400101204 محمد پارسا دینی — تمرین سری 1 آنالیز الگوریتم‌ها در ابعاد بالا

(I) Firstly, we notice that $\forall x \in \mathbb{R}: \phi(|x|) = \phi(0) + \int_0^{|x|} \phi'(t)\,dt$ which can also be formulated as: $\phi(|x|) = \phi(0) + \int_0^{\infty} \phi'(t)\, \mathbb{1}[|x| \geq t]\, dt$. Now if we take x as a random variable X, and get an expected over both sides with respect to X, we'll obtain the proof:

$$\mathbb{E}[\phi(|X|)] = \mathbb{E}\left[\phi(0) + \int_0^{\infty} \phi'(t)\, \mathbb{1}[|X| \geq t]\, dt\right] = \phi(0) + \int_0^{\infty} \phi'(t)\, \mathbb{P}[|X| \geq t]\, dt. \quad \blacksquare \text{ Q.E.D.}$$

→ note that we assumed the function $\phi(\cdot)$ was differentiable at $\mathbb{R}^+ \cup \{0\}$.

(II) $X \sim \text{subG}(\sigma^2),\ \mathbb{E}X = 0;\ \lambda \geq 0 \Rightarrow \mathbb{E}[e^{\lambda X}] \leq e^{\frac{\lambda^2 \sigma^2}{2}}$

$$\underbrace{\mathbb{P}[X \geq t] \leq e^{-\lambda t}\, \mathbb{E}[e^{\lambda X}]}_{\text{by Chernoff}} \leq e^{-\lambda t + \frac{\lambda^2 \sigma^2}{2}} \xrightarrow[\text{the bound}]{\text{tightening}} \mathbb{P}[X \geq t] \leq \inf_{\lambda \geq 0} \left\{ e^{-\lambda t + \frac{\lambda^2 \sigma^2}{2}} \right\}$$

$$\Rightarrow \mathbb{P}[X \geq t] \leq \exp\left(\inf_{\lambda \geq 0} \left\{-\lambda t + \frac{\lambda^2 \sigma^2}{2}\right\}\right) \rightarrow \frac{\partial}{\partial \lambda}\left\{-\lambda t + \frac{\lambda^2 \sigma^2}{2}\right\} = -t + \lambda \sigma^2 = 0 \rightarrow \boxed{\lambda^* = \frac{t}{\sigma^2}}$$

We can claim that λ^* is the global minima since $-\lambda t + \frac{\lambda^2 \sigma^2}{2}$ is convex with respect to λ. The same calculations for $\mathbb{P}(X \leq t) = \mathbb{P}[-X \geq -t]$ will lead to

$$\begin{cases} \mathbb{P}[X \geq t] \leq \exp\left(\frac{-t^2}{\sigma^2} + \frac{t^2}{2\sigma^2}\right) = \exp\left(-\frac{t^2}{2\sigma^2}\right) \\ \mathbb{P}[X \leq t] \leq \exp\left(-\frac{t^2}{\sigma^2} + \frac{t^2}{2\sigma^2}\right) = \exp\left(\frac{-t^2}{2\sigma^2}\right) \end{cases}$$

$$\Rightarrow \boxed{\mathbb{P}[|X| \geq t] = \mathbb{P}(X \geq t) + \mathbb{P}[X \leq t] \leq 2\exp\left(\frac{-t^2}{2\sigma^2}\right)} \quad \text{Q.E.D} \ \blacksquare$$

(III) using the lemma from (I), let $\phi(|x|) = \exp\left(\frac{x^2}{6\sigma^2}\right)$, $\phi'(x) = \frac{x}{3\sigma^2} \exp\left(\frac{x^2}{6\sigma^2}\right)$. then we get:

$$\mathbb{E}[\phi(|X|)] = \mathbb{E}\left[e^{\frac{X^2}{6\sigma^2}}\right] \leq \phi(0) + \int_0^{\infty} \frac{t}{3\sigma^2} e^{\frac{t^2}{6\sigma^2}}\, \mathbb{P}[|X| \geq t]\, dt \leq 1 + \int_0^{\infty} \frac{2t}{3\sigma^2} e^{\frac{t^2}{6\sigma^2}} e^{-\frac{t^2}{2\sigma^2}}\, dt.$$

the last inequality was due to the fact that $\mathbb{P}(|X| \geq t) \leq 2\exp\left(\frac{-t^2}{2\sigma^2}\right)$. thus:

$$\mathbb{E}\left[e^{\frac{X^2}{6\sigma^2}}\right] = 1 + 2\int_0^{\infty} \frac{t}{3\sigma^2} e^{-\frac{t^2}{3\sigma^2}}\, dt$$

when we look at $\frac{t}{3\sigma^2} e^{-\frac{t^2}{3\sigma^2}}$, it's like the pdf of $\Gamma(\alpha = 2, \beta = \frac{1}{3\sigma^2})$, therefore, we can further simplify the integral since we know that

$$\int_0^{\infty} \frac{\beta^{\alpha}}{\Gamma(\alpha)} t^{\alpha - 1} e^{-\beta t}\, dt = \frac{1}{2} \int_{-\infty}^{\infty} \underbrace{\frac{1}{\Gamma(2)\, 3\sigma^2}}_{1} t\, e^{-\frac{t}{3\sigma^2}}\, dt = \frac{1}{2}.$$ Therefore we can easily obtain that:

$$\boxed{\mathbb{E}\left[e^{\frac{X^2}{6\sigma^2}}\right] = 1 + \frac{1}{2} \times 2 = 2}. \quad \text{Q.E.D} \ \blacksquare$$

اولین سوال ۲:

(IV) $\mathbb{E}\left[e^{sX}\right] \underset{X \le |X|}{\le} \mathbb{E}\left[e^{|sX|}\right] \le \mathbb{E}\left[e^{\frac{\lambda s^2}{2}+\frac{X^2}{2\lambda}}\right] = e^{\frac{\lambda s^2}{2}} \mathbb{E}\left[e^{\frac{X^2}{2\lambda}}\right]\Bigg|_{\lambda=3\sigma^2}$

AM-GM: $\frac{\lambda s^2}{2}+\frac{X^2}{2\lambda} \ge |sX| \ (\forall \lambda>0)$

$\hookrightarrow \mathbb{E}\left[e^{sX}\right] \le e^{\frac{3\sigma^2 s^2}{2}} \times 2$ since $\mathbb{E}\left[e^{\frac{X^2}{6\sigma^2}}\right] \le 2$

this proves that X is $\sqrt{3}\sigma$ - Sub-Gaussian which shows that it's also $\sqrt{18}\sigma$ - Sub Gaussian as well since

$\mathbb{E}\left[e^{\frac{X^2}{6\sigma^2}}\right] \le 2 \longrightarrow \mathbb{E}\left[e^{sX}\right] \le 2e^{\frac{3\sigma^2 s^2}{2}} \le e^{\frac{18\sigma^2 s^2}{2}} \rightarrow \hat{\sigma} = \sqrt{18}\sigma$ Q.E.D ■

(V) $X \sim subG(\sigma)$, $\mathbb{E}X=0$. Let $k \in \mathbb{N}$, $\Phi(|x|)=|x|^{2k}$, then using the result of (I), we can get that:

$$\mathbb{E}\left[X^{2k}\right] \le 0 + \int_0^\infty 2k \cdot t^{2k-1} \mathbb{P}\left[|X| \ge t\right] dt \le \int_0^\infty 2k \cdot t^{2k-1} \cdot 2e^{-\frac{t^2}{2\sigma^2}} dt$$

Now letting $s = \frac{t^2}{2\sigma^2}$, then we'd rewrite the integral as:

$$\int_0^\infty 2k \cdot t^{2k-1} \cdot 2e^{-\frac{t^2}{2\sigma^2}} dt = \int_0^\infty 2k \cdot \frac{(2\sigma^2)^k s^k}{t} \cdot 2e^{-s} \frac{\sigma^2 ds}{t}$$

$$= 2k \cdot (2\sigma^2)^k \int_0^\infty s^{k-1} e^{-s} ds = 2k \cdot (2\sigma^2)^k \Gamma(k) \overset{k \in \mathbb{N}}{=} 2k \cdot (2\sigma^2)^k (k-1)!$$

Note that $s^{k-1}e^{-s}$ resembeled Gamma distribution pdf with $\alpha=k, \beta=1$.

thus, finally we obtained: $\boxed{\forall k \in \mathbb{N}: \mathbb{E}\left[X^{2k}\right] \le 2 \cdot (2\sigma^2)^k \cdot k!}$ Q.E.D ■

(VI)

اولین سوال:

(VI) $\mathbb{E}\left[e^{s(X^2-\mathbb{E}X^2)}\right] = \sum_{k=0}^{\infty} \frac{1}{k!}\mathbb{E}\left[(X^2-\mathbb{E}X^2)^k\right] s^k$

Taylor Series

Binomial Expansion

$$= \sum_{k=0}^{\infty} \frac{s^k}{k!} \sum_{j=0}^{k} \mathbb{E}\left[X^{2j}\right]\left[-\mathbb{E}X^2\right]^{j-k}\binom{k}{j} \longrightarrow \begin{cases}\mathbb{E}X^2 \le 4\sigma^2 \\ \mathbb{E}X^{2j} \le 2\cdot(2\sigma^2)^j j!\end{cases}$$

$$\le \sum_{k=0}^{\infty}\sum_{j=0}^{k} \frac{s^k}{k!}(4\sigma^2)^{-j+k}(2\sigma^2)^j\cdot 2\,\frac{\cancel{k!}\,\cancel{j!}}{\cancel{j!}\,(k-j)!}$$

$$= \sum_{k=0}^{\infty}\sum_{j=0}^{k} \frac{\sigma^{2k}s^k}{(k-j)!}\cdot 2^{2k-j+1}\cdot(-1)^{-j+k} = \sum_{k=0}^{\infty} s^k\sigma^{2k}2^{2k}\sum_{j=0}^{k}\frac{2^{-j+1}}{(k-j)!}(-1)^{k-j}$$

$$= \sum_{k=0}^{\infty} 2^{2k}s^k\sigma^{2k}\sum_{j=0}^{k}\frac{(-2)^{k-j}}{(k-j)!} = 1 + 2\sum_{k=2}^{\infty}(4s\sigma^2)^k\sum_{j=0}^{k}\frac{(-2)^{k-j}}{(k-j)!}\cdot (*)$$

$1+x \le e^x$

$$\le \exp\left[2\sum_{k=2}^{\infty}\sum_{j=0}^{k}(4s\sigma^2)^k\left(\frac{(-1)^{k-j}}{j!}\right)\right] \le \exp\left[2\sum_{k=2}^{\infty}(4s\sigma^2)^k e^{-1}\right]$$

(*) Now since for $k=1$: $\mathbb{E}\left[X^2-\mathbb{E}X^2\right)=0$ we neglect this on the running sum.

$$\Rightarrow \mathbb{E}\left[e^{s(X^2-\mathbb{E}X^2)}\right] \le \exp\left(2e^{-1}\sum_{k=2}^{\infty}(4s\sigma^2)^k\right)$$ will converge if $|4s\sigma^2| \le 1 \to |s| \le \frac{1}{4\sigma^2}$

$$\le \exp\left[\frac{2e^{-1}\cdot 16s^2\sigma^4}{1-4s\sigma^2}\right] \le \exp\left[\frac{32s^2\sigma^4}{1-4|s|\sigma^2}\right] \quad \forall |s| \le \frac{1}{4\sigma^2}$$

Q.E.D ■

$$\mathbb{E}\left[e^{s(X^2-\mathbb{E}X^2)}\right] \le \exp\left(\frac{32s^2\sigma^4}{1-4\sigma^2|s|}\right) \quad \forall |s| \le \frac{1}{4\sigma^2}$$

(I) $Z_i \overset{iid}{\sim} N(0,1)$, $Y = Z^2 - 1$

$$\Rightarrow \mathbb{E}[e^{sY}] = \mathbb{E}[e^{sZ^2 - s}] = e^{-s}\mathbb{E}[e^{sZ^2}] = \frac{e^{-s}}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{sz^2} e^{-\frac{z^2}{2}} dz$$

Now let $z' = \sqrt{1-2s}$ (Assuming that s lies in $(-\infty, 1/2)$); we'd obtain that:

$$\Rightarrow \mathbb{E}[e^{sY}] = \frac{e^{-s}}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-\frac{z^2}{2}(1-2s)} dz = \frac{e^{-s}}{\sqrt{2\pi}}\int_{\mathbb{R}} \frac{e^{-\frac{z'^2}{2}} dz'}{\sqrt{1-2s}} = \frac{e^{-s}}{\sqrt{1-2s}} .$$ Q.E.D

In case s lies in $(\frac{1}{2}, \infty)$; the integral will be infinite, since the exponent will be positive and thus, the integral will be infinite (∞)

(II) In order to claim that $\varphi(s) = \mathbb{E}[e^{sY}] \le e^{\frac{s^2}{1-2s}}$, it suffices to prove that $\frac{e^{-s}}{\sqrt{1-2s}} \le e^{\frac{s^2}{1-2s}}$ $\forall s \in (0, \frac{1}{2})$. Taking log from both sides implies that:

$-s - \frac{1}{2}\log(1-2s) \le \frac{s^2}{1-2s}$. Now we define $f(s) = \frac{s^2}{1-2s} + s + \frac{1}{2}\log(1-2s)$: $\forall s \in (0, \frac{1}{2})$

& we wish to show that $\forall s \in (0, \frac{1}{2})$: $f(s) \ge 0$. We know that $f(0) = 0$, so it would suffice if we just show that $\forall s \in (0, \frac{1}{2})$: $f'(s) \ge 0$:

$$f'(s) = \frac{2s(1-s)}{(1-2s)^2} + 1 + \frac{1}{2} \times \frac{1}{1-2s} = \frac{3 - 6s + 4s^2}{(1-2s)^2 \times 2} .$$

So it'd be always positive

It's obvious that $f'(s) \ge 0$ $\forall s \in (0, \frac{1}{2})$, since $4s^2 - 6s + 3$ has no real roots and $(1-2s)^2$ is always positive. Therefore, we can deduce that

$$\boxed{\varphi(s) = \mathbb{E}[e^{sY}] \le \frac{e^{-s}}{\sqrt{1-2s}} \le e^{\frac{s^2}{1-2s}}}$$ Q.E.D

(III) Firstly, we know that by Bernoulli Inequality that $1 + \frac{u}{2} \ge \sqrt{1+u}$.

$$\mathbb{P}[Y \ge 2t + 2\sqrt{t}] \underset{s>0}{\le} e^{-s(2t+2\sqrt{t})}\varphi(s) \le \exp\Big(\underbrace{-s(2t+2\sqrt{t}) + \frac{s^2}{1-2s}}_{f(s)}\Big) \quad (\star\star)$$

taking inf on $s>0 \Rightarrow \frac{\partial f(s)}{\partial s} = 0 = -2(t+\sqrt{t}) + \frac{2s(1-2s) + 2s^2}{(1-2s)^2} = 0$ which implies:

$$(\underbrace{1 + 4\sqrt{t} + 4t}_{(1+2\sqrt{t})^2})(1-2s)^2 = 1 \underset{t>0}{\overset{s<\frac{1}{2}}{\longrightarrow}} (1-2s)(1+2\sqrt{t}) = 1 \longrightarrow s = \frac{\sqrt{t}}{1+2\sqrt{t}} \quad (\star)$$

Now plugging $(\star)$ in $(\star\star)$ implies that:

$$\mathbb{P}[Y \ge 2t + 2\sqrt{t}] \le \exp\left[\frac{t}{1+2\sqrt{t}} - \frac{2t(1+\sqrt{t})}{1+2\sqrt{t}}\right] = e^{-t}$$ Q.E.D

ادامه سوال 2:

(IV) $\mathbb{P}\left[X \geq \underbrace{n + 2\sqrt{n\log\left(\frac{1}{\delta}\right)} + 2\log\left(\frac{1}{\delta}\right)}_{f(n,\delta)}\right] = \mathbb{P}\left[\sum_{i=1}^{n}(Z_i^2 - 1) \geq f(n,\delta) - n\right] = \mathbb{P}\left[\sum_{i=1}^{n} Y_i \geq f(n,\delta) - n\right]$

$\rightarrow \mathbb{P}\left[X \geq f(n,\delta)\right] \leq \mathbb{P}\left[\bigwedge_{i=1}^{n}\left(Z_i^2 \geq \frac{f(n,\delta)}{n}\right)\right] = \prod_{i=1}^{n} \mathbb{P}\left[Z_i^2 \geq \frac{f(n,\delta)}{n}\right] = \mathbb{P}\left[Z^2 \geq \frac{f(n,\delta)}{n}\right]^n$

Z_i's are independent

which results in:

$$\mathbb{P}\left[X \geq f(n,\delta)\right] \leq \left[\mathbb{P}\left(Z^2 \geq \frac{f(n,\delta)}{n}\right)\right]^n = \mathbb{P}\left[\underbrace{Z^2 - 1}_{Y} \geq \frac{2\sqrt{\log\left(\frac{1}{\delta}\right)}}{\sqrt{n}} + \frac{2\log\left(\frac{1}{\delta}\right)}{n}\right]^n$$

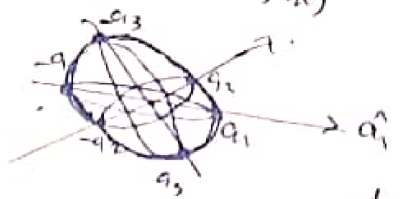
Now plugging $t = \frac{\log(1/\delta)}{n}$ in (III) leads to:

Q.E.D

$$\mathbb{P}\left[X \geq n + 2\sqrt{n\log\left(\frac{1}{\delta}\right)} + 2\log\left(\frac{1}{\delta}\right)\right] \leq \mathbb{P}\left[Y \geq 2t + 2\sqrt{t}\right]^n\Big|_{t=\frac{\log(1/\delta)}{n}} \leq e^{-nt}\Big|_{t=\frac{-\log\delta}{n}} = \delta. \blacksquare$$

سوال 3:

(I) we characterize each d-dimensional ellipsoid with params $\underline{a} = (a_1, \dots, a_d)$ as: $\sum_{k=1}^{d} \frac{x_k^2}{a_k^2} = 1 = \sum_{k=1}^{d} y_k^2$ the ellipsoid is depicted here:



(II) we can also find the volume of an ellipsoid. We know that the volume of a d-dimensional sphere of radius r is $V_{sphere}(r,d) = \frac{\pi^{d/2} r^d}{\Gamma(\frac{d}{2}+1)}$

Now let $y_k = \frac{x_k}{a_k}$, which transform the ellipsoid to the unit sphere:

$$V_{ellipsoid}(\underline{a}) = \int_{-a_d}^{a_d} \cdots \int_{-\sqrt{1-\sum_k \frac{x_k^2}{a_k^2}}}^{\sqrt{1-\sum_k \frac{x_k^2}{a_k^2}}} dx_1 \cdots dx_d = \underbrace{\int_{-1}^{+1} \int \cdots \int^{\sqrt{1-\sum_k y_k^2}} dy_1 \cdots dy_d}_{V_{sphere}(1,d)} \cdot \prod_{k=1}^{d} a_k$$

$$\Rightarrow V_{ellipsoid}(\underline{a}) = V_{sphere}(1,d) \cdot \prod_{k=1}^{d} a_k = \prod_{k=1}^{d} a_k \cdot \frac{\pi^{d/2}}{\Gamma(\frac{d}{2}+1)}.$$

(i)

now for a d-dimensional sphere of radius the volume of a thin shell of thickness εr $(\varepsilon > 0)$, near the surface is given by:

$$V_{shell}(\varepsilon, r) = V_d(r)\left(1 - \left(1 - \frac{\varepsilon}{r}\right)^d\right) \approx V_d(r)\left(1 - e^{-\frac{\varepsilon d}{r}}\right).$$

which shows that if we distribute the volume uniformly, then the probability that a point lies on the shell converges to 1, with $d \to \infty$:

$$\lim_{d\to\infty} \mathbb{P}\left[p \in V_{shell}(r,d)\right] \approx \lim_{d\to\infty} \frac{V_d(r)\left(1 - e^{-\frac{\varepsilon d}{r}}\right)}{V_d(r)} = 1.$$

Furthermore, since the ~~ellipsoid~~ sphere after a linear transformation will be an ellipsoid, the axes will be rescalled, but the concentration of measure property remains

Now we distribute the volume uniformly in our ellipsoid and our sphere.

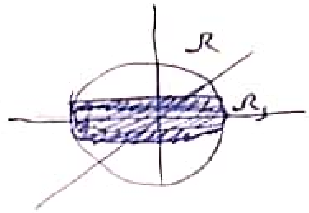
let the pdf's be: $\begin{cases} f(\vec{r}) = f(x_1, \dots, x_d) & \text{on sphere} \\ g(r) = g(x_1, \dots, x_d) & \text{on ellipsoid} \end{cases}$ $0 \le y_k = \frac{x_k}{a_k} \le 1$

$$g(y_1, \dots, y_d) = g(r) = \frac{1}{\prod_{k=1}^{d}|a_k|} f\left(\frac{r}{a}\right) = \frac{1}{\prod_{k=1}^{d}|a_k|} f\left(\frac{x_1}{a_1}, \dots, \frac{x_d}{a_d}\right) = \frac{1}{\prod_{k=1}^{d}|a_k|} f(y_1, \dots, y_d) \quad (*)$$

the (*) shows that the probability mass of the ellipsoid is just $\left(\prod_{k=1}^{d} a_k\right)$ times the probability mass function of the sphere. Thus we could imply that also in ellipsoids, the mass (volume) is concentrated on it's (thin) ~~shell~~ outest shell.

Q.E.D.

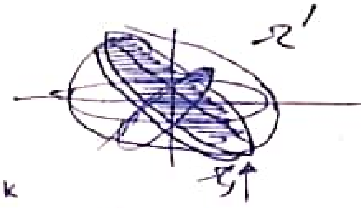
(ii) In spheres we also proved that the mass also lies in the main circle of the sphere as depicted here



(uniformly distributed)

$$\lim_{d\to\infty} \mathbb{P}[p \in \Omega_s] = \lim_{d\to\infty} \frac{|\text{Volume of } \Omega_s|}{|\text{Volume of } \Omega|} = 1 = \lim_{d\to\infty} \frac{|V_{\Omega_s}|}{|V_{\Omega}|}$$

Since we have the same thing in ellipsoids, the numerator & denominator are just scaled which doesn't change the value of ratio



(uniformly distributed)

$$\lim_{d\to\infty} \mathbb{P}[p \in \Omega'_s] = \lim_{d\to\infty} \frac{|V_{\Omega'_s}|}{|V_{\Omega'}|} = \lim_{d\to\infty} \frac{|V_{\Omega'_s}| \times \prod a_k}{|V_{\Omega'}| \times \prod a_k} = \lim_{d\to\infty} \frac{|V_{\Omega_s}|}{|V_{\Omega}|} = 1$$

Q.E.D.

So as similar to spheres, the mass (volume) of a high dimensional ellipsoid lies on its (all) main ovals as depicted above.

Scanned by CamScanner

Let X_i be a Bernoulli with param : $(\delta + \frac{1}{2})$, then we can see that : [illegible]

Each guess of our algorithm will be a Bernoulli $X_i \sim Ber(\frac{1}{2} + \delta)$. Repeating these test N times, we can deduce that our algorithm will be wrong if

$$X_1 + \dots + X_N \leq \frac{N}{2}$$

. Furthermore, let $Y_i = -X_i$. Clearly both X_i, Y_i are bounded in intervals $[0,1]$ & $[-1,0]$, respectively. We have:

$$\mathbb{P}\left[\sum_{i=1}^{N} X_i \leq \frac{N}{2}\right] = \mathbb{P}\left[\sum_{i=1}^{N}(X_i - \mathbb{E}X_i) \leq \frac{N}{2} - N\left(\frac{1}{2}+\delta\right)\right] = \mathbb{P}\left[\sum_{i=1}^{N}(X_i - \mathbb{E}X_i) \leq -N\delta\right]$$

switching to Y_i's

$$= \mathbb{P}\left[\sum_{i=1}^{N}(Y_i - \mathbb{E}Y_i) \geq \underbrace{N\delta}_{t}\right] \leq \exp\left(\frac{-2t^2}{\sum_{i=1}^{N}(-1-0)^2}\right) = \exp\left(\frac{-2N^2\delta^2}{N}\right)$$

By Hoeffding

Now we'd obtain that:

$$\mathbb{P}\left[\text{algorithm is wrong}\right] \doteq \mathbb{P}\left[\sum_{i=1}^{N} X_i \leq \frac{N}{2}\right] \leq \exp(-2N\delta^2) \leq \varepsilon$$

which leads to $\boxed{N \geq \frac{1}{2\delta^2}\log\left(\frac{1}{\varepsilon}\right)}$

Q.E.D. ■

$X_1, \ldots, X_n \sim \mathrm{subG}(\sigma^2)$

$:5$ سؤال

$$\Rightarrow \mathbb{P}\left[\max_{1\le i\le n}\{X_i - \mathbb{E}X_i\} \ge (1+\varepsilon)\sigma\sqrt{2\log n}\right] \xrightarrow{\text{Union Bound}}$$

$$= \mathbb{P}\left[\left(X_1 - \mathbb{E}X_1 \ge (1+\varepsilon)\sigma\sqrt{2\log n}\right) \vee \cdots \vee \left(X_n - \mathbb{E}X_n \ge (1+\varepsilon)\sigma\sqrt{2\log n}\right)\right] \le \sum_{i=1}^{n} \mathbb{P}\left[X_i - \mathbb{E}X_i \ge (1+\varepsilon)\sigma\sqrt{2\log n}\right]$$

Now since each X_i was a $\mathrm{subG}(\sigma^2)$ random variable, we get:

$$\sum_{i=1}^{n} \mathbb{P}\left[X_i - \mathbb{E}X_i \ge (1+\varepsilon)\sigma\sqrt{2\log n}\right] \le \exp\left(-\frac{(1+\varepsilon)^2\sigma^2 \cdot 2\log n}{2\sigma^2}\right) \times n$$

$$\Rightarrow \mathbb{P}\left[\max_i\{X_i - \mathbb{E}X_i\} \ge (1+\varepsilon)\sigma\sqrt{2\log n}\right] \le n \cdot \exp\left(\log\left(n^{-(1+\varepsilon)^2}\right)\right)$$

Now letting $n\to\infty$ will show that the Probability Converges to Zero:

$\forall \varepsilon > 0,\ (1+\varepsilon)^2 \ge 1$

$$\forall \varepsilon > 0: \lim_{n\to\infty} \mathbb{P}\left[\max_i\{X_i - \mathbb{E}X_i\} \ge \sigma(1+\varepsilon)\sqrt{2\log n}\right] \le \lim_{n\to\infty} \exp\left(\log\left(\frac{n}{n^{(1+\varepsilon)^2}}\right)\right) = \lim_{n\to\infty} \frac{n}{n^{(1+\varepsilon)^2}} = 0$$

this shows that $Y = \max_i\{X_i - \mathbb{E}X_i\}$ is subgaussian with param $\boxed{\sigma\sqrt{2\log n}}$. ∎

(I) $X_1, \dots, X_n$ (independent), $\mathbb{E}X_i = 0$, $\mathbb{E}X_i^2 = 1$ $\quad X = (X_1 - X_n)^T$, $a = (a_1, \dots, a_n)^T$

$: \delta \mathbb{R}^n$

from Jensen's inequality: $\left\| \sum_{i=1}^n a_i X_i \right\|_{L^p} \geq \left\| \sum_{i=1}^n a_i X_i \right\|_{L^2} \quad \forall p \in [2, \infty)$

Now we have:

$$\left\| \sum_{i=1}^n a_i X_i \right\|_{L^2} = \left[\mathbb{E}\left[\left(\sum_{i=1}^n a_i X_i\right)^2\right]\right]^{1/2} = \left[\mathbb{E}\left[\sum_{i=1}^n a_i^2 X_i^2\right] + \mathbb{E}\left[\sum_{(i \neq j)} a_i a_j X_i X_j\right]\right]^{1/2}$$

$$= \left[\sum_{i=1}^n a_i^2 \underbrace{(\mathbb{E}X_i^2)}_{1} + \sum_{i \neq j} a_i a_j \underbrace{\mathbb{E}X_i X_j}_{=(\mathbb{E}X_i)(\mathbb{E}X_j) = 0} \right]^{1/2} = \left[\sum_{i=1}^n a_i^2\right]^{1/2} = \|a\|_2$$

which proves one side of the inequality:

Q.E.D.

$$\sqrt{a^T a} = \|a\|_2 \leq \left\| \sum_{i=1}^n a_i X_i \right\|_{L^p} = \left(\mathbb{E}\left[|a^T X|^p\right]\right)^{1/p}$$

(II) From the properties of the subgaussian norm $\|\cdot\|_{\psi_2}$ we know that $(\forall p \geq 1)$ $\|X\|_{L^p} \leq C \|X\|_{\psi_2} \sqrt{p}$. thus, we could imply that

$$\left\| \sum_{i=1}^n a_i X_i \right\|_{L_p}^2 \leq C^2 \cdot p \left\| \sum_{i=1}^n a_i X_i \right\|_{\psi_2}^2 \leq C'^2 \cdot p \sum_{i=1}^n \|a_i X_i\|_{\psi_2}^2 = C'^2 \cdot p \sum_{i=1}^n a_i^2 \|X_i\|_{\psi_2}^2$$

Since $\|\cdot\|_{\psi}$ is a norm (it satisfies triangular inequality)

Hence: $\left\| \sum_{i=1}^n a_i X_i \right\|_{L^p}^2 \leq C'^2 \cdot p \cdot \|X_i\|_{\psi_2}^2 \left(\sum_{i=1}^n a_i^2\right) = C'' \cdot p \cdot \|X_i\|_{\psi_2}^2 \; \|a\|_2^2$

which by taking square root will result in (also taking $\max_i \|X_i\|_{\psi_2}^i$)

$$\boxed{\|a\|_2 = \sqrt{a^T a} \leq \left\| \sum_{i=1}^n a_i X_i \right\|_{L^p} = \left(\mathbb{E}\left[|a^T X|^p\right]\right)^{1/p} \leq C''' \cdot \sqrt{p} \, \|X_i\|_{\psi_{2,max}} \|a\|}$$

Q.E.D.

By Rewriting the Hölder inequality for Random L^p-norms, we get:

let $p \in (0,2]$ $\qquad \frac{r}{p}+\frac{r}{q}=1$, $|f|^r \in L^{p/r}$, $|g| \in L^{q/r}$

$$\|f \cdot g\|_{L^r}^r = \int |fg|^r = \| |f \cdot g|^r \|_{L^1} \leq \| |f|^r \|_{L^{p/r}} \, \| |g|^r \|_{L^{q/r}}$$

$$\hookrightarrow \|f \cdot g\|_{L^r}^r \leq \left(\int (|f|^r)^{p/r} \right)^{r/p} \cdot \left(\int (|g|^r)^{q/r} \right)^{r/q} = \|f\|_{L^p}^r \cdot \|g\|_{L^q}^r$$

Now take $r=2, p=q=4 \rightarrow \|XY\|_{L^2} \leq \|X\|_{L^4} \|Y\|_{L^4} = (\mathbb{E}|X|^4)^{1/4} (\mathbb{E}|Y|^4)^{1/4}$

Now let $X = |Z|^{p/4}$, $Y = |Z|^{\frac{4-p}{4}}$ implies that:

$$\|Z\|_{L^2} \leq \left[\mathbb{E}|Z|^p\right]^{1/4} \left[\mathbb{E}|Z|^{4-p}\right]^{1/4} = \|Z\|_{L^p}^{p/4} \cdot \|Z\|_{L^{4-p}}^{\frac{4-p}{4}}$$

Now if we let $Z = \sum_{i=1}^n a_i X_i$; we'd obtain that:

$$\left\| \sum_{i=1}^n a_i X_i \right\|_{L^p} \geq \frac{\left\| \sum_{i=1}^n a_i X_i \right\|_{L^2}^{4/p}}{\left\| \sum_{i=1}^n a_i X_i \right\|_{L^{4-p}}^{\frac{4-p}{p}}} \qquad (*)$$

However, we knew that: $\left\| \sum_{i=1}^n a_i X_i \right\|_{L^2} = \|a\|$ $(**)$ from Problem 6:

furthermore from subgaussian norm properties and Problem 6:

$$\left\| \sum_{i=1}^n a_i X_i \right\|_{L^{4-p}} \leq CK\sqrt{4-p}\,\|a\| \qquad (***)$$

now, combining $(*), (**), (***)$:

$$\left\| \sum_{i=1}^n a_i X_i \right\|_{L^p} \geq \|a\|^{4/p} \cdot \frac{1}{\left(CK\sqrt{4-p}\,\|a\|\right)^{\frac{4-p}{p}}} = \left(CK\sqrt{4-p}\right)^{-\frac{p}{4-p}} \|a\|$$

plugging $p=1$ in above: $\qquad c(K) = (CK\sqrt{3})^{-1/3}$

$$\boxed{\left\| \sum_{i=1}^n a_i X_i \right\|_{L^1} \geq (CK\sqrt{3})^{-1/3} \|a\| = c(K)\sqrt{a^T a}}$$

Q.E.D. ■

for $p \leq 1$ & by jensen we have: $\left\| \sum_{i=1}^n a_i X_i \right\|_{L^p} \leq \left\| \sum a_i X_i \right\|_{L^2}$

Q.E.D. ■

we also now that:

let $p=1$: $\boxed{\left\| \sum_{i=1}^n a_i X_i \right\|_{L^1} \leq \left\| \sum_{i=1}^n a_i X_i \right\|_{L^2} = \|a\| = \sqrt{a^T a}}$

Now that both sides of inequality is done, we are done by the Proof.

We know that $\bar{d} = o(\log n)$ & $G \sim G(n,p)$ is a Erdős Rényi graph. :8 ولى

thus: $\bar{p} = \frac{\bar{d}}{n-1} \to p = o\left(\frac{\log n}{n}\right) \xrightarrow{\exists \varepsilon > 0} p < \frac{\varepsilon \log n}{n} \quad (\forall n > N_0)$

the point is although vertices are not independent, any chosen set of k vertices, will be independent as $n \to \infty$. So for a fixed subset of k vertices, the probability that there's no edge between them is $(1-p)^{\binom{k}{2}} > \left(1 - \frac{\varepsilon \log n}{n}\right)^{k^2} > 1 - \frac{\varepsilon k^2 \log n}{n} \longrightarrow$

Since $p < \frac{\varepsilon \log n}{n}$ & $\binom{k}{2} < k^2$.

Now let $k = n^{1/2 - \delta}$, then as $n \to \infty$; the probability that any particular set of k vertices is independent approaches $1 \to \lim_{n \to \infty} \mathbb{P}[\underbrace{\forall i: d_i\text{'s are independent}}_{\text{event } S}] = 1$.

let event $\mathcal{E}$: There isn't a node i with $d_i = 10\bar{d}$.

$\hookrightarrow \mathbb{P}[\mathcal{E}] \leq \mathbb{P}[\mathcal{E} \mid S]\,\mathbb{P}[S] + \mathbb{P}[\mathcal{E} \mid \bar{S}]\,\mathbb{P}[\bar{S}] \lesssim \mathbb{P}[\mathcal{E} \mid S]$

By the argument the latter probability can be arbitrarily small as $n \to \infty$. Thus we must bound the former probability which is compatible due to the conditional independence!

$$\mathbb{P}[\bar{\mathcal{E}} \mid S] = \left(1 - \binom{n-k}{10\bar{d}} p^{10\bar{d}} (1-p)^{n-k-10\bar{d}}\right)^k$$

using $\binom{n}{k} \geq \left(\frac{n}{k}\right)^k$, $n-k > \frac{n-1}{2}$, $1-x \geq e^{-2x}$ $(x < c)$ $\to$

$$\mathbb{P}[\bar{\mathcal{E}} \mid S] \geq \left(\frac{(n-k)p}{10\bar{d}}\right)^{10\bar{d}} (1-p)^n \geq \left(\frac{1}{20}\right)^{10\varepsilon \log n} \left(1 - \frac{\varepsilon \log n}{n}\right)^n$$

$$\mathbb{P}[\bar{\mathcal{E}} \mid S] \geq \left(\frac{1}{20}\right)^{10\varepsilon \log n} \left(1 - \frac{\varepsilon \log n}{n}\right)^n > e^{-10\varepsilon \log(20) \log n}\, e^{-2\varepsilon \log n} = n^{-c\varepsilon}$$

$$\to \mathbb{P}[\mathcal{E} \mid S] \leq \left(1 - n^{-c\varepsilon}\right)^{n^{1/2-\delta}} = e^{-n^{\frac{1}{2} - \delta - c\varepsilon}}$$

Now assuming that δ, ε are chosen such that ~~$\mathbb{P}[\mathcal{E} \mid S]$~~ $\frac{1}{2} - \delta - c\varepsilon > 0$. the last inequality implies that the conditional probability approaches 0 as $n \to \infty$ with high probability. thus $\lim_{n \to \infty} \mathbb{P}(\mathcal{E}) = 0$ (with high probability)

So it could be found a vertex i with $d_i \geq 10\bar{d}$. Q.E.D. ∎

$$\mathbb{P}(\mathcal{E}) = e^{-n^{\frac{1}{2} - \delta - c\varepsilon}} = 0.1 \to \left(\tfrac{1}{2} - \delta - c\varepsilon\right) \log n = \log |\log(0.1)|$$

$$\to \left(\tfrac{1}{2} - \delta\right) \log n > \log |\log(0.1)| = 0.83 \checkmark$$