Convex Optimization

Project 6

Spring 1401

Due date: 31st of Khordad

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1. *Maximizing diversification ratio.* Let $x \in \mathbf{R}_+^n$, with $1^T x = 1$, denote a portfolio of n assets, with x_i the fraction of the total value (assumed positive) invested in asset i. Let $\Sigma \in \mathbf{S}_{++}^n$ denote the covariance matrix of the asset returns. The diversification ratio of the portfolio is defined as

$$D(x) = \frac{\sigma^T x}{(x^T \Sigma x)^{1/2}}$$

where $\sigma_i = (\Sigma_{ii})^{1/2}$. Note that D is defined for any $x \in \mathbf{R}_+^n$ with $\mathbf{1}^T x = 1$. We consider the problem of choosing x to maximize the diversification ratio, subject to limits on the weights,

$$\begin{array}{ll} \text{maximize} & D(x) \\ \text{subject to} & \mathbf{1}^T x = 1, \quad 0 \preceq x \preceq M, \end{array}$$

where $M \succ 0$ is a given vector of maximum allowed weights, with $\mathbf{1}^T M > 1$. Remark. (The following is not needed to solve the problem, but gives some background.) For any long-only portfolio x we have $D(x) \geq 1$. To see this we note that

$$x^T \Sigma x = \sum_{ij} x_i x_j \sigma_i \sigma_j \rho_{ij} \leq \sum_{ij} x_i x_j \sigma_i \sigma_j = \left(\sigma^T x\right)^2,$$

where $\rho_{ij} = \Sigma_{ij} / (\sigma_i \sigma_j)$ is the correlation, which satisfies $\rho_{ij} \leq 1$. The smallest possible value of diversification $D(x) = 1$ occurs only when $x = e_k$ (the k th unit vector), i.e., the portfolio is concentrated in one asset.

(a) Explain how to use convex optimization to solve the problem. We will give half credit for a solution that involves solving a quasiconvex optimization problem, and full credit to one that relies on solving one convex problem. Hints. You may need to change variables to get a one-convex-problem method. Note also that $D(tx) = D(x)$ for any $t > 0$.

(b) Use your method from part (a) to solve the problem instance with data given in max_divers_data.*. Give an optimal $x^\star$, and the associated diversification ratio $D(x^\star)$.
The (long-only) minimum variance portfolio x^{mv} is the one that minimizes $x^T \Sigma x$ subject to $0 \preceq x \preceq M, \mathbf{1}^T x = 1$. Find $D(x^{\text{mv}})$, and compare it to $D(x^\star)$. Compare the maximum diversification and minimum variances portfolios using a bar plot. (The data file contains code for creating such plots.)

Good Luck!