Subject: Hw 4 Convex  810 100 511  آرمان ...
Year 2022 Month. 4 Date. 22 ( )

1)

minimize $x^2+1$

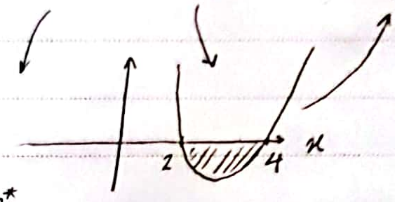
s.t. $(x-2)(x-4) \le 0$  $2 \le x \le 4$

$x \in \mathbb{R}$

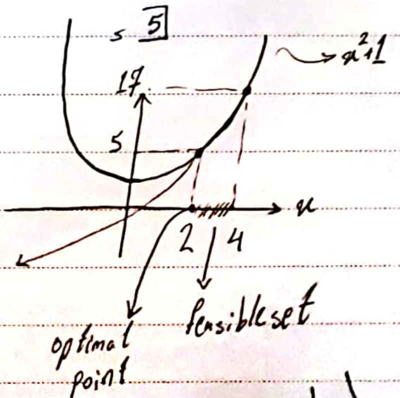
(a) feasible set $\rightarrow 2 \le x \le 4$  $x \in [2,4]$

optimal value $\rightarrow$ @ $x=2 \rightarrow x^*=2$

optimal solution $\rightarrow x^*=2 \Rightarrow x^2+1 = p^*$



(b) plot $x^2+1$ vs. $x$



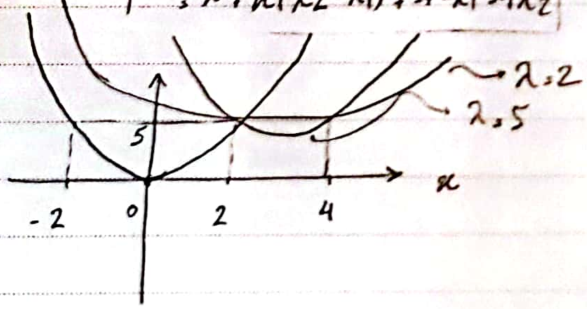

$L(x,\lambda) = \left(x^2+1+\lambda_1^T(-x+2)+\lambda_2^T(x-4)\right)$

$= x^2+1+2\lambda_1+\lambda_2 x+(-\lambda_1 x-4\lambda_2)$

$= \boxed{x^2+x(\lambda_2-\lambda_1)+1+2\lambda_1-4\lambda_2}$

$L(x,\lambda) = x^2+1+\lambda(x^2-6x+8)$

$= \boxed{x^2(1+\lambda)-6\lambda x+1+8\lambda}$




$\lambda=1 \rightarrow 2x^2-6x+9$

$\lambda=2 \rightarrow 3x^2-12x+17$

$\lambda=5 \rightarrow 6x^2-30x+41$

$\rightarrow$ plot comes with a image file

$5 \ge \inf_x\left(x^2(1+\lambda)-6\lambda x+1+8\lambda\right) \Rightarrow \inf_x\left(x^2(1+\lambda)-6\lambda x-4+8\lambda\right) \le 0$

با توجه به نمودارها این رابطه برای هر λ برقرار است!

تمام نمودارهای لاگرانژین از نقطه‌ی $p^*$ عبور می‌کنند ولی حتماً مقدار کمتر از آن در بازه‌ی [2,4] دارند

$g(x,\lambda) = \inf_x(L(x,\lambda)) = 1+8\lambda+\inf_x\left(x^2(1+\lambda)-6\lambda x\right)$

(c)

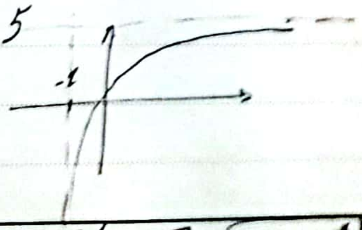
$\frac{d}{dx}=0 \quad 2x(1+\lambda)-6\lambda=0$

$\Rightarrow 2x = \frac{6\lambda}{1+\lambda} \Rightarrow \boxed{x^* = \frac{3\lambda}{1+\lambda}}$

$g(x,\lambda) = 1+8\lambda+\frac{9\lambda^2}{1+\lambda}-\frac{18\lambda^2}{1+\lambda} = \boxed{\frac{-\lambda^2+9\lambda+1}{1+\lambda}}$

maximize $g(\lambda)$
$\lambda \ge 0$

$g(\lambda) = \begin{cases} \frac{-\lambda^2+9\lambda+1}{1+\lambda} & \text{o.w} \\ -\infty & \lambda \le -1 \end{cases} \rightarrow$




$\nabla=0 \Rightarrow \lambda^2+2\lambda-8=0$ @ $\lambda=2$, $-4$ $\rightarrow \frac{-4+18+1}{3} = \boxed{5}$ $\boxed{g(2)=p^*}$ $\boxed{2d^*}$

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d) $p^*(u) \longrightarrow \begin{cases} \text{maximize} & x^2+1 \\ \text{s.t} & (x-2)(x-4) \leqslant u \end{cases}$

$$\begin{cases} x^2+1 \\ x^2-6x+8-u \leqslant 0 \end{cases} \longrightarrow \left[ \frac{6-\sqrt{4+4u}}{2}, \frac{6+\sqrt{4+4u}}{2} \right]$$

$$\frac{6 \pm \sqrt{36-32+4u}}{2}$$

$$= \boxed{\left[ 3-\sqrt{1+u}, 3+\sqrt{1+u} \right]}$$

$$\begin{cases} \Delta \geqslant 0 \Rightarrow 1+u \geqslant 0 \Rightarrow \boxed{u \geqslant -1} \Rightarrow u < -1 \longrightarrow \text{infeasible} \\ 3-\sqrt{1+u} \leqslant 0 \Rightarrow 9 \leqslant 1+u \Rightarrow \boxed{u \geqslant 8} \longrightarrow 0 \in [-, +] \Rightarrow \boxed{p^* = 1}, x^* = 0 \\ \text{for } u < 8 \Rightarrow p^* = (3-\sqrt{1+u})^2 + 1 = 9 + 1+u - 6\sqrt{1+u} + 1 = \boxed{11+u-6\sqrt{1+u}} \end{cases}$$

$u \geqslant -1$; $x^* = 3-\sqrt{1+u} \geqslant 0$

$$\Rightarrow p^*(u) = \begin{cases} -\infty & u \leqslant -1 \\ 11+u-6\sqrt{1+u} & -1 \leqslant u < 8 \\ 1 & u \geqslant 8 \end{cases} \longrightarrow \frac{dp^*(u)}{du} = 1 - \frac{3}{\sqrt{1+u}} \Rightarrow$$

$$\frac{d}{du} p^*(0) = 1 - 3 = \boxed{-2} = \lambda^* \rightarrow$$ با قبل مطابقت دارد!

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2) Numerical perturbation analysis example: $\min \; x_1^2 + 2x_2^2 - x_1x_2 - x_1$

s.t.

a QP

$x_1 + 2x_2 \leq u_1$

$x_1 - 4x_2 \leq u_2$

$5x_1 + 76x_2 \leq 1$

(a) solve for $u_1 = -2$, $u_2 = -3$

$x^TAx$, $A = \begin{bmatrix} 1 & -1/2 \\ -1/2 & 2 \end{bmatrix}$

$[x_1 \; x_2] \begin{bmatrix} 1 & -1/2 \\ -1/2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \left(x_1 - \frac{x_2}{2} \quad -\frac{x_1}{2} + 2x_2\right) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$1{,}2 \quad 2{,}2$

$= x_1^2 - \frac{x_1x_2}{2} - \frac{x_1x_2}{2} + 2x_2^2$

$f_0 = x^TAx + c^Tx$ where $c = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

s.t. $Sx \leq b$,

$S = \begin{bmatrix} 1 & 2 \\ 1 & -4 \\ 5 & 76 \end{bmatrix}$, $b = \begin{bmatrix} u_1 \\ u_2 \\ 1 \end{bmatrix}$ $\Rightarrow$ dual Problem $g(\lambda) = \inf_x \left( x^TAx + c^Tx + \lambda^T(Sx - b) \right)$

$= \inf_x \left( x^TAx + (c + \lambda S^T)^T x - \lambda^T b \right) = \inf_x \left( x^TAx + (c + \lambda S^T)^T x \right) - \lambda^T b$

$\nabla_x L(x,\lambda) = 0 \Rightarrow 2Ax + c + \lambda S^T = 0 \Rightarrow Ax = -\frac{(c + \lambda S^T)}{2} \Rightarrow x^* = \boxed{-\frac{A^+}{2}(c + \lambda S^T)}$

$\Rightarrow$ dual Problem: $\begin{cases} \max \; -\lambda^T b + \underbrace{\frac{1}{2}(c + \lambda S^T)^T A^{+T} A \frac{A^+}{2}(c + \lambda S^T) - (c + \lambda S^T)^T \frac{A^+}{2}(c + \lambda S^T)}_{-\frac{1}{4}(c + \lambda S^T)^T A^+ (c + \lambda S^T)} \\ \lambda \geq 0 \end{cases}$

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3) Robust LP with Polyhedral cost uncertainty

$$\min \ \sup_{c \in C} \ c^T x$$
$$\text{s.t.} \quad Ax \geq b$$

$C = \{c \mid Fc \leq g\}$

$Ax \geq b$ are feasible

$C \neq \emptyset$

(a) why f is convex? $\Rightarrow$ $f(x)$ is a function of $x$ and the operation $\sup_{c \in C} c^T x$ is based on vector $c$ so the convexity holds for variable $x$.

(b)
$$\min \ -c^T x \quad \text{s.t.} \quad Fc \leq g \leq 0$$
$$Fc - g \leq 0$$

$\Rightarrow L(c,\lambda) = -c^T x + \lambda^T(Fc - g)$

$\Rightarrow g(\lambda) = \inf_c (L(c,\lambda))$

$$\Rightarrow g(\lambda) = \inf_c (-c^T x + \lambda^T F c) - \lambda^T g = \inf_c (-x^T + \lambda^T F) c - \lambda^T g$$

$$\Rightarrow g(\lambda) = \begin{cases} -\lambda^T g \\ \text{s.t.} \quad x = F^T \lambda \end{cases}$$

$\Rightarrow$ dual problem:

$$\begin{cases} \max & -\lambda^T g \\ \text{s.t.} & x = F^T \lambda \\ & \lambda \geq 0 \end{cases} \quad \text{OR} \quad \begin{cases} \min & \lambda^T g \\ \text{s.t.} & \lambda \geq 0 \\ & x = F^T \lambda \end{cases}$$

Optimal value of the dual is $f(x)$ due to zero duality gap based on feasibility & convexity of the primal problem.

(c) باتوجه به فرم convex مسئله میتوان گفت که LP را برای دو متغیر $x$ و $\lambda$ انجام میدهیم:

$$\begin{cases} \min & \lambda^T g \quad \text{on } x, \lambda \\ \text{s.t.} & Ax \geq b \\ & \lambda \geq 0 \\ & F^T \lambda = x \end{cases}$$

$B \rightarrow$ Bandwidth

$f_{min} \rightarrow$ lowest freq.

## 4) Band Limited signal Recovery from zero-crossing

$$y_t = \sum_{j=1}^{B} a_j \cos\left(\frac{2\pi}{n}(f_{min}+j-1)t\right) + b_j \sin\left(\frac{2\pi}{n}(f_{min}+j-1)t\right) \quad ; t=1,\dots,n$$

given $s = \text{sign}(y)$ and $f_{min}, B$

(a)

با توجه به اینکه متغیرهای ما $a$, $b$ هستند، پس تابع objective را به صورت برداری در $\begin{bmatrix} a \\ b \end{bmatrix}$ می‌نویسیم!

$$\hat{y} = A\begin{bmatrix} a \\ b \end{bmatrix} \quad ; \quad A = [C \quad C_s]$$

$$C: {}_{n\times B}[C_{tj}] \rightarrow \cos\left(\frac{2\pi}{n}(f_{min}+j-1)t\right)$$

$$C_s: [C_{s\,tj}] \rightarrow \sin\left(\frac{2\pi}{n}(f_{min}+j-1)t\right)$$

(b):

$$\|\hat{y}\|_1 = n$$

$$s_t \cdot y_t \geq 0 \Rightarrow \underbrace{s_t^T A_t}_{} \begin{bmatrix} a \\ b \end{bmatrix} \geq 0 \quad , \quad \left\|A\begin{bmatrix} a \\ b \end{bmatrix}\right\|_1 = n \xrightarrow{} \boxed{s^T A\begin{bmatrix} a \\ b \end{bmatrix} = n}$$

sign

سطر $t$ ام $y$ ← سطر $t$ ام $A$

تک تک نقطه‌ها را اگر در علامتشان ضرب کنیم که نهایت کار ما قدر مطلق را بگیرد!

فرم نهایی مسئله $\Rightarrow$

$$\begin{cases} \min\limits_{a,b} \left\|A\begin{bmatrix} a \\ b \end{bmatrix}\right\|_2 = \|\hat{y}\|_2 \\ \text{s.t.} \quad s^T A\begin{bmatrix} a \\ b \end{bmatrix} = n \quad , \quad s_t^T A_t \begin{bmatrix} a \\ b \end{bmatrix} \geq 0 \quad ; t=1,\dots,n \end{cases}$$

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5) point $a \in \mathbb{R}^n$ in $\ell_1$-norm unit ball

$$\min \quad (1/2)\|x-a\|_2^2$$
$$\text{s.t.} \quad \|x\|_1 \leq 1$$

$$\rightarrow \mathcal{L}(x,\lambda) = \frac{1}{2}\|x-a\|_2^2 + \lambda^T \|x\|_1 - \lambda^T$$

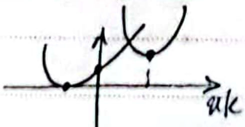
$\|x\|_1 \rightarrow$ مشتق پذیر نیست پس باید تفکیک کنیم $\Rightarrow \sum_k \left( \frac{(x_k-a_k)^2}{2} + \lambda|x_k| \right) - \lambda$

$$\text{dual problem} \rightarrow g(\lambda) = \inf_x (\mathcal{L}(x,\lambda)) = \inf_{\substack{x_k \\ k=1:n}} \left( \frac{(x_k-a_k)^2}{2} + \lambda|x_k| \right)$$

$$= \begin{cases} x_k \geq 0 & \frac{(x_k-a_k)^2}{2} + \lambda x_k = \frac{x_k^2}{2} - (a_k-\lambda)x_k + \frac{1}{2}a_k^2 \\ x_k \leq 0 & \frac{(x_k-a_k)^2}{2} - \lambda x_k = \frac{x_k^2}{2} - (a_k+\lambda)x_k + \frac{1}{2}a_k^2 \end{cases} \Rightarrow \frac{d}{dx_k} = 0 \Rightarrow$$

$$\begin{cases} x_k - (a_k-\lambda) = 0 \\ x_k - (a_k+\lambda) = 0 \end{cases} \Rightarrow \begin{cases} x_k = a_k - \lambda \\ x_k = a_k + \lambda \end{cases}$$

الان خوب 2 حالت داریم یا $a_k-\lambda$ مثبته هست و اگه داگر منفی باشه خوب تناقضه و $x_k=0$ جواب ما میشه که در زیرش میشه



که توی شکل $x_k$ ممکنه جای $x_p^*$ قرار بگیره

$$\Rightarrow g(\lambda) = \begin{cases} x_k \geq 0 & x_k = a_k - \lambda \begin{cases} a_k - \lambda \geq 0 \rightarrow x_k^* = a_k - \lambda \\ a_k - \lambda \leq 0 \rightarrow x_k^* = 0 \end{cases} \\ x_k \leq 0 & x_k = a_k + \lambda \begin{cases} a_k + \lambda \leq 0 \rightarrow x_k^* = a_k + \lambda \\ a_k + \lambda \geq 0 \rightarrow x_k^* = 0 \end{cases} \end{cases} \quad \text{(I)}$$

dual problem

$$\max \sum_{k=1}^{n} g_k(\lambda) - \lambda$$
$$\text{s.t.} \quad \lambda \geq 0$$

dual feasibility

$$\Rightarrow g_k(\lambda) = \begin{cases} \frac{1}{2}a_k^2 & a_k < \lambda, \ a_k > -\lambda \\ -\frac{\lambda^2}{2} + \lambda|a_k| & a_k \geq \lambda, \ a_k < -\lambda \end{cases} \Rightarrow \boxed{g(\lambda) = \sum_{k=1}^{n} g_k(\lambda) - \lambda}$$

$\lambda \rightarrow$ is known $\Leftarrow$ از روی (I) می‌توان مقدار $x_k^*$ را پیدا کرد!

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6) non-convex least squares, binary constraints

$$\text{maximize} \quad \|Ax-b\|_2^2$$
$$\text{s.t.} \quad x_k^2 = 1, \quad k = 1, 2, \dots, n$$

$\begin{cases} A \in R^{m \times n} \\ b \in R^m \end{cases} \rightarrow x \in R^{n \times 1}$

$\text{rank}(A) = n$

$A^TA \rightarrow$ non-singular

$-x^TA^Tb - b^TAx = -2b^TAx$

(a) $L(x,v) = \underbrace{\|Ax-b\|_2^2}_{(Ax-b)^T(Ax-b)} + \underbrace{\sum_k v_k(x_k^2-1)}_{x^T\text{diag}(v)x \; - 1^Tv} = (x^TA^T - b^T)(Ax-b) + x^T\text{diag}(v)x - 1^Tv$

as in slide $\boxed{5\text{-}7}$, $\boxed{5\text{-}14}$

$= x^T(A^TA + \text{diag}(v))x + b^Tb - 1^Tv - 2b^TAx$

$$\boxed{= x^T(A^TA + \text{diag}(v))x - 2b^TAx + b^Tb - 1^Tv}$$

$g(v) = \inf_x (L(x,v))$

$\nabla_x L = 0 \Rightarrow 2(A^TA + \text{diag}(v))x - 2b^TA = 0$

$\Rightarrow x^* = (A^TA + \text{diag}(v))^{+}A^Tb$

$\Rightarrow L(x^*, v) = b^TA(A^TA + \text{diag}(v))^{+}(A^TA + \text{diag}(v))(A^TA + \text{diag}(v))A^Tb - 2b^TA(A^TA + \text{diag}(v))^{+}A^Tb$

$$g(v) = \begin{cases} -1^Tv + b^Tb - b^TA(A^TA + \text{diag}(v))^{+}A^Tb \\ -\infty \end{cases}$$

, $A^TA + \text{diag}(v) \succeq 0 \rightarrow$ چون که یه سری x رو میبره منفی الی نهایت

, $A^Tb \in R(A^TA + \text{diag}(v))$ اون موقع $-\infty$ هم جوابه!

slide $\boxed{5\text{-}14}$

$x^T(A+\lambda I)x$

چون که $\ker(A+\lambda I) \perp R(A+\lambda I)$

که یه xهایی میشه پیدا کرد که

صفر و ولی $b^Tx \neq 0 \Rightarrow \boxed{-\infty}$

$x \rightarrow -\infty$

$$\Rightarrow g(v) = \begin{cases} -1^Tv + b^Tb - b^TA(A^TA + \text{diag}(v))^{+}A^Tb & A^TA + \text{diag}(v) \succeq 0, \; A^Tb \in R(\cdot\cdot) \\ -\infty & \text{o.w.} \end{cases}$$

Schur complement

$\Rightarrow$ maximize $-1^Tv - t + b^Tb$

$$\text{s.t.} \quad \begin{bmatrix} A^TA + \text{diag}(v) & -A^Tb \\ -b^TA & t \end{bmatrix} \succeq 0, \quad A^TA + \text{diag}(v) \succeq 0, \quad t - b^TA(A^TA + \text{diag}(v))^{+}A^Tb \succeq 0$$

(b)

dual $\rightsquigarrow$ minimize $1^T v + t - b^T b$

s.t. $\begin{bmatrix} A^T A + \text{diag}(v) & -A^T b \\ -b^T A & t \end{bmatrix} \succeq 0$

$\Rightarrow \mathcal{L}(v, t, \phi) = 1^T v + t - b^T b - tr\left(\phi \cdot \begin{bmatrix} A^T A + \text{diag}(v) & -A^T b \\ -b^T A & t \end{bmatrix}\right)$

معرفی $\phi$ را به صورت $\begin{bmatrix} Z & z \\ z^T & \lambda \end{bmatrix}$ در نظر می‌گیریم!

$\Rightarrow \mathcal{L}(v, t, Z, z, \lambda) = 1^T v + t - b^T b + tr\left(\begin{bmatrix} Z(A^T A + \text{diag}(v)) - b^T A z & 0 \\ 0 & -z^T A^T b + \lambda t \end{bmatrix}\right)$

$\Rightarrow \mathcal{L}(v, t, Z, z, \lambda) = \underline{1^T v} + \underline{t} - \underline{b^T b} - tr(Z(A^T A + \underline{\text{diag}(v)})) + 2\underline{z^T A^T b} - \underline{t\lambda}$

$= \underbrace{(1 - \text{diag}(Z))^T v} + \underbrace{t(1 - \lambda)} - \underline{b^T b} + 2\underline{z^T A^T b} - tr(Z A^T A)$

$t \to -\infty \Rightarrow$ unbounded below $\to \lambda = 1$, $\text{diag}(Z) = 1$

$v \to -\infty$

$\Rightarrow$ dual problem : $\begin{cases} \max & -b^T b - tr(Z A^T A) + 2 b^T A z \\ \text{s.t.} & \text{diag}(Z) = 1 \\ & \begin{bmatrix} Z & z \\ z^T & 1 \end{bmatrix} \succeq 0 \end{cases}$ $\to Z \succeq 0,\ 1 \geq 0,\ 1 - z^T Z z \geq 0$

$\Rightarrow 1 \geq z^T Z z$, if $Z = zz^T$

$\text{rank}\left(\begin{bmatrix} Z & z \\ z^T & 1 \end{bmatrix}\right) = 1$

$\hookrightarrow Z = Z z^T \Rightarrow \boxed{\text{ردیف پایینی در } z \text{ ضرب شده } Z}$

relaxation of original problem

$\Rightarrow \begin{cases} \max & -b^T b - tr(Z A^T A) + 2 b^T A z \\ \text{s.t.} & \text{diag}(Z) = 1 \\ & Z \succeq zz^T \end{cases}$

$Z \succeq zz^T$

weaker constraint

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$z = Ev$

6)(C) $z \to$ first moment of random vector $v \in R^n$

$Z \to$ second

$Z = Evv^T$

Shor (4) $\equiv \min \; E\|Av-b\|_2^2$

s.t. $Ev_k^2 = 1, \; k = 1, 2, \ldots$

$$E\underbrace{\|Av-b\|_2^2}_{(Av-b)^T(Av-b)} = E(v^TA^TAv + b^Tb - 2b^TAv)$$

$(v^TA^T - b^T)(Av-b)$

$$= E(tr(v^TA^TAv) + b^Tb - 2b^TAv)$$

$E(c) = c$ (A)

له اگر A یک عدد باشد

$v^TA^TAv \to$ یک عدد

$$= tr(E\underbrace{(v^TA^TAv)}_{E(Avv^TA^T)}) + b^Tb - 2b^TAEv$$

$$= tr(A\underbrace{E(vv^T)}_{Z}A^T) + b^Tb - 2b^TAz \Rightarrow tr(AZA^T) - 2b^TAz + b^Tb$$

$$Ev_k^2 = Z_{kk} = 1 \equiv diag(Z) = 1 \checkmark$$