پاسخ تمرین ITSL　　محمد ساداتی ۴۰۰۱۰۱۲۰۴　　سوال 1:　　بنام یزدان توانا

$$\chi^2(P,Q) = \sup_{h:\mathbb{R}\to\mathbb{R}} \mathbb{E}_P(h(x)) - \mathbb{E}_Q\left[\frac{h(x)^2}{4}\right] + 1 \overset{(*)}{=} \sup_{h:\mathbb{R}\to\mathbb{R}} \mathbb{E}_P[h(x)] - \mathbb{E}_Q[h(x)] - \frac{1}{4}\operatorname{Var}_{X\sim Q}(h(x))$$

$$h : h+c \to \mathbb{E}_P[h(x)] + c - \mathbb{E}_Q\left[\frac{h(x)^2}{4}\right] + \frac{c^2}{4} - \frac{2c}{4}\mathbb{E}_Q[h(x)] + 1$$

$$\sup_c \hookrightarrow \frac{\partial}{\partial c} = 0 \to 1 - \frac{c}{2} - \frac{1}{2}\mathbb{E}_Q[h(x)] = 0 \to c^* = \mathbb{E}_Q[h(x)] + 2$$

Plugging c^* into $(*)$ implies that: (this section is done in the next page!)

$$\chi^2(P,Q) = \sup_{h:\mathbb{R}\to\mathbb{R}} \mathbb{E}_P[h(x)] - \mathbb{E}_Q[h(x)] - \frac{1}{4}\underbrace{\operatorname{Var}_{X\sim Q}(h(x))}_{\left(\mathbb{E}_Q[h(x)^2] - \mathbb{E}_Q[h(x)]^2\right)}$$

Now we have $Q_{XY} = Q_X Q_Y$ and a function $h(x,y)$ over $\mathbb{R}^2$. So we reduce the space of all functions $h(x,y) : \mathbb{R}^2 \to \mathbb{R}$ to all separable functions $h(x,y) = a(x) + b(y)$. thus we get

$$\sup_{h(x,y)} \mathbb{E}_{P_{XY}}[h(x,y)] - \mathbb{E}_{Q_{XY}}[h(x,y)] - \frac{1}{4}\operatorname{Var}_{Y,X\sim Q_{XY}}(h(X,Y)) \geq$$

$$\sup_{a(x),b(y)} \mathbb{E}_P[a(x)+b(y)] - \mathbb{E}_Q[a(x)+b(y)] - \frac{1}{4}\operatorname{Var}_{Y,X\sim Q}[a(x)+b(y)]$$

$$= \sup_{a(x),b(y)} \begin{cases} \mathbb{E}_P[a(x)] - \mathbb{E}_Q(a(x)) - \frac{1}{4}\left[\mathbb{E}_Q[a(x)^2] - (\mathbb{E}_Q[a(x)])^2\right] + \\ \mathbb{E}_P[b(y)] - \mathbb{E}_Q(b(y)) - \frac{1}{4}\left[\mathbb{E}_Q(b(y)^2) - \mathbb{E}_Q[b(y)]^2\right] \\ -\frac{1}{4}\Big[\underbrace{\mathbb{E}_{Q_{XY}}[a(x)b(y)] - \mathbb{E}_{Q_X}[a(x)]\,\mathbb{E}_{Q_Y}[b(y)]}\Big] \to (\text{zero}) \end{cases}$$

Since the term $\mathbb{E}_{Q_{XY}}(ab) = \mathbb{E}_{Q_X}(a)\,\mathbb{E}_{Q_Y}(b)$ for the fact that h is separable, so is $Q_{XY} = Q_X Q_Y$. therefore, we get:

$$\chi^2(P_{XY}, Q_X Q_Y) \geq \sup_{a(x),b(y)} \begin{cases} \mathbb{E}_P(b(y)) - \mathbb{E}_Q(b(y)) - \frac{1}{4}\overbrace{\left[\mathbb{E}_Q[b(y)^2] - (\mathbb{E}_Q\, b(y))^2\right]}^{\operatorname{Var}_Q(b(y))} \\ \mathbb{E}_P(a(x)) - \mathbb{E}_Q(a(x)) - \frac{1}{4}\left[\mathbb{E}_Q[a(x)^2] - (\mathbb{E}_Q[a(x)])^2\right] \end{cases}$$

$$= \sup_{a(x)} \mathbb{E}_P[a(x)] - \mathbb{E}_Q(a(x)) - \frac{1}{4}\operatorname{Var}_Q(a(x))$$

$$+ \sup_{b(y)} \mathbb{E}_P(b(y)) - \mathbb{E}_Q(b(y)) - \frac{1}{4}\operatorname{Var}_Q(b(y))$$

$$= \chi^2(P_X \| Q_X) + \chi^2(P_Y \| Q_Y)$$