Convex Optimization
Project 4

Spring 1401
Due date: 13th of Khordad

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1. *Optimizing the sequence of commitments in an alternative investment.* In an alternative investment, the investor makes commitments each period for an amount that she will invest. Over the next few years, the investor puts money into the investment in response to capital calls, up to the amount of previous commitments. The investor receives money from the investment in later years through distributions. Examples of alternative investments include private equity, venture capital, and infrastructure projects. Alternative investments are found in the portfolios of insurance companies, retirement funds, and university endowments. ('Alternative' refers to the investment not being the more usual stocks, bonds, currencies, and financial derivatives.)

   We consider time periods $t = 1, \ldots, T$, which are typically quarters. We first describe some critical quantities.

   - $c_t \geq 0$ denotes the amount that the investor commits in period t.
   - $p_t \geq 0$ denotes the amount that the investor pays in to the investment in response to capital calls in period t.
   - $d_t \geq 0$ denotes the amount that the investor receives in distributions from the investment in period t.
   - $n_t \geq 0$ denotes the net asset value (NAV) of the investment in period t.
   - $u_t \geq 0$ denotes the total amount of uncalled commitments, i.e., the difference between the total so far committed and the total so far that has been called (and paid into the investment).

   The units for all of these is typically millions of USD. Among these quantities, the only ones we have direct control over are the commitments c_t; the others are functions of these.

   A simple dynamical model of these variables is

   $$n_{t+1} = (1+r)n_t + p_t - d_t, \quad u_{t+1} = u_t - p_t + c_t, \quad t = 1, \ldots, T,$$

   where $r \geq 0$ is the per-period return, with initial conditions $n_1 = u_1 = 0$. (Note that n and u are $(T+1)$-vectors, whereas c, d, and p are T-vectors.) In words: the value of the investment increases by its return, plus the amount paid in, minus the amount distributed; the total uncalled commitments is decreased by the capital calls, and increased by new commitments. The calls and distributions are modeled as

   $$p_t = \gamma^{\text{call}} u_t, \quad d_t = \gamma^{\text{dist}} n_t, \quad t = 1, \ldots, T,$$

   where $\gamma^{\text{call}} \in (0,1)$ and $\gamma^{\text{dist}} \in (0,1)$ are the call and distribution intensities, respectively. The parameters r, γ^{call}, and γ^{dist} are given. Your job is to choose the sequence of commitments $c = (c_1, \ldots, c_T)$.

   The commitments and the capital calls are limited by $c_t \leq c^{\max}$ and $p_t \leq p^{\max}$, for $t = 1, \ldots, T$, where $c^{\max} > 0$ and $p^{\max} > 0$ are given. In addition we have a total budget $B > 0$ for commitments, with $\mathbf{1}^T c \leq B$. Our objective is to minimize

   $$\frac{1}{T+1} \sum_{t=1}^{T+1} \left(n_t - n^{\text{des}}\right)^2 + \lambda \frac{1}{T-1} \sum_{t=1}^{T-1} (c_{t+1} - c_t)^2$$

   where $n^{\text{des}} > 0$ is a given positive target NAV, and $\lambda > 0$ is a parameter. The first term in the objective is the mean-square tracking error, and the second term, the mean-square difference in commitments, encourages smooth sequences of commitments.

(a) Optimized commitments. Explain how to solve this problem with convex optimization. Solve this problem with parameters $T=40$ (ten years), $r=0.04$ (4% quarterly return),

$$\gamma^{\text{call}} = .23, \quad \gamma^{\text{dist}} = .15, \quad c^{\max} = 4, \quad p^{\max} = 3, \quad B = 85, \quad n^{\text{des}} = 15, \quad \lambda = 5.$$

Plot $c, p, d, n,$ and u versus t. Give the root-mean-square (RMS) tracking error, i.e., the squareroot of the mean-square tracking error, for the optimal commitments.

(b) Constant commitment based on steady-state. By solving the dynamics equations with all quantities constant, we find that $c^{\text{ss}} = \left(\gamma^{\text{dist}} - r\right) n^{\text{des}}$ is the value of a constant commitment (i.e., the same each period) that gives $n_t = n^{\text{des}}$ asymptotically, in steady-state. Plot the same quantities as in part (a) for the constant commitment $c_t = c^{\text{ss}}$ for $t = 1, \ldots, T$. Give the RMS tracking error. Hint. A quick and simple (but not computationally efficient) way to do the simulation is to modify the code for part (a), adding the constraint that $c_t = c^{\text{ss}}, t = 1, \ldots, T$.

Give a very brief description of what you see, comparing the optimal sequence of commitments found in part (a) and the constant commitments found in part (b).

Good Luck!