تمرین سری پنجم ریاضیات مهندسی دکتر کربلایی — 400101204 — محمدیار ساردنی

سوال ۱ : بخش اول:

let $u(x,y,t)=X(x)Y(y)T(t) \longrightarrow$

جایگذاری در

$u_{tt}=u_{xx}+u_{yy} \longrightarrow T''(t)X(x)Y(y)=X''(x)Y(y)T(t)+Y''(y)T(t)X(x)$

$$\longrightarrow \frac{T''(t)}{T(t)}=\frac{X''(x)}{X(x)}+\frac{Y''(y)}{Y(y)}$$

چون که هر کسر تابعی از یک متغیر است نتیجه می‌گیریم که هر کسر باید برابر یک عبارت ثابت باشد.

$$\# \quad \frac{X''(x)}{X(x)}=K_1=-\lambda_1^2 \longrightarrow X''(x)+\lambda_1^2X(x)=0 \longrightarrow X(x)=A\sin\lambda_1 x+B\cos\lambda_1 x$$

چون که شرایط اولیه برای x صفر است می‌دانیم تنها در حالتی که $K_1=-\lambda^2$ جواب غیرصفر بدیهی دارد.

$u(0,y,t)=Y(y)T(t)(B)=0 \longrightarrow B=0$

$u(\pi,y,t)=Y(y)T(t)(A\sin\lambda_1\pi)=0 \longrightarrow \lambda_1\pi=n\pi \xrightarrow{(n\in\mathbb{N})} \lambda_1(n)=n$

$$\# \quad \frac{Y''(y)}{Y(y)}=K_2=-\lambda_2^2 \longrightarrow Y''(y)+\lambda_2^2Y(y)=0 \longrightarrow Y(y)=C\sin\lambda_2 y+D\cos\lambda_2 y$$

$u(x,0,t)=X(x)T(t)(D)=0 \longrightarrow D=0$

$u(x,\pi,t)=X(x)T(t)(C\sin\lambda_2\pi)=0 \longrightarrow \lambda_2\pi=m\pi \xrightarrow{(m\in\mathbb{N})} \lambda_2(m)=m$

$$\# \quad \frac{T''(t)}{T(t)}=K_1+K_2=-\lambda_1^2-\lambda_2^2=-n^2-m^2 \longrightarrow T''(t)+(n^2+m^2)T(t)=0$$

$\hookrightarrow T(t)=M\sin\sqrt{n^2+m^2}\,t+N\cos\sqrt{n^2+m^2}\,t$

ادامه سوال 1.1:

$$u_t(x,y,0)=0 \to X(x)Y(y)\left(\frac{M}{\sqrt{m^2+n^2}}\right)=0 \to M=0$$

$$\Rightarrow u(x,y,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} X_n(x)Y_n(y)T_{m,n}(t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} F_{n,m}\sin nx\,\sin my\,\cos\sqrt{m^2+n^2}\,t$$

$$F_{n,m}=\frac{4}{\pi^2}\int_0^{\pi}\int_0^{\pi}\sin 2x\sin y\sin nx\sin my\,dy\,dx=\frac{4}{\pi^2}\int_0^{\pi}\sin 2x\sin nx\,dx\int_0^{\pi}\sin y\sin my\,dy$$

$$\int_0^{\pi}\sin y\sin my\,dy=\begin{cases} m=1\to\frac{\pi}{2} \\ m=-1\to-\pi/2 \\ \text{else}\to 0\end{cases}$$ یعنی

اما برای $\int_0^{\pi}\sin 2x\sin nx\,dx=\left\{\begin{array}{ll} n=2 & \frac{\pi}{2} \\ \text{else} & 0 \\ n=-2 & -\frac{\pi}{2}\end{array}\right.$

که در نتیجه می شود

$$F_{n,m}=\begin{cases}1 & n=2,m=1\\ 0 & \text{otherwise}\end{cases}$$

که در پایان خواهیم داشت:

$$\boxed{u(x,y,t)=\sin 2x\sin y\cos\sqrt{5}\,t}$$

سوال 1.2

$$u_{rr}+\frac{1}{r^2}u_r=u_{tt}\qquad \text{let}\quad u(r,\varphi)=R(r)\,\Phi(\varphi)$$

از جایگذاری در معادله خواهیم داشت:

$$\frac{r^2R''(r)+R'(r)}{r^2}\,T(t)=T''(t)\,R(r)\ \longrightarrow\ \frac{r^2R''(r)+R'(r)}{r^2R(r)}=\frac{T''(t)}{T(t)}=k$$

از دو معادله فوق به معادلات زیر می‌رسیم

$$\begin{cases} r^2R''(r)+rR'(r)+k^2r^2R(r)=0\\ T''(t)+k^2T(t)=0\end{cases}$$

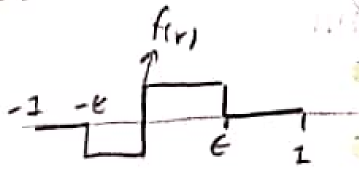
با تغییر متغیر از معادله اول به معادله بسل درجه صفر می‌رسیم $x=kr$

$$T_n(t)=B_n\cos(\alpha_n t)+C_n\sin(\alpha_n t)\longleftarrow R_n(kr)=A_n\cdot J_0(\alpha_n\cdot r)$$

تابع جواب در معادلات با توجه به اینکه مسئله ما در $r=0$ محدود باشد را داریم. در کل خواهیم داشت:

$$u(r,t)=\sum_{n=1}^{\infty}F_n\cdot J_0(\alpha_n\cdot r)\left[\sin(\alpha_n t)+B\cos(\alpha_n t)\right]\qquad u(r,0)=0\rightarrow B_n=0$$

$$\rightarrow u(r,t)=\sum_{n=1}^{\infty}F_n\cdot J_0(\alpha_n r)\sin(\alpha_n t)=f(r)=\begin{cases}0 & \epsilon<r<1\\ \dfrac{P}{\pi\epsilon^2\rho} & 0<r<\epsilon\end{cases}$$



از آنجا که $J_n(x)$ ها توابع متعامد هستند می‌توانیم ضرایب F_n را از انتگرال‌گیری مشخص کنیم

$$F_n=\frac{2}{J_1^2(\alpha_n)}\int_0^{\epsilon}\frac{P}{\pi\epsilon^2\rho}\,r\,J_0(\alpha_n\cdot r)\,dr=\frac{2P}{J_1^2(\alpha_n)\,\pi\epsilon^2\rho}\left[\frac{r\,J_1(\alpha_n r)}{\alpha_n}\right]\Big|_0^{\epsilon}=$$

$$=\frac{2P}{\alpha_n\cdot J_1^2(\alpha_n)\,\epsilon^2\rho}\left[\epsilon J_1(\alpha_n\epsilon)\right]=\frac{2P\,J_1(\alpha_n\epsilon)}{\alpha_n\cdot J_1^2(\alpha_n)\cdot\pi\epsilon\rho}$$

α_n امین ریشه بسل صفر

$$\Rightarrow\ \boxed{u(r,t)=\sum_{n=1}^{\infty}\frac{2P\,J_1(\alpha_n\epsilon)}{\alpha_n\cdot J_1^2(\alpha_n)\cdot\pi\epsilon\rho}\,J_0(\alpha_n r)\sin(\alpha_n t)}$$

سؤال 2:

$$u_t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u_r\right) \Rightarrow \text{let } u(r,t) = R(r)T(t) \Rightarrow \frac{T'(t)}{T(t)} = \frac{1}{R(r)\cdot r^2}\left(r^2 R''(r) + 2r R'(r)\right) = -k^2$$

$$\text{let } f(r) = r \cdot R(r) \Rightarrow \begin{matrix}\text{writing the equation} \\ \text{with respect to } f(r)\end{matrix} \Rightarrow \begin{cases} R'(r) = \dfrac{r f'(r) - f(r)}{r^2} \\ R''(r) = \dfrac{r^3 f''(r) - 2r^2 f'(r) + 2r f(r)}{r^4} \end{cases}$$

$$\Rightarrow r\left(f''(r) + k^2 f(r)\right) = 0 \rightarrow f(r) = A\cos kr + B\sin kr \Rightarrow R(r) = A\frac{\cos kr}{r} + B\frac{\sin kr}{r}$$

ازآنجاکه باید $u(0,t)$ کراندار باشد باید داشته باشیم $A=0$ در نتیجه $R(r) = B\frac{\sin kr}{r}$. از روش دیگر با حل معادله

برای $T(t)$ خواهیم داشت: $T'(t) + k^2 T(t) = 0 \rightarrow T(t) = Ce^{-k^2t^2}$

$$\cancel{\Rightarrow u(r,t)} \quad u(1,t) = R(r)T(t) = BCe^{-k^2t^2} \quad \left(\frac{\sin k}{1}\right) = 0 \rightarrow \sin k = 0 \rightarrow k_n = n\pi \quad (n \in \mathbb{N})$$

$$k_n = n\pi$$

$$u(r,t) = \sum_{n=1}^{\infty} \frac{F_n \cdot e^{-k_n^2 t^2} \cdot \sin(n\pi r)}{r}$$

با استفاده از گسترش فرد تابع $r = f(r)$ که $f:[0,1]\to\mathbb{R}$ توسیع است:

$$u(r,0) = 1 = \sum_{n=1}^{\infty} F_n \cdot \frac{\sin n\pi r}{r} \rightarrow \sum_{n=1}^{\infty} F_n \sin n\pi r = r \rightarrow F_n = \frac{2}{1}\int_0^1 r \sin n\pi r \, dr$$

$$F_n = 2\left(\frac{r\cos(n\pi r)}{n\pi} - \frac{\sin(n\pi r)}{n^2\pi^2}\right)\Bigg|_{r=0}^{r=1} = \frac{-2\cos n\pi}{n\pi} = \frac{(-1)^{n+1} \times 2}{n\pi}$$

از آنجا به دست می‌آید که:

$$\boxed{u(r,t) = \sum_{n=1}^{\infty} \frac{2\cdot(-1)^{n+1}}{n\pi} \cdot \frac{e^{-n^2\pi^2 t}\sin(n\pi r)}{r}}$$

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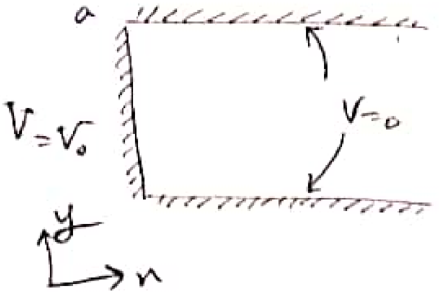
سوال 3:



$$\left.\begin{array}{l} u(x,0) = u(x,a) = 0 \\ u(0,y) = V_0 \end{array}\right\} \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 = \nabla^2 u$$

let $u(x,y) = X(x)\,Y(y) \xrightarrow{\text{جداسازی}} X''(x)Y(y) + X(x)Y''(y) = 0 \rightarrow \frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = 0$

$$\frac{X''(x)}{X(x)} = \frac{-Y''(y)}{Y(y)} = K$$

Case I: $K = 0 \rightarrow Y''(y) = 0 \rightarrow Y(y) = Ay + B \rightarrow u(x,0) = X(x)(A\cdot 0 + B) = 0$

$\rightarrow B = 0 \rightarrow$

$u(x,a) = X(x)Y(a) = X(x)(Aa) = 0 \rightarrow A = 0$

Case II: $K = -\lambda^2 \rightarrow Y''(y) - \lambda^2 Y(y) = 0 \rightarrow Y(y) = Ae^{-\lambda y} + Be^{\lambda y}$

$u(x,0) = X(x)Y(0) = A + B = 0 \rightarrow u(x,a) = A(e^{-\lambda a} - e^{\lambda a}) = 0 \rightarrow A = 0$

Case III: $K = \lambda^2 \rightarrow Y''(y) + \lambda^2 Y(y) = 0 \rightarrow Y(y) = A\sin\lambda y + B\cos\lambda y$

$u(x,0) = X(x)Y(0) = (B) = 0 \rightarrow B = 0$

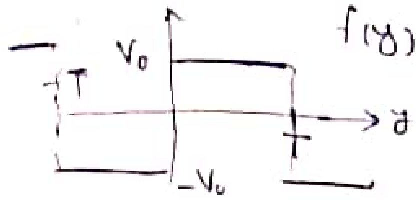
$u(x,a) = X(x)Y(a) = A\sin\lambda a = 0 \rightarrow \lambda a = n\pi \xrightarrow{n\in\mathbb{Z}} \boxed{\lambda_n = \frac{n\pi}{a}}$

حالا سراغ $X(x)$ می‌رویم:

$X''(x) - \lambda^2 X(x) = 0 \rightarrow X(x) = Ae^{-\lambda x} + Be^{\lambda x}$

$u(0,y) = Y(y)(A+B) = V_0 \rightarrow B = 0$, چون $\lim_{x\to\infty} u(x,t)$ bounded $\rightarrow \boxed{A = \frac{V_0}{a}}$

$$u(x,y) = \sum_{n=1}^{\infty} X_n(x)Y_n(y) = \sum_{n=1}^{\infty} \underbrace{D_n \cdot e^{-\lambda_n x}}_{F_n} \sin(\lambda_n y), \quad \lambda_n = \frac{n\pi}{a}$$



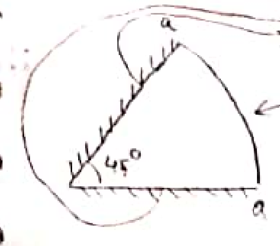
$$u(x,y)\Big|_{x=0} = \sum_{n=1}^{\infty} D_n \sin\lambda_n y, \quad T = \left(\frac{\pi}{a}\right)^{-1}, \quad D_0 = 0$$

$$D_n = \frac{2}{\pi}\int_0^{T/2} f(t)\sin\left(\frac{n\pi}{a}t\right)dt = \frac{-2V_0}{\pi n}\left((-1)^n - 1\right)$$

ادامه سوال 3 :

حال که D_n را حساب کردیم $u(x,y)$ به دست می‌آید:

$$\boxed{u(x,y) = -\frac{2V_0}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n} e^{-\left(\frac{n\pi x}{a}\right)} \sin\left(\frac{n\pi}{a} y\right)}$$



$u(r,0) = u(r,\frac{\pi}{4}) = 0$

$u(a,\varphi) = 1$

در مختصات قطبی

$\nabla^2 u = 0 \to u_{xx} + u_{yy} = 0 \to \frac{1}{r^2}u_{\varphi\varphi} + \frac{1}{r}\frac{\partial}{\partial r}(r u_r) = 0$

50

let $u(r,\varphi) = R(r)\Phi(\varphi) \to \frac{1}{r}\frac{\partial}{\partial r}(rR'(r)\Phi(\varphi)) + \frac{1}{r^2}\Phi''(\varphi)R(r) = 0$

$\longrightarrow \frac{r}{R(r)}\frac{d}{dr}\left(rR'(r)\right) = \frac{-\Phi''(\varphi)}{\Phi(\varphi)} = k$

(I) In Case $k=0 \to \Phi''(\varphi) = 0 \to \Phi(\varphi) = A\varphi + B \to u(r,0) = B\cdot R(r) = 0 \to B = 0$

$u(r,\frac{\pi}{4}) = A\cdot\frac{\pi}{4}\cdot R(r) = 0 \to A = 0$. در این حالت جواب بدیهی صفر خواهیم داشت

(II) In Case $k<0 \to \exists\lambda>0 : k=-\lambda^2 \to \Phi''(\varphi) - \lambda^2\Phi(\varphi) = 0 \to \Phi(\varphi) = Ae^{-\lambda\varphi} + Be^{\lambda\varphi}$

$u(r,0) = R(r)(A+B) = 0 \to A+B = 0 \to u(r,\frac{\pi}{4}) = A\cdot R(r)\Big(\underbrace{e^{-\lambda\frac{\pi}{4}} - e^{\lambda\frac{\pi}{4}}}_{\text{ناصفر} \leftarrow \lambda\neq 0}\Big) \Longrightarrow \to A = 0 = B$ جواب بدیهی صفر

(III) In Case $k>0 \to \exists\lambda>0 : k=\lambda^2 \to \Phi''(\varphi) - \lambda^2\Phi(\varphi) = 0 \to$

$\Phi(\varphi) = A\cos\lambda\varphi + B\sin\lambda\varphi$

$u(r,0) = B\cdot R(r) = 0 \to B = 0 \to u(r,\frac{\pi}{4}) = A\cdot R(r)\sin\frac{\lambda\pi}{4} = 0 \xrightarrow{A\neq 0} \frac{\lambda\pi}{4} = n\pi \ (n\in\mathbb{N}) \to \lambda_n = 4n$

الان که $\Phi(\varphi)$ را داریم، $R(r)$ را حساب می‌کنیم:

$\frac{r}{R(r)}\frac{d}{dr}(rR'(r)) = \lambda^2 \to r^2R''(r) + rR'(r) - \lambda^2R(r) = 0$

let $R(r) = r^\alpha \to \alpha(\alpha-1)r^\alpha + \alpha r^\alpha - 16n^2r^\alpha = 0 \to \alpha^2 - \alpha + \alpha - 16n^2 = 0 \to \alpha = \pm 4n \ (n\in\mathbb{N})$

اما از آنجایی که در $r=0$ باید هم‌چنین کراندار باشد، باید $\alpha = 4n$ باشد چرا که $R(r)$ برای حالت $\alpha = -4n$ بی‌کران خواهد بود

درحالتی که $n=0$ باشد هم داریم:

if $n=0 \to \frac{r}{R(r)}\frac{d}{dr}(rR'(r)) = 0 \to R(r) = C + D\ln r$

که ایجاب می‌کند $D=0$ چرا که در $r=0$ دوباره جواب بی‌کران خواهد شد. پس در پایان فرم $u(r,\varphi)$ به صورت زیر خواهد بود:

$u(r,\varphi) = a_0 + \sum_{n=1}^{\infty}\left(a_n\cos 4n\varphi + b_n\sin 4n\varphi\right)r^{4n}$

$\left.\begin{array}{l} u(r,0) = 0 \\ u(r,\frac{\pi}{4}) = 0 \end{array}\right\}\ \begin{array}{l} a_0 = 0 \\ a_n = 0 \end{array}$

$g(\varphi) = 2$ گسترش فرد

$\Rightarrow u(a,\varphi) = \sum_{n=1}^{\infty} a^n\sin 4n\varphi = 2$

$$\boxed{u(r,\varphi) = \sum_{n=1}^{\infty}\frac{8\cdot r^{4n}}{a^{4n}\cdot n\pi}\sin(n\varphi)\left(1-(-1)^n\right)}$$

$\hookrightarrow b_n = \frac{2}{a^{4n}\,\frac{\pi}{4}}\int_0^{\pi/4} 2\sin 4n\varphi\, d\varphi = \frac{8}{\pi\cdot a^{4n}}\left(\frac{-\cos 4n\varphi}{n}\right)\Big|_0^{\pi/4} = \frac{8\cdot a^{-4n}}{4\pi n}\left(1-(-1)^n\right)$

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(I) $u_{xx} = u_{tt}$, $t, x > 0$, $u(0,t) = \begin{cases} f(t) & 0 < t \le T \\ 0 & t \ge T \end{cases}$: سؤال 6

$u_t \Big|_{t=0} = 0$, $u(x,0) = 0$, $\lim\limits_{x \to \infty} u(x,t) = 0$

$$\mathcal{L}\{u_{xx}\} = \int_0^\infty e^{-st} \frac{\partial^2}{\partial x^2} u(x,t)\, dt = \frac{\partial^2}{\partial x^2} \int_0^\infty e^{-st} u(x,t)\, dt = \frac{\partial^2}{\partial x^2} U(x,s)$$

$$\mathcal{L}\{u_{tt}\} = s^2 U(x,s) - \cancel{s\, u(x,0)} - \cancel{u_t(x,0)} = s^2 U(x,s)$$

$$\Rightarrow u_{xx} = u_{tt} \xrightarrow[\text{form}]{\text{Laplace}} \frac{\partial^2}{\partial x^2} U(x,s) = s^2 U(x,s) \longrightarrow \text{let } X(x)\, S(s) = U(x,s)$$

$$\longrightarrow X''(x) - s^2 X(x) = 0 \longrightarrow U(x,s) = A(s)\, e^{-sx} + B(s)\, e^{+sx} \qquad (*)$$

$$\lim_{x \to \infty} U(x,s) = 0 \longrightarrow \lim_{x \to \infty} B(s)\, e^{sx} = 0 \longrightarrow B(s) = 0$$

$$u(0,t) = f(t) \longrightarrow U(0,s) = \mathcal{L}\{f(t)\} = F(s) \overset{(*)}{\Rightarrow} U(0,s) = A(s) = F(s)$$

$$\hookrightarrow u(x,t) = \mathcal{L}^{-1}\{F(s)\, e^{-sx}\} = f(-x+t) = f(t-x) \qquad \forall\, t > 0$$

$$= \mathcal{L}^{-1}\left\{\left(\int_0^\infty e^{-st} f(t)\, dt\right) e^{-sx}\right\}$$

(II) $u_{xx} = u_t$, $t, x > 0$, $u(0,t) = f(t)$, $u(x,0) = 0$, $\lim\limits_{x \to \infty} u(x,t) =$ bounded

$$\mathcal{L}\{u_{xx}\} = \frac{\partial^2}{\partial x^2} U(x,s) \quad , \mathcal{L}\{u_t\} = s\, U(x,s) - \cancel{u(x,0)}$$

$$\lim_{x \to \infty} U(x,s) = \lim_{x \to \infty} \int_0^\infty e^{-st} u(x,t)\, dt = \int_0^\infty e^{-st} \lim_{x \to \infty} u(x,t)\, dt = \text{صفر}$$

$$\frac{\partial^2 U(x,s)}{\partial x^2} = s\, U(x,s) \longrightarrow U(x,s) = A e^{-x\sqrt{s}} + B e^{x\sqrt{s}}$$

$$\left.\right\} \begin{matrix} B(s) = 0 \\ A(s) = F(s) \end{matrix}$$

$$U(0,s) = F(s) \; , B = 0 \Rightarrow U(x,s) = F(s)\, e^{-x\sqrt{s}}$$

$$u(x,t) = \mathcal{L}^{-1}\left\{F(s)\, e^{-x\sqrt{s}}\right\} = \mathcal{L}^{-1}\left\{\left(\int_0^\infty f(t)\, e^{-st}\, dt\right) e^{-x\sqrt{s}}\right\}$$