GSP - HW4

تمرین سری 4 GSP

[illegible] 400101204

$\vec{h_i}=V_i^H\vec{x_i}$

$\vec{x_i}=V_i\vec{h_i},\ \vec{h_i}\sim\mathcal{N}(0,\Lambda_i^{+})\rightarrow \mathbb{E}\,h_ih_i^H=\Lambda_i^{+},\quad \mathbb{E}_{i\neq j}[h_ih_j]=0$

$\vec{x}=\sum_{i=1}^{k}x_i,\ n\sim\mathcal{N}(0,\sigma^2 I)$

:سؤال 1

MAP estimation: $\max_{\vec{h}_{1:k}}\mathbb{P}[h_{1:k}|x]=\max_{\vec{h}_{1:k}}\dfrac{\mathbb{P}(\vec{x}|h_{1:k})P(h_{1:k})}{\int_{\mathbb{R}^k}\mathbb{P}[x|h_{1:k}]\mathbb{P}(h_{1:k})dh_{1:k}}$

$=\max\ \mathbb{P}(\vec{x}|\vec{h}_{1:k})\mathbb{P}(h_{1:k})\qquad (*)$

(i) $\mathbb{P}(h_{1:k})=\prod_{i=1}^{k}P(h_i)=\prod_{i=1}^{k}\overbrace{\dfrac{1}{(2\pi)^{M/2}|\Lambda_i|^{1/2}}}^{\text{Constant}}\overbrace{\exp\left(-\frac{1}{2}h_i^T\Lambda_ih_i\right)}^{\text{important}}$

(ii) $\mathbb{P}(\vec{x}|h_{1:k})=\mathbb{P}(x|x_{1:k})=\underbrace{\dfrac{1}{(2\pi\sigma^2)^{M/2}}}_{\text{Constant}}\exp\left(-\dfrac{(x-\sum V_i\vec{h_i})^T(x-\sum\overbrace{V_i\vec{h_i}}^{\vec{x_i}})}{\underbrace{2\sigma^2}_{\text{important}}}\right)$

$(*),(i),(ii)\longrightarrow \arg\max_{h_{1:k}}\mathbb{P}(h_{1:k}|\vec{x})=\arg\min_{\vec{x}}\left[\frac{-1}{2\sigma^2}\|x-\sum_{i=1}^{k}x_i\|_2^2+\sum_{i=1}^{k}x_i^HLx_i\right]$

$=\arg\min_{x_{1:k}}\left\{\|x-\sum_{i=1}^{k}x_i\|_2^2+\sigma^2\sum_{i=1}^{k}x_i^HLx_i\right\}$ Q.E.D. ■

:سؤال 2

Ⓘ $\vec{x_i}\in\{0,1\}^N\qquad(\vec{x_i})(k)\sim Ber(p)\quad,\ w_{ij}=\begin{cases}0 & i=j\\ \dfrac{1}{1+\sum_{k=1}^{M}((\vec{x}_k)(i)-(\vec{x}_k)(j))^2} & i\neq j\end{cases}$

$\vec{y}\sim\mathcal{N}(0,\sigma^2 I)\qquad W=W^T,\ W_{ii}=0$

$\Rightarrow \vec{y}^HL\vec{y}=\sum_{i<k}w_{ik}|\vec{y}(i)-\vec{y}(k)|^2=\frac{1}{2}\sum_{i,k}w_{ik}|\vec{y}(i)-\vec{y}(k)|^2$

$\Rightarrow \mathbb{E}[\vec{y}^HL\vec{y}|L]=\mathbb{E}[\vec{y}^HL\vec{y}|w_{11},\dots,w_{nn}]=\frac{1}{2}\sum_{i,k}w_{ik}\mathbb{E}[|\vec{y}(i)-y(k)|^2]$

$\forall i,k:\longrightarrow \mathbb{E}[|\vec{y}(i)-\vec{y}(k)|^2]=\mathbb{E}\vec{y}(i)^2+\mathbb{E}\vec{y}(k)^2-2\underbrace{\mathbb{E}\vec{y}(i)\vec{y}(k)}_{0}=2\sigma^2$

which suggest $\mathbb{E}[\vec{y}^HL\vec{y}|L]=\frac{2\sigma^2}{2}\sum_{i,k}\mathbb{E}[w_{ij}]$

so now we'd go to get the distribution of w_{ij} and $|\vec{x}_k(i)-x_k(j)|^2=S_{ij}$

$x_k(j)$ \ $x_k(i)$	$\bar{p}$: 0	p: 1
$\bar{p}$: 0	0	1
p: 1	1	0

$\Longrightarrow \underset{|x_k(i)-x_k(j)|^2}{S_{ij}^{k}}\sim Ber(2p\bar{p})\longrightarrow\sum_{k=1}^{M}|x_k(i)-x_k(j)|^2\sim Bin(M,\tilde{p})$

where $\tilde{p}=2p(1-p)=2p\bar{p}$

let $\bar{p}=1-p$

so we Calculate $\mathbb{E}W_{ij}$ for all i,j: ~~[illegible]~~

$$\tilde{q} = 1-\tilde{p} = 1-2p(1-p)$$

$$\mathbb{E}[W_{ij}] = \mathbb{E}_{Z\sim Bin(M,\tilde{p})}\left[\frac{1}{1+Z}\right] = \sum_{k=0}^{M} \frac{\binom{M}{k}\tilde{p}^{k}\tilde{q}^{M-k}}{1+k} = f(\tilde{p},\tilde{q}) \quad (*)$$

[illegible]

$$\rightarrow \frac{\partial f}{\partial \tilde{p}} = \sum_{k=0}^{M-1} \frac{k}{1+k}\binom{M}{k}\tilde{p}^{k-1}\tilde{q}^{M-k} \quad (**) \Rightarrow (*), (**) \text{ Can lead to :}$$

$$\frac{\partial f}{\partial \tilde{p}}\cdot\tilde{p} + f(\tilde{p},\tilde{q}) = \sum_{k=0}^{M}\frac{\overset{1}{\cancel{1+k}}}{\cancel{1+k}}\binom{M}{k}\tilde{p}^{k}\tilde{q}^{-k} = \sum_{k=0}^{M}\binom{M}{k}\tilde{p}^{k}\tilde{q}^{k} = (\tilde{p}+\tilde{q})^{M} = 1$$

$$\frac{\partial f}{\partial \tilde{p}}\cdot\tilde{p} + f = \frac{\partial}{\partial\tilde{p}}\left(\tilde{p}\cdot f(\tilde{p},\tilde{q})\right) \xrightarrow[\text{integration}]{} \tilde{p}\cdot f(\tilde{p},\tilde{q}) = \frac{1}{M+1}\tilde{q}^{M+1} + \underbrace{h(\tilde{q})}_{\text{a function of } \tilde{q}.} \quad (***)$$

Now let $\tilde{p}=0$ in $f(\tilde{p},\tilde{q})\big|_{\tilde{p}=0} = \tilde{q}^{M} \Rightarrow \tilde{p}\cdot f(\tilde{p},\tilde{q})\big|_{\tilde{p}=0} = 0 \Rightarrow$ which suggests

$$h(\tilde{q}) = \frac{-1}{M+1}\tilde{q}^{M+1} + f(\tilde{p},\tilde{q}) \quad (****) \Rightarrow$$

$$\overset{(****)(***)}{\Longrightarrow} f(\tilde{p},\tilde{q}) = \frac{(\tilde{p}+\tilde{q})^{M+1} - \tilde{q}^{M+1}}{(M+1)\tilde{p}} = \frac{1-\tilde{q}^{M+1}}{(M+1)\tilde{p}} = \frac{1-[2p(1-p)]^{M+1}}{2(M+1)p(1-p)} \quad (\ast\ast\ast\ast\ast)$$

So now from $(\ast\ast\ast\ast\ast)$ and before, we can deduce that

$$\mathbb{E}[y^H \Sigma y] = N\sigma^2\,\mathbb{E}[W_{ij}] = N\sigma^2\cdot\frac{1-\tilde{q}^{M+1}}{(M+1)\tilde{p}} = \boxed{N\sigma^2\cdot\frac{1-[2p(1-p)]^{M+1}}{(M+1)2p(1-p)}}$$

(II) from the first hw, we know that $\sum_{\substack{S\in[k]\\|S|=k}}\prod_{i\in S}\lambda_i = \sum_{\substack{S\in K\\|S|=k}}\det(A(S,S))$

$$\Rightarrow \sum_{i\neq j}\lambda_i\lambda_j = \sum_{\substack{S\in[N]\\|S|=2}}\det(W(S,S)) \rightarrow \left\{\begin{array}{l}\text{because } \sum_{i\neq j}\lambda_i\lambda_j \text{ is the 2nd coeff}\\ \text{in } P(\lambda) \text{ of } W \text{ matrix.}\end{array}\right.$$

However, $W(S,S) = \begin{vmatrix} W_{ii} & W_{ij} \\ W_{ji} & W_{jj}\end{vmatrix} = -W_{ij}^2$

$$\Rightarrow \mathbb{E}\left[\sum_{i\neq j}\lambda_i\lambda_j\right] = \mathbb{E}\left[-\sum_{i\neq j}W_{ij}^2\right] = -\binom{N}{2}\mathbb{E}[W_{ij}^2]$$

Now writing $\mathbb{E}\left[\frac{1}{1+S^2}\right]$ is very hard differential equation. thus in order to get $\frac{1}{1+S^2}$, we may use its taylor series and hence (we only use the second moment:

$$\Rightarrow \mathbb{E}\left[\frac{1}{1+S^2}\right] = Var\left(\frac{1}{1+S}\right) + \left(\mathbb{E}\left(\frac{1}{1+S}\right)\right)^2 \longrightarrow \text{thus next page}$$

سوال 3:

(I)

$$P_{t+1}(x=k) = \sum_{l} P_t(x=l)\,P(l\to k) = \frac{1}{2}P_t(x=k) + \frac{1}{2}\sum_{l=1}^{n} \frac{W_{kl}}{d(l)} P_t(x=l)$$

equations above describe the markov chain process of the problem.

$$\Rightarrow P_{t+1}(x=k) = \frac{1}{2}P_t(x=k) + \frac{1}{2}\sum_{l}(D^{-1}W)^{T}_{kl}\,P(x=l) \longrightarrow \vec{P}_{t+1} = \frac{1}{2}(I + D^{-1}W^{T})\vec{P_t}$$

$\Rightarrow \vec{P}_{t+1} = \tilde{W}\vec{P_t}$, where $\tilde{W} = \frac{1}{2}(I + D^{-1}W^{T}) = \frac{1}{2}D^{1/2}(I+A)D^{-1/2}$, where $A = D^{-1/2}WD^{-1/2}$.

Now if we suppose that $\omega_1 \geq \dots \geq \omega_n$ are the eigenvalues associated with eigenvectors $\psi_1, \dots, \psi_n$; then $\tilde{W}\psi_i = \omega_i\psi_i = \frac{1}{2}D^{1/2}(I+A)D^{-1/2}\psi_i = \omega_i\psi_i \Rightarrow (I+A)(D^{-1/2}\psi_i) = 2\omega_i(D^{-1/2}\psi_i)$

$\Rightarrow$ which suggests $\overset{\forall i}{}$ $A(D^{-1/2}\psi_i) = \underbrace{(2\omega_i - 1)}_{\lambda_i}\underbrace{(D^{-1/2}\psi_i)}_{\phi_i} \rightarrow \begin{cases} \forall i \rightarrow 2\omega_i - 1 \in \Lambda_A ,\ \omega_i \in \Lambda_{\tilde{W}} \\ \hookrightarrow \phi_i = D^{-1/2}\psi_i \end{cases}$

(II) from Peron-Frobenius, we know that:

$$\boxed{\forall i\ \ \mu_1(i) > 0;\ \ \mu_1 \geq -\mu_n\ ,\ \mu_1 > \mu_2\ \text{if}\ M_{ij} \geq 0}$$

$$A\phi_i = \mu_i\phi_i \xrightarrow[\vec{x} = \sqrt{\vec{d}}]{A = D^{-1/2}WD^{-1/2}} x = [\sqrt{d_1}, \dots, \sqrt{d(n)}]^{T} \Rightarrow A\vec{x} = D^{-1/2}WD^{-1/2}\vec{x} = \vec{x}$$

thus $\vec{x}$ is an eigenvector of A. with $\mu = 1$. $\rightarrow \forall i : x(i) > 0 \rightarrow \begin{cases} x = u_1 = \phi_1 \\ 1 = \mu_1 = 2\omega_1 - 1 \\ \omega_1 = 1 \hookleftarrow \end{cases}$

$$\vec{V}_1 = D^{-1/2}\vec{\psi}_1 \Rightarrow \vec{\psi}_1 = D^{1/2}\phi_1 = D^{1/2}[\sqrt{d(1)}, \dots, \sqrt{d(n)}]^{T} = (d(1), \dots, d(n))^{T}$$

$$\mu_1 = 1 \xrightarrow[\text{Peron}]{\text{Ferobenius}} \left\{\begin{matrix} \mu_2 = 2\omega_2 - 1 \geq 2\omega_i - 1 \\ \mu_2 < \mu_1 = 1 \end{matrix}\right\} \boxed{\forall i : \omega_i \leq 1} \quad (*)$$

$$\left\{ \mu_n \geq -\mu_1 = -1 \rightarrow 2\omega_i - 1 \geq -1 \rightarrow \boxed{\omega_i \geq 0}\ \forall i \quad (**) \right.$$

Now, we can add these two $(*)$, $(**)$ and get: $\begin{cases} \omega_1 = 1 \ ;\ \vec{d} = W\vec{\mathbb{1}}_{n\times 1} \\ \vec{\phi}_1 = \vec{d} = [d(1), \dots, d(n)] \\ 0 \leq \omega_i < 1 \ ,\ i \neq 1 \end{cases}$

(III) $\vec{P}_{t+1} = \tilde{W}\vec{P}_t$, $0 \leq \omega_i \leq 1$, $\tilde{W}^{t} = \psi W^{t}\psi^{H} \Rightarrow \lim_{t\to\infty}\tilde{W}^{t} = \psi \lim_{t\to\infty} W^{t}\psi^{H} = \psi\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 0 \end{pmatrix}\psi^{H}$

since $\tilde{W}$ has only one "1" eigenvalue & the rest are of less size than 1.

$\exists \vec{\pi} :$ $\pi = \lim_{t\to\infty} P_t = \left(\lim_{t\to\infty}\tilde{W}^{t}\right)\underset{\delta_a}{\underset{\sim}{P_0}} \Rightarrow \tilde{W}\pi = \pi \Rightarrow \pi = \phi_1 = \vec{d}$

$|\pi| < \infty$ (due to $t\to\infty$)

ادامه سوال 3:

and if we normalize $\pi \rightarrow$ we get $\pi = \frac{[d(1), \dots, d(n)]^T}{\sum_{i=1}^{n} d_i}$

$\Rightarrow \boxed{\vec{\pi} = \frac{\vec{d}}{\|\vec{d}\|_1}}$ No matter where we start, $(P_0 = \delta_a)$, we will (end up at) converge to the vector $\vec{\pi}$ in probability.

(IV) $\vec{P_t} = \tilde{W}^t P_0 = \tilde{W}^t \vec{\delta_a} = D^{1/2} \cdot \left(\frac{I+A}{2}\right)^t D^{-1/2} \vec{\delta_a} \Rightarrow D^{-1/2}\vec{P_t} = \left(\frac{I+A}{2}\right)^t (D^{-1/2}\vec{\delta_a})$

$D^{-1/2}\vec{\pi} = \lim_{t \to \infty} \left(\frac{I+A}{2}\right)^t (D^{-1/2}\vec{\delta_a}) \longrightarrow D^{-1/2}(\vec{P_t} - \vec{\pi}) = \left[\underbrace{\left(\frac{I+A}{2}\right)}_{B}^t - \lim_{t \to \infty}\left(\frac{I+A}{2}\right)^t\right] D^{-1/2}\vec{\delta_a}$

$\longrightarrow \; = \sigma_{max}\left\{B^t - \lim_{t \to \infty} B^t\right\} = \lambda_{max}\{\dots\}$

$\Rightarrow \lambda_i\left(\left(\frac{I+A}{2}\right)^t - \lim_{t \to \infty}\left(\frac{I+A}{2}\right)^t\right) = \begin{cases} 0 & i=0 \\ \omega_i^t & \text{o.w.} \end{cases} \Rightarrow \lambda_{max} = \omega_2^t$

$\|D^{-1/2}(\vec{P_t} - \vec{\pi})\|_2 = \|[\dots] D^{-1/2}\vec{\delta_a}\|_2 \leq \|\dots\|_2 \cdot \|D^{-1/2}\delta_a\|_2 = \omega_2^t \|D^{-1/2}\vec{\delta_a}\|_2$

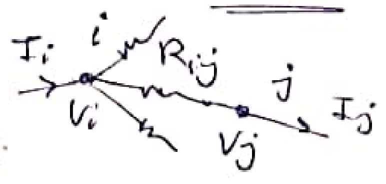
$\|D^{-1/2}\vec{\delta_a}\|_2 = \frac{1}{\sqrt{d(a)}} \longrightarrow \|D^{-1/2}v\|_2 \geq \|(D^{-1/2}v)(b)\|_2 = \frac{1}{\sqrt{d(b)}}|v(b)|$

$\xrightarrow{\text{thus}} \|D^{-1/2}(\vec{P_t} - \vec{\pi})\|_2 \leq \frac{1}{\sqrt{d(a)}}\omega_2^t \xrightarrow{@b} \boxed{P_1^{(b)} - \pi(b) \leq \sqrt{\frac{d(b)}{d(a)}}\,\omega_2^t}$

۹-۹.۱۵ ■

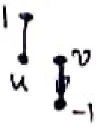
سوال 4:

(I) $w_{ij} = \frac{1}{R_{ij}}$



$$\left.\begin{aligned} &\text{if } I_i = I_j = I \Rightarrow V_i - V_j = R_{eff}(i,j)\, I \\ &\text{KCL} \rightarrow I_i = \sum_{j=1}^{n} \frac{V_i - V_j}{R_{ij}} = \sum_{j=1}^{n}(V_i - V_j) w_{ij} \end{aligned}\right\} \begin{aligned} \vec{I} &= L\vec{V} \\ \vec{V} &= L^+\vec{I} \end{aligned}$$

let $\vec{I} = \vec{\delta_u} - \vec{\delta_v}$



$\Rightarrow \vec{V}(u) - \vec{V}(v) = R_{eff}(u,v) = (\delta_u - \delta_v)^T \vec{V}$

since $L\vec{V} = \vec{I} = \vec{\delta_u} - \vec{\delta_u} \Longrightarrow \vec{V} = L^+(\vec{\delta_u} - \vec{\delta_v}) \longrightarrow \boxed{R_{eff}(u,v) = (\vec{\delta_u} - \vec{\delta_v})^T L^+ (\vec{\delta_u} - \vec{\delta_v})}$

■ ۹-۴.۱۵

Scanned by CamScanner

So we have a subset $B \subseteq \mathcal{V}$ of arbitrary voltages. ⓘⓘ ادامه سوال 4:

let $LV = I \xrightarrow{@u} (LV)_{(u)} = I_{(u)} = 0 = d(u)\vec{v}_{(u)} - (W\vec{v})_{(u)} = d(u)\vec{v}_{(u)} - \sum_{s=1}^{n} w_{su}\vec{v}_{(s)}$

$\rightarrow$ Thus, we can deduce that $\vec{v}_{(u)} = \frac{1}{d(u)} \sum_{s,u} w_{su}\vec{v}_{(s)}$ Q.E.D. ∎

let $f(v) = v^T L v = \sum_{i,j} w_{ij} \| \vec{v}_i - \vec{v}_j \|_2^2 \longrightarrow \nabla f(v)|_{N_B^v} = 0$

$\Longrightarrow \sum_{s,u} 2 w_{us} (\vec{v}_{(u)} - \vec{v}_{(s)}) = 0 \rightarrow \vec{v}_{(u)} \sum w_{su} = \sum w_{su} v_{(s)}$ | the same equation as before

$$\begin{cases} \nabla_v (\vec{v}^T L \vec{v}) = 0 \\ \vec{v} \in \mathcal{V}/B \end{cases}$$

ⓘⓘⓘ We need to find L_B: $v_{(B)}^T L_B v_B = \vec{v}^T L_G \vec{v}$

$\forall u \in \mathcal{V}/B \rightarrow (L\vec{v})_{(u)} = 0$ (harmonic)

$\vec{v}^T L \vec{v} = \sum_u \vec{v}_{(u)} (L\vec{v})_{(u)} = \sum_{u \in B} \vec{v}_{(u)} (L\vec{v})_{(u)} + \sum_{v \in \mathcal{V}/B} \vec{v}_{(u)} \overset{0}{\cancel{(L\vec{v})_{(u)}}} = \sum_{u \in B} \sum_s \vec{v}_{(u)} L_{us} v_{(s)}$

consider only removing node $k \longrightarrow B = \mathcal{V}/\{k\}$

$\longrightarrow \vec{v}^T L \vec{v} = \sum_{\substack{u \in \mathcal{V} \\ u \neq k}} \sum_s \vec{v}_{(u)} L_{us} v_{(s)} = \cancel{\sum_{\substack{u \in \mathcal{V} \\ u \neq k}} \sum_s} \sum_{\substack{u \in \mathcal{V} \\ u \neq k}} \vec{v}_{(u)} \left[L_{uk} \vec{v}_{(k)} + \sum_{\substack{s \in \mathcal{V} \\ s \neq k}} L_{us} \vec{v}_{(s)} \right]$

$(L\vec{v})_{(k)} = 0 \longrightarrow \vec{v}_{(k)} = \frac{1}{d(k)} \sum_{\substack{s \in \mathcal{V} \\ s \neq k}} w_{sk} \vec{v}_{(s)}$

$\Rightarrow \vec{v}^T L \vec{v} = \sum_{\substack{u \in \mathcal{V} \\ u \neq k}} \vec{v}_{(u)} \sum_{\substack{s \in \mathcal{V} \\ s \neq k}} \left(L_{su} + \frac{L_{uk}}{d(k)} w_{sk} \right) \vec{v}_{(s)} = \sum_{u,s \in B} \vec{v}_{(u)} \left(L_{su} + \frac{w_{sk}}{d(k)} L_{ku} \right) \vec{v}_{(s)}$

$= v_{(B)}^T L' \vec{v}_{(B)} \rightarrow$ if L' here is valid, then we have an algorithm to remove nodes.

$$\sum_{\substack{j=1 \\ j \neq k}}^{n} L'_{ij} = L_{ii} + \sum_{i \neq k} \left(L_{ij} + \frac{L_{kj}}{d(k)} w_{ik} \right) = L_{ii} + \sum_{i \neq k} L_{ij} + L_{kj} \sum \frac{w_{ik}}{d(k)} = 0$$

thus, since the sum of a row of L' is zero $L'\vec{\mathbb{1}} = \vec{0}$, thus, L' is a valid Laplacian. So inductively we can remove nodes and we can get $L^{(2)}$ from $L^{(1)} = L'$ and so on and so forth.

$\Longrightarrow$ So as we said, we can find L_B such that: $\vec{v}^T L \vec{v} = \vec{v}_{(B)}^T L_B \vec{v}_B$

Q.E.D. ●

ادامه سوال 4:

$$f(\mathcal{L}^{n\times n}, k) = \mathcal{L}'^{(n-1)\times(n-1)}$$

↑ node to remove

$$\longrightarrow \boxed{(\mathcal{L}_B)_{(s,u)} = \mathcal{L}_{(s,u)} + \sum_{k\in V/B} \frac{w_{sk}}{d(k)} \mathcal{L}_{(k,u)}}$$

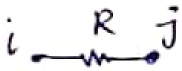
$\mathcal{L}_B$ for removal of V/B.

node removal algorithm ⟶

$$\begin{cases} \mathcal{L}'_{su} = \mathcal{L}_{su} + \dfrac{w_{sk}}{d(k)} \mathcal{L}_{ku} \\ s,u \in B \end{cases}$$

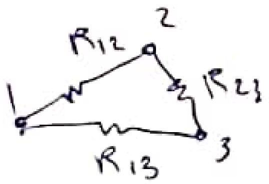
Ⓘⓥ if we can prove the triangle inequality for three nodes, then we can show that $d = \frac{1}{R_{ij}}$ is a meter.

for two nodes:



$$\mathcal{L} = \begin{bmatrix} 1/R & -1/R \\ -1/R & 1/R \end{bmatrix} \Rightarrow \mathcal{L}^{+} = \begin{bmatrix} R & -R \\ -R & R \end{bmatrix} \frac{1}{4}$$

$$\longrightarrow R_{eff}(1,2) = \frac{1}{4}(1 \quad -1)\begin{bmatrix} R & -R \\ -R & R \end{bmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = R \,. \checkmark$$



$$\left.\begin{aligned} R_{eq}^1 &= R_1 \| (R_2+R_3) \\ R_{eq}^2 &= R_2 \| (R_1+R_3) \\ R_{eq}^3 &= R_3 \| (R_2+R_1) \end{aligned}\right\} \quad R_{eq}^1 + R_{eq}^2 = \underbrace{\frac{R_3(R_1+R_2)}{R_1+R_2+R_3}}_{R_{eq}^3} + \frac{2R_1R_2}{R_1+R_2+R_3}$$

$$\downarrow$$

$\Longrightarrow$ $R_{eq}^1 + R_{eq}^2 \geq R_{eq}^3$ which proves that $\frac{1}{R}$ is a valid metric. Q.E.D

جواب 5:

Ⓘ $C_\psi = \int_0 \frac{\overline{\hat{\psi}(w)}}{w} dw$, let $x, \psi \in L^2(R)$, $0 < C_\psi < \infty$

$$W_f(a,b) = \int_{-\infty}^{\infty} x(t) \frac{1}{\sqrt{a}} \overline{\psi\left(\frac{t-b}{a}\right)} dt = \int_{-\infty}^{\infty} \hat{x}(w) \frac{1}{\sqrt{a}} . a \overline{\hat{\psi}(aw) e^{-j2\pi bw}} dw$$

$$= \sqrt{a} \int_{-\infty}^{\infty} \hat{x}(w) e^{j2\pi bw} \overline{\hat{\psi}(aw)} dw$$

$$\Rightarrow \frac{1}{C_\psi} \int_0^{\infty} W_f(a,t) \frac{da}{a^{3/2}} = \frac{1}{C_\psi} \int_0^{\infty} \frac{\sqrt{a}}{a^{3/2}} \int_{-\infty}^{\infty} \hat{x}(w) e^{j2\pi bw} \overline{\hat{\psi}(aw)} dw da \Big|_{b=t}$$

$$= \frac{1}{C_\psi} \int_{-\infty}^{\infty} \hat{x}(w) e^{j2\pi bw} \underbrace{\int_0^{\infty} \frac{\overline{\hat{\psi}(aw)}}{a} da}_{C_\psi} dw \Big|_{b=t} = \frac{C_\psi}{C_\psi} \int_{-\infty}^{\infty} \hat{x}(w) e^{j2\pi bw} dw \Big|_{b=t} = x(b) = \boxed{x(t)}$$

let b=t

Q.E.D.

Ⓘⓘ By definition of wavelet transform.

$$\begin{cases} W_f(a,b) = \int_{-\infty}^{\infty} f(t) \overline{\psi_{a,b}(t)} dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(w) \sqrt{|a|} \overline{\hat{\psi}(aw)} e^{jbw} dw \\ W_g(a,b) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{\hat{g}(w)} \sqrt{|a|} \hat{\psi}(aw) e^{-jbw} dw \end{cases}$$

therefore

$$\Rightarrow \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W_f(a,b) \overline{W_g(a,b)} \frac{da\, db}{a^2} = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{da\, db}{a^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |a| \hat{f}(w) \overline{\hat{g}(w)} \overline{\hat{\psi}(a\varepsilon)} \hat{\psi}(aw) e^{jb(\varepsilon - w)} d\varepsilon\, dw.$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{da}{|a|} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{f}(\varepsilon) \overline{\hat{g}(w)} \overline{\hat{\psi}(a\varepsilon)} \hat{\psi}(aw) \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{jb(\varepsilon - w)} db \right) d\varepsilon\, dw$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{da}{|a|} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{f}(\varepsilon) \overline{\hat{g}(w)} \overline{\hat{\psi}(a\varepsilon)} \hat{\psi}(aw) \delta(\varepsilon - w) d\varepsilon\, dw$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{da}{|a|} \int_{-\infty}^{\infty} \hat{f}(w) \overline{\hat{g}(w)} |\hat{\psi}(aw)|^2 dw$$

note that $\left(\int_{-\infty}^{\infty} f(t) \overline{g(t)} dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(w) \overline{\hat{g}(w)} dw \right.$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(w) \overline{\hat{g}(w)} dw . \underbrace{\int_{-\infty}^{\infty} \frac{|\hat{\psi}(aw)|^2}{|a|} da}_{C_\psi} = C_\psi \int_{-\infty}^{\infty} f(t) \overline{g(t)} dt$$

$$\Rightarrow \boxed{\langle f, g \rangle = \int_{-\infty}^{\infty} f(t) \overline{g(t)} dt = \frac{1}{C_\psi} \int_0^{\infty} \int_{-\infty}^{\infty} W_f(u,s) \overline{W_g(u,s)} \frac{du\, ds}{s^2}}$$

$u = b$

$s = a$

Q.E.D.

ادامه سوال 5:

$$C_\psi \langle f, f\rangle = C_\psi \cdot \int_{-\infty}^{\infty} f(t)\overline{f(t)}\,dT = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_f(a,b)\overline{W_f(a,b)}\,\frac{da\,db}{b^2}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_f(a,b)\int_{-\infty}^{\infty} f(t)\,\overline{\psi_{a,b}(t)}\,dt\,\frac{da\,db}{a^2}$$

$$= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_f(a,b)\,\psi_{a,b}(t)\,\frac{da\,db}{a^2}\right)\overline{f(t)}\,dt = \left\langle f(t), \underset{\psi_{a,b}(t)}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_f(a,b)\,\frac{da\,db}{a^2}}\right\rangle$$

$$\Rightarrow \begin{cases} C_\psi \cdot f(t) = \displaystyle\iint_{-\infty}^{\infty} W_f(a,b)\,\psi_{a,b}(t)\,\frac{da\,db}{a^2} \\ \|f\|^2 = \displaystyle\frac{1}{C_\psi}\int_{-\infty}^{\infty} |W_f(a,b)|^2 \cdot \frac{da\,db}{a^2} \end{cases}$$

Q.E.D. ■

سوال 6:

since $s < d(i,j) \to W^s(i,j) = 0 \to L'(i,j) = \begin{cases} d(i) & i=j \\ w(i,j) & i\neq j \end{cases}$

(I)

$$\Rightarrow \underset{i\neq j}{L^s(i,j)} = \underset{l_1\neq l_2\neq\cdots\neq l_{s-1}}{\sum_{l_1}\sum_{l_2}\cdots\sum_{l_{s-1}}} L_{i l_1} L_{l_1 l_2}\cdots L_{l_{s-1} j} = (-1)^{d(i,j)}\,W^s(i,j) = 0$$

$\forall s < d(i,j)$

(II)

$\forall i \in \{1,\dots,n\}$

$$|\psi_{s,n}(i) - \tilde\psi_{s,n}(i)| = \left|\sqrt{s}\,U \underbrace{\begin{bmatrix} g(s\lambda_1)-\tilde g(s\lambda_1) & & 0 \\ & \ddots & \\ 0 & & g(s\lambda_n)-\tilde g(s\lambda_n) \end{bmatrix}}_{\le M(s)\,I} \delta_n\right| \le \sqrt{s}\cdot M(s)\,\|\delta_n\| = \sqrt{s}\,M(s)$$

Q.E.D. ■

$$\Rightarrow \|\vec\psi_{s,n} - \tilde{\vec\psi}_{s,n}(i)\| = \sqrt{\sum_{i=1}^{n} |\psi_{s,n}(i) - \tilde\psi_{s,n}(i)|^2} \le \sqrt{N\cdot s\,M(t)^2} = \sqrt{Ns}\,M(s)$$

(III)

$$M(t) = \sup_{\lambda\in[0,t'\lambda_n]} |g(s\lambda) - \tilde g(s\lambda)|$$

$$\begin{cases} g^{(k)}(0) = C,\ g^{(r)}(0) = 0 \quad \forall r<k \\ |g^{(k+1)}(\lambda)| \le B \quad \forall \lambda\in(0,t'\lambda_N) \\ \tilde g(\lambda) = \dfrac{C(t\lambda)^k}{k!} = \dfrac{g^{(k)}(0)\,(t\lambda)^k}{k!} \end{cases}$$

$$g(\lambda) = \frac{\lambda^k g^{(k)}(0)}{k!} + \frac{\lambda^{k+1} g^{(k+1)}(0)}{(k+1)!} + o(\lambda^{k+2})$$

Q.E.D. ■

$$\sup_{\lambda} |g(t\lambda) - \tilde g(t\lambda)| = \left|\frac{(t\lambda)^{k+1}}{(k+1)!}\,g^{(k+1)}(0) + \underbrace{o(\lambda^{k+2})}_{\approx 0}\right| \le \sup_{\lambda\in(0,t\lambda_N)}\left\{\frac{B\,(t\lambda)^{(k+1)}}{(k+1)!}\right\} = \frac{B(t\lambda_n)^{k+1}}{(k+1)!}$$