquad\quad v \qquad\quad w$$

we can see that for the matrix $\mathcal{L}$, the vector:

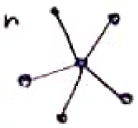
$$h = [0, 0, 0, \cdots, \overset{u\text{-th}}{1}, 0, \cdots, 0, \overset{v\text{-th}}{1}, 0, \cdots, 0]$$

is an eigenvalue with $\lambda = 1$.

$\mathcal{L}h = h \rightarrow \lambda = 1$

the other eigenvalues will be the basis of $\mathbb{R}^n \setminus \operatorname{span}(h)$. $\rightarrow \{\psi_1, \cdots, \psi_{n-1}\} \xrightarrow{\text{base}} \mathbb{R}^n \setminus \operatorname{span}(h)$

Ⓘⓘⓘ S_n

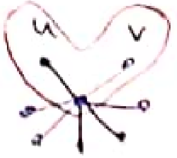


$$\mathcal{L} = \begin{bmatrix} n-1 & -1 & -1 & \cdots & -1 \\ -1 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & 1 \end{bmatrix} = I + \overbrace{\begin{bmatrix} n-2 & -1 & \cdots & -1 \\ -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & \cdots & 0 \end{bmatrix}}^{A} \quad \begin{cases} \Lambda_{\mathcal{L}} = 1 + \Lambda_A \\ \text{Eigen}(\mathcal{L}) = \text{Eigen}(A) \end{cases}$$

since the $\operatorname{rank}(A) = 2$, then A has $n-2$ zero eigenvalues. $\mathcal{L}$ has one $\overset{\text{zero}}{\text{eigenvalue}}$, which suggests $\Lambda_A \ni \{-1\}$. $\operatorname{Tr}\{A\} = \sum_{\lambda \in \Lambda_L} \lambda = \underbrace{0 + 0 + \cdots + 0}_{n-2} + (-1) + \lambda_{max} = n-2$.

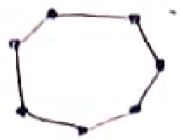
$\lambda_{max} = (n-1) \rightarrow \Lambda_A = \{\underbrace{0, 0, \cdots, 0}_{n-2}, -1, n-1\} \rightarrow \Lambda_L = \{\underbrace{1, 1, \cdots, 1}_{n-2}, 0, n\}$.

Now using the result of the previous section, we can let u,v as below, and have these eigenvectors:



$$u_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, u_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \cdots, u_{n-1} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ -1 \end{bmatrix}, \quad u_n = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix}$$

$$u_1 = \begin{bmatrix} \frac{n(1-n)}{n^2-n-1} \\ \frac{n^2}{n^2-n-1} \\ \vdots \\ \frac{n^2}{n^2-n-1} \end{bmatrix} \quad \lambda_2 = 1, \quad \lambda_3 = 1, \ldots, \lambda_{n-1} = 1, \quad \lambda_n = 0$$



$$\mathcal{L} = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ -1 & \cdots & \cdots & -1 & 2 \end{bmatrix}$$

(Fv)

$0 \leq i < n$

$$\lambda_i \in \mathrm{DFT}([2,-1,0 \ldots, -1]) \rightarrow \lambda_i = 2 + e^{-j\frac{2\pi}{n}i} + e^{-j\frac{2\pi}{n}i(n-1)}$$

Q.E.D. ■

$$\rightarrow \lambda_i = 2 + e^{-j\frac{2\pi}{n}i} + e^{+j\frac{2\pi}{n}i} = 2 + 2\cos\left(\frac{\pi}{n}i\right) = 2\left(1 - \cos\left(\frac{2\pi i}{n}\right)\right) \forall i \in \{0, \ldots, n-1\}$$

the set of eigenvalues are $\left\{u_K : u_K = \begin{bmatrix} e^{-j\frac{2\pi}{n}K(1-1)} \\ e^{-j\frac{2\pi}{n}K(2-1)} \\ \vdots \\ e^{-j\frac{2\pi}{n}K(n-1)} \end{bmatrix}\right\}$ since $e^{j\theta} = \cos\theta + j\sin\theta$:

$$u_K = \mathrm{Re}\{u_K\} + j\,\mathrm{Im}\{u_K\}$$

$$= \begin{bmatrix} \vdots \\ \cos\left(\frac{2\pi}{n}K(i-1)\right) \\ \vdots \end{bmatrix} + j\begin{bmatrix} \vdots \\ \sin\left(\frac{2\pi}{n}K(i-1)\right) \\ \vdots \end{bmatrix}$$

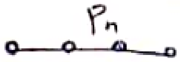
since $\lambda_i \in \mathbb{R} \rightarrow \mathrm{Re}\{u_K\}, \mathrm{Im}\{u_K\} \in \mathrm{Eigen}(\mathcal{L})$

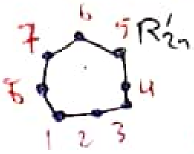
So each of the set of vectors above are eigenvalues. ~~furthermore since~~ ~~cos[illegible] sinθ~~ thus $x_K(i) = \cos\left(\frac{2\pi}{n}Ki\right)$, $y_K(i) = \sin\left(\frac{2\pi}{n}Ki\right)$ are the eigenvectors for the same λ.

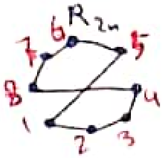
Q.E.D. ■

سؤال
مسئله 2 :

Ⓥ







$$\mathcal{L}'(R_{2n}) = \left[\begin{array}{cccc|cccc} 2 & -1 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{array}\right]$$

$$\mathcal{L}(P_n) = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

$$\mathcal{L}(R_{2n}) = \left[\begin{array}{cccc|cccc} & & & & \multicolumn{4}{c}{\overbrace{\hphantom{-1\;0\;0\;0}}^{A}} \\ 2 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 & 0 & -1 \\ \hline -1 & 0 & 0 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1 & 2 \\ \multicolumn{4}{c}{\underbrace{\hphantom{-1\;0\;0\;0}}_{A}} & \multicolumn{4}{c}{\underbrace{\hphantom{-1\;0\;0\;0}}_{\mathcal{L}(P_n)-A}} \end{array}\right] = \mathcal{L}(P_n) \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + A \otimes \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\boxed{[I_n \mid I_n]\, \mathcal{L}(R_{2n}) \begin{bmatrix} I_n \\ I_n \end{bmatrix} = 2\mathcal{L}(P_n)}$$

Q.E.D ■

$$[I_n \mid I_n] \left[\begin{array}{c|c} \mathcal{L}(P_n)-A & A \\ \hline A & \mathcal{L}(P_n)-A \end{array}\right] \left[\begin{array}{c} I_n \\ \hline I_n \end{array}\right] = [I_n \mid I_n] \begin{bmatrix} \mathcal{L}(P_n) \\ \mathcal{L}(P_n) \end{bmatrix} = \mathcal{L}(P_n) + \mathcal{L}(P_n) = 2\mathcal{L}(P_n)$$

let $\psi = \begin{bmatrix} \phi \\ \phi \end{bmatrix} = \phi \otimes \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ where $\phi \in \text{Eigen}(\mathcal{L}(P_n))$: $\mathcal{L}(P_n)\phi = \lambda\phi$:

$$\phi^T [I_n \mid I_n]\, \mathcal{L}(R_{2n}) \begin{bmatrix} I_n \\ I_n \end{bmatrix} \phi = \psi^T \mathcal{L} \psi = 2\phi^T \mathcal{L}(P_n)\phi = 2\lambda\|\phi\|^2 = \lambda\|\psi\|^2$$

$\longrightarrow \psi^T \mathcal{L} \psi = \lambda\|\psi\|^2$ shows that :

Q.E.D ■

$$\boxed{\forall \lambda \in \Lambda_{P_n}, \phi \in \text{Eigen}(P_n, \lambda) : \left(\psi^T \mathcal{L} \psi = \lambda\|\psi\|^2\right) \equiv \left\{\lambda, \psi = \begin{pmatrix} \phi \\ \phi \end{pmatrix}\right\} \in \Lambda_{\mathcal{L}}, \text{Eigen}(\mathcal{L}_{R_{2n}})}$$

(we could also do this $\phi[I_n \mid I_n]\, \mathcal{L}(R_{2n}) \begin{bmatrix} I_n \\ I_n \end{bmatrix} \phi = [I_n \mid I_n]\, \mathcal{L}(R_{2n}) \psi = \lambda\|\psi\|^2$
to make sure λ, ψ are (eigen(vectors, eigenvalue) pair.)

$$v_j(R_{2n}) = \begin{bmatrix} \omega^{0j} \\ \omega^{1j} \\ \vdots \\ \omega^{nj} \\ \vdots \\ \omega^{(2n-1)j} \end{bmatrix} = \begin{pmatrix} \phi_j \\ \omega^{nj}\phi_j \end{pmatrix} = \begin{pmatrix} \phi_j \\ \phi_j \end{pmatrix} \text{ if } \omega^{nj} = 1 \rightarrow j \overset{2}{\equiv} 0$$

$\omega = e^{j\frac{2\pi}{n}}$

Let $\Phi_n = \begin{bmatrix} \omega \\ \vdots \end{bmatrix} \longrightarrow$ so the eigenvectors of R_{2n} with even indices are of the form $\begin{pmatrix} \phi \\ \phi \end{pmatrix}$. for these eigenvectors $\boxed{\lambda_k = 2(1 - \cos(\frac{2\pi}{n}k)), 0 \leq k < n}$

Furthermore: $v_j(R_{2n}) \in \text{Eigen}(R_{2n}), j \overset{2}{\equiv} 0$ then:

$$\begin{cases} \text{Re}\{v_j(R_{2n})\} \in \text{Eigen}(R_{2n}) \\ \text{Im}\{v_j(R_{2n})\} \in \text{Eigen}(R_{2n}) \end{cases}$$

Q.E.D. ■

which suggests that for $\begin{cases} x_k(i) = \cos(\frac{2\pi}{n}ki) \\ y_k(i) = \sin(\frac{2\pi}{n}ki) \end{cases}$ we have $\begin{cases} \begin{pmatrix} x_k \\ x_k \end{pmatrix}, \begin{pmatrix} y_k \\ y_k \end{pmatrix}. \\ \in \text{Eigen}(R_{2n}) \end{cases}$

Note that we only found n (eigenvector, eigenvalue) for $\mathcal{L}(R_{2n})$.

$$\forall j \overset{2}{\equiv} 0 \longrightarrow \begin{cases} v_j = \begin{pmatrix} x_{j/2} \\ x_{j/2} \end{pmatrix} \\ \lambda_j = 2(1 - \cos(\frac{\pi j}{n})) \end{cases} \quad \begin{cases} v_j = \begin{pmatrix} y_{j/2} \\ y_{j/2} \end{pmatrix} \\ \lambda_j = 2(1 - \cos(\frac{\pi j}{n})) \end{cases} \quad \begin{cases} v_j = \begin{bmatrix} \omega^{0j} \\ \omega^{1j} \\ \vdots \\ \omega^{(2n-1)j} \end{bmatrix} \\ \lambda_j = 2(1 - \cos(\frac{\pi j}{n})) \end{cases}$$

(I) Suppose we have graphs $G(n, V_1, B_1)$ & $H(m, V_2, B_2)$. then we know :30/[illegible] that for their Cartesian Product $S = G \times H$, we have $A_S = A_G \otimes I_m + I_n \otimes A_H$ and also $D_S = D_G \otimes I_m + I_n \otimes D_S$. therefore, we get:

$$\mathcal{L}_S = D_S - A_S = (\overbrace{D_G - A_G}^{\mathcal{L}_G}) \otimes I_m + I_n \otimes (\overbrace{D_H - A_H}^{\mathcal{L}_H}) = \mathcal{L}_G \otimes I_m + I_n \otimes \mathcal{L}_H .$$

Now we want to show that $\beta_{ij}(a,b) = \psi_i(a)\phi_j(b)$ is an eigenvector associated with the eigenvalue $\alpha(i,j) = \lambda_i + \gamma_j$ where $1 \le i \le n$, $1 \le j \le m$. We can easily see that ➡ $\forall a,b: \beta_{ij}(a,b) = \psi_i(a)\phi_j(b) \xrightarrow[\beta_{ij} \in \mathbb{R}^{mn}]{} \beta_{ij} = \psi_i \otimes \phi_j$. So we prove our claim now:

$$\mathcal{L}_S \beta_{ij} = (\mathcal{L}_G \otimes I_m + I_n \otimes \mathcal{L}_H)(\psi_i \otimes \phi_j) = (\mathcal{L}_G \otimes I_m)(\psi_i \otimes \phi_j) + (I_n \otimes \mathcal{L}_H)(\psi_i \otimes \phi_j)$$

$$= (\mathcal{L}_G \psi_i) \otimes (I_m \phi_j) + (I_n \psi_i) \otimes (\mathcal{L}_H \phi_j)$$

Q.E.D ■

$$= \lambda_i (\psi_i \otimes \phi_j) + \gamma_j (\psi_i \otimes \phi_j) = (\lambda_i + \gamma_j)(\psi_i \otimes \phi_j) = (\lambda_i + \gamma_j)\beta_{ij}.$$

Now that $\forall i \in [n], j \in [m]$: $\mathcal{L}_S \beta_{ij} = (\lambda_i + \gamma_j)\beta_{ij}$, we can see that the vector β_{ij} are eigenvalues of $\mathcal{L}_S$ with the eigenvalues $(\lambda_i + \gamma_j)$ $\forall i,j$.

(II) First, we notice that: $(\hat{a}_{i,j} = A_H(i,j))$, $(e_{ij} = A_S(i,j))$

$$\{(\hat{a}_{i,j} = 1) \wedge (\hat{a}_{i',j'} = 1)\} \Leftrightarrow \{e_{ij} = 1\}, \quad \{(\hat{a}_{i,j'} = 1) \wedge (\hat{a}_{i',j'} = 1)\} \Leftrightarrow \{e_{ij} = -1\}$$

thus, we can see that if we decompose A_S into $A_S^{(+)}, A_S^{(-)}$ such that: $A_S = A_S^{(+)} - A_S^{(-)}$, then we could see this equivalency: $\left(\begin{array}{l} \forall i,j: \\ A_S^{(+)}(i,j) \ge 0 \\ A_S^{(-)}(i,j) \ge 0 \end{array} \right)$

$$(\hat{a}_{i,j} = 1) \wedge (\hat{a}_{i',j'} = 1) \equiv \{A_S^{(+)}(i,j) = 1\} \overset{\equiv}{\Leftrightarrow} \{e_{ij} = 1\}$$

$$(\hat{a}_{i,j'} = 1) \wedge (\hat{a}_{i',j} = 1) \equiv \{A_S^{(-)}(i,j) = 1\} \equiv \{e_{ij} = -1\}$$

thus, we can see A_H as below: Since $A_H(i,j) = \hat{a}(i,j)$ is 1 whenever $A_S^{(+)}(i,j)$ is 1 and visaversa. As you can see, we can write A_H as block matrices of $A_S^{(+)}, A_S^{(-)}$.

$$A_H = \begin{array}{c|c|c} & 1\ 2 \cdots n & 1'\ 2' \cdots n' \\ \hline 1, 2, \vdots, n & A_S^{(+)} & A_S^{(-)} \\ \hline 1', 2', \vdots, n' & A_S^{(-)} & A_S^{(+)} \end{array}$$

Furthermore, we can obtain $A_S^{(+)}, A_S^{(-)}$ in terms of A_G, A_S:

$$A_S^{(+)} = \frac{1}{2}(A_S + A_G) \quad , \quad A_S^{(-)} = \frac{1}{2}(A_G - A_S) .. \qquad (*)$$

We can also write A_H as below:

$$A_H = \left[\begin{array}{c|c} A_S^{(+)} & A_S^{(-)} \\ \hline A_S^{(-)} & A_S^{(+)} \end{array}\right] = A_S^{(+)} \otimes \begin{bmatrix}1&0\\0&1\end{bmatrix} + A_S^{(-)} \otimes \begin{bmatrix}0&1\\1&0\end{bmatrix} = \xrightarrow{(*)}$$

$$= \frac{A_S}{2} \otimes \left(\begin{bmatrix}1&0\\0&1\end{bmatrix} + \begin{bmatrix}0&1\\1&0\end{bmatrix}\right) + \frac{A_G}{2}\otimes\left(\begin{bmatrix}1&0\\0&1\end{bmatrix} + \begin{bmatrix}0&1\\1&0\end{bmatrix}\right)$$

$$= A_S \otimes \underbrace{\begin{bmatrix}1/2 & -1/2\\ -1/2 & 1/2\end{bmatrix}}_{M_S} + A_G \otimes \underbrace{\begin{bmatrix}1/2&1/2\\1/2&1/2\end{bmatrix}}_{M_G} = A_G \otimes M_G + A_S \otimes M_S$$

$M_G \to \lambda_G \in \{0,1\} \to u_1 = (1,1)^T \,, \; u_2 = (1,-1)^T$

$M_S \to \lambda_S \in \{0,-1\} \to u_1 = (1,1)^T \,, \; u_2 = (1,-1)^T$

Now let $\{\psi_i\}_{i=1}^n, \{\phi_i\}_{i=1}^n$ be the eigenvectors of A_S, A_G; respectively. let:

① $\beta_{ij} = \psi_i \otimes u_j$ [illegible] else' let $j=2 \to u_j = (1,-1)^T$

$$A_H\beta_{ij}\Big|_{j=2} = (A_S\otimes M_S)(\psi_i\otimes u_j) + (A_G\otimes M_G)(\psi_i\otimes u_j)\Big|_{j=2}$$

$$= (A_S\psi_i)\otimes(M_S u_j) + \cancel{(A_G\psi_i)\otimes(\underbrace{M_G u_j}_{0})}\Big|_{j=2}$$

$$= (A_S\psi_i)\otimes(M_S u_j)\Big|_{j=2} = (\lambda_i\psi_i)\otimes\begin{pmatrix}1\\1\end{pmatrix}$$

So we got $A_H\beta_{i2} = A_H\left(\psi_i\otimes\begin{pmatrix}1\\1\end{pmatrix}\right) = \lambda_i\left(\psi_i\otimes\begin{pmatrix}1\\1\end{pmatrix}\right) \Rightarrow \begin{cases}\psi_i\otimes\begin{pmatrix}1\\1\end{pmatrix} \in \text{Eigen}(A_H)\\ \lambda_i\in\Lambda_{A_H}\end{cases}$

② on the contrary we have: let $\tilde\beta_{ij} = \phi_i\otimes u_j$ (let $j=1$), $(u_j = (1,1)^T)$

$$A_H\tilde\beta_{i,j}\Big|_{j=2} = (A_S\otimes M_S)(\phi_i\otimes u_j) + (A_G\otimes M_G)(\phi_i\otimes u_j)\Big|_{j=2} .$$

$$= \cancel{(A_S\psi_i)\otimes(\underbrace{M_S u_j}_{0})} + (A_G\phi_i)\otimes(M_G u_j)\Big|_{j=2}$$

$$= (A_G\phi_i)\otimes(M_G u_j)\Big|_{j=2} = (\gamma_i\phi_i)\otimes\begin{pmatrix}1\\1\end{pmatrix}$$

which results in $A_H\tilde\beta_{i,1} = A_H\left(\phi_i\otimes\begin{pmatrix}1\\1\end{pmatrix}\right) = \gamma_i\left(\phi_i\otimes\begin{pmatrix}1\\1\end{pmatrix}\right) \Rightarrow \begin{cases}\phi_i\otimes\begin{pmatrix}1\\1\end{pmatrix} \in \text{Eigen}(A\\ \gamma_i\in\Lambda_{A_H}\end{cases}$

Up until now, we got that $(\lambda_i, \psi_i \otimes \binom{1}{1})$, $(\gamma_i, \phi_i \otimes \binom{1}{1})_{(\forall i)}$ are the (eigenvalues, eigenvectors) of A_H. Since we have atmost $2n$ eigenvalues for A_H, and for each (λ_i, ψ_i) & (γ_i, ϕ_i) we can find exactly one (eigenvalue - eigenvector) pair for A_H, we can deduce that $\Lambda_{A_H} = \{\lambda_i \in \text{Eigen}^V(A_S)\} \cup \{\gamma_i \in \text{Eigen}^V(A_G)\}$. which proves the theorem.

Q.E.D. ■

$$\begin{cases} \Lambda_{A_H} = \{\lambda_i\}_{i=1}^{n} \cup \{\Phi_i\}_{i=1}^{n} \\ \text{eigen}(A_H) = \left\{ \psi_i \otimes \binom{1}{1} = \binom{\psi_i}{\psi_i} \right\}_{i=1}^{n} \cup \left\{ \Phi_i \otimes \binom{1}{1} = \binom{\Phi_i}{\Phi_i} \right\}_{i=1}^{n} \end{cases}$$

(I) Gershgorin theorem states that: every eigenvalue of matrix A lies on union of disks at complex plane characterized by $D_i\left(a_{ii}, \sum_{j \neq i} |a_{ij}|\right)$ 40/3

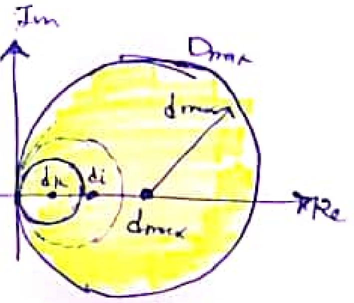
Proof: $\left.\begin{matrix}\forall \lambda \in \Lambda_A \\ x \in \text{Eigen}(A)\end{matrix}\right\} \exists i = \arg\max_k \{|x_k|\} \rightarrow Ax = \lambda x \rightarrow (Ax)_{(i)} = \lambda x_{(i)}$

$$\rightarrow \sum_{j \neq i} a_{ij} x_j = (\lambda - a_{ii}) x_i$$

$$|\lambda - a_{ii}| \le \left| \sum_{j \neq i} \frac{a_{ij} x_j}{x_i} \right| \le \sum_{j \neq i} |a_{ij}| \left| \frac{x_j}{x_i} \right| \le \sum_{j \neq i} |a_{ij}| = R_i$$

Triangle inequality $\quad |x_j| \le |x_i|$

Now, for a laplacian matrix, we know that $a_{ii} = d(i) = \sum_{j \neq i} |l_{ij}|$ → the degree of i-th node. So these disks are centered at positive half of real axis. since $\forall i: a_{ii} \ge 0$. So each disk D_i is centered at $d(i)$ and with radius $d(i)$. So we can deduce that all of these disks D_i are covered completely in a disk with d_{max}. So all eigenvalues $\lambda \in \Lambda_L$ lie in the circle $D(d_{max}, d_{max})$



$\forall \lambda \in \Lambda_L : |\lambda - d_{max}| \le d_{max} \rightarrow 0 \le \lambda \le 2d_{max}$ → Now taking the biggest eigenvalue, implies that: $\boxed{\lambda_n \le 2d_{max}}$ Q.E.D. ■

(II) we know that $\lambda_1 = 0$, thus $\lambda_2 = \inf_{x \in \mathbb{R}^n \setminus \text{Span}(v_1)} \left\{ \frac{x^T L x}{x^T x} \right\} \le \frac{y^T L y}{y^T y}$

now let $y = [0, \dots, 0, \overset{i}{1}, 0, \dots, 0, \overset{j}{1}, 0, \dots, 0]$, since $l_{ij} = l_{ji} = 0$ (i, j are disconnected)

$y = (\hat{e}_i + \hat{e}_j) \rightarrow y^T L y = \hat{e}_i^T L \hat{e}_i + \hat{e}_j^T L \hat{e}_j + \hat{e}_i^T L \hat{e}_j + \hat{e}_j^T L \hat{e}_i$, $y^T y = 2$

$\rightarrow y^T L y = l_{ii} + l_{ij} + l_{ji} + l_{jj} = \sum_{k \neq i} -l_{ki} + \sum_{k \neq j} -l_{kj} = d(i) + d(j)$

which results in:

■ Q.E.D.
$$\begin{cases} \lambda_2 \le \frac{y^T L y}{y^T y} = \frac{d(i) + d(j)}{2} & i, j \text{ are disconnected} \\ \lambda_2 < \frac{d(i) + d(j)}{2} & i, j \text{ are connected} \end{cases}$$

(III) (i) $\lambda_2 = \min_{x \perp \mathbb{1}_{n\times 1}} \left\{ \frac{n^T \mathcal{L} x}{x^T x} \right\} \le \frac{y^T \mathcal{L} y}{y^T y}$ $\quad \forall y : y^T \mathbb{1}_{n\times 1} = 0$ با استفاده از ۴ :

let $y = \left(\frac{-1}{n}, \frac{-1}{n}, \cdots, \underset{i \nearrow}{\frac{n-1}{n}}, \cdots, \frac{-1}{n} \right) \to y^T y = \frac{n-1}{n^2} + \frac{(n-1)^2}{n} = \frac{n-1}{n}$

$$y^T \mathcal{L} y = y^T \begin{bmatrix} \frac{-1}{n} \sum_{k=1}^{n} l_{1k} + l_{1i} \\ \frac{-1}{n} \sum_{k=1}^{n} l_{2k} + l_{2i} \\ \vdots \\ \underbrace{\frac{-1}{n} \sum_{k=1}^{n} l_{nk}}_{0} + l_{ni} \end{bmatrix} = y^T \begin{bmatrix} l_{1i} \\ \vdots \\ l_{ni} \end{bmatrix} = \frac{-1}{n} \overbrace{\sum_{k=1}^{n} l_{ki}}^{0} + l_{ii} = d(i)$$

Q.E.D ■

finally, we get : $\lambda_2 \le \left. \frac{x^T \mathcal{L} x}{x^T x} \right|_{x=y} = \frac{n}{n-1} d(i) \xrightarrow[\forall i]{\inf} \boxed{\lambda_2 \le \frac{n}{n-1} d_{min}}$

(ii) $\lambda_n = \max_{x \in \mathbb{R}^n} \left\{ \frac{x^T \mathcal{L} n}{n^T n} \right\} \ge \frac{y^T \mathcal{L} y}{y^T y}$ $\to$ we define the same y here :

$y^T y = \frac{n-1}{n}$, $y^T \mathcal{L} y = d(i) \to \lambda_n \ge \frac{n}{n-1} d(i) \xrightarrow{\sup} \boxed{\lambda_n \ge \frac{n}{n-1} d_{max}}$

Q.E.D. ■

(II) $S = S^T$, $\mu_1 \ge \mu_2 \ge \cdots \ge \mu_n$, $S = Q \Lambda Q^T$, $\Lambda = diag(\mu)$

$d_i = S_{ii} = \sum_{j=1}^{n} q_{ij}^2 u_j$. Since q_i are orthogonal $\to \|q_i\| = \sum_{j=1}^{n} q_{ij}^2 = 1$.

[illegible]

Induction on n : let $A = \begin{bmatrix} s_{22} & \cdots & s_{2n} \\ \vdots & \ddots & \\ s_{n2} & & s_{nn} \end{bmatrix} \to \sum_{i=2}^{n} d_i = tr\{A\} = \sum_{i=1}^{n-1} \gamma_i \underset{\substack{\uparrow \\ \text{Interlacing theorem}}}{\le} \sum_{i=2}^{n} \mu_i$

and since by the other theorem we know $\mu_1 \ge \gamma_1$,
we can say that $\sum_{i=1}^{n} \gamma_i = tr\{A\} + \gamma_1 = \sum_{i=1}^{n} d_i \le \sum_{i=1}^{n} \mu_i$

we can do the same process to reach t-th step s.
we get :

$$\sum_{i=1}^{1} \gamma_i \le \sum_{i=1}^{t} \mu_i$$

Q.E.D. ■

(I) the same theorem in Part 4 will prove that $\lambda_{max}^{(M)} \leq d_{max}$: 25 ()2

let $\hat{D} = \bigcup_{i=1}^{n} Disk(0, d(i))$. Since $M_{ii} = 0$ $\forall i$, we can say that all eigenvalues lie in $\hat{D}$ since

$\forall \lambda, \exists i$ $|\lambda - \overset{0}{\cancel{M_{ii}}}| \leq \sum_{j \neq i} M_{ij} = d(i) \rightarrow \forall \lambda \in \Lambda_M: \overset{\exists i}{|\lambda - 0| \leq d(i)} \xrightarrow{\text{thus}}$

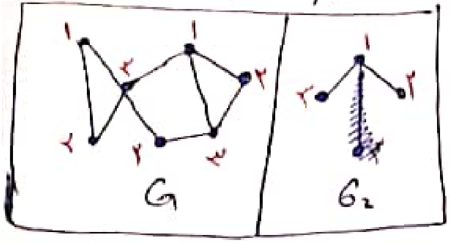
$\left.\begin{array}{l}\Lambda_M \in \hat{D}. \text{ Now let } \lambda = \lambda_{max} \\ \hat{D} \subseteq Disk(0, d_{max})\end{array}\right\} \rightarrow |\lambda_{max}^{(M)}| \leq d_{max} \rightarrow \boxed{\lambda_{max}^{(M)} \leq d_{max}}$ Q.E.D ■

(II) since $\lambda_{max}^{(M)} = \sup \frac{x^T M x}{x^T x}$, then $\forall y \in \mathbb{R}^n$: $\lambda_{max}^{(M)} \geq \frac{y^T M y}{y^T y}$.

let $\left.\begin{array}{l} y = \mathbb{1}_{n \times 1} \\ y^T y = n \end{array}\right\} \rightarrow \lambda_{max}^{(M)} = \frac{1}{n} \sum_{i,j=1}^{n} M_{ij} = \frac{1}{n} \sum_{i=1}^{n} \left(\sum_{\substack{j=1 \\ j \neq i}}^{n} M_{ij} \right) = \frac{1}{n} \sum_{i=1}^{n} d(i) = d_{avg}$

which leads to $\boxed{\lambda_{max}^{(M)} \geq d_{avg} = \frac{2e}{n}}$

(III) Suppose we have a graph G_1, and we eliminate nodes from G_1 such that $\chi(G_1) = \chi(G_2)$, and G_2 can't get smaller any further. $(G = G_1)$



(*)
then, we can see that $\forall i \in (n): d(i) \geq \chi(G_2) - 1$.
this is true since the number of colores is bounded by the degree of node 'i' plus 1.

$(*) \rightarrow \frac{1}{n} \sum_{i=1}^{n} d(i) = d_{avg} \geq \frac{1}{n} \sum_{i=1}^{n} (\chi(G_2) - 1) = \chi(G_2) - 1 \Rightarrow d_{avg} \geq \chi(G_2) - 1$

Now since $\left\{\begin{array}{l} \chi(G_1) = \chi(G_2) \\ d_{avg} \leq \lambda_{max}^{(M)} \\ \lambda_{max}^{(M_1)} \geq \lambda_{max}^{(M_2)} \end{array}\right.$ $\rightarrow \left\lfloor \lambda_{max}^{(M_{G_1})} \right\rfloor \geq \left\lfloor \lambda_{max}^{(M_{G_2})} \right\rfloor \geq \overset{\lfloor d_{avg} \rfloor}{\overset{\shortparallel}{d_{avg}}} \geq \chi(G_2) - 1$

from II → ($d_{avg} \leq \lambda_{max}^{(M)}$)

from Interlacting theorem → ($\lambda_{max}^{(M_1)} \geq \lambda_{max}^{(M_2)}$)

$\rightarrow \left\lfloor \lambda_{max}^{(M_{G_1})} \right\rfloor \geq \chi(G_1) - 1$

which proves: $\boxed{\chi(G_1) \leq \left\lfloor \lambda_{max}^{(G)} \right\rfloor + 1}$ Q.E.D. ■

ادامه سوال 5:

(III) let us assume that the graph G, is k-colorable, then we can write A as:

$$A = \begin{bmatrix} \varphi_{1,1} & M_{12} & \cdots & M_{1n} \\ M_{1,2}^T & \varphi_{2,2} & \ddots & \vdots \\ \vdots & & & \varphi_{k,k} \\ M_{1,n}^T & \cdots & & \end{bmatrix}$$

from the last assignment we know that:

(*)

$$(k-1)\lambda_{min}^{(A)} + \lambda_{max}^{(A)} \leq \underbrace{\sum_{i=1}^{n} \lambda_{max}^{(M_{ii})}}_{0}$$

Since $\forall i: \lambda_{max}(M_{ii}) = \lambda_{max}(\varphi_{ii}) = 0 \longrightarrow$ from (*) we get:

(**) $k \geq -\frac{\lambda_{max}^{(A)}}{\lambda_{min}^{(A)}} + 1$. Since this (**) equation $\overset{\text{holds}}{\vee}$ for all $k \geq \chi(G)$,

getting min of k will result in $\boxed{k \geq \chi(G) \geq -\frac{\lambda_{max}^{(A)}}{\lambda_{min}^{(A)}} + 1}$ Q.E.D. ■