Neuroscience of Learning, Memory, Cognition

Theory Homework 1

Due date: Aban 18th (Question 1), 1404

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Notes

- This assignment involves computational tasks that may be completed using programming languages such as Python or MATLAB.
- The use of AI tools is strictly prohibited.
-Include the results of your code in your answer document (PDF), and upload your source code files to CW together with the PDF in a single ZIP archive.
- If you have any questions or ambiguities regarding the assignments, please contact the teaching assistants through Telegram Seyyed Hossein Jahromi, Reza Nayeb Habib ,or via email.

Question 1: Fixed Point Stability [55 pts]

This question is due Aban 18th

In this section, you will study fixed points and their stability. This concept will be used in modeling neurons and their spiking behaviour depending on their input current. First, we explain the underlying mathematics, and then discuss how it connects to the neuron model and helps us describe neural behaviour.

1.1 Taylor expansion

For this problem we assume the following system of differential equations:

$$\begin{cases} \dot{x} = f(x, y) \\ \dot{y} = g(x, y) \end{cases}$$

Let us also assume that (x_0, y_0) is a fixed point in this system, such that:

$$\begin{cases} \dot{x}\big|_{x_0} = f(x_0, y_0) = 0 \\ \dot{y}\big|_{y_0} = g(x_0, y_0) = 0 \end{cases}$$

(a) [5 pts] Write down the Taylor expansion of $f(x, y)$ and $g(x, y)$ near the fixed point.

(b) [5 pts] Now only keep up to the first derivative terms of the Taylor expansion you wrote in part a and substitute them into the differential equation such that you have a linear first order set of differential equations(ODEs). After that, apply the change of variables $u = x - x_0$ and $w = y - y_0$ to final equations for simplicity.

1.2 Eigenvalue and reversal of vector

Assume we have the following differential equation:

$$\dot{\mathbf{x}} = A\mathbf{x}$$

where A is a 2x2 matrix with eigenvalues λ_1 and λ_2 and eigenvectors v_1 and v_2.

(c) [15 pts] Prove that the above ODE system has solutions of form $e_i^{\lambda} v_i$, $i = 1, 2$.

(Hint: Try expressing $\mathbf{x}$ as a linear combination of $\mathbf{v_1}$ and $\mathbf{v_2}$)

1.3 Characteristic equation

What you found in part (b) can be written using the Jacobian matrix (L) as defined below:

$$L = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x}\Big|_{(x_0,y_0)} & \frac{\partial f}{\partial y}\Big|_{(x_0,y_0)} \\ \frac{\partial g}{\partial x}\Big|_{(x_0,y_0)} & \frac{\partial g}{\partial y}\Big|_{(x_0,y_0)} \end{bmatrix}$$

$$\begin{bmatrix} \dot{u} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} u \\ w \end{bmatrix}$$

or in matrix form:

$$\dot{\mathbf{u}} = \mathbf{L}\mathbf{u}$$

We want to find the eigenvalues of $\mathbf{L}$ by solving $\det(L - \lambda I) = 0$. Following parameters are used, instead of raw values of a,b,c,d:

$$\tau = \text{trace}(L) = a + d$$

$$\Delta = \det(L) = ad - bc$$

(d) [10 pts] Find the eigenvalues of L in terms of τ and Δ.

(e) [20 pts] Draw the vector field of the differential equation around the Fixed point using part (d) in each of the following conditions. Then assign each of these vector fields to one of the areas in Figure 1:

- $\tau^2 - 4\Delta > 0$ & $\tau < 0$ & $\Delta > 0$
- $\tau^2 - 4\Delta > 0$ & $\tau > 0$ & $\Delta > 0$
- $\tau^2 - 4\Delta < 0$ & $\tau < 0$ & $\Delta > 0$
- $\tau^2 - 4\Delta < 0$ & $\tau > 0$ & $\Delta > 0$
- $\Delta < 0$

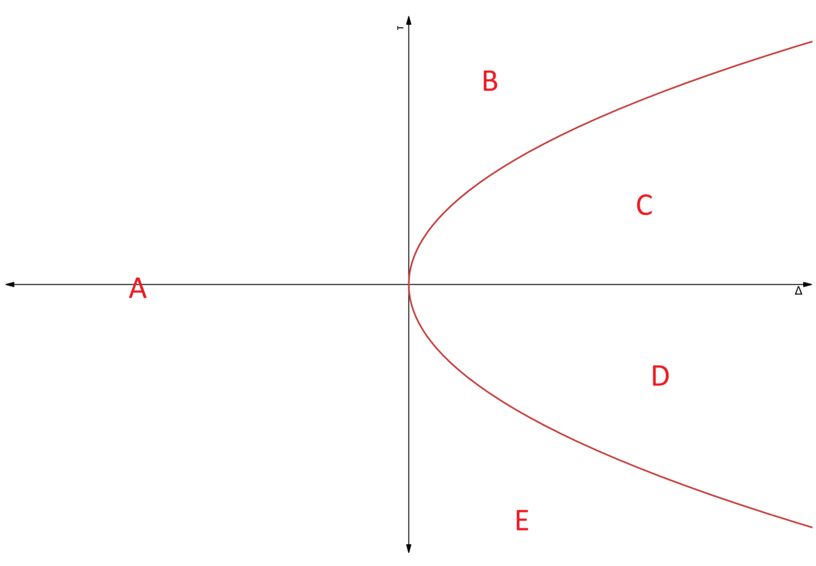

Figure 1: Classification of fixed points

Question 2: Simple Dynamic Model [45 pts]

You shouldn't upload this question yet the deadline of this question will be announced later
Dr. Eugene Izhikevich has proposed the following spiking model for a neuron:

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -\alpha V + \beta V^2 + \gamma - u + I(t)$$

$$\frac{\mathrm{d}u}{\mathrm{d}t} = a(-u + bV)$$

The model includes the following thresholding condition:

$$\begin{cases} \text{if } V = 30\,\text{mV} \\ \text{then } V \to c \textbf{ and } u \to u + d \end{cases}$$

meaning when V reaches 30mV the parameter V resets and d is added to u.

where V is the potential of a neuron's membrane and u is a hidden variable composed of the effects of potassium and sodium (The detailed biophysical interpretation is beyond the scope of this question.). Now we want to see how we can simulate Regular spiking, Fast Spiking and Chattering using this simple model by feeding it different parameters.
For each set of parameters described below do the following:

(a) Find the stable point (numerically on paper) and determine its type [stable node, unstable node, stable focus, unstable focus, saddle node] (without considering the effect of the thresholding condition, also assume $t \to \infty$ meaning $u_s(t) = 1$ for simplicity)

(b) Draw the vector field of V and u using a software simulation program of your choice (suggested options: python or MATLAB)

(c) Draw V(t) over time starting from $t = 0$ for a few spikes or cycles, using a software simulation program of your choice. (suggested options: python or MATLAB)

($u_s(t)$ is the heaviside step function)

2.1 [15 pts] Regular spiking

$a = 0.02,\ b = 0.2,\ c = -65,\ d = 8,\ I(t) = 10u_s(t)$

2.2 [15 pts] Fast spiking

$a = 0.1,\ b = 0.2,\ c = -65,\ d = 2,\ I(t) = 10u_s(t)$

2.3 [15 pts] Chattering

$a = 0.02,\ b = 0.2,\ c = -50,\ d = 2,\ I(t) = 10u_s(t)$

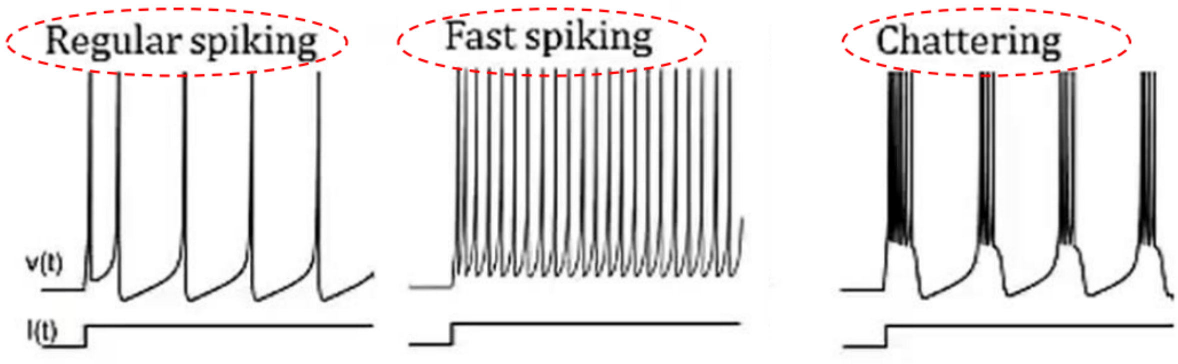

Figure 2: V(t) over time for Regular Spiking (Left), Fast Spiking (Middle) and Chattering (Right)