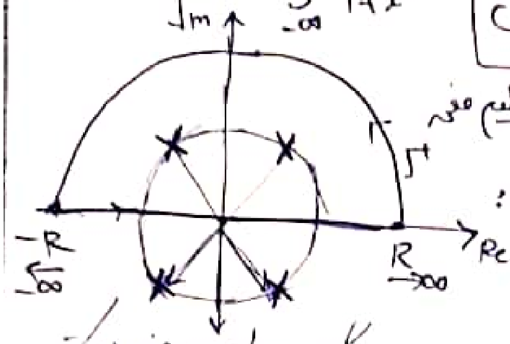
سوال 1: تمرین سری و ریاضیات مهندسی — محمدپارسا ارزنی ۴۰۰۱۰۱۲۰۵

(I) $I_1 = \int_{-\infty}^{\infty} \frac{1}{1+x^4} dx$



قطب‌های تابع $f(z) = \frac{1}{1+z^4}$ به صورت $e^{j\pi/4}, e^{j3\pi/4}, e^{j5\pi/4}, e^{j7\pi/4}$ است. دارای قطب ساده است که ۲ قطب در نیم صفحه بالای محور افقی و نقاط باقیمانده قطب‌ها در درون مسئله نخواهد بود پس داریم:

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4} = 2\pi j \sum_{z = e^{j\pi/4}, e^{j3\pi/4}} \text{Res}\, f(z) = \lim_{R\to\infty} \int_{C_R} f(z)\, dz$$

اندازه $|f(z)|$ از مرتبه $\frac{M}{R^4}$ است و بنابراین انتگرال روی کمان $\to$ صفر میل می‌کند.

$$\underset{z=e^{j\pi/4}}{\text{Res}}\, f(z) = \frac{1}{4z^3}\Big|_{z=e^{j\pi/4}} = \frac{1}{4} e^{-\frac{3}{4}\pi j}$$

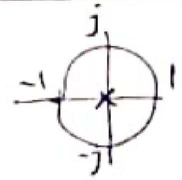
$$\underset{z=e^{j\frac{3\pi}{4}}}{\text{Res}}\, f(z) = \frac{1}{4z^3}\Big|_{z=e^{j3\pi/4}} = \frac{1}{4} e^{-\frac{9}{4}\pi j}$$

$$I_1 = \frac{2\pi j}{4}\left(e^{-\frac{3\pi j}{4}} + e^{-\frac{9\pi j}{4}}\right) = \frac{2\pi j}{4}\left(-2j\cos\left(\frac{\pi}{4}\right)\right) = \frac{\pi\sqrt{2}}{2} = \boxed{\frac{\pi}{\sqrt{2}}}$$

(II) $I_2 = \int_0^{2\pi} \sin^4\theta \, d\theta$

let $z = e^{j\theta} \to \sin\theta = \frac{1}{2j}\left(z - \frac{1}{z}\right)$

$$d\theta = \frac{dz}{je^{j\theta}} = \frac{dz}{jz}$$



$$I_2 = \oint_C \frac{1}{16}\left(z - \frac{1}{z}\right)^4 \frac{dz}{jz} = \frac{1}{16}\oint_C \frac{1}{jz}\left(\frac{z^2-1}{z}\right)^4 dz = \frac{-j}{16}\oint_C \frac{(z^2-1)^4}{z^5} dz$$

C ← دایره واحد

$$I_2 = \left(\frac{-j}{16}\right) 2\pi j \underset{z=0}{\text{Res}}\left(\frac{(z^2-1)^4}{z^5}\right) = \left(2\pi j \times \frac{-j}{16}\right)\left(\frac{1}{4!}\lim_{z\to 0}\frac{d^4}{dz^4}\left[(z^2-1)^4\right]\right)$$

قطب از مرتبه ۵ است.

$$= \left(\frac{\pi}{8\times 4!}\right)\left(48(35z^4 - 30z^2 + 3)\Big|_{z=0}\right) = \frac{48\times 3\times\pi}{8\times 4!} = \boxed{\frac{3\pi}{4}}$$

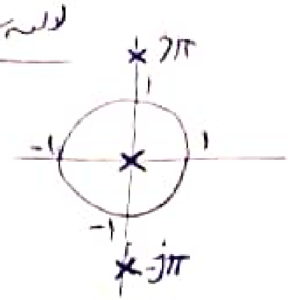
$$= \boxed{\frac{3\pi}{4} \simeq 2.35}$$

ب ← انتگرال به روش مرسوم تدریس:

integral by parts

$$\int_0^{2\pi} \sin^4\theta\, d\theta = \left(\frac{\cos\theta\sin^3\theta}{4}\right)\Big|_0^{2\pi} + \frac{3}{4}\pi = \boxed{\frac{3\pi}{4}}$$

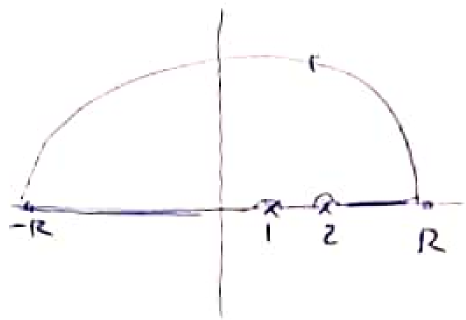
(III) $I_3 = \oint\limits_{|z|=1} \dfrac{\tanh(z+1)}{e^{z} \sin z}\, dz = \dfrac{\tanh(z+1)\, e^{-z}}{\cos z}\Bigg|_{z=0}$ لو السؤال:



$$\left.\begin{aligned} & e^{z} = 1 \rightarrow z = 0 \\ & e^{z} = -1 \rightarrow z = kj\pi \\ & e^{2z+2} = -1 \rightarrow z = \frac{kj\pi - 2}{2} \end{aligned}\right\} \text{قطب}$$

$$= \frac{\tanh(1)\, e^{-1}}{1} = 2.07$$

(IV) $I_4 = \displaystyle\int_{-\infty}^{\infty} \frac{e^{jx}}{(x-1)(x-2)}\, dx = \pi j \left(\operatorname*{Res}_{z=1} f(z) + \operatorname*{Res}_{z=2} f(z) \right)$



let $f(z) = \dfrac{e^{jz}}{(z-1)(z-2)}$

$$\operatorname*{Res}_{z=1} f(z) = \frac{e^{jz}}{\frac{d}{dz}\left[(z-1)(z-2)\right]} = \frac{e^{jz}}{2z-3}\Bigg|_{z=1} = \frac{e^{j}}{-1}$$

$$\operatorname*{Res}_{z=2} f(z) = \frac{e^{jz}}{\frac{d}{dz}\left[(z-1)(z-2)\right]} = \frac{e^{jz}}{2z-3}\Bigg|_{z=1} = \frac{e^{2j}}{1}$$

$$I_4 = \pi j \left(e^{2j} - e^{j} \right)$$

$$= \pi j (\cos 2 - \cos 1)$$

$$- \pi (\sin 2 - \sin 1)$$

تمرین سری دو ریاضیات مهندسی ۴۰۰۱۶۱۲۰۲ — محمدرضا لبینی ۴۰۰۱۰۱۲۰۲

سوال 2:

(I) $y[n]=\sum_{k=-\infty}^{n} x[k] \longrightarrow Y[z]=\sum_{n=-\infty}^{+\infty} y[n]z^{-n}=\sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{n} x[k]\, z^{-n}$

$$y[n]=\sum_{k=-\infty}^{\infty} x[k]u[n-k] \Leftrightarrow y[n]=x[n]*u[n] \Rightarrow Y(z)=X(z)\cdot U(z)$$

$$\Rightarrow Y(z)=X(z)\times\frac{1}{1-z^{-1}}=\frac{X(z)}{1-z^{-1}} \Rightarrow y[n]\leftrightarrow \frac{X(z)}{1-z^{-1}}$$

(II) $X(z)=\sum_{n=-\infty}^{\infty} x[n]z^{-n} \rightarrow \frac{dX(z)}{dz}=\sum_{n=-\infty}^{\infty}(-n)\, x[n]z^{-n-1}=\left(\sum_{n=-\infty}^{\infty}[-nx[n]]\,z^{-n}\right)\frac{1}{z}$

$$\Rightarrow n\,x[n]\leftrightarrow -z\frac{dX(z)}{dz}=\sum_{n=-\infty}^{\infty}\left(n\,x[n]\,z^{-n}\right)$$

(III) $y[n]=x_1[n]*x_2[n] \Rightarrow Y(z)=\sum_{n=-\infty}^{\infty} y[n]z^{-n}=\sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} x_1[k]\,x_2[n-k]\,z^{-n}$

$$=\sum_{k=-\infty}^{\infty} z^{-k}x_1[k]\sum_{n=-\infty}^{\infty} x_2[n-k]z^{-n+k}=\left(\sum_{k=-\infty}^{\infty} z^{-k}x_1[k]\right)\cdot\left(\sum_{n=-\infty}^{\infty} z^{-n}x_2[n]\right)=X_1(z)\cdot X_2(z)$$

سوال 3:

(I) $x_1[n]=a^n u[n]+b^n u[n]+c^n u[-n-1] \Leftrightarrow \frac{1}{1-az^{-1}}+\frac{1}{1-bz^{-1}}-\frac{1}{1-cz^{-1}}$

$$ROC \longrightarrow |z|>\max\{|a|,|b|\}\;|c|$$

(II) $x_2[n]=n^2a^n u[n]$, $a^n u[n]\leftrightarrow f(z)=\frac{1}{1-az^{-1}}$,

ROC تعیین نمی‌کند.

$$n\cdot\left[a^n u[n]\right]\leftrightarrow -z\frac{d}{dz}f(z) \Rightarrow n^2\left[a^n u[n]\right]\leftrightarrow -z\frac{d}{dz}\left(-z\frac{d}{dz}f(z)\right)=$$

$$n^2a^n u[n]\leftrightarrow -z\frac{d}{dz}\left(\frac{-az}{(a-z)^2}\right)=\frac{-az(a+z)}{(a-z)^3}=\boxed{\frac{az^{-1}(1+az^{-1})}{(1-az^{-1})^3}}$$

(III) $x_3[n]=\left(\frac{1}{2}\right)^{n+1}u[n+3]=4\left(\frac{1}{2}\right)^{n+3}u[n+3]\Leftrightarrow 4z^{-3}\,\mathcal{Z}\left\{\left(\frac{1}{2}\right)^n u[n]\right\}$

$$x_3[n]\leftrightarrow 4z^{-3}\frac{1}{1-\frac{1}{2}z^{-1}}=\frac{4(z^{-1})^3}{1-\frac{1}{2}z^{-1}}\qquad |z|<\frac{1}{2}$$

IV $x_4[n] = \left(\frac{1}{3}\right)^{|n|} = \left(\frac{1}{3}\right)^{n} u[n] + \left(\frac{1}{3}\right)^{-n} u[-n-1]$

$x_4[n] \Leftrightarrow \frac{1}{1-\frac{1}{3}z^{-1}} - \frac{1}{1-3z^{-1}} \quad \left(|z| > \left|\frac{1}{3}\right|,\ |z| < |3|\right) \rightarrow \left(\frac{1}{3} < z < 3\right)$

V $x_5[n] = \frac{u[n]}{n!} \longrightarrow X(z) = \sum_{n=-\infty}^{\infty} \frac{u[n]\, z^{-n}}{n!} = \sum_{n=0}^{\infty} \frac{z^{-n}}{n!} = \boxed{e^{\frac{1}{z}}}$

ROC $\rightarrow$ all points of the z complex plane! $(|z| < \infty)$

سوال 5:

$y[n] - \frac{29}{4} y[n-1] + \frac{7}{4} y[n-2] = 2x[n] - \frac{29}{4} x[n-1]$

از گرفتن تبدیل z داریم:

$Y(z)\left(1 - \frac{29}{4} z^{-1} + \frac{7}{4} z^{-2}\right) = X(z)\left(2 - \frac{29}{4} z^{-1}\right)$

تابع تبدیل این سیستم را داریم

$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - \frac{29}{4} z^{-1} + \frac{7}{4} z^{-2}}{2 - \frac{29}{4} z^{-1}} = \frac{4z^2 - 29z + 7}{8z^2 - 29z}$

حال اگر $\delta[n] = x[n]$ در این صورت $X(z) = 1$ و پاسخ ضربه برابر خود $H(z)$ می‌شود

$\hookrightarrow Y(z)\Big|_{X(z)=1} = H(z) = \frac{4z^2 - 29z + 7}{8z^2 - 29z} = -\frac{7}{29} z^{-1} + \frac{2916}{841}\left(\frac{1}{29z^{-1} - 8}\right) + \frac{785}{841}$

$\Rightarrow y[n] = -\frac{7}{29}\delta[n-1] + \frac{2916}{(841)(8)}\left(\frac{29}{8}\right)^{n} u[n] + \frac{785}{841}\delta[n]$

1) $X(z) = \dfrac{4-8z^{-1}+6z^{-2}}{z(1+z^{-1})(1-2z^{-1})^2} = \dfrac{2}{1+z} - \dfrac{2}{z-2} + \dfrac{2}{(z-2)^2}$:4 dB

$X(z) \leftrightarrow 2(-1)^n u[n-1] - 2^{n+1} u[n-1] - (n-1)2^n u[n-2]$

$\text{RoC} = (|z|<\infty) \cap (|z|>2) \cap (|z|>2) \rightarrow (2<|z|<\infty)$

$\dfrac{-2}{z-2} \leftrightarrow -2^{n+1}u[n-1] \xRightarrow{\text{derivative Property}} \dfrac{-2z}{(z-2)^2} \leftrightarrow -n2^{n+1}u[n-1] \xRightarrow{\text{Shift Property}} -(n-1)2^n u[n-2]$

II) $X(z) = \log(1-3z) \xrightarrow{\text{derivative Property}} \dfrac{dX(z)}{dz} = \dfrac{-3}{1-3z} \leftrightarrow \dfrac{-3}{1-3z} = -\sum 3^{n+1} z^{-n}$

$\longrightarrow \displaystyle\int_0^z \frac{-3}{1-3z}\,dz = \int_0^z \sum_{n=-\infty}^{\infty} -3^{n+1} z^{-n}\,dz = -\sum_{n=0}^{\infty} \frac{3^{n+1} z^{1-n}}{1-n} = \sum_{n=1}^{\infty} \frac{+3^{n+2} z^{-n}}{n}$

$= -3z + 9\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{n} z^{-n} \Rightarrow X(z) \leftrightarrow -3\delta[n+1] + 9\left(\frac{3^n}{n}\right)u[n]$

$\Rightarrow \text{RoC} \Rightarrow |z|>3$

III) $X(z) = e^{z^2} = \displaystyle\sum_{n=0}^{\infty} \frac{z^{2n}}{n!} = \sum_{-\infty}^{0} \frac{z^{-2n}}{-n!} = \sum_{\substack{n \overset{2}{=} 0 \\ n=-\infty}}^{0} \frac{+z^{-n}}{\left(\frac{n}{2}\right)!}$

$X(z) \leftrightarrow \begin{cases} \dfrac{1}{\left(\frac{n}{2}\right)!} u[n] & n \overset{2}{=} 0 \\ 0 & n \overset{2}{\neq} 0 \end{cases}$ $\text{RoC} \Rightarrow (|z|<\infty)$

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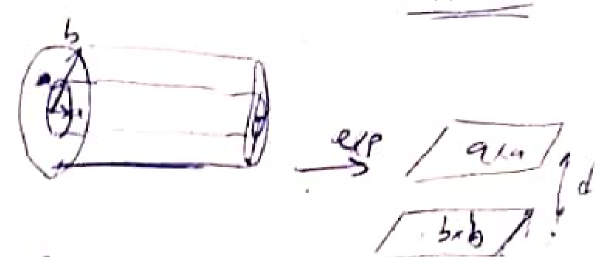
سوال 7:

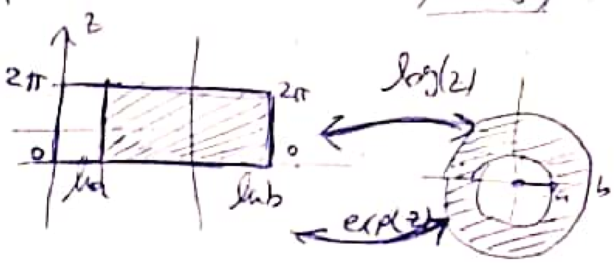
با استفاده از رابطه گوس
ظرفیت خازن

$$C=\frac{q}{|\Delta V|}=\frac{2\pi\epsilon_0 L}{\ln\frac{b}{a}}$$

$$\Delta V=-\int \vec{E}\cdot d\vec{r}=\int\frac{-\lambda}{2\pi\epsilon_0 r}\,dr=\frac{-\lambda}{2\pi\epsilon_0}\ln\left|\frac{b}{a}\right|\Bigg|_{\lambda=1}$$



$$C_0=\frac{C}{L}=\frac{2\pi\epsilon_0}{\ln\left|\frac{b}{a}\right|}\longrightarrow \text{ظرفیت واحد طول}$$



$$C=\frac{A\epsilon_0}{d}=\frac{2\pi(1)\epsilon_0}{\ln b-\ln a}=\frac{2\pi\epsilon_0}{\ln(b/a)}$$

نکته: با تبدیل نگاشت $\ln(z)$، استوانه به یک صفحه تخت تبدیل می‌شود و همان طور که نشان داده می‌شود ظرفیت خازن واحد در سطح $\frac{2\pi\epsilon_0}{\ln b/a}$ به دست آمد که انتظار می‌رفت.

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