تمرین سری ۴ SLT [illegible] ۱۴۰۲/۱۰/۴

سوال ۲،

$X_1, \dots, X_n \overset{iid}{\sim} P_{\theta_v}$ یا $P_{\theta_v + 2\delta}$

(I)

$$\inf_{\hat\theta} \sup_{\theta_v, \theta_v + 2\delta} \left\{ \mathbb{E}\left\{ \ell(\rho(\hat\theta, \theta)) \right\} \right\} \geq \Phi(\delta)\, \mathbb{P}[\hat v \neq v] \geq \frac{\Phi(\delta)}{2}\left(1 - TV(P_{\theta_v}^{\otimes n}, P_{\theta_v + 2\delta}^{\otimes n})\right)$$

$$TV \leq \sqrt{1 - BC^2}.$$

$$\left.\begin{array}{l} BC(P_{\theta_v}^{\otimes n}, P_{\theta_v + 2\delta}^{\otimes n}) = BC(P_{\theta_v}, P_{\theta_v + 2\delta})^n \\ BC(P_{\theta_v}, P_{\theta_v + 2\delta}) = \displaystyle\int_{\theta + 2\delta}^{\theta + 1} 1\, dx = 1 - 2\delta \end{array}\right\} TV \leq \sqrt{1 - (1 - 2\delta)^{2n}} \leq \sqrt{1 - e^{-4\delta n}}$$

$$\mathbb{E}\{\ell(\theta, \hat\theta)\} \leq \frac{\Phi(\delta)}{2}\left(1 - \underbrace{\sqrt{1 - e^{-4\delta n}}}_{= c = O(1)}\right) \longrightarrow \delta = \frac{c}{n}$$

$$\inf \sup \mathbb{E}(\ell(\theta, \hat\theta)) = O(\delta^2) = O\left(\frac{c}{n^2}\right)$$

(II) $\hat\theta = \min\{X_1, \dots, X_n\}$

$$F_{\hat\theta}(x) = P_\theta(X_{min} < x) = 1 - P_\theta(X_{min} > x) = 1 - P_\theta(X_i > x, \forall i)$$

$$= 1 - \prod_{i=1}^n P_\theta(X_i > x) = 1 - P_\theta(X > x)^n = 1 - (1 - \underbrace{F_\theta(x)}_{cdf})^n$$

$$\text{pdf} \leftarrow f_{\hat\theta}(x) = \frac{\partial}{\partial x} F_{\hat\theta}(x) = n f_\theta(x)(1 - F_\theta(x))^{n-1} = \begin{cases} 0 & (x \geq \theta + 1) \vee (x \leq \theta) \\ n(1 - x + \theta)^{n-1} & \theta \leq x \leq \theta + 1 \end{cases}$$

$$\forall \theta: \mathbb{E}_{P_\theta}\left[(\theta - \hat\theta)^2\right] = \int_\theta^{\theta + 1} (x - \theta)^2 n (1 - x + \theta)^{n-1} dx \xrightarrow{u = 1 - x + \theta} = \int_0^1 n(1 - u)^2 u^{n-1} du$$

$$= n \int_0^1 u^{n+1} - 2u^n + u^{n-1} = n\left(\frac{1}{n+2} - \frac{2}{n+1} + \frac{1}{n}\right) = \frac{2}{(n+1)(n+2)} \leq \frac{2}{n^2}$$

$$\sup \mathbb{E}(\dots) \simeq O\left(\frac{1}{n^2}\right)$$

(III) خیر اگر مانند این که بالا سریع تغییر کرد با میانگین ۱ توزیع دهیم از امور واریانس کاهش پیدا می‌کند.
که $O(\frac{1}{n})$ است ولی نمی‌توانیم را از $O(\frac{1}{n^2})$ است که بیشتر است

<u>سوال ۲</u> ← واریانس عبارت تاثیری نمی‌کند

(I) $D_f(P_{MX} T_{\hat{M}|X}, P_M Q_X T_{\hat{M}|X}) - D_f(P_{MX}, P_M Q_X) = \mathbb{E}_{P_M Q_X T_{\hat{M}|X}}\left[f\left(\frac{P_{M|X} \cancel{T_{\hat{M}|X}}}{P_M Q_X \cancel{T_{\hat{M}|X}}} \right)\right]$

$= D_f(P_{MX}, P_M Q_X)$

(II) $D_f(P_{MX}, P_M Q_X) \geq D_f(P_{M\hat{M}}, P_M Q_{\hat{M}})$

$X \to \boxed{T_{\hat{M}|X}} \to \hat{M}$

$D_f(P_{MX} T_{\hat{M}|X}, P_M Q_X T_{\hat{M}|X}) = D_f(P_{M,X,\hat{M}}, P_M Q_{X,\hat{M}}) \leq D_f(P_{M,\hat{M}}, P_M Q_{\hat{M}})$

(III)

طبق مسئله داریم:

$P_{M,\hat{M}} \to \boxed{\mathbb{1}(M \neq \hat{M})} \to Ber(P_e)$

$P_M Q_{\hat{M}} \to \boxed{\mathbb{1}(M \neq \hat{M})} \to Ber(1-\frac{1}{m})$

Data Processing inequality: $D_f(P_{M,\hat{M}}, P_M Q_{\hat{M}}) \geq D_f(Ber(P_e), Ber(1-\frac{1}{m}))$

(IV)

$D_f(P_{MX}, P_M Q_X) \geq D_f(Ber(P_e), Ber(1-\frac{1}{m}))$

$\inf_{Q_X} D_f(P_{MX}, P_M Q_X) = I(X;M) \geq D_f(Ber(P_e), Ber(1-\frac{1}{m}))$

thus, $\forall Q_X$: $D_f(P_{MX}, P_M Q_X) = \mathbb{E}_m\left[D_f(P_{X|M=m}, Q_X | P_M) \right] = \sum_m^{m} \frac{1}{m} D_f(P_{X|M=m}, Q_X)$

$\leq \frac{1}{m} \sum \max_m D_f(P_{X|M=m}, Q_X) = \max_m D_f(P_{X|M=m}, Q_X)$

$\Rightarrow \inf_{Q_X} \sup_m D_f(P_{X|M=m}, Q_X) \geq D_f(Ber(P_e), Ber(1-\frac{1}{m}))$

سوال ۳:

for $\theta_0, \theta_1, \theta \in \Theta$; let $\Delta \leq l(\theta_0,\theta), l(\theta_1,\theta)$

(I)

$$\tilde{\theta} = \begin{cases} \theta_0 & \Pr = \dfrac{l(\theta_1,\hat{\theta})}{l(\theta_0,\hat{\theta})+l(\theta_1,\hat{\theta})} \\ \theta_1 & \Pr = \dfrac{l(\theta_0,\hat{\theta})}{l(\theta_0,\hat{\theta})+l(\theta_1,\hat{\theta})} \end{cases}$$

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therefore

$$\Rightarrow \mathbb{E}_{\theta_0}\left[l(\tilde{\theta},\theta_0)\right] = l(\theta_0,\theta_1)\, \mathbb{E}_{\theta_0}\left[\frac{l(\theta_0,\hat{\theta})}{l(\theta_0,\hat{\theta})+l(\theta_1,\hat{\theta})}\right] \leq \frac{l(\theta_0,\theta_1)\,\mathbb{E}_{\theta_0}\left[l(\theta_0,\hat{\theta})\right]}{\Delta}$$

$$\frac{l(\theta_0,\theta_1)}{\Delta}\,\mathbb{E}\left[l(\hat{\theta},\theta)\right] \geq \mathbb{E}\left[l(\theta,\tilde{\theta})\right] \geq \mathbb{P}\left[\hat{\theta}\neq\theta\right] l(\theta_1,\theta_0)$$

since we know that $\quad \mathbb{P}[\theta\neq\theta] \geq \frac{1}{2}\left(1-TV(P_{\theta_0},P_{\theta_1})\right)$

we get $\quad \mathbb{P}[\hat{\theta}\neq\tilde{\theta}]\, l(\theta_1,\theta_0) \geq \frac{1}{2}\left(1-\mathbb{E}_{\mu\theta_1}\{TV(P_{\theta_0},P_{\theta_1})\}\right)$ دلیل نامساوی جنسن $\mathbb{E}$

$$\geq \frac{1}{2}\left[1-TV\{P_{\theta_0},\mathbb{E}_{\mu}[P_{\theta}]\}\right]$$

therefore overall we get that:

$$\Rightarrow \mathbb{E}\left[l(\theta,\hat{\theta})\right] \geq \frac{\Delta}{2}\left[1-TV\left[P_{\theta_0},\mathbb{E}_{\mu\theta_1}\left[P_{\theta_1}\right]\right]\right]$$

(II) by Pinsker we have that: $\quad TV(P,Q)^2 \leq \frac{1}{2} D_{KL}(P\|Q)$

furthermore $\quad D_{KL}(P\|Q) \leq D_{\chi^2}(P,Q)$

will lead to $\quad TV^2(P,Q) \leq \frac{1}{2} D_{KL}(P\|Q)$

thus $\boxed{C=\frac{1}{2}}$

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(III) $1+\mathbb{E}_Q\left[\frac{|\mathbb{E}(P_\theta(x)) - Q_x(x)|^2}{Q_x(x)}\right] = 1+\int_{x\in\mathcal{D}} \frac{\mathbb{E}_\mu[P_\theta(x)]^2 - 2\mathbb{E}_\mu[P_\theta(x)]\, Q_x(x) + Q_x(x)^2}{Q_x(x)}\, dx$

$= 1+\int \frac{\mathbb{E}_\mu[P_\theta^2(x)]}{Q_x(x)} dx \cancel{+2\int \frac{P_\theta(x) Q(x)}{Q(x)} dx} + \int Q(x)\, dx$
$-2\mathbb{E}_\mu[P_\theta(x)]$

$= 1+1+\int \frac{\mathbb{E}_\mu[P_\theta(x)]\,\mathbb{E}_\mu[P_\theta(x)]}{Q_x(x)} dx - 2\underbrace{\mathbb{E}_\mu[P_\theta(x)]}_{1}$

$= \int \frac{\mathbb{E}_\mu[P_\theta(x)]\,\mathbb{E}_\mu[P_{\theta'}(x)]\, dx}{Q(x)} = \mathbb{E}_{\mu\theta}\left[\int \frac{P_\theta(x) P_{\theta'}(x)}{Q(x)}\, dx\right]$

$\xRightarrow{\text{at last}} \inf_{\hat\theta} \sup_{\theta} \mathbb{E}_\theta[\ell(\theta,\hat\theta)] \geq \frac{\Delta}{2}\left(1-\frac{1}{2}\mathbb{E}_\mu\left[\int \frac{P_{\theta_1}(x) P_{\theta'}(x)}{P_{\theta_0}(x)}\, dx - 1\right]\right)^{1/2}$

$TV^2(\mathbb{E}_{\mu,\theta_1} P_{\theta_1}, P_{\theta_0}) \leq \frac{1}{2} D_{\chi^2}(\mathbb{E}_{\mu_\theta}\{P_\theta\}, P_{\theta_0}) \leq \frac{1}{2}\mathbb{E}\left[\frac{P_{\theta_1}(x) P_{\theta_2}(x)}{P_{\theta_0}(x)}\, dx\right] - \frac{1}{2}$

(IV)

سوال ۴:

① $X_i \sim P_\theta \quad , \theta \in [-\alpha, \alpha]$

$$R_n^*(\theta) = \inf_{\hat\theta} \sup_{\theta \in (-\alpha,\alpha)} \mathbb{E}_\theta[\|\theta - \hat\theta\|_2^2] \geq \sup_{\theta \in (-\alpha,\alpha)} \inf_{\hat\theta} \mathbb{E}_\theta[\|\hat\theta - \theta\|_2^2]$$

$$\geq \inf_{\hat\theta} \mathbb{E}_\pi(\|\theta - \hat\theta\|_2^2)$$

$$\forall \theta: \quad \mathbb{E}_\pi(\|\theta - \hat\theta\|_2^2) \geq \mathrm{Var}_Q(\theta - \hat\theta)$$

$$\chi^2(P_{\theta, X^n}, Q_{\theta, X^n}) \geq \chi^2(P_{\theta,\hat\theta}, Q_{\theta,\hat\theta}) \geq \chi^2(P_{\theta-\hat\theta}, Q_{\theta-\hat\theta}) \xrightarrow{\text{HCR}} \geq \frac{(\mathbb{E}_P(\theta - \hat\theta) - \mathbb{E}_Q(\theta - \hat\theta))^2}{\mathrm{Var}_Q(\hat\theta - \theta)}$$

$$\text{if } P_x = Q_x \rightarrow \mathbb{E}_P[\hat\theta(X^n)] = \mathbb{E}_Q[\hat\theta(X^n)], \mathbb{E}_P[\theta] = \mathbb{E}_Q[\theta] + \delta$$

$$R_\pi^* \geq \sup \frac{\delta^2}{\chi^2(P_{\theta, X^n}, Q_{\theta, X^n})}$$

tensoring: $\chi^2(P_{\theta, X^n}, Q_{\theta, X^n}) = \chi^2(P_\theta, Q_\theta) \rightarrow$ which results in

using taylor series

$$\mathbb{E}_Q\Big[\chi^2(P_{X^n|\theta}, Q_{X^n|\theta}) \underbrace{\Big(\frac{dP_\theta}{dQ_\theta}\Big)^2}_{o(1)}\Big] \Rightarrow \lim_{\delta \to 0} \chi^2(P_\theta, Q_\theta) = (nI(\theta) + o(1))\delta^2$$

$$\chi^2(P_{X^n|\theta}, Q_{X^n|\theta}) = (nI(\theta) + o(1))\delta^2$$

$I_\pi \rightarrow$ fischer information of π

$$\Rightarrow R_\pi^* \geq \frac{1}{\alpha^{-2} + nI} = \frac{\alpha^2}{1 + n\alpha^2 I}$$

$$R_\pi^* \geq \frac{1}{J_\pi + \mathbb{E}_\pi(nI(\theta))} = \frac{1}{I(\pi) + nI}$$

$$\left.\begin{array}{l} \pi' \sim U(-\alpha, 2\delta + \alpha) \\ \pi \sim U(-\alpha, \alpha) \end{array}\right\} \quad D_{\chi^2}(P_\theta, P_\theta) = \int_{-\alpha}^{\alpha} \frac{\left(\frac{1}{2\alpha} - \frac{1}{2\alpha + 2\delta}\right)^2}{\frac{1}{2\alpha}} dx$$

$$= 2\alpha \times 2\alpha \times \left(\frac{2\delta}{2\alpha(2\delta + 2\alpha)}\right)^2 \approx \frac{\delta^2}{\alpha^2} \longrightarrow I_\pi = \alpha^{-2}$$

$$\varepsilon^2\alpha^2 = \frac{1-\varepsilon^2}{n\bar I} \longrightarrow n\bar I\alpha^2\varepsilon^2 = 1-\varepsilon^2 \qquad \min_{0<\varepsilon<1} \max\left(\varepsilon^2\alpha^2, \frac{1-\varepsilon^2}{n\bar I}\right)$$

boundary $\varepsilon^2\alpha^2 = \frac{1-\varepsilon^2}{n\bar I} \longrightarrow \varepsilon = \sqrt{\frac{1}{1+n\bar I\alpha^2}}$

$$\min_{0\le\varepsilon\le 1} \max\left\{\varepsilon^2\alpha^2, \frac{1-\varepsilon^2}{n\bar I}\right\} = \varepsilon^2\alpha^2 = \frac{\alpha^2}{1+n\bar I\alpha^2}$$

$$R_\pi^*(\theta) \ge R_\pi^* \ge \frac{\alpha^2}{1+n\bar I\alpha^2} = \min_{0\le\varepsilon\le 1} \max\left\{\varepsilon^2\alpha^2, \frac{1-\varepsilon^2}{n\bar I}\right\}$$

Rate distortion

II

$$\frac{\alpha^2}{1+n\bar I\alpha^2} = \frac{1}{\alpha^{-2}+n\bar I} = \frac{1}{(\alpha^{-1}+\sqrt{n\bar I})^2 (\sqrt{n\bar I}\alpha^{-1}} \ge \frac{1}{(\alpha^{-1}+\sqrt{n\bar I})^2}$$

therefore $\longrightarrow R_n^*(\theta) \ge \frac{1}{(\alpha^{-1}+\sqrt{n\bar I})^2}$

III

$$\inf_{\hat\theta} \sup_{\theta\in[-\alpha+\theta_0, \alpha+\theta_0]} \left\{ \mathbb{E}_\theta\left[\|\theta-\hat\theta\|_2^2\right] = R_n^*(\theta) \ge \frac{1}{(\alpha^{-1}+\sqrt{n\bar I})^2}\right.$$

where $\bar I = \frac{1}{2\alpha}\int_{\theta_0-\alpha}^{\theta_0+\alpha} I(\theta)\,d\theta$ is the avg of $I(\theta)$ in $(\theta_0-\alpha, \theta_0+\alpha)$

$$\alpha = n^{-1/4}, \quad \bar I = \frac{1}{2\alpha}\int_{-\alpha+\theta_0}^{\alpha+\theta_0} I(\theta)\,d\theta \xrightarrow{\text{shift, change vars}} \frac{1}{2\alpha}\int_{-\alpha}^{\alpha} I(\theta_0)+O(\alpha) = I(\theta_0)+O(n^{-1/4})$$

$$\Rightarrow \inf_{\hat\theta} \sup_{\theta\in[\theta_0-n^{-1/4}, \theta_0+n^{-1/4}]} \left\{ \mathbb{E}_\theta\left[\|\hat\theta-\theta\|_2^2\right] \ge \frac{1}{(\alpha^{-1}+\sqrt{n\bar I})^2} = \frac{1}{(n^{1/4}+n^{1/2}I^{1/2}(\theta_0)+o(n^{1/4}))^2}\right.$$

$$\Rightarrow R_n^*(\theta) \ge \frac{o(1)}{nI(\theta_0)}$$

Q.E.D. ■

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