we say $\vec{b}$ is more than $\vec{a}$ iff: $(a \geq b) \overset{\text{fix } B_j}{\longleftrightarrow} \frac{1}{m} \sum_{i \in B_j} \left(\| X_i - b\|^2 - \|X_i - a\|_2^2 \right)$ for most of j's $\in \{1 \dots k\}$

then by expanding we get

$$\delta = \frac{1}{m} \sum_{i \in B_j} \left(\|X_i - b\|_2^2 - \|X_i - a\|_2^2 \right) = \frac{1}{m} \sum_{i \in B_j} \left(\|2a^T X_i + \|b\|_2^2 - \|a\|_2^2 + \cancel{\|X_i\|_2^2} - \cancel{\|X_i\|_2^2} - 2b^T X_i \right)$$

$$= \|b\|_2^2 - \|a\|_2^2 + \frac{2}{m} \sum_{i \in B_j} \langle X_i, a-b \rangle$$

$\boxed{\text{تمرین ۸: ادامه‌ی}}$

from dedication we used $Z_j = \frac{1}{m} \sum_{i \in B_j} X_i$, we can easily see that if $a \leq b$:

① $\delta = 2\langle Z_j, a-b \rangle + \|b\|_2^2 - \|a\|_2^2 + \|Z_j\|_2^2 - \|Z_j\|_2^2 = \|Z_j - b\|_2^2 - \|Z_j - a\|_2^2$.

So we found that if $\boxed{a \leq b \Rightarrow \underset{j \in \{1 \dots k\}}{\text{for most}} \ \|Z_j - a\|_2^2 \geq \|Z_j - b\|_2^2}$

$\hookrightarrow \forall j \in |J|$ where $|J| \geq k/2$

② from the Previous Parts we can say that

$b \geq a \overset{\text{if and only if}}{\longleftrightarrow}, a \in S_b \Leftrightarrow b \notin S_a$

we also know that: $|\langle Z_i - \mu, v \rangle|$

③ $\left.\begin{array}{l} \forall v \in S^{d-1}, \\ \text{for at least } 0.7 \text{ of Blocks} \end{array}\right\} P\left[\left| \frac{1}{m} \sum_{i \in B_j} \langle X_i - \mu, v \rangle \right| \leq \frac{r^2}{4} \right] \geq 1 - \delta$

④ $\boxed{\text{if } a \geq b \text{ then } \|Z_j - a\|_2^2 \leq \|Z_j - b\|_2^2 \Leftrightarrow \|Z_j - b + (b-a)\|_2^2 \leq \|Z_j - b\|_2^2 \Leftrightarrow \langle Z_j - b, a - b \rangle \geq \frac{\|b-a\|_2^2}{2}}$

⑤ now if we let $a = \mu$, $b = a$, $v = b - a = a - \mu$ $\rightarrow$ we'll get that

$$\mu \geq a) \xrightarrow{\text{if}} \langle Z_j - \mu, v \rangle \leq \frac{\|v\|_2^2}{2} \leq \frac{r^2}{2}$$

$$\forall a: \|a - \mu\| \geq r \rightarrow \|v\|_2 \leq r$$

So we can say that $\mu \geq a \rightarrow \begin{cases} \langle Z_j - \mu, v \rangle \leq \frac{\|v\|_2^2}{2} \\ \text{for most of } j\text{'s} \in J \end{cases}$

So without the loss of Generality we take the assumption that $\|v\| = r$ which has no harm and it's kind of worst case.

(a) if for $X_1, \ldots, X_N \to \left|\frac{1}{m}\sum_{i \in B_j} \langle X_i - \mu, v \rangle\right|, \langle Z_j, v \rangle \le \frac{r^2}{2}$
$\forall v : \|v\| \le r$

then the definitions in this part will suggest that $\begin{cases} \vec{\mu} \succeq \vec{a} \\ \forall a, a \notin B_r(\mu) \\ \|a - \mu\| \ge r \end{cases}$

thus we can say that $\forall v \in S^d: r \to \begin{cases} |\langle Z_j - \mu, v \rangle| \le \frac{r^2}{2} \\ \text{for out of blocks with Prob} \ge 1-\delta \end{cases}$

$\to$ $\boxed{\text{in this case } \mu \succeq a \ \ \forall a : \|\vec{a} - \vec{\mu}\| \ge r \text{ with Probability at least } 1-\delta}$ Q.E.D. ■

(ii) *

in the first section we proved that:

$$b \succeq a \iff b \notin S_a \to b \in \bar{S}_a \to a \in S_b$$

furthermore, with probability of at least $1-\delta$, $\forall a : \|\vec{a} - \vec{\mu}\| \ge r$
which suggests $\mu \in \bar{S}_a, a \notin S_\mu$ and due to the definition we get:

$\forall a \in \bar{S}_\mu : \|a - \mu\|_2 \le r$
we also know $\mu \in \bar{S}_a \to \operatorname{diam}(\bar{S}_a) \ge \|\vec{a} - \vec{\mu}\|_2$ $\Big\}$ ~~diam$(\bar{S}_\mu) \ge$~~ $\operatorname{diam}(\bar{S}_\mu) \le \|a\| + \|\mu\| \le 2r$ $(*)$

$\xrightarrow{\text{and}}$ $\hat{\mu}_N = \underset{a \in \mathbb{R}}{\arg\min} \{\operatorname{diam}(\bar{S}_a)\} \xrightarrow{(*)} \boxed{\operatorname{diam}(\bar{S}_{\hat{\mu}_N}) \le \operatorname{diam}(\bar{S}_\mu) \le 2r}$ $(**)$

So by letting $a = \mu \to$ we can get that
① if $(\mu \succeq \hat{\mu}_N) \Longrightarrow \mu \in \bar{S}_{\hat{\mu}_N}$
② if $(\hat{\mu}_N \succeq \mu) \Longrightarrow \hat{\mu}_N \in \bar{S}_\mu$
$(\mu \succeq \hat{\mu}_N) \vee (\hat{\mu}_N \succeq \mu) = 1$ (true)
$\to \left[(\mu \in \bar{S}_{\hat{\mu}_N}, \operatorname{diam}(\bar{S}_{\hat{\mu}_N}) \le 2r) \vee (\hat{\mu}_N \in \bar{S}_\mu, \operatorname{diam}(\bar{S}_\mu) \le 2r)\right]$
(1) True $=$

Thus, we can deduce that $\|\mu - \hat{\mu}_N\| \le \operatorname{diam}(\bar{S}_\mu) \le 2r$

$$\|\mu - \hat{\mu}_N\| \le 2 \max\left\{400\sqrt{\frac{\operatorname{Tr}\{\Sigma\}}{N}}, 240\sqrt{\frac{\|\Sigma\| \log(\frac{1}{\delta})}{N}}\right\}$$

Q.E.D. ■