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Communication Systems: Transforms & Equations

Content:

Fourier Transform Properties:

Property	Mathematical Description
1. Linearity	$ag_1(t) + bg_2(t) \rightleftharpoons aG_1(f) + bG_2(f)$ where a and b are constants
2. Time scaling	$g(at) \rightleftharpoons \frac{1}{\lvert a \rvert} G\left(\frac{f}{a}\right)$ where a is a constant
3. Duality	If $g(t) \rightleftharpoons G(f)$, then $G(t) \rightleftharpoons g(-f)$
4. Time shifting	$g(t - t_0) \rightleftharpoons G(f)\exp(-j2\pi f t_0)$
5. Frequency shifting	$\exp(j2\pi f_c t)g(t) \rightleftharpoons G(f - f_c)$
6. Area under $g(t)$	$\int_{-\infty}^{\infty} g(t)dt = G(0)$
7. Area under $G(f)$.	$g(0) = \int_{-\infty}^{\infty} G(f)df$
8. Differentiation in the time domain	$\frac{d}{dt} g(t) \rightleftharpoons j2\pi f G(f)$
9. Integration in the time domain	$\int_{-\infty}^{t} g(\tau)d\tau \rightleftharpoons \frac{1}{j2\pi f} G(f) + \frac{G(0)}{2}\delta(f)$
10. Conjugate functions	If $g(t) \rightleftharpoons G(f)$, then $g^*(t) \rightleftharpoons G^*(-f)$
11. Multiplication in the time domain	$g_1(t)g_2(t) \rightleftharpoons \int_{-\infty}^{\infty} G_1(\lambda)G_2(f - \lambda)d\lambda$
12. Convolution in the time domain	$\int_{-\infty}^{\infty} g_1(\tau)g_2(t - \tau)d\tau \rightleftharpoons G_1(f)G_2(f)$.

Operation	$g(t)$	$G(f)$
Superposition	$g_1(t) + g_2(t)$	$G_1(f) + G_2(f)$
Scalar multiplication	$kg(t)$	$kG(f)$
Duality	$G(t)$	$g(-f)$
Time scaling	$g(at)$	$\frac{1}{\lvert a \rvert} G\left(\frac{f}{a}\right)$
Time shifting	$g(t - t_0)$	$G(f)e^{-j2\pi f t_0}$
Frequency shifting	$g(t)e^{j2\pi f_0 t}$	$G(f - f_0)$
Time convolution	$g_1(t) * g_2(t)$	$G_1(f)G_2(f)$
Frequency convolution	$g_1(t)g_2(t)$	$G_1(f) * G_2(f)$
Time differentiation	$\frac{d^n g(t)}{dt^n}$	$(j2\pi f)^n G(f)$
Time integration	$\int_{-\infty}^{t} g(x)\, dx$	$\frac{G(f)}{j2\pi f} + \frac{1}{2}G(0)\delta(f)$

Fourier Transform Pairs:

$x(t)$	$X(f)$
$\delta(t)$	1
1	$\delta(f)$
$\delta(t-t_0)$	$e^{-j2\pi f t_0}$
$e^{j2\pi f_0 t}$	$\delta(f-f_0)$
$\cos(2\pi f_0 t)$	$\frac{1}{2}\left[\delta(f-f_0)+\delta(f+f_0)\right]$
$\sin(2\pi f_0 t)$	$\frac{1}{2j}\left[\delta(f-f_0)-\delta(f+f_0)\right]$
$rect(t)$	$\mathrm{sinc}(f)$
$\mathrm{sinc}(t)$	$rect(f)$
$\Lambda(t)$	$\mathrm{sinc}^2(f)$
$\mathrm{sinc}^2(t)$	$\Lambda(f)$
$e^{-\alpha t}u(t), \quad \alpha>0$	$\frac{1}{\alpha+j2\pi f}$
$te^{-\alpha t}u(t), \quad \alpha>0$	$\frac{1}{(\alpha+j2\pi f)^2}$
$e^{-\alpha\lvert t\rvert}, \quad \alpha>0$	$\frac{2\alpha}{(\alpha^2+(2\pi f)^2}$
$e^{-\pi t^2}$	$e^{-\alpha f^2}$
$\mathrm{sgn}(t)$	$\frac{1}{j\pi f}$
$u(t)$	$\frac{1}{2}\delta(f)+\frac{1}{j2\pi f}$
$\frac{d}{dt}\delta(t)$	$j2\pi f$
$\sum_{n=-\infty}^{\infty}\delta(t-nT_0)$	$\frac{1}{T_0}\sum_{n=-\infty}^{\infty}\delta\left(f-\frac{n}{T_0}\right)$

Time Function	Fourier Transform
$\delta(t)$	1
1	$\delta(f)$
$\delta(t-t_0)$	$\exp(-j2\pi f t_0)$
$\exp(j2\pi f_c t)$	$\delta(f-f_c)$
$\cos(2\pi f_c t)$	$\frac{1}{2}[\delta(f-f_c)+\delta(f+f_c)]$
$\sin(2\pi f_c t)$	$\frac{1}{2j}[\delta(f-f_c)-\delta(f+f_c)]$
$\mathrm{sgn}(t)$	$\frac{1}{j\pi f}$
$\frac{1}{\pi t}$	$-j\,\mathrm{sgn}(f)$
$u(t)$	$\frac{1}{2}\delta(f)+\frac{1}{j2\pi f}$
$\sum_{n=-\infty}^{\infty}\delta(t-nT_0)$	$\frac{1}{T_0}\sum_{n=-\infty}^{\infty}\delta\left(f-\frac{n}{T_0}\right)$
$\mathrm{rect}\left(\frac{t}{T}\right)$	$T\,\mathrm{sinc}(fT)$
$\mathrm{sinc}(2Wt)$	$\frac{1}{2W}\mathrm{rect}\left(\frac{f}{2W}\right)$
$\exp(-at)u(t), \quad a>0$	$\frac{1}{a+j2\pi f}$
$\exp(-\pi t^2)$	$\exp(-\pi f^2)$
$\begin{cases}1-\frac{\lvert t\rvert}{T}, & \lvert t\rvert<T\\ 0, & \lvert t\rvert\ge T\end{cases}$	$T\,\mathrm{sinc}^2(fT)$

	$g(t)$	$G(f)$	
1	$e^{-at}u(t)$	$\dfrac{1}{a+j2\pi f}$	$a>0$
2	$e^{at}u(-t)$	$\dfrac{1}{a-j2\pi f}$	$a>0$
3	$e^{-a\|t\|}$	$\dfrac{2a}{a^2+(2\pi f)^2}$	$a>0$
4	$te^{-at}u(t)$	$\dfrac{1}{(a+j2\pi f)^2}$	$a>0$
5	$t^n e^{-at}u(t)$	$\dfrac{n!}{(a+j2\pi f)^{n+1}}$	$a>0$
6	$\delta(t)$	1	
7	1	$\delta(f)$	
8	$e^{j2\pi f_0 t}$	$\delta(f-f_0)$	
9	$\cos 2\pi f_0 t$	$0.5\,[\delta(f+f_0)+\delta(f-f_0)]$	
10	$\sin 2\pi f_0 t$	$j0.5\,[\delta(f+f_0)-\delta(f-f_0)]$	
11	$u(t)$	$\frac{1}{2}\delta(f)+\dfrac{1}{j2\pi f}$	
12	$\operatorname{sgn} t$	$\dfrac{2}{j2\pi f}$	
13	$\cos 2\pi f_0 t\, u(t)$	$\frac{1}{4}[\delta(f-f_0)+\delta(f+f_0)]+\dfrac{j2\pi f}{(2\pi f_0)^2-(2\pi f)^2}$	
14	$\sin 2\pi f_0 t\, u(t)$	$\frac{1}{4j}[\delta(f-f_0)-\delta(f+f_0)]+\dfrac{2\pi f_0}{(2\pi f_0)^2-(2\pi f)^2}$	
15	$e^{-at}\sin 2\pi f_0 t\, u(t)$	$\dfrac{2\pi f_0}{(a+j2\pi f)^2+4\pi^2 f_0^2}$	$a>0$
16	$e^{-at}\cos 2\pi f_0 t\, u(t)$	$\dfrac{a+j2\pi f}{(a+j2\pi f)^2+4\pi^2 f_0^2}$	$a>0$
17	$\Pi\left(\frac{t}{\tau}\right)$	$\tau \operatorname{sinc}(\pi f\tau)$	
18	$2B\operatorname{sinc}(2\pi Bt)$	$\Pi\left(\dfrac{f}{2B}\right)$	
19	$\Delta\left(\frac{t}{\tau}\right)$	$\frac{\tau}{2}\operatorname{sinc}^2\left(\dfrac{\pi f\tau}{2}\right)$	
20	$B\operatorname{sinc}^2(\pi Bt)$	$\Delta\left(\dfrac{f}{2B}\right)$	
21	$\sum_{n=-\infty}^{\infty}\delta(t-nT)$	$f_0\sum_{n=-\infty}^{\infty}\delta(f-nf_0)$	$f_0=\dfrac{1}{T}$
22	$e^{-t^2/2\sigma^2}$	$\sigma\sqrt{2\pi}e^{-2(\sigma\pi f)^2}$	

Trigonometric Identities:

$\sin^2\theta + \cos^2\theta = 1$	$1 + \tan^2\theta = \sec^2\theta$
$\sin(\frac{\pi}{2} - \theta) = \cos\theta$	$\cos(\frac{\pi}{2} - \theta) = \sin\theta$
$\sec(\frac{\pi}{2} - \theta) = \csc\theta$	$\csc(\frac{\pi}{2} - \theta) = \sec\theta$
$\sin(-\theta) = -\sin\theta$	$\cos(-\theta) = \cos\theta$
$\sin 2\theta = 2\sin\theta\cos\theta$	$\cos 2\theta = \cos^2 - \sin^2 = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$
$\sin^2\theta = \dfrac{1 - \cos 2\theta}{2}$	$\cos^2\theta = \dfrac{1 + \cos 2\theta}{2}$
$\sin\alpha + \sin\beta = 2\sin(\frac{\alpha+\beta}{2})\cos(\frac{\alpha-\beta}{2})$	$\sin\alpha - \sin\beta = 2\cos(\frac{\alpha+\beta}{2})\sin(\frac{\alpha-\beta}{2})$
$\cos\alpha + \cos\beta = 2\cos(\frac{\alpha+\beta}{2})\cos(\frac{\alpha-\beta}{2})$	$\cos\alpha - \cos\beta = -2\sin(\frac{\alpha+\beta}{2})\sin(\frac{\alpha-\beta}{2})$
$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$	$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)]$
$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$	$1 + \cot^2 = \csc^2$
$e^{j\theta} = \cos\theta + j\sin\theta$	$\cos\theta = \dfrac{e^{j\theta} + e^{-j\theta}}{2}$
$e^{-j\theta} = \cos\theta - j\sin\theta$	$\sin\theta = \dfrac{e^{j\theta} - e^{-j\theta}}{2j}$
$\tan(\frac{\pi}{2} - \theta) = \cot\theta$	$\cot(\frac{\pi}{2} - \theta) = \tan\theta$
$\tan(-\theta) = -\tan\theta$	
$\tan^2\theta = \dfrac{1 - \cos 2\theta}{1 + \cos 2\theta}$	$\tan 2\theta = \dfrac{2\tan\theta}{1 - tan^2\theta}$

$$\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y)$$

$$\sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y)$$

Common Integrals:

$\int \cos(x)dx$	$\sin(x)$
$\int \sin(x)dx$	$-\cos(x)$
$\int x\cos(x)dx$	$\cos(x) + x\sin(x)$
$\int x\sin(x)dx$	$\sin(x) - x\cos(x)$
$\int x^2 \cos(x)dx$	$2x\cos(x) + (x^2 - 2)\sin(x)$
$\int x^2 \sin(x)dx$	$2x\sin(x) - (x^2 - 2)\cos(x)$
$\int e^{\alpha x} dx$	$\frac{e^{\alpha x}}{a}$
$\int xe^{\alpha x} dx$	$e^{\alpha x}\left[\frac{x}{a} - \frac{1}{a^2}\right]$
$\int x^2 e^{\alpha x} dx$	$e^{\alpha x}\left[\frac{x^2}{a} - \frac{2x}{a^2} - \frac{2}{a^3}\right]$
$\int \frac{dx}{\alpha + \beta x}$	$\frac{1}{\beta}\ln\lvert\alpha + \beta x\rvert$
$\int \frac{dx}{\alpha^2 + \beta^2 x^2}$	$\frac{1}{\alpha\beta}\tan^{-1}(\frac{\beta x}{\alpha})$

Integration Properties:

Table of Properties of Integrals

	Rule	Conditions
1	$\int a\,dx = ax$	
2 Homogeniety	$\int af(x)\,dx = a\int f(x)\,dx$	
3 Associativity	$\int (f \pm g \pm h \pm \cdots)\,dx = \int f\,dx \pm \int g\,dx \pm \int h\,dx \pm \cdots$	
4 Integration by Parts	$\int_a^b fg'\,dx = [fg]_a^b - \int_a^b gf'\,dx$	
4 General Integration by Parts	$\int f^{(n)}g\,dx = f^{(n-1)}g' - f^{(n-2)}g'' + \ldots + (-1)^n \int fg^{(n)}\,dx$	
5	$\int f(ax)\,dx = \frac{1}{a}\int f(x)\,dx$	
6 Substitution Rule	$\int g\{f(x)\}\,dx = \int g(u)\frac{dx}{du}\,du = \int \frac{g(u)}{f'(x)}\,du$	$u = f(x)$
7	$\int x^n\,dx = \frac{x^{n+1}}{n+1}$	$n \neq -1$
8	$\int \frac{1}{x}\,dx = \ln\lvert x\rvert$	
9	$\int e^x\,dx = e^x$	
10	$\int a^x\,dx = \frac{a^x}{\ln\lvert a\rvert}$	$a \neq 1$
Notes:	1. *f*, *g*, *h* are functions of *x* 2. *a*, *n* are constants. 3. The constant of integration, *C* has been omitted from this table. It should be included in the working of the equation if applicable.	

Hilbert Transform:

	Signal $s(t)$	Hilbert transform $\mathcal{H}\{s\}(t)$
	$\sin(t)$	$-\cos(t)$
	$\cos(t)$	$\sin(t)$
	$\frac{1}{t^2+1}$	$\frac{t}{t^2+1}$
sinc(t)	$\frac{\sin(t)}{t}$	$\frac{1-\cos(t)}{t}$
rect(t)	$\sqcap(t)$	$\frac{1}{\pi}\ln\left\lvert\frac{t+\frac{1}{2}}{t-\frac{1}{2}}\right\rvert$
	$\delta(t)$	$\frac{1}{\pi t}$
	$\chi_{[a,b]}(x)$	$\frac{1}{\pi}\log\left\lvert\frac{x-a}{x-b}\right\rvert$

Linear AM Modulation:

Type of Modulation	In-Phase Component $s_I(t)$	Quadrature Component $s_Q(t)$	Comments
DSB-SC	$m(t)$	0	$m(t)$ = message signal
SSB:[a]			
(a) Upper sideband transmitted	$\frac{1}{2}m(t)$	$\frac{1}{2}\hat{m}(t)$	$\hat{m}(t)$ = Hilbert transform of $m(t)$
(b) Lower sideband transmitted	$\frac{1}{2}m(t)$	$-\frac{1}{2}\hat{m}(t)$	
VSB:			
(a) Vestige of lower sideband transmitted	$\frac{1}{2}m(t)$	$\frac{1}{2}m'(t)$	$m'(t)$ = output of the filter of frequency response $H_Q(f)$ due to $m(t)$. For the definition of $H_Q(f)$, see Eq. (2.16)
(b) Vestige of upper sideband transmitted	$\frac{1}{2}m(t)$	$-\frac{1}{2}m'(t)$	

Where $S(t) = A_c \cdot S_I(t) \cdot \cos(2\pi f_c t) - A_c \cdot S_Q(t) \cdot \sin(2\pi f_c t)$,

Example: LSSB S(t)

$$S(t) = A_c \cdot \left(\frac{1}{2}m(t)\right) \cdot \cos(2\pi f_c t) - A_c \cdot \left(\frac{-1}{2}\hat{m}(t)\right) \cdot \sin(2\pi f_c t) = \frac{A_c}{2}m(t) \cdot \cos(2\pi f_c t) + \frac{A_c}{2}\hat{m}(t) \cdot \sin(2\pi f_c t)$$

Misc. Equations:

Pre-envelope:	$g_+(t) = g(t) + j\hat{g}(t)$	$g_-(t) = g(t) - j\hat{g}(t)$	$\hat{g}(t)$: Hilbert transform of $g(t)$
	$2g(t) = g_+(t) + g_-(t)$	$g_+(t)$: pre-envelope for positive frequencies $g_-(t)$: pre-envelope for negative frequencies	
Complex envelope: ($g(t)$ is a narrow-band signal)	$\tilde{g}(t) = g_+(t) \cdot e^{-j2\pi f_c t}$	$g_+(t) = \tilde{g}(t) \cdot e^{j2\pi f_c t}$	$g(t) = Re\{\tilde{g}(t) \cdot e^{j2\pi f_c t}\}$ $g(t) = Re\{g_+(t)\}$
	$\tilde{g}(t) = g_I(t) + jg_Q(t)$	The complex envelope of a bandpass signal is a low-pass signal.	
Canonical Signal Representation:	$g(t) = g_I(t)\cos(2\pi f_c t) - g_Q(t)\sin(2\pi f_c t)$	$g_I(t)$: In-phase component $g_Q(t)$: Quadrature component	
	Natural Envelope: $a(t) = \sqrt{[g_I(t)]^2 + [g_Q(t)]^2}$	Phase of $g(t)$: $\phi(t) = \tan^{-1}\left(\frac{g_Q(t)}{g_I(t)}\right)$	
	$\tilde{g}(t) = a(t) \cdot e^{j\phi(t)}$ $g(t) = a(t)\cos(2\pi f_c t + \phi(t))$		