نازان — پاسخ درس ITSL — محمدپارسا دینی — ۴۰۰۱۰۱۲۰۴ — سوال 2: — بنام ایزد یکتا

(I) $Y \sim \mathcal{N}(0, \beta I) \in \mathbb{R}^{\ell}$

$$D_{KL}(P_X \| P_Y) = \mathbb{E}_{X\sim P_X}\left[-\log\left(\frac{P_Y}{P_X}\right)\right] = \mathbb{E}_{X\sim P_X}\left[-\log\left(\frac{1}{P_X}\right) + \log P_Y\right] = -h(X) + \mathbb{E}_{X\sim P_X}\left[\log P_Y\right]$$

$$= -h(X) + \mathbb{E}_{X\sim P_X}\left[\frac{\ell}{2}\log 2\pi\beta + \frac{1}{2\beta}\|X\|_2^2\right] = -h(X) + \frac{\ell}{2}\log 2\pi\beta + \frac{1}{2\beta}\mathbb{E}\left[\|X\|_2^2\right]$$

(II) let $\mathbb{E}\left[\|X\|_2^2\right] \leq t$ since from (I) we get:

$h(Y) = -D_{KL}(P_X \| P_X) + \frac{\ell}{2}\log(2\pi\beta) + \frac{1}{2\beta}\mathbb{E}\left[\|X\|_2^2\right]$. In order to maximize $h(Y)$, one can increase $\mathbb{E}\|X\|_2^2$ to its max, & also decrease $D_{KL}(P_X \| P_X)$ since $D_{KL} \geq 0$ its min $= 0$, therefore greedy thinking we will get $\mathbb{E}\left[\|X\|^2\right] = t$, $P_X = P_Y$ or $D_{KL}(P_X \| P_X) = 0$. therefore $Y$ must take Normal distribution (its mean ($\mu$) plays no role. so let it be 0.) $\Rightarrow$ $Y \sim \mathcal{N}(0, \beta I)$

$$\frac{\partial h(Y)}{\partial \beta} = \frac{\partial}{\partial \beta}\left[\frac{+\ell}{2}\log(2\pi\beta) + \frac{+1}{2\beta}\underbrace{\mathbb{E}\left[\|X\|^2\right]}_{t}\right] = 0$$

$$\hookrightarrow \frac{+2\pi\ell}{4\pi\beta} + \frac{-t}{2\beta^2} = 0 \longrightarrow \frac{t}{2\beta^2} = \frac{\ell}{2\beta} \longrightarrow \boxed{\beta^* = \frac{t}{\ell}}$$

thus: $$h_{max}(Y) = \frac{\ell}{2}\log(2\pi\beta) + \frac{t}{2\beta} = \frac{\ell}{2}\log\left(\frac{2\pi t}{\ell}\right) + \frac{\ell}{2} = \boxed{\frac{\ell}{2}\log\left(\frac{2\pi t e}{\ell}\right)}$$

(III) since $I(X;Y) = h(X) - h(Y|X)$ & the fact that $Y = AX + Z$, we can deduce that $h(Y|X) = h(Z)$, therefore $\boxed{I(X;Y) = h(Y) - h(Z)}$

Q.E.D. ■

1 نمره

(I) We can see that the output of multiplication of two Signed - Permutation Matrices A, B ; are always in $\{0, \pm 1\}$. So we must just Prove that in each row & each column of C, there is only one nonzero element. تمرین

We know that each row & column of A, B is one of the $\{\pm e_1, \pm e_2, \dots, \pm e_l\}$, so when considering $A b_i = \begin{bmatrix} a_1 & \cdots & a_l \end{bmatrix}^T b_i$, there will be one and exactly one row of A (let it be $a_k$) that $a_k^T b_i \neq 0$. And this will prove that each column of C can not be zero and it will be in the set $\{\pm e_1, \dots, \pm e_l\}$.

We can do the same thing for $a_i^T B = a_i^T \begin{bmatrix} b_1 & \cdots & b_l \end{bmatrix}$ and it will be suffice to prove it.

2 نمره

So the multiplication of two Signed - Permutation Matrices will be a Signed - Permutation matrix as its elements $c_{ij} \in \{0, \pm 1\}$ & each row & column has exactly one non-zero element. Q.E.D ∎

(II) let $x$ be $k$-sparse vector ($\|x\|_0 = k$) & $B \in \mathcal{B}$ a Signed-Permutation Matrix.

Since $\forall i, \exists j : y_i = b_i^T x = \pm e_j^T x = \pm x_j$, we can deduce that $y = Bx$ has the exact elements with ~~different~~ the same absolute value & in a different order since a permutation matrix was applied.

therefore, $Y = BX$ and $X$ have the exact number of zero elements.

therefore $Y = BX$ is a $k$-sparse vector since $\|x\|_0 = \|Y\|_0 \leq k$. ■

Q.E.D

(III) Since each row of $B$ is in the set $\{\pm e_1, \dots, \pm e_\ell\}$, then we have

$$b_i^T b_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \quad \text{. so let } C = BB^T \rightarrow C_{ij} = b_i^T b_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

therefore we can see that $BB^T = I$, QED ■ $\rightarrow$ also $B^T B = I$ for the same rationale. $B^T = B^{-1}$

(IV) $\|x - x'\|_2^2 = (x - x')^T (x - x') = x^T x + x'^T x' - 2x'^T x.$

$$\|Bx - Bx'\|_2^2 = (Bx - Bx')^T (Bx - Bx') = x^T \underbrace{B^T B}_{I} x + x'^T \underbrace{B^T B}_{I} x' - 2x'^T \underbrace{B^T B}_{I} x$$

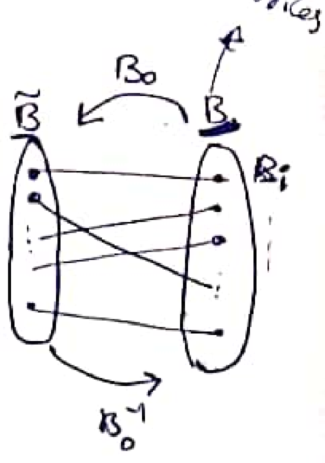
so $\|x - x'\|_2^2 = \|Bx - Bx'\|_2^2 = x^T x - x^T x' - 2x^T x'$ Q.E.D ■

$\|\cdot\| \geq 0$ $\|x - x'\|_2 = \|Bx - Bx'\|_2$

(V) Permutation Matrices $\rightarrow \ell!$ with $\{0, \pm 1\}$ elements

Signed-Permutation Matrices $\rightarrow 2^\ell \times \ell!$

(VI) So in this graph which is depicted down below we can see that each $B_{0i}$ will correspond to exactly one element of $\tilde{B}$, since $B, B_0$ are full rank, they have inverse matrices, therefore $\tilde{B}$ can correspond to only one $B_{0i}$ via $B_0^{-1}$. Now since the distribution of $B_0$'s was uniform over $B_0$, we can deduce that $\tilde{B}$ will have uniform distribution over its elements, since we have an injective transform from $B_0$ to $\tilde{B}$




Q.E.D ■ $\frac{1}{|B|} = f_{B_0}(B) = f_{B_0}(B_0^{-1}\tilde{B}) = g_{\tilde{B}}(\tilde{B}) = \frac{1}{|B|}$

(VI) In this $B$ is a random matrix. It's obvious that $\mathbb{E}[b_{ij}]=0$ for each element of $B$. let $y = Bx$, then

$$y_i = b_i^T x \Rightarrow \mathbb{E}[y_i] = \mathbb{E}[b_i^T x] = \mathbb{E}\Big[\sum_{j=1}^{l} b_{ij} x_j\Big] = \sum_{j=1}^{l} \underbrace{\mathbb{E}[b_{ij}]}_{0} x_j = 0$$

so $\forall i$ $\mathbb{E}\, y_i = 0$ which leads to $\mathbb{E}[y] = \mathbb{E}[Bx] = 0$.

~~for the second part we can realize that each element which does not lie on the main diagonal is zero (proof).~~

~~let $i \neq j \to c_{ij} = b_i x x^T b_j = \sum_{k=1}^{l} b_{ik} b_{kj}$~~

~~$C = Bxx^TB^T$~~

let $C = \mathbb{E}[Bxx^TB^T]$

Since from (V) we got that applying a matrix $B_0$ to $B$ will not change its distribution & expectation value since $B_0$ is a full rank matrix. so by multiplying $C$ from both sides by $B_0$ we will get:

$B_0 B_0^T = I$

$$B_0 C B_0^T = B_0\, \mathbb{E}[Bxx^TB^T] B_0^T = \mathbb{E}\Big[\underbrace{B_0 B}_{\tilde{B}}\, xx^T \underbrace{B^T B_0^T}_{\tilde{B}^T}\Big] = \mathbb{E}[\tilde{B} x x^T \tilde{B}] = C$$

$B \sim Unif(\frac{1}{|B|})$ $\quad$ $\tilde{B} \sim Unif(\frac{1}{|B|})$

Since $B_0^T = B_0^{-1}$ and $B_0 C B_0^T = C \rightarrow B_0 C \underbrace{B_0^T B_0}_{I} = C B_0 \rightarrow C B_0 = B_0 C$

Main Proof:

(*) now will prove that for each $c_{ij}$ where $i \neq j$; $c_{ij} = 0$.

$$c_{ij} = b_i^T b_j (x_i x_j) = x_i x_j \times b_i^T b_j = x_i x_j \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases} = x_i^2$$

$b_i \to$ row

now since there is symmetry we can tell that on the diameters each $x_i \in \{x_1 - , x_l\}$ ~~hence~~ has the same probability so the expected value of each $c_{ii}$ will be: $\quad I = i \sim Unif(\{1,2,..,l\})$

$$\left.\begin{array}{l} \mathbb{E}[c_{ii}] = \frac{1}{l} \sum_{i=1}^{l} x_i^2 = \frac{1}{l} \|x\|_2^2 \\ i \neq j: \ \mathbb{E}[c_{ij}] = 0 \end{array}\right\} \quad C = \mathbb{E}[Bxx^TB^T] = \alpha I \ \text{ where } \alpha = \frac{\|x\|_2^2}{l}$$

Q.E.D ■

ادامه: << In this phase we used the method of a paper by Emmanuel J. Candès & Mark A. Davenport :

[[How well can we estimate a sparse vector]]>>

$\|X\|_0 \le K$

if $X_i \in (0,1) \rightarrow \|X\| \le 1$

we can see that each $x$ with $k$-sparsity idles into a sphere of radius 1 in $d$-dimensions.. so now we will find $f_X(x)$ where $x$ can have up to '$k$', nonzero elements. so let $\bar{X} = [\underbrace{1,1,\dots,1}_{k}, 0, \dots 0] \longrightarrow$ so the first $k$-elements of $B$ will matter

بدون نقض کلیت مسئله

$$f_X(X) = \mathbb{P}[X = x] = \mathbb{P}[B\bar{x} = x] = \mathbb{P}[B \in \{B \text{ is in first } k \text{ columns of } B\}]$$

for the first $k$ columns

$$= \frac{k!(l-k)!\,2^{l-k}}{2^l \times l!}$$

$\rightarrow$ $l$-$k$ ~~element~~ columns of $B$ does not play a role so $(l-k)! \times 2^{l-k}$ different metrices

$\rightarrow$ here the degree of freedom is up to $l$. $\rightarrow$ so $l! \times 2^l$ different metrices

$$f_X(X) = \frac{k!(l-k)!}{l!} \times 2^{-k} = \frac{2^{-k}}{\binom{l}{k}} = \frac{1}{2^k \binom{l}{k}}$$

since we wanted to count the number of $x$'s into the sphere which is $2^k \times \binom{l}{k} = |A|_{k,l}$

Now we let $\epsilon = 1/2$.

$$A_{k,l} = \left\{ x \mid x = \frac{1}{\sqrt{k}}(x_1, \dots, x_l),\ \forall i\ x_i \in \{0, \pm 1\},\ \|X_0\| = k \right\}$$

let $N_\epsilon(P)$ denote the number of points with distance at most $\epsilon$ with respect to $P$. now if we sum all the points in $A$, we will ensure that this number exceeds $|A|$. thus (let us assume that the points are positioned these points cover all $B$ with $\epsilon = \frac{1}{2}$ maximaly.)

$$\sum_{P \in B} N_\epsilon(P) \ge |A|$$

$$|B| = \mathcal{P}(A, \|\cdot\|_2, \epsilon = \tfrac{1}{2}), \quad B \subseteq A$$

$$\hookrightarrow \mathcal{P}(A, \|\cdot\|_2, \epsilon) \times |\bar{N}_\epsilon| \ge |A| \Rightarrow \mathcal{P}(A, \|\cdot\|_2, \epsilon) \ge \frac{|A|}{|\bar{N}_\epsilon|}$$

ince $\bar{N}_\epsilon = \{\hat{x} \mid \hat{x} \in A,\ \|\hat{x} - x\|_2^2 \le \epsilon\} \subseteq \{\hat{x} \mid \hat{x} \in A,\ \frac{1}{k}\|x - \hat{x}\|_0 \le \epsilon\}$ چرا

what we did was to use the inequality to manipulate norm "2" to norm "0".

صفحه 5

در صفحه بعد اینجا را ادامه دادیم
در اصلاح در دلیل نیست را فوق را باید بزنیم.

where $\mathcal{U} = \{x \in \{0, +\frac{1}{\sqrt{k}}, -\frac{1}{\sqrt{k}}\}^d , \|x\|_0 = k\}$ , $|\mathcal{U}| = \binom{d}{k} 2^k$

$\forall x \,\&\, x' \in \mathcal{U}$ ; $\frac{1}{k}\|x'-x\|_0 \le \|x'-x\|_2^2$ & thus if $\|x'-x\|_2^2 \le \varepsilon = \frac{1}{2}$ , then $\|x'-x\|_0 \le \frac{k}{2}$ . from this we observe that for any fixed $x \in \mathcal{U}$

$$\left|\{x' \in \mathcal{U} : \|x'-x\|_2^2 \le \varepsilon\}\right| \underset{\subseteq}{\le} \left|\{x' \in \mathcal{U} : \|x'-x\|_0 \le \tfrac{k}{2}\}\right| \le \binom{n}{k/2} 3^{k/2}$$

each element is either $\{\pm\frac{1}{\sqrt{k}}, 0\}$

suppose we wanted to construct $\mathcal{X}$ by picking elements of $\mathcal{U}$ at random. when adding the $j^{th}$ point point to $\mathcal{X}$ (denotably $x_j$); the probability that $x_j$ violates ~~...~~ $\forall x_i, x_j \in \mathcal{X}\ \|x_i - x_j\|_2^2 \ge \varepsilon$ is bounded by:

$$\frac{(j-1)\binom{d}{k/2} 3^{k/2}}{\binom{l}{k} 2^k} \longrightarrow \text{by union bound}$$

$$P_1 \le \frac{|\mathcal{X}|^2}{2} \frac{\binom{l}{k/2} 3^{k/2}}{\binom{l}{k} 2^k}$$

so we get $\mathcal{P}(\mathcal{A}, \|\cdot\|_2, \varepsilon) \ge \dfrac{|\mathcal{A}|}{|\mathcal{N}_\varepsilon|}$

$$\frac{|\mathcal{N}_\varepsilon|}{|\mathcal{A}|} \le \frac{\binom{l}{k/2} 3^{k/2}}{\binom{l}{k} 2^k} = \left(\frac{3}{4}\right)^{k/2} \frac{\binom{l}{k/2}}{\binom{l}{k}} = \left(\frac{3}{4}\right)^{k/2} \frac{k!\,(l-k)!}{(\frac{k}{2})!\,(l-\frac{k}{2})!} = \left(\frac{3}{4}\right)^{k/2} \times \prod_{i=1}^{\lfloor k/2 \rfloor} \frac{\lfloor k/2 \rfloor + i}{l-k+i}$$

$$\le \left(\frac{3}{4}\right)^{k/2} \left(\frac{k/2 + k/2}{l - k + \frac{k}{2}}\right)^{k/2} = \left(\frac{3}{4}\right)^{k/2} \left(\frac{l}{k} - \frac{1}{2}\right)^{k/2}$$

& since $k \le \frac{l}{2} \rightarrow 2k \le l \xrightarrow{+2l} 4l - 2k \ge 3l$

6 صفحه

$$\mathcal{P}(\mathcal{A}, \|\cdot\|_2, \varepsilon) \ge \left(\frac{4l-2k}{3k}\right)^{k/2} \ge \left(\frac{3l}{9k}\right)^{k/2} \ge \left(\frac{l}{k}\right)^{k/2}$$

finally $\Rightarrow$ $\boxed{\mathcal{P}(\mathcal{A}, \|\cdot\|_2, \varepsilon) \ge \left(\frac{l}{k}\right)^{k/2}}$ Q.E.D

$$\mathcal{E} = \min_{\hat{X}(Y)} \max_{x \in S_u^d} \mathbb{E}\left[\|\hat{X} - X\|^2\right]$$

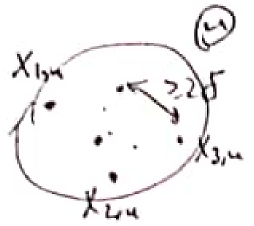
for a given $U = u$:

$$J - X - Y_{|U} \quad \hat{X}$$

: (ch7, 6)

here we have observed $Y|U$ & estimated $X$. Furthermore, Instead of one $\varepsilon$-packing set, we have $|\mathcal{U}|$ $\varepsilon$-packing sets from our parametric set $\Theta$.

let $T_u = \{x_{1,u}, \dots, x_{m,u}\}$ be the points such that they have at least $2\delta$ distance




let us consider only this case, we know from Fano's inequality that

$$\mathcal{E} \geq \delta^2 \cdot \mathbb{E}_{U \sim \pi_u}\left[\mathbb{P}[J \neq \hat{J}]\right] \overset{\text{by}}{\geq} \delta^2 \mathbb{E}_{U \sim \pi_u}\left(1 - \frac{I(J; Y|\overset{U=u}{U}) + \log 2}{\log M}\right)$$

thus:

$$\Rightarrow \mathcal{E} \geq \delta^2 \mathbb{E}_{U \sim \pi_u}\left[1 - \frac{I(J; Y|\overset{u}{U}) + \log 2}{\log M}\right] \overset{(\ast\ast)}{=} \delta^2 \cdot 1 - \frac{\mathbb{E}_{U \sim \pi_u}\left[I(J; Y|U=u)\right] + \log 2}{\log M}$$

$$\Rightarrow \mathcal{E} \geq \delta^2 \left(1 - \frac{I(J; Y|U) + \log 2}{\log M}\right) \qquad (\ast)$$

since by strong DPI we have $I(J; \underset{Y|U}{Y}) \leq I(X; Y)$, from $(\ast)$ we can deduce that:

$$\mathcal{E} \geq \delta^2 \left(1 - \frac{I(X; Y) + \log 2}{\log M}\right)$$

Q.E.D. ∎

$$(\ast\ast) \quad \mathbb{E}_{\pi_u}\left[I(J; Y|U=u)\right] = I(J; Y|U) \leq I(X; Y)$$

$\Phi(\delta) = \delta^2$

با این فرض سؤال
حل شده است

صفحه ۲

ادامه:

Yet we observed $E = \min_{\hat{x}(y)} \max_{x} \mathbb{E}\left[\|x - \hat{x}(y)\|_2^2\right]$,

$E$ is bounded by $\delta^2\left(1 - \frac{I(X;Y) + \log 2}{\log M}\right)$. In previous parts we found that: (we use the things we did in Assouad method)

$$\begin{cases} I(X;Y) = h(Y) - h(Z) \\ M \geq \left(\frac{d}{k}\right)^{k/2} \end{cases} \qquad \begin{cases} \underset{\max}{\text{argmax}}\{h(Y)\} = Y \sim N(0, \frac{t}{e} I) \\ \mathbb{E}\|T\|^2 \leq t \end{cases}$$

So we wish to minimize the R.H.S. by maximizing $I(X;Y)$ which is done when $h(Y)$ is maximized $\to$ $Y \sim N(0, \beta I)$.

$$\mathbb{E}\left[\|Y\|_2^2\right] = \mathbb{E}\left[\|AX + Z\|_2^2\right] = \mathbb{E}\left[x^T A^T A x + 2x^T A Z + Z^T Z\right] \qquad \mathbb{E} Z = 0$$

since $\mathbb{E}\left[Z^T Z\right] = n\sigma^2 \longrightarrow \mathbb{E}\left[\|Y\|_2^2\right] = n\sigma^2 + \mathbb{E}\left[x^T A^T A x\right]$

$$\mathbb{E}\left(\|Y\|_2^2\right) \leq n\sigma^2 + \left(\mathbb{E}\|x\|^2\right) \sigma_{max}(A^T A) \leq n\sigma^2 + \left(\mathbb{E}\|X\|_2^2\right) \|A\|_F^2$$

$$\leftrightarrow \mathbb{E}\left[\|Y\|_2^2\right] \leq n\sigma^2 + 8\delta^2 \|v_i\|_2^2 \|A\|_F^2 = n\sigma^2 + 8\delta^2 \|A\|_F$$

the main part

مهم

$$\Longrightarrow I(X;Y) = h(Y) - h(Z) \leq \frac{n}{2}\log\left(\frac{2\pi t e}{n}\right) - \frac{n}{2}\log(2\pi\sigma^2 e)$$

$$\Rightarrow I(X;Y) \leq \frac{n}{2}\log\left(\frac{t}{n\sigma^2}\right) = \frac{n}{2}\log\left(1 + \frac{8\delta^2\|A\|_F^2}{n\sigma^2}\right) \xrightarrow[\text{if } x > 0]{\log(1+x) \leq x}$$

$$\Rightarrow I(X;Y) \leq \frac{n}{2}\left(\frac{8\delta^2\|A\|_F^2}{n\sigma^2}\right) = \frac{4\delta^2\|A\|_F^2}{\sigma^2} \qquad (*)$$

now plugging $(*)$ into the main assertion from $4^{th}$ phase implies that

$$E \geq \delta^2\left(1 - \frac{I(X;Y) + \overset{\to \text{ dump } \log 2}{\log 2}}{\log M}\right) \geq \delta^2\left(1 - \underbrace{\frac{2}{k\log(\frac{d}{k})}\left(\frac{4\delta^2\|A\|_F^2}{\sigma^2}\right)}_{\text{let } = \frac{1}{2}}\right)$$

ادامه قضیه

$$\frac{1}{2} = 1 - \frac{2}{k\log\left(\frac{d}{k}\right)}\left(\frac{4\delta^2\|A\|_F^2}{\sigma^2}\right) \qquad c = O(1)$$

$$\hookrightarrow 4\delta^2\|A\|_F^2 = 4k\log\left(\frac{d}{k}\right) \rightarrow \delta^2 = \frac{k\log\left(\frac{d}{k}\right)\sigma^2}{\|A\|_F^2} \times c$$

$$\boxed{E \geq \delta^2 \times \frac{1}{2} = \frac{c\,k\log\left(\frac{d}{k}\right)\sigma^2}{\|A\|_F^2}}$$

Q.E.D. ■

صفحه ۹