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3.2 $y'' - y = -1 \quad , \quad 0 < x < 1$

$y'(0) = 0 \; ; \; y(1) = 2 \; ; \; \Delta x = 0.25$

$h = 1/4$

CD ⤳ $\dfrac{y(i+1) - 2y_i + y_{i-1}}{h^2} - y_i = -1 \quad , \; FD: \dfrac{y_1 - y_0}{h} = 0 \Rightarrow \boxed{y_1 = y_0}$ $\quad \boxed{1/h^2 = 16}$

for

$\Rightarrow y(1) = 2 \Rightarrow \boxed{y_4 = 2} \Rightarrow$

$i = 1 \rightarrow y_2(16) - \underset{17}{\cancel{32}} y_1 + \cancel{16 y_0} - \cancel{y_1} = -1$

$i = 2 \quad 16 y_3 - 32 y_2 + 16 y_1 - y_2 = -1$

$i = 3 \quad 16 y_4 - 32 y_3 + 16 y_2 - y_3 = -1$

$$\Rightarrow \begin{bmatrix} -17 & 16 & 0 & 0 \\ 16 & -33 & 16 & 0 \\ 0 & 16 & -33 & 16 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \\ 2 \end{bmatrix} \Rightarrow \boxed{y = \checkmark}$$

(3.8) Crank-Nicholson implicit algo for: hyperbolic Equation

$\phi_{xx} = a^2 \phi_{yy} \; , \; a^2 = \text{Constant}$

$\Delta x = \Delta y = \Delta$

replace the CD with average of Central Differences!

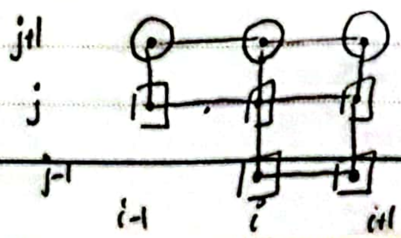
$$\Rightarrow \frac{1}{2}\left[\frac{\phi(i+1,j) - 2\phi(i,j) + \phi(i-1,j)}{\Delta^2} + \frac{\phi(i+1,j+1) - 2\phi(i,j+1) + \phi(i-1,j+1)}{\Delta^2}\right] = \frac{a^2}{2\Delta^2} \cdot \left[\phi(i,j+1) - 2\phi(i,j) + \phi(i,j-1) + \phi(i+1,j+1) - 2\phi(i+1,j) + \phi(i+1,j-1)\right]$$

$$\Rightarrow \phi(i+1,j) - 2\phi(i,j) + \phi(i-1,j) + \phi(i+1,j+1) - 2\phi(i,j+1) + \phi(i-1,j+1)$$
$$- a^2\left[-2\phi(i+1,j) - 2\phi(i,j) + \phi(i,j-1) + \phi(i+1,j+1) + \phi(i,j+1) + \phi(i+1,j-1)\right] = 0$$

$$\Rightarrow \underset{(1)}{\phi(i,j+1)}\left[-2 - a^2\right] + \underset{(2)}{\phi(i+1,j+1)}\left[1 - a^2\right] + \underset{(3)}{\phi(i-1,j+1)} = \overset{(4)}{\phi(i+1,j)}\left[-1 + 2a^2\right] + \overset{(5)}{\phi(i,j)}\left[+2 + 2a^2\right]$$
$$+ \underset{(6)}{\phi(i-1,j)}(-1) + \underset{(7)}{\phi(i,j-1)}\left[-a^2\right] + \underset{(8)}{\phi(i+1,j-1)}(-a^2)$$

Implicit Formula of Crank-Nicholson

for $\phi_{xx} = a^2 \phi_{yy}$



PAPCO

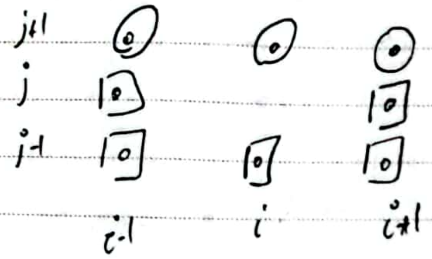
در سطر دوم میانگین گیری از $j+1$, $j-1$ ، همچنین $i+1$, $i-1$ از است!

$$\Rightarrow \frac{1}{2\Delta^2}\Big[\phi(i+1,j+1) - 2\phi(i,j+1) + \phi(i-1,j+1) + \phi(i+1,j-1) - 2\phi(i,j-1) + \phi(i-1,j-1)\Big] = \frac{a^2}{2\Delta^2}\Big[$$

$$\phi(i+1,j+1) - 2\phi(i+1,j) + \phi(i+1,j-1) + \phi(i-1,j+1) - 2\phi(i-1,j) + \phi(i-1,j-1)\Big]$$

$$\Rightarrow \underset{1}{[1-a^2]\phi(i+1,j+1)} - \underset{2}{2\phi(i,j+1)} + \underset{3}{\phi(i-1,j+1)[1-a^2]} = \underset{5}{[1-a^2]\phi(i+1,j-1)}$$

$$+\underset{4}{[1-a^2]\phi(i-1,j-1)} + \underset{6}{2\phi(i,j-1)} - 2a^2\big[\underset{7}{\phi(i+1,j)} + \underset{8}{\phi(i-1,j)}\big]$$



(3.9)

$$\frac{d^2}{dx^2}\phi = x+1 \; ; \quad 0<x<1$$

$\phi(0) = 0$ $\quad \Delta = 0.25$

$\phi(1) = 1$ $\quad \phi(0.5) = ?$

$$\frac{\phi(i+1) - 2\phi(i) + \phi(i-1)}{\Delta x^2} = x+1 \Rightarrow \phi(i+1) - 2\phi(i) + \phi(i-1) = \Delta x^2(x+1)$$

$i=1 \quad \phi_2 - 2\phi_1 + \phi_0 = h^2(x_1+1)$ ✗

$i=2 \quad \phi_3 - 2\phi_2 + \phi_1 = h^2(x_2+1)$

$i=3 \quad \phi_4 - 2\phi_3 + \phi_2 = h^2(x_3+1)$

$i=4 \quad \phi_5 - 2\phi_4 + \phi_3 = h^2(x_4+1)$

$i=5 \quad \phi_6 - 2\phi_5 + \phi_4 = h^2(x_5+1)$ ✗

(1) (2) (3) (4) (5)
$0,\ 0.25,\ 0.5,\ 0.75,\ 1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \phi \end{bmatrix} = \begin{bmatrix} 0 \\ (x+1)h^2 \\ \\ \\ 1 \end{bmatrix}$$

Matlab code

↳ $\boxed{\phi(0.5) = 0.3438}$

Analytical solution : $\phi(x) = \iint x+1 \, dx$

wolfram Alpha $= \int \frac{1}{2}x^2 + x + c = \frac{1}{6}x^3 + \frac{x^2}{2} + c_1x + c_2$

$\Rightarrow c_2 = 0 ,\ \boxed{c_1 = 1/3}$

$\boxed{\phi(0.5) = 0.3125}$

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(3.10) Prove 4th order approx. of Laplace's Eq. $\phi_{xx}+\phi_{yy}=0$

$60\,\phi(i,j)-16\left[\phi(i+1,j)+\phi(i-1,j)+\phi(i,j+1)+\phi(i,j-1)\right]+\phi(i+2,j)+\phi(i-2,j)$

$+\phi(i,2+j)+\phi(i,j-2)=0$

$f_{-2}=f(x-2\Delta)=f(x)-2\Delta x\,f'+\frac{4}{2}\Delta x^2 f''-8/6\,\Delta x^3 f'''+\frac{16}{24}\Delta x^4 f^{(4)}$

$f_2=f(x+2\Delta)=f(x)+2\Delta x\,f'+\frac{4}{2}\Delta x^2 f''+\frac{8}{6}\Delta x^3 f'''+\frac{16}{24}\Delta x^4 f^{(4)}$

$f1=f(x+\Delta x)=f+\Delta x f'+\frac{\Delta x^2}{2}f''+\frac{\Delta x^3}{6}f'''+\frac{\Delta x^4}{24}f^{(4)}$

$f_{-1}=f(x-\Delta x)=f-\Delta x f'+\frac{\Delta x^2}{2}f''-\frac{\Delta x^3}{6}f'''+\frac{\Delta x^4}{24}f^{(4)}$

$\Rightarrow f_1+f_{-1}=2f+\Delta x^2 f''+\frac{\Delta x^4}{12}f^{(4)}+O(\Delta x^6)$ (2)

$\Rightarrow f_2+f_{-2}=2f+4\Delta x^2 f''+\frac{16}{12}\Delta x^4 f^{(4)}+O(\Delta x^6)$ (1)

$\Rightarrow (1)-16\times(2)\Rightarrow$

$f''_x=\frac{1}{12\Delta x^2}\left[-f_{-2}+16f_{-1}-30f+16f_1-f_2\right]+O(\Delta x^4)$

$f''_y=\frac{1}{12\Delta y^2}\left[-f_{-2}+16f_{-1}-30f+16f_1-f_2\right]+O(\Delta x^4)$

$\Rightarrow f''_x+f''_y=0$

$\Rightarrow 60f(i,j)-16\left[f(i-1,j)+f(i+1,j)\right.$

$\left.+f(i,j+1)+f(i,j-1)\right]+f(i-2,j)+f(i+2,j)+f(i,j-2)+f(i,j+2)=0$ ✓

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$\boxed{3.16}$

$\nabla^2 V = -\rho_v/\epsilon$, $0 \le x \le 1$, $\rho_v = xyz^2$ nC/m²

$0 \le y \le 1$ $\boxed{V_0 = 100\,V}$

$0 \le z \le 1$ $\epsilon = 2\epsilon_0$

$V(0,y,z) = 0 = V(1,y,z)$

analytical solution: $V_t = V_1 + V_2$ $V(x,0,z) = 0 = V(x,1,z)$

$V(x,y,0) = 0 = V(x,y,1)$

$$, V_1 = \frac{16V_0}{\pi^2} \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\frac{\sin(j\pi x)\sin(k\pi y)\sinh(\pi z\sqrt{j^2+k^2})}{jk\sinh \pi\sqrt{j^2+k^2}}\right] ; \quad j = 2m+1, \; k = 2n+1$$

$$V_2 = \frac{144}{\pi^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty} \frac{(-1)^{m+n}\left[\frac{2[(-1)^p - 1]}{p^3\pi^3} - \frac{(-1)^p}{p\pi}\right]}{mn(m^2+n^2+p^2)} \cdot \sin(m\pi x)\sin(n\pi y)\sinh(p\pi z)$$

$\boxed{V_t = V_1 + V_2}$

decompose

=> FD sol: $\nabla^2 V = V_{xx} + V_{yy} + V_{zz} = g(x,y,z)$ for $\boxed{\Delta y = \Delta x = \Delta z = h}$

=> $V(i,j,k) = \frac{1}{6}\Big[V(i+1,j,k) + V(i-1,j,k) + V(i,j+1,k) + V(i,j-1,k) +$

Central Diff

$V(i,j,k+1) + V(i,j,k-1) - h^2 g(i,j,k)\Big]$ → ادامه در صفحه بعد

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(3.11) $\Delta x \neq \Delta y \rightarrow$ figure 3.18a $\rightarrow$ Eq. 3.49 $\rightarrow$

$$V_0 = \frac{V_1}{2(1+\alpha)} + \frac{V_2}{2(1+\alpha)} + \frac{V_3}{2(1+1/\alpha)} + \frac{V_4}{2(1+1/\alpha)} \quad ; \alpha = (\Delta x/\Delta y)^2$$

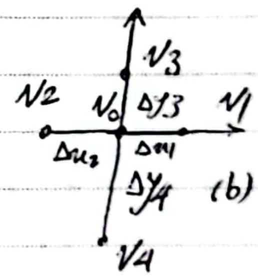
$$\nabla^2 V = \frac{\partial^2}{\partial x^2}V + \frac{\partial^2}{\partial y^2}V = 0 \Rightarrow \text{for } \Delta x \neq \Delta y \Rightarrow \frac{V_1 - 2V_0 + V_2}{(\Delta x)^2} + \frac{V_3 - 2V_0 + V_4}{(\Delta y)^2} = 0$$

$$\Rightarrow 2V_0\left[1 + (\Delta x/\Delta y)^2\right] = V_1 + V_2 + \left(\frac{\Delta x}{\Delta y}\right)^2 (V_3 + V_4) \rightarrow \text{for } \alpha = \left(\frac{\Delta x}{\Delta y}\right)^2$$

$$\Rightarrow V_0 = \frac{V_1}{2(1+\alpha)} + \frac{V_2}{2(1+\alpha)} + \frac{V_3 + V_4}{2(1+1/\alpha)} \Rightarrow \checkmark$$

(b) FD of $V_{xx} \rightarrow$

$$V_{xx} = \frac{\frac{V_1 - V_0}{\Delta x_1} - \frac{V_0 - V_2}{\Delta x_2}}{\frac{\Delta x_1 + \Delta x_2}{2}}$$



$$\Rightarrow V_{xx} = \frac{2\left[-(\Delta x_1 + \Delta x_2)V_0 + \Delta x_2 V_1 + \Delta x_1 V_2\right]}{\Delta x_1 \Delta x_2 (\Delta x_1 + \Delta x_2)}$$

بهمین ترتیب

$$\Rightarrow V_{yy} = \frac{2\left[-(\Delta y_3 + \Delta y_4)V_0 + \Delta y_4 V_3 + \Delta y_3 V_4\right]}{\Delta y_3 \Delta y_4 (\Delta y_3 + \Delta y_4)}$$

$V_{xx} + V_{yy} = 0 \Rightarrow$

$$V_0\left[\frac{1}{\Delta x_1 \Delta x_2} + \frac{1}{\Delta y_3 \Delta y_4}\right] = \frac{V_1}{\Delta x_1(\Delta x_1 + \Delta x_2)} + \frac{V_2}{\Delta x_2(\Delta x_1 + \Delta x_2)} + \frac{V_3}{\Delta y_3(\Delta y_3 + \Delta y_4)}$$

$$+ \frac{V_4}{\Delta y_4(\Delta y_3 + \Delta y_4)} \Rightarrow V_0 = \frac{V_1}{(1 + \Delta x_1/\Delta x_2)\left(1 + \frac{\Delta x_1 \Delta x_2}{\Delta y_3 \Delta y_4}\right)} + \frac{V_2}{(1 + \Delta x_2/\Delta x_1)\left(1 + \frac{\Delta x_1 \Delta x_2}{\Delta y_3 \Delta y_4}\right)}$$

$$+ \frac{V_3}{(1 + \Delta y_3/\Delta y_4)\left(1 + \frac{\Delta y_3 \Delta y_4}{\Delta x_1 \Delta x_2}\right)} + \frac{V_4}{\left(1 + \frac{\Delta y_4}{\Delta y_3}\right)\left(1 + \frac{\Delta y_3 \Delta y_4}{\Delta x_1 \Delta x_2}\right)} \checkmark$$

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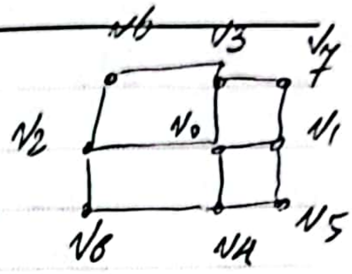
(C) nine-point Molecule ~ 3.48C → $V_0 = \frac{1}{8}\sum_{i=1}^{8} V_i$



دوبار فرمول 331 را اشتباه کنم →

① $V_0 = \frac{1}{4}[V_3+V_1+V_2+V_4]$

② $V_0 = \frac{1}{4}[V_5+V_6+V_7+V_8]$

$+ \quad 2V_0 = \frac{1}{4}[V_1+\dots+V_8]$

$\Rightarrow \boxed{V_0 = \frac{1}{8}\sum_{i=1}^{8} V_i}$ ✓

(3.19) $\frac{\partial}{\partial t}U = \frac{\partial^2}{\partial x^2}U + \frac{\partial^2}{\partial y^2}U$, $0 \le x \le 1$, $0 \le y \le 1$, $t > 0$

i. $U_{i,j}^{n+1} = [1 + r(\delta_x^2 + \delta_y^2)]U_{ij}^n$?

ii. $U_{i,j}^{n+1} = (1 + r\delta_x^2)(1 + r\delta_y^2)U_{ij}^n$

$r = \frac{\Delta t}{h^2}$, $h = \Delta x = \Delta y$

i → $r \le 1/4$, ii → $r \le 1/2$

Von-Newman Method → $U_{i,l}^n = A^n e^{jk_x i x} e^{jk_y l y}$

$\Rightarrow$ (i) $\Rightarrow \frac{A^{n+1} e^{jk_x i x} e^{jk_y l y} - A^n e^{jk_x i x} e^{jk_y l y}}{\Delta t} = [A^n e^{jk_x(i-1)x} e^{jk_y l y} - 2A^n e^{jk_x i x} e^{jk_y l y}$

FD

$+ A^n e^{jk_x(i+1)x} e^{jk_y l y} + A^n e^{jk_x i x} e^{jk_y(l-1)y} - 2A^n e^{jk_x i x} e^{jk_y l y} + A^n e^{jk_x i x} e^{jk_y(l+1)y}]$

CD CD

$\div A^n e^{jk_x i x} e^{jk_y l y}$

$\Rightarrow \frac{A^{n+1}}{A^n} = 1 + r[-4 + 2\cos(k_x x) + 2\cos(k_y y)] \Rightarrow |g| \le 1 \Rightarrow |1 - 2r(-2 + \cos k_x x + \cos k_y y)| \le 1$

$\Rightarrow 1 - 2r(2+1+1) \ge -1 \Rightarrow \boxed{r \le 1/4}$ ✓

(ii) $\xrightarrow{\div e^{jo}}$ $A^{n+1} = (1 + 2r\cos k_x x - 2r)(1 + 2r\cos k_y y - 2r)A^n \Rightarrow -1 \le g$ $g = (1+D_1 r)(1+D_2 r)$

$|g| \le 1$ → $-1 \le g \le 1$, $g \le 1$ ✓

$\underbrace{1 + 2r(\cos k_x x - 1)}_{D_1}$, $\underbrace{1 + 2r(\cos k_y y - 1)}_{D_2}$ ⇒ r → real ⇒ { D_1 min, D_2 max → ; D_1 max, D_2 min → }

max → 0, min → −4r

1 → $\boxed{r \le 1/2}$ ✓