Subject: Hw8 - Convex

Year. Month. Date (,)

1) Optimal spacecraft landing / thrust profile: $\boxed{m\ddot{p} = f - mge_3;}$

$\boxed{m>0}$

$p(t) \in \mathbb{R}^3 \longrightarrow$ position

$p_3(t) \longrightarrow$ height

$f(t) \in \mathbb{R}^3 \longrightarrow$ thrust force

$\boxed{g>0} \longrightarrow$ gravitational acceleration

$m \longrightarrow$ is constant

$p(T^{td}) = 0$

$\dot{p}(T^{td}) = 0$

$T^{td} =$ touch down time

$\boxed{\alpha > 0 \longrightarrow \text{minimum glide slope}}$

$p(0) \longrightarrow$ initial point } given

$\dot{p}(0) \longrightarrow$ initial velocity }

$\gamma > 0 \longrightarrow$ fuel consumption coefficient

$p_3(t) \geq \alpha \|(p_1(t), p_2(t))\|_2$

$\|f(t)\|_2 \leq F^{max}$

$\int_0^{T^{td}} \gamma \|f(t)\|_2 dt$

$f(t) = f_k \qquad t \in [(k-1)h, kh)$

$k = 1, \dots, k$

$T^{td} = kh$

$v_{k+1} = v_k + (h/m) f_k - hge_3 ; \quad p_{k+1} = p_k + (h/2)(v_k + v_{k+1})$

$\dot{p}((k-1)h) \qquad p((k-1)h)$

a) objective $\min \sum_{k=1}^{k} \|f_k\|_2$ s.t. $\Rightarrow$ objective (summation of norm2) is convex $\Rightarrow$ constraints are convex or affine $\Rightarrow$ we have a convex optimization problem

Constraints $v_{k+1} = v_k + (\frac{h}{m}) f_k - hge_3;$

$p_{k+1} = p_k + (\frac{h}{2})[v_k + v_{k+1}];$

$\|f_k\|_2 \leq F^{max};$

$p_{k+1} = 0;$

$p_1 = p(0);$

$v_{k+1} = 0;$

$v_1 = \dot{p}(0)$

$(p_k)_3 \geq \alpha \|((p_k)_1, (p_k)_2)\|_2$

b) minimize touch down time, k, h fixed $\longrightarrow$ if we can solve more than (1) convex problem $\Rightarrow$ due to our constraints $\longrightarrow$ starting with a big k and reducing k if the problem's answer is reachable $\Rightarrow$ if we hit the answer $= \inf \Rightarrow$ k is too small $\Rightarrow$ $k+1 \longrightarrow$ minimum available $k \Rightarrow$ minimum touch down time

PAPCO

Subject: Year. Month. Date. / /

C) in matlab ✓

3) Simple portfolio optimization

Page 155, 185-186

- $1^T x = 1$ | • No (additional) constraints

minimum-risk Portfolios

↳ same expected return as $\boxed{x = \frac{1}{n} \cdot 1}$

done in matlab ✓

maximizing algebraic

4) Connectivity of a graph

$G = (N, E) \rightarrow$ weighted undirected graph

$n = |N|$ nodes

$m = |E|$ edges

$w_1, \dots, w_m \in R_+$ weights

$A = [a_1 \dots a_m] \in R^{n \times m} \rightarrow$ incidence matrix

$(a_k)_i = 1$ اگه k از i بره

$(a_k)_j = -1$ به j بره

$$L = \sum_{k=1}^{m} w_k . a_k . a_k^T = A \operatorname{diag}(w) A^T$$

$L \succeq 0$; $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$

$\lambda_1 = 0$ / algebraic Connectivity of G

the better connected the graph ⟸ the larger λ_2

$$\boxed{\begin{aligned} &\text{maximize} \quad \lambda_2 \\ &\text{s.t.} \quad w \succeq 0, \; Fw \preceq g \end{aligned}}$$

(a) we have to maximize λ_2 >> λ_2 shall be a concave function of $\boxed{w}$

↳ knowing that λ_1 is a concave function of w

↳ minimum eigen value

form a new Matrix >> λ_2 becomes its minimum eigen value

eigen vector $1 \leftrightarrow$ eigne value λ_1

↳ we have to consider a space orthogonal to 1 to have $\boxed{\text{a subspace } 1^{\perp}}$ >> $\lambda_2 = \lambda_{min}(Q^T L Q)$

$Q \in R^{n \times (n-1)} \longrightarrow$ ستون‌های این ماتریس $\longrightarrow$ فضای متعامد بر $N(1^T) = 1^\perp$ را تشکیل می‌دهند!

$\rightarrow Q^T L Q \rightarrow \lambda_2, \dots, \lambda_n \mid \boxed{\lambda_2 = \lambda_{min}(Q^T L Q) \text{ is concave}}$

if L is symetric, positive semidefinite function $\Rightarrow Q^T L Q \rightarrow$ convex $\Big\} \Rightarrow (Q^T L Q)$ is convex

$L \succeq 0$

$\lambda_{max} = \sup_{\|x\|=1} x^T A x$ is convex $\mid \boxed{\lambda_{min} \text{ is Concave}}$

(b) Numerical example ; $F = 1^T$, $g = 1$; Compare $w^* \rightarrow$ optimal with $w = (1/m) 1$

Done in Matlab. ✓

5) radiation treatment ; $b_j \rightarrow$ the level of beam $\boxed{j}$

$0 \leqslant b_j \leqslant B^{max} \longrightarrow$ maximum possible beam level

$\boxed{d_i = \sum_{j=1}^{n} A_{ij} b_j}$ $d_i \geqslant D^{target}$; $d_i \leqslant D^{other} \longrightarrow$ not feasible generally

$A \in R_+^{m \times n} \rightarrow$ known $\qquad \Rightarrow E = \sum_{i \notin T} \left((d_i - D^{other})_+ \right)^2$

Penalty

(a) to prove convexity of the problem, we need to show that objective is convex $\longrightarrow$ The fact that summation of squares of difference $\rightarrow$ from a known vector is convex $\Rightarrow$ proves the convexity of the problem $\rightarrow$ due to the fact that

Constraints are all convex [some are affine] ✓

b) in matlab

Subject:
Year. Month. Date. ( )

6) $v = (r_{j+1} - r_i)_+ = \max\{r_{j+1} - r_i, 0\}$

$r_i \geq r_{j+1}$; $\quad J = \sum_{k=1}^{m} \phi(v^{(k)})$; $\quad v^{(k)} \xrightarrow[\text{violation}]{\text{Preference}} (i^{(k)}, j^{(k)})$

$r \in R^n$

ϕ: nondecreasing convex penalty function

$\phi(u) = 0 \quad u \leq 0$

(a) (i) it is better to use $\| \ \|_2^2$ or $(\ )^2$ element wise simply because large error is damned for this ϕ! $\rightarrow$ for example quadratic form penalty!

(ii) we should, in this case, take a linear cost function like

$\boxed{\phi(u) = u}$

(b) in matlab

7) Optimizing Processor Speed

$E = \sum_{i=1}^{n} \frac{\alpha_i}{s_i} f(s_i)$

$s_1, \dots, s_n \quad ; s_{min} \leq s \leq s_{max}$

minimize completion time T

time task i: $\tau_i = \alpha_i / s_i$; $\Rightarrow \boxed{s_i = \frac{\alpha_i}{\tau_i}}$

$E \leq E_{max}$

$P_i = f(s_i)$; $f: R \to R$ { positive, increasing, convex

Energy as a function of s_i is not generally convex $\Rightarrow E = \sum_{i=1}^{n} \tau_i \underbrace{f(\frac{\alpha_i}{\tau_i})}_{\text{perspective of } f}$

$\Rightarrow$ the objective is convex $\rightarrow$ summation of convex functions!

$t f(u/t) \rightarrow \{ t = \tau_i, u = \alpha_i$

PAPCO

⇒ G is convex function of τ_i

Constraints will be: $S_{min} \le S_i \le S_{max}$ → $\boxed{\frac{\alpha_i}{S_{max}} \le \tau_i \le \frac{\alpha_i}{S_{min}}}$ convex

precedence: ⇒ Considering idea of epigraph ⇒ $T \le \max_i \{t_i\}$

if to (i,j)

⇒ $\boxed{t_j \ge t_i + \tau_j}$ ⇒ $\boxed{t_i \ge \tau_i}$ ⇒ $\boxed{\min \max_i (t_i)}$

t_i: تا task i انجام بشه

τ_j: خود تسک j

t_i: upper bound روی τ_i

τ_i: زمان خود تسک i

s.t.

$$\begin{cases} \sum_{i=1}^{n} \tau_i f\left(\frac{\alpha_i}{\tau_i}\right) \le E_{max} \\ \frac{\alpha_i}{S_{max}} \le \tau_i \le \frac{\alpha_i}{S_{min}} \\ t_i \ge \tau_i \\ t_j \ge t_i + \tau_j \end{cases}$$

اگر بخوایم ماکسیمم یه vector رو min کنیم، کافیه همه‌ی درایه‌هاشو min کنیم!

⇒ Problem is convex with convex constraints!

b) in matlab!