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سوال 1

(1) $\begin{cases} y'' + y' + \lambda y = 0 \\ y(0) = y(\pi) = 0 \end{cases} \Rightarrow$ معادله مشخصه $\quad s^2 + s + \lambda = 0$

eigen Values

$$S_{1,2} = \frac{-1 \pm \sqrt{1 - 4\lambda}}{2}$$

$$S_1 = S_2 = \frac{-1}{2}$$

Case (I) : $\lambda = \frac{1}{4} \Rightarrow y(t) = c_1 e^{-\frac{t}{2}} + c_2 t e^{-\frac{t}{2}} \xrightarrow{t=0} c_1 = 0$

$t = \frac{\pi}{\cancel{2}} \Rightarrow 0 = c_2 \pi e^{-\pi/2} \xrightarrow{e^{x} \neq 0} c_2 = 0 \longrightarrow$ تنها جواب ، جواب بدیهی صفر می‌باشد.

Case (II) : $\lambda \neq \frac{1}{4} \Rightarrow y(t) = c_1 e^{s_1 t} + c_2 t e^{s_2 t} \xrightarrow{t=0} c_1 = 0$

$t = \pi \Rightarrow 0 = c_2 \pi e^{-s\pi} \xrightarrow{e^{-s\pi} \neq 0} c_2 = 0 \longrightarrow$ تنها جواب در این حالت ، جواب بدیهی صفر می‌باشد.

Case (III) : $\lambda > \frac{1}{4} \Rightarrow$ if $\underbrace{\boxed{\lambda = \frac{-1}{2} \pm \frac{j}{2}\sqrt{\frac{1-4\lambda}{-1}}}}_{}$ $= \left(\frac{-1}{2}, \frac{\sqrt{4\lambda - 1}}{2}\right) = (\alpha, \beta) = \alpha + j\beta$

eigen Values

$y(t) = c_1 e^{\alpha t} \sin\beta t + c_2 e^{\alpha t} \cos\beta t \Rightarrow t = 0 \Rightarrow c_2 = 0$

$t = \pi \Rightarrow c_1 e^{\alpha\pi} \cdot \sin\beta\pi = 0 \underset{c_2 \neq 0}{\Rightarrow} \sin\beta\pi = 0 \Rightarrow \beta = \sqrt{\frac{4\lambda - 1}{4}} = k \in \mathbb{N}$

$\beta\pi = k\pi$ where $k \in \mathbb{Z}^+$

$$\begin{cases} \lambda = \dfrac{4k^2 + 1}{4} \quad (k \in \mathbb{N}) \\ \lambda \neq \dfrac{4k^2 + 1}{4} \quad (k \in \mathbb{N}) \Rightarrow c_1 = c_2 = 0 \end{cases}$$

$\Rightarrow y(t) = c_1 e^{-\frac{t}{2}} \sin\left(\sqrt{\frac{4\lambda - 1}{4}}\, t\right)$

if $\lambda = \frac{4k^2+1}{4}$, $k \in \mathbb{N}$

$$\Rightarrow \begin{cases} y(t) = c_1 e^{-\frac{t}{2}} \sin(\lambda_n t) & \longleftarrow \text{eigen functions} \\ \lambda_n = \dfrac{4n^2 + 1}{4} \quad n \in \mathbb{N} \end{cases}$$

III $\begin{cases} y'' + \lambda y = 0 \\ 0 < x < \pi \\ y'(0) = y'(\pi) = 0 \end{cases} \xrightarrow{\text{معادله مشخصه}} s^2 + \lambda = 0 \Rightarrow s = \pm\sqrt{-\lambda}$

Case (I): $\lambda = 0 \Rightarrow y(t) = c_1 + c_2 t \Rightarrow y'(t) = c_2 \xrightarrow{t=0} c_2 = 0$

$t = \pi \Rightarrow y'(\pi$

Case (II): $\lambda < 0 \Rightarrow \underset{p>0}{\lambda = -p^2} \Rightarrow s = p, -p \Rightarrow y(t) = c_1 e^{pt} + c_2 e^{-pt}$

$y'(t) = p\left(c_1 e^{pt} - c_2 e^{-pt}\right) \longrightarrow y'(0) = p(c_1 - c_2) = 0 \longrightarrow c_1 = c_2$

$y'(\pi) = p\left(c_1 e^{\pi p} - c_2 e^{-p\pi}\right) = p c_1 \left(e^{p\pi} - e^{-p\pi}\right) \xrightarrow{p \neq 0} c_1 = 0 \Rightarrow c_2 = 0$

در این حالت هم جواب بدیهی صفر داریم.

Case (III): $\lambda > 0 \Rightarrow \underset{p>0}{\lambda = p^2} \Rightarrow s = \pm pj \Rightarrow y(t) = c_1 \sin pt + c_2 \cos pt$

$y'(t) = p(c_1 \cos pt - c_2 \sin pt) \Rightarrow y'(0) = p \cdot c_1 = 0 \xrightarrow{p \neq 0} c_1 = 0$

$y'(\pi) = 0 = p(c_2 \sin p\pi) \xrightarrow{p \neq 0} \begin{cases} c_2 = 0 \longrightarrow \text{جواب بدیهی صفر} \\ c_2 \neq 0 \xrightarrow{p \neq 0} \underset{k \in \mathbb{Z}}{p\pi = k\pi} \longrightarrow p \in \mathbb{Z} \end{cases}$

$$\boxed{\begin{array}{c} y(t) = c_2 \cos(\sqrt{\lambda}\, t) \\ \sqrt{\lambda} \in \mathbb{N} \end{array}} \longrightarrow \begin{cases} y(t) = c_2 \cdot \cos(p_n t) \\ n \in \mathbb{N} \quad p_n = \sqrt{n^2} \end{cases} \longleftarrow \boxed{\text{eigen function}}$$

سوال ۲:

(I) $u_{xyy} = u_x$, let $u_x = P \Rightarrow \dfrac{\partial^2 P}{\partial y^2} = P$

معادله مشخصه
$\Rightarrow s^2 - 1 = 0 \longrightarrow s = \pm 1 \Rightarrow P = \underbrace{c_1 e^{-y} + c_2 e^{y}}_{\text{تابعی از } x} = u_x$

$u(x,y) = \int P\,dx = e^{-y}\int c_1(x)\,dx + e^{y}\int c_2(x)\,dx = d_1(x)e^{-y} + d_2(x)e^{y}$

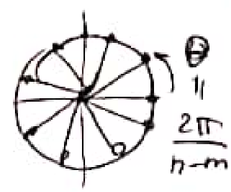
(II) $u_{xxxx} = u_{xx}$, let $u_{xx} = P \Rightarrow \dfrac{\partial^2 P}{\partial x^2} = P \xrightarrow[\text{معادله مشخصه}]{s^2 - 1 = 0} \Rightarrow s = \pm 1$ roots of characteristic equation

$\Rightarrow P = c_1(y)e^{-x} + c_2(y)e^{x} \Rightarrow u = \iint P\,dx\,dx = \iint (c_1 e^{-x} + c_2 e^{x})\,dx\,dx$

$u(x,y) = d_4(y) + d_3(y)\cdot x + d_1(y)e^{-x} + d_1(y)e^{x}$

(III) $\dfrac{\partial^n u}{\partial x^n} = \dfrac{\partial^m u}{\partial x^m}$, let $P = \dfrac{\partial^m u}{\partial x^m} \Rightarrow \dfrac{\partial^{n-m}}{\partial x^{n-m}} P = P \xRightarrow{\text{معادله مشخصه}} s^{n-m} = 1$

S_k's are the $(n-m)$'s roots of unity $\Rightarrow S_k = e^{\frac{2k\pi}{n-m}}$ $\forall\, 0 \le k < n-m$



$S_k = \cos\left(\dfrac{2k\pi}{n-m}\right) + j\sin\left(\dfrac{2k\pi}{n-m}\right)$

$P = \dfrac{\partial^{n-m}}{\partial x^{n-m}} u(x,y) = \sum_{k=0}^{n-m-1} c_k e^{S_k x} \Rightarrow u = \sum_{k=0}^{n-m-1} \underbrace{\iiint\cdots\int}_{m\text{ بار}} e^{S_k x} c_k\,dx$

$= \sum_{k=0}^{n-m-1} \dfrac{A_k(y)}{(S_k)^m} e^{S_k x}$

سؤال 2

I) $u_x + u_y = 2(x+y)u$ let $u(x,y) = X(x)Y(y)$

با جایگذاری u در معادله فوق $\Rightarrow X'(x)Y(y) + X(x)Y'(y) = 2(x+y)X(x)Y(y)$

$\Rightarrow \frac{X'(x)}{X(x)} + \frac{Y'(y)}{Y(y)} = 2(x+y)$ از آنجایی که $\frac{X'(x)}{X(x)}$ تنها تابع x و $\frac{Y'(y)}{Y(y)}$ تنها تابع y است

نتیجه می‌گیریم در سمت راست نیز $\frac{X'(x)}{X(x)} = 2x+\lambda$ ، همچنین $\frac{Y'(y)}{Y(y)} = 2y-\lambda$. پس داریم خواهیم داشت:

$$\begin{cases} X'(x) = 2x X(x) + \lambda X(x) \\ Y'(y) = 2y Y(y) - \lambda Y(y) \end{cases}$$

حال باید معادله دیفرانسیل ode ، $y' = 2xy$ را حل کنیم:

$$y' = 2xy + \lambda y \to \overbrace{y(\lambda + 2x)}^{M} dx + \overbrace{(-1)}^{N} dy = 0$$

عامل انتگرال ساز: $\mu(x) = e^{\int \frac{M_y - N_x}{-N} dx} = e^{\int -2x-\lambda\, dx} = e^{-\lambda x^2 - \lambda x}$

$$\Rightarrow \underbrace{-2xy e^{-x^2}}_{M} dx + \underbrace{e^{-x^2} dy}_{N} = 0 \Rightarrow \begin{cases} \int M dx = y e^{-x^2} \\ \int N dy = y e^{-x^2} \end{cases} \quad y e^{-x^2 - \lambda x} = c \to \text{some real constant}$$

$$y(x) = c \cdot e^{x^2}$$

از حل معادله دیفرانسیل فوق بدست می‌آید:

$$\Rightarrow \begin{cases} X(x) = c_1 e^{x^2} \\ Y(y) = c_2 e^{y^2} \end{cases} \Rightarrow \boxed{u(x,y) = X(x)Y(y) = \underbrace{(c_1 \cdot c_2)}_{K} e^{\lambda x + x^2} e^{y^2 - \lambda y} = K e^{x^2 + y^2 + \lambda(x-y)}}$$

some real constant

II) $x^2 u_{xy} + 3y^2 u = 0$ let $u(x,y) = X(x)Y(y)$

$$x^2 X'(x)Y'(y) + 3y^2 X(x)Y(y) = 0 \Rightarrow \frac{x^2 X'(x)}{X(x)} + \frac{Y'(y)}{Y(y)} + 3y^2 = 0$$

از اینجایی که بخشی از معادله فقط به y وابستگی دارد ($\frac{Y'(y)}{Y(y)} + 3y^2$) و بخش دیگر فقط به x وابستگی دارد ($x^2 \frac{X'(x)}{X(x)}$) نتیجه می‌گیریم که

$$\begin{cases} x^2 X'(x) = X(x) \\ Y'(y) = -3y^2 Y(y) \end{cases}$$

حال دو معادله فوق را حل خواهیم کرد:

(i) $y' = -3x^2y \to \underbrace{1}_{N}dy + \underbrace{3x^2y}_{M}dx$

[illegible] $\mu(x) = e^{\int \frac{M_y - N_x}{N}dx} = e^{\int 3x^2 dx} = e^{x^3}$

C is some real constant

$\Rightarrow \underbrace{e^{x^3}}_{N}dy + \underbrace{e^{x^3}\cdot 3x^2y}_{M}dx = 0 \to \begin{cases} \int M dx = ye^{x^3} \\ \int N dy = ye^{x^3} \end{cases} \Rightarrow ye^{x^3} = c \to \boxed{y = ce^{-x^3}}$

(ii) $y' = \frac{y}{x^2} \longrightarrow \underbrace{1}_{N}dy - \underbrace{\frac{y}{x^2}}_{M}dx = 0$

[illegible] $\mu(x) = e^{\int \frac{M_y - N_x}{-N}dx} = e^{\int \frac{-\frac{1}{x^2}}{-1}dx} = e^{-\frac{1}{x}}$

$c \in \mathbb{R}$

$\Rightarrow e^{-\frac{1}{x}}dy - \frac{y}{x^2}e^{-\frac{1}{x}}dx = 0 \Rightarrow \begin{cases} \int M dx = ye^{-\frac{1}{x}} \\ \int N dy = ye^{-\frac{1}{x}} \end{cases} \Rightarrow y\cdot e^{+\frac{1}{x}} = c$

$\boxed{y = c\cdot e^{+\frac{1}{x}}}$

$$\begin{cases} \boxed{\mu(x,y) = X(x)Y(y) = \underbrace{(c_1\cdot c_2)}_{k\in\mathbb{R}} e^{+\frac{1}{x}} e^{-\frac{y^3}{1}} = ke^{+\frac{1}{x} - \frac{y^3}{1}}} \\ X(x) = c_1\cdot e^{-\frac{1}{x}} \\ Y(y) = c_2\cdot e^{-y^3} \end{cases}$$

سوال 3: (بخش الف)

$t>0\ ,\quad 0<x<\pi\ ,\quad u_{tt}=c^2u_{xx}-h^2u$

$u(0,t)=0\ ,\quad u(\pi,t)=0\ ,\quad u(x,0)=\pi e^{x}\ ,\quad u_t(x,0)=0$

let $u(x,t)=X(x)T(t) \xrightarrow{\text{جایگذاری}} T''(t)X(x)=c^2X''(x)T(t)-h^2X(x)T(t)$

$$\Rightarrow \frac{T''(t)+h^2T(t)}{c^2T(t)}=\frac{X''(x)}{X(x)}=A=\text{const}$$

چون که یک سمت تساوی فقط تابعی از x و سمت دیگر معادله تنها تابعی از t است، آنگاه این عبارت ثابت است.

می‌دانیم که به ازای $A=0$ ، $A>0$ تنها جواب، جواب بدیهی صفر برای $X(x)$ است که منجر به صفر شدن $u(x,t)$ منجر می‌شود بنابراین $A=-p^2$ فرض می‌کنیم که $p\in\mathbb{R}, p>0$.

if $A=0 \longrightarrow \begin{cases} X(x)=ax+b \\ X(0)=b=0 \end{cases} \longrightarrow X(\pi)=a\pi+\cancel{b}=0 \longrightarrow a=0$

if $\underset{p\in\mathbb{R}}{\underset{p>0}{A=p^2}} \longrightarrow \begin{cases} X(x)=ae^{px}+be^{-px} \\ X(0)=a+b=0 \longrightarrow a=-b \end{cases} \longrightarrow X(\pi)=a(e^{p\pi}-e^{-p\pi})=0 \longrightarrow a=0$

e^{x} یک تابع یک به یک است و چون که همواره $-p\pi\neq p\pi$ بنابراین صفر نمی‌شود.

if $\underset{p>0,\ p\in\mathbb{R}}{A=-p^2} \longrightarrow X(x)=a\cos px+b\sin px \longrightarrow X(0)=a\cdot 1=0 \longrightarrow a=0$

$X(\pi)=0=b\sin p\pi \longrightarrow p\pi \underset{p>0}{\overset{k\in\mathbb{Z}}{=}} k\pi \longrightarrow p=k\in\mathbb{N} \longrightarrow X_n(x)=B_n\sin nx$

$$\frac{T(t)h^2+T''(t)}{c^2T(t)}=-p^2=A \longrightarrow T''(t)+T(t)(h^2+c^2p^2)=0$$

$$\longrightarrow T_n(t)=a_n\sin\left(\sqrt{h^2+c^2p^2}\,t\right)+b_n\cos\left(\sqrt{h^2+c^2p^2}\,t\right)$$

$$\Rightarrow u(x,t)=\sum_{n=1}^{\infty}X_n(x)T_n(t)=\sum_{n=1}^{\infty}u_n(x,t)=\sum_{n=1}^{\infty}B_n\sin nx\left(a_n\sin\sqrt{h^2+c^2n^2}\,t+b_n\cos\sqrt{h^2+c^2n^2}\,t\right)$$

$$u(x,0)=\sum_{n=1}^{\infty}\overbrace{(B_nb_n)}^{\phi_n}\sin nx=\pi e^{x}\ ,\quad \forall x\in[0,\pi)$$

حال می‌دانیم ϕ_n ضرایب سری فوریه گسترش فرد تابع πe^{x} که بازه $(0,\pi)$ به بازه $[-\pi,\pi)$ است.

ادامه سؤال 3 (تکلیف اول)

$\forall n \in \mathbb{N}$

$$\phi_n = \frac{2}{\pi}\int_0^{\pi} \pi e^x \sin nx \, dx = \frac{2n^2}{n^2+1}\left(e^x \sin nx - n\cos nx \cdot e^x\right)\Big|_{x=0}^{x=\pi}$$

$$\phi_n = \frac{2}{n^2+1}\left(n\cdot\left(\cos n\pi \cdot e^{\pi}+1\right)\right) = \frac{2n}{n^2+1}\left(e^{\pi}\cdot(-1)^{n+1}+1\right)$$

$$\Longrightarrow u_t(x,t)\Big|_{t=0} = \frac{\partial}{\partial t}\sum_{n=1}^{\infty} B_n \sin nx\left(a_n \sin\left(\sqrt{h^2+c^2n^2}\cdot t\right) + b_n \cos\left(\sqrt{h^2+c^2n^2}\cdot t\right)\right)\Big|_{t=0}$$

$$= \sum_{n=1}^{\infty} \frac{B_n \sin nx}{\sqrt{h^2+c^2n^2}}\left(a_n \cos\sqrt{h^2+c^2n^2}\cdot t + b_n \sin\sqrt{h^2+c^2n^2}\cdot t\right)\Big|_{t=0}$$

$$= \sum_{n=1}^{\infty} \frac{B_n \sin nx}{\sqrt{h^2+c^2n^2}}\left(a_n\right) = 0 \xrightarrow[\forall x \in \mathbb{R}]{B_n \neq 0} a_n = 0$$

با استفاده از تبدیل ضرب به جمع خواهیم داشت:

$$u(x,t) = \sum_{n=1}^{\infty} \frac{\phi_n}{2}\left[\sin\left(nx - \sqrt{h^2+c^2n^2}\cdot t\right) + \sin\left(nx + \sqrt{h^2+c^2n^2}\cdot t\right)\right]$$

$$= \sum_{n=1}^{\infty} \frac{n}{n^2+1}\left(e^{\pi}\cdot(-1)^{n+1}+1\right)\left[\sin\left(nx - \sqrt{h^2+c^2n^2}\cdot t\right) + \sin\left(nx + \sqrt{h^2+c^2n^2}\cdot t\right)\right]$$

3.2

$0<x<\pi \qquad t>0 \qquad u(x,0)=0 \qquad u(0,t)=1 \qquad u_x(1,t)=0$

$u_{tt}=9u_{xx}+\underbrace{(1-2x)}_{Q(x)}$ let $u(x,t)=w(x,t)+V(x)$

$\begin{cases} u_t(1,t)=0 \to u(1,t)=\lambda \\ w(x,0)=0-V(x) \\ w(0,t)=1-1=0 \\ w(1,t)=\lambda-(\frac{-1}{2}+\frac{1}{3}+1)=\lambda-5/6 \end{cases}$

$\hookrightarrow (V_{tt}+w_{tt})=9(V_{xx}+w_{xx})+(1-2x)$

$\hookrightarrow 0=9V_{xx}+(1-2x)$

$\to V'(x)=\frac{-x}{9}+\frac{x^2}{9}+c_1$

$\to V(x)=\frac{-x^2}{9\times 2}+\frac{x^3}{9\times 3}+c_1x+c_0 \to \begin{cases} V_t=c_1=0 \\ V(0,t)=1 \end{cases}$

$w_{tt}=9w_{xx}+\left[(1-2x)+V_{xx}\right]$

$T''(t)X(x)=9X''(x)T(t) \to \frac{T''(t)}{T(t)}=\frac{9X''(x)}{X(x)}=P=\alpha^2$

$\begin{cases} X(x)=A\cos\frac{Px}{3}+B\sin\frac{Px}{3} \\ T(t)=C\sin Pt+D\cos Pt \qquad (*) \end{cases}$

$k\in\mathbb{Z}$

$w(0,t)=T(t)(A)=0\to A=0 \Rightarrow \sin\frac{Px}{3}=0 \Rightarrow \frac{P}{3}=k\pi \to P=3k\pi$

$w(x,0)=\left(B\sin\frac{Px}{3}\right)\left(D\cos Pt\right)=-V(x)=\frac{x^2}{18}-\frac{x^3}{27}-1$

درابن D تابعی از x نخواهد داشت دار وقت اگر طرفین معادله تعداد نامحدودی جواب ریشه دارد. بنابراین D تابعیت از n نگیرد درحالتی که سبب می‌شود در (*) T(t) تابعی از x بگیرد که غلط است. درنتیجه D=0 بوده است.

$w(x,t)=\sum_{n=1}^{\infty} b_n\sin n\pi x \Rightarrow$

$b_n=\frac{2}{1}\int_0^1 f(x)\sin n\pi x\,dx=2\int_0^1\left(\frac{x^3}{27}-\frac{x^2}{18}+1\right)\sin n\pi x\,dx$

$=\frac{-(53\pi^3n^3-6\pi n)(1-(-1)^n)+\pi^3n^3+12\pi n}{27\pi^4n^4}$

ادامه سوال 3 :

بخش ۲

$$w_t\Big|_{t=0} = \sum_{n=1}^{\infty} 3n\, d_n \sin n\pi x = 0 \longrightarrow d_n = 0$$

$$w(x,t) = \sum_{n=1}^{\infty} \delta_n \cos 3n\pi t \sin n\pi x$$

حال با استفاده از $u(x,t)$ و $v(x)$ ، w داریم:

$$u(x,t) = v(x) + w(x,t)$$

$$= \left(\frac{-x^2}{18} + \frac{x^3}{27} + 1\right) + \sum_{n=1}^{\infty} \frac{(-53\pi^3 n^3 - 6\pi n)(1-(-1)^n) + \pi^3 n^3 + 72\pi n}{27\pi^4 n^4} (\sin n\pi x) \times (\cos 3n\pi t)$$

سؤال 3 (کتبی سری ۰):

$u(0,t)=0 \qquad 0\le x\le\pi$

$u(\pi,t)=0 \qquad t\ge 0$

$u_{xx}=t^2u_t$

$\lim_{t\to\infty} u(x,t)=4\sin^3 2x-\sin 4x\cos 6x=f(x)$

let $u(x,t)=X(x)\,T(t)$

$$X''(x)T(t)=t^2T'(t)X(x)\longrightarrow \frac{X''(x)}{X(x)}=\frac{T(t)}{t^2T'(t)}=k=\text{constant}$$

(i) if $k=0\rightarrow X''(x)=0\rightarrow X(x)=ax+b$ $\left.\begin{matrix}\\ X(0)=0\rightarrow b=0\end{matrix}\right\}$ $X(\pi)=a\pi=0\rightarrow a=0$

جواب بدیهی صفر

(ii) if $k>0$, $k=\lambda^2$, $\lambda\in\mathbb{R}$, $\lambda>0$ $\Big\}$ $X''(x)-\lambda^2X(x)=0\rightarrow X(x)=Ae^{\lambda x}+Be^{-\lambda x}$

$X(0)=0\rightarrow A+B=0$

$X(\pi)=A(e^{\lambda\pi}-e^{-\lambda\pi})=0 \xRightarrow{\lambda\neq 0} A=0=B$

تابع e صفر نیست

جواب بدیهی صفر

(iii) if $k<0$, $k=-\lambda^2$, $\lambda\in\mathbb{R}$, $\lambda>0$ $\Big\}$ $X''(x)+\lambda^2X(x)=0\rightarrow X(x)=A\sin\lambda x+B\cos\lambda x$

$X(0)=0\rightarrow B(1)=0\rightarrow B=0$

$X(\pi)=0\rightarrow A\sin\lambda\pi=0\rightarrow \lambda\pi=n\pi\rightarrow\lambda=n$

$n\in\mathbb{Z}$

حال که معادله را برای $X(x)$ حل کردیم برای $T(t)$ حل خواهیم کرد

$$\frac{T(t)}{t^2T'(t)}=-\lambda^2\rightarrow\frac{T'(t)}{T(t)}=\frac{-\lambda^2}{t^2}\xrightarrow{\text{انتگرال}}\ln T(t)=\frac{+\lambda^2}{t}\Rightarrow T_n(t)=e^{\frac{\lambda_n^2}{t}}$$

$$\text{since }\lambda_n=n\Rightarrow T_n(t)=e^{\frac{n^2}{t}}$$

$$\Rightarrow u(x,t)=\sum_{n=1}^{\infty}u_n(x,t)=\sum_{n=1}^{\infty}X_n(x)T_n(t)=\sum_{n=1}^{\infty}A_n\sin\lambda_n x\,e^{-\frac{n^2}{t}}$$

$\lambda_n=n$

$$\lim_{t\to\infty}u(x,t)=\sum_{n=1}^{\infty}A_n\sin nx=4\sin^3 2x-\sin 4x\cos 6x=-\sin 6x+3\sin 2x-\frac{\sin 10x}{2}$$

بنابراین از آنجایی که A_n ضرایب سری فوریه $f(x)$ هستند و سری فوریه تابع یکتایی دارد خواهیم داشت:

$\forall n\in\mathbb{N}-\{2,6,10\}: A_n=0\rightarrow A_2=+3\ ,\ A_6=-1\ ,\ A_{10}=\frac{-1}{2}$

ادامه سوال ۳ (حل مسئله اول)

$\forall n \in \mathbb{N}$

$$\phi_n = \frac{2}{\pi}\int_0^{\pi} \pi e^{x} \sin nx \, dx = \frac{2n^2}{n^2+1}\left(e^{x}\sin nx + n\cos nx . e^{x}\right)\Big|_{x=0}^{x=\pi}$$

$$\phi_n = \frac{2}{n^2+1}\left(n\cdot\left(\cos n\pi . e^{\pi}+1\right)\right) = \frac{2n}{n^2+1}\left(e^{\pi}\cdot(-1)^{n+1}+1\right)$$

$$\xrightarrow{(\ast\ast)} u_t(x,t)\Big|_{t=0} = \frac{\partial}{\partial t}\sum_{n=1}^{\infty} B_n \sin nx\left(a_n \sin\sqrt{h^2+c^2n^2}\,t\right) + b_n \cos\left(\sqrt{h^2+c^2n^2}\,t\right)\Big|_{t=0}$$

$$= \sum_{n=1}^{\infty} \frac{B_n \sin nx}{\sqrt{h^2+c^2n^2}}\left(a_n \cos\sqrt{h^2+c^2n^2}\,t\right) + b_n \sin\sqrt{h^2+c^2n^2}\,t\Big)\Big|_{t=0}$$

$$= \sum_{n=1}^{\infty} \frac{B_n \sin nx}{\sqrt{h^2+c^2n^2}}\left(a_n\right) = 0 \xrightarrow[\forall x \in \mathbb{R}]{B_n \neq 0} a_n = 0$$

با استفاده از تبدیل ضرب به جمع خواهیم داشت:

$$u(x,t) = \sum_{n=1}^{\infty}\frac{\phi_n}{2}\left[\sin\left(nx - \sqrt{h^2+c^2n^2}\,t\right) + \sin\left(nx + \sqrt{h^2+c^2n^2}\,t\right)\right]$$

$$= \sum_{n=1}^{\infty}\frac{n}{n^2+1}\left(e^{\pi}\cdot(-1)^{n+1}+1\right)\left[\sin\left(nx-\sqrt{h^2+c^2n^2}\,t\right) + \sin\left(nx+\sqrt{h^2+c^2n^2}\,t\right)\right]$$

$$\frac{-2}{13} - \frac{1}{27} \quad \left(\ \frac{1}{9}\left(\ \frac{1}{1}\right)\right.$$

تمرین 4:

(I) $v(x,t) = e^{\lambda t} u(x,t) \longrightarrow u(x,t) = v(x,t)\, e^{-\lambda t}$

$$u_t = v_t e^{-\lambda t} - \lambda v e^{-\lambda t}$$

$$u_{tt} = v_{tt} e^{-\lambda t} + v_t(-\lambda e^{-\lambda t}) - \lambda\left(v_t e^{-\lambda t} + v(-\lambda e^{-\lambda t})\right)$$

$$u_{xx} = v_{xx} e^{-\lambda t}$$

plugging in the equation $\Rightarrow$

$$0 = v_{tt} e^{-\lambda t} - \cancel{\lambda v_t e^{-\lambda t}} + \cancel{\lambda v_t e^{-\lambda t}} - v_{xx} e^{-\lambda t} + 2\lambda v_t e^{-\lambda t} - 2\lambda^2 v e^{-\lambda t} + \lambda^2 v e^{-\lambda t}$$

$$\hookrightarrow e^{-\lambda t}(v_{tt} - v_{xx}) = 0 \Rightarrow v_{tt} = v_{xx}$$

if $v(x,t) = X(x)T(t) \Rightarrow X''(x)T(t) = T''(t)X(x) \Rightarrow \dfrac{X''(x)}{X(x)} = \dfrac{T''(t)}{T(t)} = k$

چون که دو طرف از معادله فوق نسبت به یک متغیر x و t وابسته است، بنابراین حاصل کسر ثابت است

(II) $y^2 u_x^2 + x^2 u_y^2 = x^2 y^2 u^2$ $\qquad u(x,0) = e^{x^2}$ $\qquad u(x,y) = e^{f(x)+g(y)}$

$$\left.\begin{aligned} u_x &= f'(x) e^{f(x)+g(y)} \\ u_y &= g'(y) e^{f(x)+g(y)} \end{aligned}\right\} \quad y^2 f'(x) e^{2f(x)+2g(y)} + x^2 g'(y) e^{2f(x)+2g(y)} = x^2 y^2 e^{2f(x)+2g(y)}$$

$$e^{2f(x)+2g(y)} \neq 0 \Rightarrow y^2 f'(x)^2 + x^2 g'(y)^2 = x^2 y^2$$

$$\Rightarrow \frac{f'^2(x)}{x^2} + \frac{g'^2(y)}{y^2} = 1 \quad (*)$$

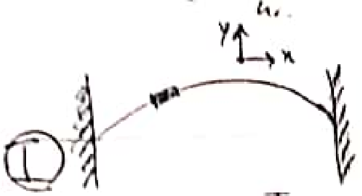
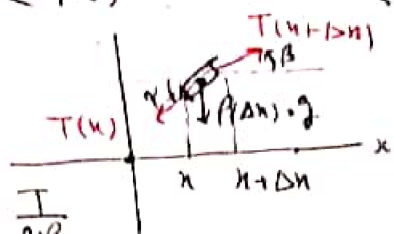
$u(x,0) = e^{x^2} = e^{f(x)+g(0)}$ $\xrightarrow{\forall x}$ $f(x) = x^2 - g(0) = x^2 - c$ $\quad$ (c is constant) $c \in \mathbb{R}$

$\rightarrow f'(x) = 2x \overset{(*)}{\Rightarrow} \dfrac{4x^2}{x^2} + \dfrac{g'^2(y)}{y^2} = 1 \rightarrow \underline{g'^2(y)} = -3y^2$

$\rightarrow g'(y) = -i\sqrt{3}\, y \Rightarrow g(y) = \dfrac{-i\sqrt{3}}{2} y^2 + d$ $\quad$ (d is constant, $d \in \mathbb{R}$)

$$\Rightarrow \begin{cases} f(x) = x^2 - g(0) = x^2 - d \\ g(y) = \dfrac{-i\sqrt{3}\, y^2}{2} + d \end{cases} \Rightarrow \boxed{u(x,y) = e^{x^2 - \frac{i\sqrt{3}}{2} y^2}}$$

سوال ۵:

I  $\rightarrow$ 

$T(x)=\frac{T}{\cos\alpha}$, $T(x+\Delta x)=\frac{T}{\cos\beta}$

$u_t(x,t)$ = سرعت

$u_{tt}(x,t)$ = شتاب

$$\sum F_x = m\, u_{xx}(x,t) = \rho\cdot\Delta x\cdot g\cdot u_{xx}(x,t)$$

$$= -T(x)\cos\alpha + T(x+\Delta x)\cos\beta = \rho(\Delta x)g\, u_{tt}(x,t) + k(\Delta x)u_t(x,t) +$$

$$\cancel{T(x+\Delta x)\cos\beta = T(x)\cos\alpha = T}$$

$$\sum F_y = T(x+\Delta x)\sin\beta - T(x)\sin\alpha - \cancel{\rho\Delta x g} = \cancel{\rho\Delta x}\, u_{tt} \Rightarrow \frac{T(x+\Delta x)}{T(x)} = \frac{T\sin\beta}{T\sin\alpha}$$

$$T\frac{\sin\alpha}{\cos\alpha} + T\frac{\sin\beta}{\cos\beta} = T(\tan\alpha+\tan\beta) = \rho\Delta x\left(g u_{tt} + \frac{k}{\rho}u_t\right)$$

$$\tan\alpha = \frac{\partial u}{\partial x}\Big|_x \qquad \Rightarrow T\left(\frac{u_x(x+\Delta x,t) - u_x(x,t)}{\Delta x}\right) = \rho\left(g u_{tt} + \frac{k}{\rho}u_t\right)$$

$$\tan\beta = \frac{\partial u}{\partial x}\Big|_{x+\Delta x}$$

$$T u_{xx}(x,t) = \rho u_{tt} + k u_t$$

II
$$\begin{cases} u(0,t) = u(L,t) = 0 \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases} \Rightarrow T u_{xx} = \rho u_{tt} + k u_t + \cancel{\rho g}$$

ادامه سوال ۵:

$$u_{xx} = u_{tt} + u_t$$

let $u(x,t) = X(x)T(t) \xrightarrow{\text{معادله اصلی}} X''(x)T(t) = X(x)T''(t) + X(x)T'(t)$

$$\Rightarrow \frac{X''(x)}{X(x)} = \frac{T''(t)+T'(t)}{T(t)} = K$$

اما می‌دانیم معادله $X''(x)+kX(x)=0$ صرفاً به ازای k های منفی جواب نابدیهی غیر صفر دارد. بنابراین فرض می‌کنیم که $K=-p^2$ که $P>0, P\in\mathbb{R}$. بدست می‌آید

$\cancel{X_n(x) = A_n \sin\frac{n\pi}{L}x + B_n\cos\frac{n\pi}{L}x}$

$$X(x) = A\cos px + B\sin px$$

$$T''(t)+T'(t)-kT(t) = T''(t)+T'(t)-P^2T(t)=0$$ (ادامه ی الف ۲)

ریشه‌های معادله‌ی مشخصه $\Rightarrow \quad S^2+S+P^2=0 \longrightarrow S=\dfrac{-1\pm\sqrt{1+4P}}{2}$

$$\Rightarrow T(t)= Ce^{-\frac{t}{2}}\sin\left(\sqrt{\frac{1+4P}{4}}\,t\right)+De^{-\frac{t}{2}}\cos\left(\sqrt{\frac{1+4P}{4}}\,t\right)$$

$u(0,t)=0$

$\hookrightarrow X(0)=0 \longrightarrow A\cos px+B\sin px\Big|_{x=0}=A\cos px \longrightarrow A=0$

$$\sin p_n l=0 \xrightarrow{n\in\mathbb{N}} p_n l=n\pi \longrightarrow p_n=\frac{n\pi}{l}$$

$u(l,t)=0$

$$P_n=\frac{n\pi}{l}=n$$

$$u(x,t)=\sum_{n=1}^{\infty}u_n(x,t)=\sum_{n=1}^{\infty}X_n(x)T_n(t)=\sum_{n=1}^{\infty}\sin\frac{n\pi x}{l}\left(e^{-\frac{t}{2}}\right)\left(A_n\cos\sqrt{\frac{1+4P_n}{4}}\,t+B_n\sin\sqrt{\frac{1+4P_n}{4}}\,t\right)$$

$$u(x,0)=\sum_{n=1}^{\infty}\sin nx\,(A_n)=f(x)$$

ضریب A_n از ضریب سری فوریه گسترش فرد یافته تابع $f(x)$ بدست می‌آید

$$\hookrightarrow A_n=\frac{2}{l}\int_0^l f(x)\sin nx\,dx$$

$$u_t(x,t)=\sum_{n=1}^{\infty}-e^{-\frac{t}{2}}\sin nx\left[\left(2\sqrt{\frac{1+4n}{4}}\sin\sqrt{\frac{1+4n}{4}}\,t+\cos\sqrt{\frac{1+4n}{4}}\right)^{A_n}+\left(\sin\sqrt{\frac{1+4n}{4}}\,t-2\sqrt{\frac{1+4n}{4}}\cos\sqrt{\frac{1+4n}{4}}\right)^{B_n}\right]$$

$$u_t(x,t)\Big|_{t=0}=\sum_{n=1}^{\infty}-\sin nx\left[A_n\cancel{\cos\sqrt{\frac{1+4n}{4n}}\,t}^{\,1}+(-2)\sqrt{\frac{1+4n}{4}}\,\cancel{\cos\sqrt{\frac{1+4n}{4}}\,t}^{\,1}B_n\right]=g(x)$$

$$-A_n+2B_n\sqrt{\frac{1+4n}{4}}=\frac{2}{l}\int_0^l g(x)\sin nx\,dx$$

بنابراین ضرایب A_n و B_n به دست می‌آید.

$$A_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx \, dx$$

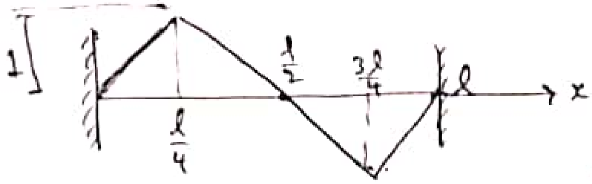
$$B_n = \frac{1}{\sqrt{4n+1}}\left(\frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx + \frac{2}{\pi}\int_0^{\pi} g(x)\sin nx\, dx\right) = \frac{2}{\pi\sqrt{4n+1}}\int_0^{\pi}\left(f(x)+g(x)\right)\sin nx\, dx$$

ادامه سوال 5:

$$u(x,t) = \sum_{n=1}^{\infty} \sin nx \cdot e^{-\frac{t}{2}}\left(A_n \cos\sqrt{\frac{4n+1}{4}}\,t + B_n \sin\sqrt{\frac{4n+1}{4}}\,t\right)$$

$$u(x,t) = \sum_{n=1}^{\infty} \sin nx \cdot e^{-\frac{t}{2}}\left(\frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx \cdot \cos\sqrt{\frac{4n+1}{4}}\,t + \frac{2}{\pi}\int_0^{\pi}\frac{\left(f(x)+g(x)\right)\sin nx\, dx}{\sqrt{4n+1}} \cdot \sin\sqrt{\frac{4n+1}{4}}\,t\right)$$

سوال 6



$u(0,t)=0 \qquad u_t(x,0)=g(x)$

$u(l,t)=0 \qquad u(x,0)=f(x)$

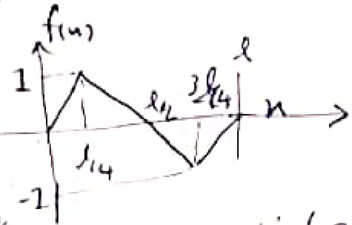
می دانیم که جواب معادله موج به عبارت

$$u(x,t)=\sum_{n=1}^{\infty} a_n \sin\frac{n\pi}{l}x \cos\frac{n\pi ct}{l} + b_n \sin\frac{n\pi x}{l} \cos\frac{n\pi ct}{l} \qquad 0\le x\le l$$

است. اگر سرعت اولیه که در قسمت صفر باشد در این صورت $g(x)=0 \rightarrow b_n=0$

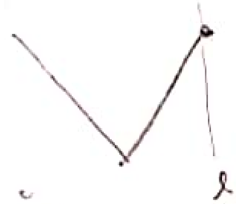
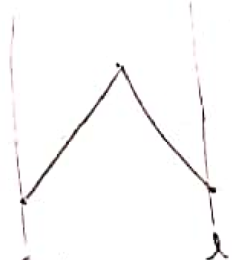
$$u(x,t)=\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi ct}{l}\right)$$

$$u(x,0)=\sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l} = f(x) =$$



بنابراین a_n ها ضرایب سری فوریه سینوسی $f^*(x)$ می باشند که $f^*(x)$ گسترش فرد تابع $f(x)$ در بازه $[-l,l]$ است.

$u(x, t=\frac{l}{4c})$

(i)  $f^*(x+\frac{l}{4c})$ +  $f^*(x-\frac{l}{4c})$ = 

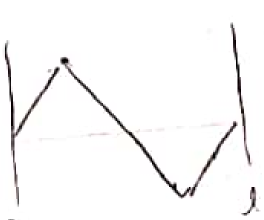
$u(x, t=\frac{2l}{4c})$

(ii)  $\frac{f^*(x+\frac{2l}{4c})}{2}$ +  $\frac{f^*(x-\frac{2l}{4c})}{2}$ = 

$u(x, t=\frac{3l}{4c})$

(iii)  $\frac{1}{2}f^*(x+\frac{3l}{4c})$ +  $\frac{1}{2}f^*(x-\frac{3l}{4c})$ = 

$u(x, t=\frac{l}{c})$

(iv)  $\frac{1}{2}f^*(x+\frac{4l}{4c})$ +  $\frac{1}{2}f^*(x-\frac{4l}{4c})$ = 

سؤال 7:

$$u_t = a^2 u_{xx} + \underbrace{t(x^2 - \pi x)}_{Q(x,t)}$$

$0 < x < \pi \qquad t > 0$

$u(x,0) = x$

$u(0,t) = u(\pi, t) = 0 \quad \rightarrow l = \pi$

$$q_n(t) = \frac{2}{l}\int_0^l Q(x,t) \sin\frac{n\pi x}{l}\,dx = \frac{2}{\pi}\int_0^{\pi} t(x^2 - \pi x) \sin nx \,dx$$

$$= \frac{2t}{\pi}\left[-\left(\frac{x^2 - \pi x}{n}\right)\cos nx + \left(\frac{2x - \pi}{n^2}\right)\sin nx + \frac{2}{n^3}\cos nx\right]\Bigg|_{x=0}^{x=\pi}$$

$$= \frac{2t}{\pi}\left(\frac{2}{n^3}\right)(\cos n\pi - 1) = \frac{4t}{\pi n^3}\left((-1)^n - 1\right)$$

let $\displaystyle u(x,t) = \sum_{n=1}^{\infty} \gamma_n(t) \sin\frac{n\pi x}{l}$.

using $u_t = a^2 u_{xx} + \overbrace{t(x^2 - \pi x)}^{Q(x,t)}$

$$u_t = \sum_{n=1}^{\infty} \gamma_n'(t) \sin nx = a^2 \sum_{n=1}^{\infty} \gamma_n(t)(-n^2 \sin nx) + Q(x,t)$$

$$\sum_{n=1}^{\infty}\left(\frac{d}{dt}\gamma_n(t) + a^2 n^2 \gamma_n(t)\right)\sin nx = Q(x,t) = \sum_{n=1}^{\infty} q_n(t) \sin nx$$

$$\rightarrow \gamma_n(0) = \frac{2}{l}\int_0^l x \sin nx = \frac{-2}{n}(-1)^n$$

$$\frac{d}{dt}\gamma_n(t) + a^2 n^2 \gamma_n(t) = q_n(t) = \frac{4t}{\pi n^3}\left((-1)^n - 1\right)$$

$$\Rightarrow \gamma_n(t) = A_n e^{-a^2 n^2 t} + (B_n t + C_n)$$

if $\gamma_n(t) = (B_n t + C_n) \Rightarrow \quad B_n + a^2 n^2 (B_n t + C_n) = \dfrac{4t}{\pi n^3}\left((-1)^n - 1\right)$

$$\Rightarrow B_n = a^2 n^2 C_n, \quad a^2 n^2 B_n = \frac{4t}{\pi n^3}\left((-1)^n - 1\right) \rightarrow$$

السؤال 7 :

$$C_n = \frac{4t}{\pi a^4 n^7}\left((-1)^n - 1\right) \quad , B_n = \frac{4t}{\pi a^2 n^5}\left((-1)^n - 1\right)$$

now we will obtain that:

$$u(x,t) = \sum_{n=1}^{\infty} \gamma_n(t) \overset{\sin n}{} = \sum_{n=1}^{\infty}\left(A_n e^{-a^2 n^2 t} + \frac{4\left((-1)^n - 1\right)}{\pi n^3}\left(t + a^2 n^2\right)\right) \sin nx$$

$$u(x,t)\Big|_{t=0} = \sum_{n=1}^{\infty} \gamma_n(0) \sin nx = \sum_{n=1}^{\infty} +\frac{2}{n}(-1)^{n+1} = \sum_{n=1}^{\infty}\left(A_n + \frac{4\left((-1)^n - 1\right)a^2 n^2}{\pi n^3}\right)$$

$$\Rightarrow A_n = \frac{2}{n}(-1)^{n+1} - \frac{4\left((-1)^n - 1\right)a^2 n^2}{\pi n^3} = \frac{2(-1)^{n+1} + \left(4(-1)^n - 4\right)a^2}{n}$$

$$\Rightarrow A_n = \frac{(-1)^n\left(-2 + 4a^2\right) - 4a^2}{n}$$

eventually $u(x,t)$ can be written in this form as below:

$$u(x,t) = \sum_{n=1}^{\infty}\left[\left(\frac{(-1)^n\left(-2+4a^2\right) - 4a^2}{n}\right) e^{-a^2 n^2 t} + \frac{4\left((-1)^n - 1\right)}{\pi n^3}\left(t + a^2 n^2\right)\right] \sin nx$$

سوال 8:

قسمت اول:

$$\left.\begin{aligned} &u_t = t\,u_{xx} \\ &-\infty < x < \infty \\ &t > 0 \\ &u(x,0) = e^{-x^2} = f(x) \end{aligned}\right\} \qquad \lim_{|t|,|x|\to+\infty} |u(x,t)| < \infty$$

let $u(x,t) = X(x)T(t) \longrightarrow X(x)T'(t) = t\,X''(x)T(t) \longrightarrow \dfrac{T'(t)}{t\,T(t)} = \dfrac{X''(x)}{X(x)} = k = \text{const}$

L.H.S. و R.H.S. تساوی فوق به ترتیب تابعیت از x و t هستند بنابراین حاصل k باید مستقل از x و t باشد
طبق قید وسعت u(x,t) دارای نوسان باید k<0 باشد .

$$\left.\begin{aligned} &k = -\lambda^2 < 0 \\ &\exists\, \lambda > 0,\ \lambda \in \mathbb{R} \end{aligned}\right\} \quad X''(x) + \lambda^2 X(x) = 0 \longrightarrow X(x) = A(\lambda)\cos\lambda x + B(\lambda)\sin\lambda x$$

$$\longrightarrow T'(t) + \lambda^2 t\,T(t) = 0 \longrightarrow \frac{T'(t)}{T(t)} = -\lambda^2 t \to \ln T(t) = -\frac{\lambda^2 t^2}{2}$$

$$T(t) = e^{-\frac{\lambda^2 t^2}{2}}$$

$$u(x,t) = X(x)T(t) = \int_0^\infty \left[A(\lambda)\cos\lambda x + B(\lambda)\sin\lambda x\right] e^{-\frac{\lambda^2 t^2}{2}}\, d\lambda$$

$$u(x,t)\Big|_{t=0} = \int_0^\infty \left[A(\lambda)\cos\lambda x + B(\lambda)\sin\lambda x\right] d\lambda = f(x) = e^{-x^2}$$

حال می‌دانیم که A(λ) و B(λ) ضرایب انتگرال فوریه تابع f(x) هستند:

$$\left.\begin{aligned} A(\lambda) &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos\lambda x\, dx \\ B(\lambda) &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin\lambda x\, dx \end{aligned}\right\} \quad g(x) = e^{-x^2} \longleftrightarrow \frac{e^{-\frac{\lambda^2}{4}}}{\sqrt{2}} = G(\lambda) = \sqrt{\frac{\pi}{2}}\left(A(\lambda) - jB(\lambda)\right) \Rightarrow$$

$$\begin{cases} B(\lambda) = 0 \\ A(\lambda) = \dfrac{e^{-\frac{\lambda^2}{4}}}{\sqrt{\pi}} \end{cases}$$

$$u(x,t) = \int_0^\infty \frac{e^{-\frac{\lambda^2}{4}}}{\sqrt{\pi}} \cdot \cos\lambda x \cdot e^{-\frac{\lambda^2 t^2}{2}}\, d\lambda = \int_0^\infty \frac{e^{-\frac{\lambda^2}{4}(1+2t^2)} \cdot \cos\lambda x}{\sqrt{\pi}}\, d\lambda$$

می‌دانیم که انتگرال $\int_{-\infty}^{\infty} \frac{1}{c\sqrt{4\pi t}} \exp\left(\frac{-(x-x')^2}{4c^2t}\right) dx'$ برابر یک است چون که $u(x,t)=1$.

حال از قضیه رایلی داریم:

$$I(\tau) = \int_{-\infty}^{\tau} \left(\frac{\sqrt{2\pi}}{c\sqrt{4\pi t}}\,\delta(\omega')\right)\left(\frac{e^{\frac{\omega'^2}{4\left(\frac{1}{4c^2t}\right)}}\, e^{-\ldots}}{\sqrt{2\,\frac{1}{4c^2t}}}\right) d\omega'$$

از فرمول فوق داریم:

$$I(\tau) = \begin{cases} \frac{1}{c\sqrt{2t}}\sqrt{2c^2t} = 1 & \tau > 0 \\ 0 & \tau < 0 \end{cases}$$

شکل آن تابع هویساید مانند می‌آید.