Convex Optimization
Homework 3

Spring 1401
Due date: 19th of Farvardin

---

1. Consider the optimization problem

$$\text{minimize} \quad f_0(x) = -\sum_{i=1}^{m} \log\left(b_i - a_i^T x\right)$$

with domain $\operatorname{dom} f_0 = \{x \mid Ax \prec b\}$, where $A \in \mathbf{R}^{m\times n}$ (with rows a_i^T). We assume that $\operatorname{dom} f_0$ is nonempty. Prove the following facts (which include the results quoted without proof on page 141).

   (a) $\operatorname{dom} f_0$ is unbounded if and only if there exists a $v \neq 0$ with $Av \preceq 0$.

   (b) f_0 is unbounded below if and only if there exists a v with $Av \preceq 0, Av \neq 0$. Hint. There exists a v such that $Av \preceq 0, Av \neq 0$ if and only if there exists no $z \succ 0$ such that $A^T z = 0$. This follows from the theorem of alternatives in example 2.21, page 50 .

   (c) If f_0 is bounded below then its minimum is attained, i.e., there exists an x that satisfies the optimality condition (4.23).

   (d) The optimal set is affine: $X_{\text{opt}} = \{x^\star + v \mid Av = 0\}$, where $x^\star$ is any optimal point.

2. Some simple LPs. Give an explicit solution of each of the following LPs.

   (a) Minimizing a linear function over an affine set.

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \end{array}$$

   (b) Minimizing a linear function over a halfspace.

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a^T x \leq b \end{array}$$

   where $a \neq 0$.

   (c) Minimizing a linear function over a rectangle.

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & l \preceq x \preceq u, \end{array}$$

   where l and u satisfy $l \preceq u$.

   (d) Minimizing a linear function over the probability simplex.

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \mathbf{1}^T x = 1, \quad x \succeq 0 \end{array}$$

   What happens if the equality constraint is replaced by an inequality $\mathbf{1}^T x \leq 1$? We can interpret this LP as a simple portfolio optimization problem. The vector x represents the allocation of our total budget over different assets, with x_i the fraction invested in asset i. The return of each investment is fixed and given by $-c_i$, so our total return (which we want to maximize) is $-c^T x$. If we replace the budget constraint $\mathbf{1}^T x = 1$ with an inequality $\mathbf{1}^T x \leq 1$, we have the option of not investing a portion of the total budget.

3. Problems involving ℓ_1 - and $\ell_\infty - norms$. Formulate the following problems as LPs. Explain in detail the relation between the optimal solution of each problem and the solution of its equivalent LP.

   (a) Minimize $\|Ax - b\|_\infty$ (ℓ_∞-norm approximation).

   (b) Minimize $\|Ax - b\|_1$ (ℓ_1-norm approximation).

   (c) Minimize $\|Ax - b\|_1$ subject to $\|x\|_\infty \leq 1$.

4. Minimum fuel optimal control. We consider a linear dynamical system with state $x(t) \in \mathbf{R}^n, t = 0, \ldots, N$, and actuator or input signal $u(t) \in \mathbf{R}$, for $t = 0, \ldots, N-1$. The dynamics of the system is given by the linear recurrence

$$x(t+1) = Ax(t) + bu(t), \quad t = 0, \ldots, N-1$$

   where $A \in \mathbf{R}^{n \times n}$ and $b \in \mathbf{R}^n$ are given. We assume that the initial state is zero, i.e., $x(0) = 0$.

   The minimum fuel optimal control problem is to choose the inputs $u(0), \ldots, u(N-1)$ so as to minimize the total fuel consumed, which is given by

$$F = \sum_{t=0}^{N-1} f(u(t))$$

   subject to the constraint that $x(N) = x_{\text{des}}$, where N is the (given) time horizon, and $x_{\text{des}} \in \mathbf{R}^n$ is the (given) desired final or target state. The function $f : \mathbf{R} \to \mathbf{R}$ is the $fuel$ use map for the actuator, and gives the amount of fuel used as a function of the actuator signal amplitude. In this problem we use

$$f(a) = \begin{cases} |a| & |a| \leq 1 \\ 2|a| - 1 & |a| > 1 \end{cases}$$

   This means that fuel use is proportional to the absolute value of the actuator signal, for actuator signals between -1 and 1 ; for larger actuator signals the marginal fuel efficiency is half.

   Formulate the minimum fuel optimal control problem as an LP.

5. 'Hello World' in CVX^*. Use CVXPY, Convex.jl, or CVX to verify the optimal values you obtained (analytically) for exercise 4.1 in Convex Optimization.

6. The illumination problem. In lecture 1 we encountered the function

$$f(p) = \max_{i=1,\ldots,n} \left| \log a_i^T p - \log I_{\text{des}} \right|$$

   where $a_i \in \mathbf{R}^m$, and $I_{\text{des}} > 0$ are given, and $p \in \mathbf{R}_+^m$.

   (a) Show that $\exp f$ is convex on $\left\{ p \mid a_i^T p > 0, i = 1, \ldots, n \right\}$.

   (b) Show that the constraint 'no more than half of the total power is in any 10 lamps' is convex (i.e., the set of vectors p that satisfy the constraint is convex).

   (c) Show that the constraint 'no more than half of the lamps are on' is (in general) not convex.

7. Formulate the following optimization problems as semidefinite programs. The variable is $x \in \mathbf{R}^n$; $F(x)$ is defined as

$$F(x) = F_0 + x_1 F_1 + x_2 F_2 + \cdots + x_n F_n$$

   with $F_i \in \mathbf{S}^m$. The domain of f in each subproblem is $\operatorname{dom} f = \{x \in \mathbf{R}^n \mid F(x) \succ 0\}$.

   (a) Minimize $f(x) = c^T F(x)^{-1} c$ where $c \in \mathbf{R}^m$.

   (b) Minimize $f(x) = \max_{i=1,\ldots,K} c_i^T F(x)^{-1} c_i$ where $c_i \in \mathbf{R}^m, i = 1, \ldots, K$.

   (c) Minimize $f(x) = \sup_{\|c\|_2 \leq 1} c^T F(x)^{-1} c$.

(d) Minimize $f(x) = \mathbf{E}\left(c^T F(x)^{-1} c\right)$ where c is a random vector with mean $\mathbf{E}c = \bar{c}$ and covariance $\mathbf{E}(c - \bar{c})(c - \bar{c})^T = S$.

8. (a) Minimum fuel optimal control. We consider a linear dynamical system with state $x(t) \in \mathbf{R}^n, t = 0, \ldots, N$, and actuator or input signal $u(t) \in \mathbf{R}$, for $t = 0, \ldots, N-1$. The dynamics of the system is given by the linear recurrence

$$x(t+1) = Ax(t) + bu(t), \quad t = 0, \ldots, N-1$$

where $A \in \mathbf{R}^{n \times n}$ and $b \in \mathbf{R}^n$ are given. We assume that the initial state is zero, i.e., $x(0) = 0$. The minimum fuel optimal control problem is to choose the inputs $u(0), \ldots, u(N-1)$ so as to minimize the total fuel consumed, which is given by

$$F = \sum_{t=0}^{N-1} f(u(t))$$

subject to the constraint that $x(N) = x_{\text{des}}$, where N is the (given) time horizon, and $x_{\text{des}} \in \mathbf{R}^n$ is the (given) desired final or target state. The function $f : \mathbf{R} \rightarrow \mathbf{R}$ is the fuel use map for the actuator, and gives the amount of fuel used as a function of the actuator signal amplitude. In this problem we use

$$f(a) = \begin{cases} |a| & |a| \leq 1 \\ 2|a| - 1 & |a| > 1 \end{cases}$$

This means that fuel use is proportional to the absolute value of the actuator signal, for actuator signals between -1 and 1 ; for larger actuator signals the marginal fuel efficiency is half.

Formulate the minimum fuel optimal control problem as an LP.

(b) Solve the minimum fuel optimal control problem for the instance with problem data

$$A = \begin{bmatrix} -1 & 0.4 & 0.8 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \\ 0.3 \end{bmatrix}, \quad x_{\text{des}} = \begin{bmatrix} 7 \\ 2 \\ -6 \end{bmatrix}, \quad N = 30$$

You can do this by forming the LP you found or more directly using CVX . Plot the actuator signal $u(t)$ as a function of time t.

9. (a) Relaxation of Boolean LP. In a Boolean linear program, the variable x is constrained to have components equal to zero or one:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b \\ & x_i \in \{0, 1\}, \quad i = 1, \ldots, n. \end{array}$$

In general, such problems are very difficult to solve, even though the feasible set is finite (containing at most 2^n points). In a general method called relaxation, the constraint that x_i be zero or one is replaced with the linear inequalities $0 \leq x_i \leq 1$:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b \\ & 0 \leq x_i \leq 1, \quad i = 1, \ldots, n \end{array}$$

We refer to this problem as the *LP* relaxation of the Boolean LP (4.67). The LP relaxation is far easier to solve than the original Boolean LP.

i. Show that the optimal value of the LP relaxation (4.68) is a lower bound on the optimal value of the Boolean LP (4.67). What can you say about the Boolean LP if the LP relaxation is infeasible?

ii. It sometimes happens that the LP relaxation has a solution with $x_i \in \{0, 1\}$. What can you say in this case?

(b) Heuristic suboptimal solution for Boolean LP.

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b \\ & x_i \in \{0,1\}, \quad i = 1, \ldots, n \end{array}$$

with optimal value $p^\star$. Let x^{rlx} be a solution of the LP relaxation

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b \\ & 0 \preceq x \preceq \mathbf{1} \end{array}$$

so $L = c^T x^{\text{rlx}}$ is a lower bound on $p^\star$. The relaxed solution x^{rlx} can also be used to guess a Boolean point $\hat{x}$, by rounding its entries, based on a threshold $t \in [0,1]$:

$$\hat{x}_i = \begin{cases} 1 & x_i^{\text{rlx}} \geq t \\ 0 & \text{otherwise} \end{cases}$$

for $i = 1, \ldots, n$. Evidently $\hat{x}$ is Boolean (i.e., has entries in $\{0,1\}$). If it is feasible for the Boolean LP, i.e., if $A\hat{x} \preceq b$, then it can be considered a guess at a good, if not optimal, point for the Boolean LP. Its objective value, $U = c^T \hat{x}$, is an upper bound on $p^\star$. If U and L are close, then $\hat{x}$ is nearly optimal; specifically, $\hat{x}$ cannot be more than $(U - L)$-suboptimal for the Boolean LP.

This rounding need not work; indeed, it can happen that for all threshold values, $\hat{x}$ is infeasible. But for some problem instances, it can work well.

Of course, there are many variations on this simple scheme for (possibly) constructing a feasible, good point from x^{rlx}.

Finally, we get to the problem. Generate problem data using one of the following.

Matlab code:

```
rand('state',0);
n=100;
m=300;
A=rand(m,n);
b=A*ones(n,1)/2;
c=-rand(n,1);
```

Python code:

```
import numpy as np
np.random.seed(0)
(m, n) = (300, 100)
A = np.random.rand(m, n); A = np.asmatrix(A)
b = A.dot(np.ones((n, 1)))/2; b = np.asmatrix(b)
c = -np.random.rand(n, 1); c = np.asmatrix(c)
```

Julia code:

```
srand(0);
n=100;
m=300;
A=rand(m,n);
b=A*ones(n,1)/2;
c=-rand(n,1);
```

You can think of x_i as a job we either accept or decline, and $-c_i$ as the (positive) revenue we generate if we accept job i. We can think of $Ax \preceq b$ as a set of limits on m resources. A_{ij}, which is positive, is the amount of resource i consumed if we accept job j; b_i, which is positive, is the amount of resource i available.

Find a solution of the relaxed LP and examine its entries. Note the associated lower bound L. Carry out threshold rounding for (say) 100 values of t, uniformly spaced over $[0,1]$. For each value of t, note the objective value $c^T\hat{x}$ and the maximum constraint violation $\max_i(A\hat{x}-b)_i$. Plot the objective value and the maximum violation versus t. Be sure to indicate on the plot the values of t for which $\hat{x}$ is feasible, and those for which it is not.

Find a value of t for which $\hat{x}$ is feasible, and gives minimum objective value, and note the associated upper bound U. Give the gap $U-L$ between the upper bound on $p^\star$ and the lower bound on $p^\star$.

In Matlab, if you define vectors obj and maxviol, you can find the upper bound as

$$| \; U = \min(obj(f \text{ ind } (\text{maxviol} <= 0))) \; |$$

Good Luck!