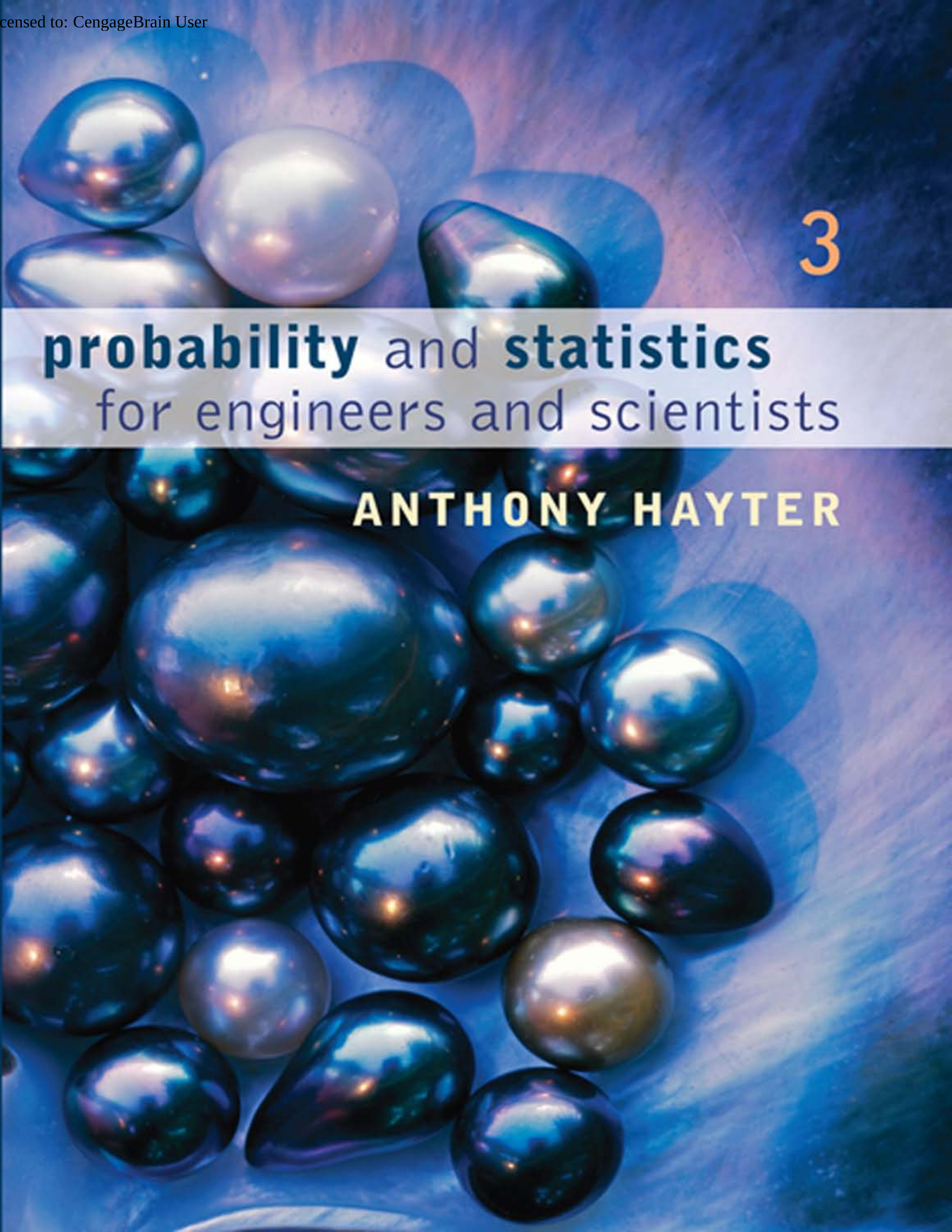

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probability and statistics for engineers and scientists

ANTHONY HAYTER

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CHAPTER ONE

Probability Theory

1.1 Probabilities

1.1.1 Introduction

Jointly with statistics, probability theory is a branch of mathematics that has been developed to deal with **uncertainty**. Classical mathematical theory had been successful in describing the world as a series of fixed and real observable events, yet before the seventeenth century it was largely inadequate in coping with processes or experiments that involved uncertain or random outcomes. Spurred initially by the mathematician's desire to analyze gambling games and later by the scientific analysis of mortality tables within the medical profession, the theory of probability has been developed as a scientific tool dealing with **chance**.

Today, probability theory is recognized as one of the most interesting and also one of the most useful areas of mathematics. It provides the basis for the science of statistical inference through experimentation and data analysis—an area of crucial importance in an increasingly quantitative world. Through its applications to problems such as the assessment of system reliability, the interpretation of measurement accuracy, and the maintenance of suitable quality controls, probability theory is particularly relevant to the engineering sciences today.

1.1.2 Sample Spaces

An **experiment** can in general be thought of as any process or procedure for which more than one **outcome** is possible. The goal of probability theory is to provide a mathematical structure for understanding or explaining the chances or likelihoods of the various outcomes actually occurring. A first step in the development of this theory is the construction of a list of the possible experimental outcomes. This collection of outcomes is called the **sample space** or **state space** and is denoted by $\mathcal{S}$.

Sample Space

The **sample space** $\mathcal{S}$ of an experiment is a set consisting of all of the possible experimental outcomes.

The following examples help illustrate the concept of a sample space.

Example 1
Machine Breakdowns

An engineer in charge of the maintenance of a particular machine notices that its breakdowns can be characterized as due to an electrical failure within the machine, a mechanical failure of some component of the machine, or operator misuse. When the machine is running, the engineer is uncertain what will be the cause of the next breakdown. The problem can be thought of as an experiment with the sample space

$$\mathcal{S} = \{\text{electrical, mechanical, misuse}\}$$

Example 2

Defective Computer Chips

A company sells computer chips in boxes of 500, and each chip can be classified as either satisfactory or defective. The number of defective chips in a particular box is uncertain, and the sample space is

$$\mathcal{S} = \{\text{0 defectives, 1 defective, 2 defectives, 3 defectives, 4 defectives}, \ldots, \text{499 defectives, 500 defectives}\}$$

Example 3

Software Errors

The control of errors in computer software products is obviously of great importance. The number of separate errors in a particular piece of software can be viewed as having a sample space

$$\mathcal{S} = \{\text{0 errors, 1 error, 2 errors, 3 errors, 4 errors, 5 errors}, \ldots\}$$

In practice there will be an upper bound on the possible number of errors in the software, although conceptually it is all right to allow the sample space to consist of all of the positive integers.

Example 4

Power Plant Operation

$\mathcal{S}$

(0, 0, 0)	(1, 0, 0)
(0, 0, 1)	(1, 0, 1)
(0, 1, 0)	(1, 1, 0)
(0, 1, 1)	(1, 1, 1)

FIGURE 1.1

Sample space for power plant example

A manager supervises the operation of three power plants, plant X, plant Y, and plant Z. At any given time, each of the three plants can be classified as either generating electricity (1) or being idle (0). With the notation (0, 1, 0) used to represent the situation where plant Y is generating electricity but plants X and Z are both idle, the sample space for the status of the three plants at a particular point in time is

$$\mathcal{S} = \{(0, 0, 0)\ (0, 0, 1)\ (0, 1, 0)\ (0, 1, 1)\ (1, 0, 0)\ (1, 0, 1)\ (1, 1, 0)\ (1, 1, 1)\}$$

It is often helpful to portray a sample space as a diagram. Figure 1.1 shows a diagram of the sample space for this example, where the sample space is represented by a box containing the eight individual outcomes. Diagrams of this kind are known as **Venn diagrams**.

GAMES OF CHANCE

Games of chance commonly involve the toss of a coin, the roll of a die, or the use of a pack of cards. The toss of a single coin has a sample space

$$\mathcal{S} = \{\text{head, tail}\}$$

and the toss of two coins (or one coin twice) has a sample space

$$\mathcal{S} = \{(\text{head, head})\ (\text{head, tail})\ (\text{tail, head})\ (\text{tail, tail})\}$$

where (head, tail), say, represents the event that the first coin resulted in a head and the second coin resulted in a tail. Notice that (head, tail) and (tail, head) are two distinct outcomes since observing a head on the first coin and a tail on the second coin is different from observing a tail on the first coin and a head on the second coin.

A usual six-sided die has a sample space

$$\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$$

FIGURE 1.2

Sample space for rolling two dice

$\mathcal{S}$

(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

FIGURE 1.3

Sample space for choosing one card

$\mathcal{S}$

A♡	2♡	3♡	4♡	5♡	6♡	7♡	8♡	9♡	10♡	J♡	Q♡	K♡
A♣	2♣	3♣	4♣	5♣	6♣	7♣	8♣	9♣	10♣	J♣	Q♣	K♣
A♢	2♢	3♢	4♢	5♢	6♢	7♢	8♢	9♢	10♢	J♢	Q♢	K♢
A♠	2♠	3♠	4♠	5♠	6♠	7♠	8♠	9♠	10♠	J♠	Q♠	K♠

FIGURE 1.4

Sample space for choosing two cards with replacement

$\mathcal{S}$

(A♡, A♡)	(A♡, 2♡)	(A♡, 3♡)	⋯	(A♡, Q♠)	(A♡, K♠)
(2♡, A♡)	(2♡, 2♡)	(2♡, 3♡)	⋯	(2♡, Q♠)	(2♡, K♠)
(3♡, A♡)	(3♡, 2♡)	(3♡, 3♡)	⋯	(3♡, Q♠)	(3♡, K♠)
⋮	⋮	⋮		⋮	⋮
(Q♠, A♡)	(Q♠, 2♡)	(Q♠, 3♡)	⋯	(Q♠, Q♠)	(Q♠, K♠)
(K♠, A♡)	(K♠, 2♡)	(K♠, 3♡)	⋯	(K♠, Q♠)	(K♠, K♠)

If two dice are rolled (or, equivalently, if one die is rolled twice), then the sample space is shown in Figure 1.2, where (1, 2) represents the event that the first die recorded a 1 and the second die recorded a 2. Again, notice that the events (1, 2) and (2, 1) are both included in the sample space because they represent two distinct events. This can be seen by considering one die to be red and the other die to be blue, and by distinguishing between obtaining a 1 on the red die and a 2 on the blue die and obtaining a 2 on the red die and a 1 on the blue die.

If a card is chosen from an ordinary pack of 52 playing cards, the sample space consists of the 52 individual cards as shown in Figure 1.3. If two cards are drawn, then it is necessary to consider whether they are drawn with or without **replacement**. If the drawing is performed *with replacement*, so that the initial card drawn is returned to the pack and the second drawing is from a full pack of 52 cards, then the sample space consists of events such as (6♡, 8♣), where the first card drawn is 6♡ and the second card drawn is 8♣. Altogether there will be $52 \times 52 = 2704$ elements of the sample space, including events such as (A♡, A♡), where the A♡ is drawn twice. This sample space is shown in Figure 1.4.

FIGURE 1.5

Sample space for choosing two cards without replacement

$\mathcal{S}$

	(A♡, 2♡)	(A♡, 3♡)	···	(A♡, Q♠)	(A♡, K♠)
(2♡, A♡)		(2♡, 3♡)	···	(2♡, Q♠)	(2♡, K♠)
(3♡, A♡)	(3♡, 2♡)		···	(3♡, Q♠)	(3♡, K♠)
⋮	⋮	⋮		⋮	⋮
(Q♠, A♡)	(Q♠, 2♡)	(Q♠, 3♡)	···		(Q♠, K♠)
(K♠, A♡)	(K♠, 2♡)	(K♠, 3♡)	···	(K♠, Q♠)	

If two cards are drawn *without replacement*, so that the second card is drawn from a reduced pack of 51 cards, then the sample space will be a subset of that above, as shown in Figure 1.5. Specifically, events such as $(A\heartsuit, A\heartsuit)$, where a particular card is drawn twice, will not be in the sample space. The total number of elements in this new sample space will therefore be $2704 - 52 = 2652$.

1.1.3 Probability Values

The likelihoods of particular experimental outcomes actually occurring are found by assigning a set of **probability values** to each of the elements of the sample space. Specifically, each outcome in the sample space is assigned a probability value that is a number between zero and one. The probabilities are chosen so that the sum of the probability values over all of the elements in the sample space is one.

Probabilities

A set of **probability values** for an experiment with a sample space $\mathcal{S} = \{O_1, O_2, \ldots, O_n\}$ consists of some probabilities

$$p_1, p_2, \ldots, p_n$$

that satisfy

$$0 \leq p_1 \leq 1, 0 \leq p_2 \leq 1, \ldots, 0 \leq p_n \leq 1$$

and

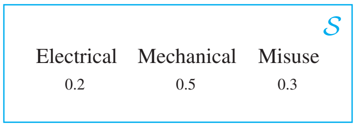
$$p_1 + p_2 + \cdots + p_n = 1$$

The probability of outcome O_i occurring is said to be p_i, and this is written $P(O_i) = p_i$.

An intuitive interpretation of a set of probability values is that the *larger* the probability value of a particular outcome, the *more likely* it is to happen. If two outcomes have identical probability values assigned to them, then they can be thought of as being equally likely to occur. On the other hand, if one outcome has a larger probability value assigned to it than another outcome, then the first outcome can be thought of as being more likely to occur.

FIGURE 1.6

Probability values for machine breakdown example



If a particular outcome has a probability value of one, then the interpretation is that it is certain to occur, so that there is actually no uncertainty in the experiment. In this case all of the other outcomes must necessarily have probability values of zero.

The following examples illustrate the assignment of probability values.

Example 1
Machine Breakdowns

Suppose that the machine breakdowns occur with probability values of $P(\text{electrical}) = 0.2$, $P(\text{mechanical}) = 0.5$, and $P(\text{misuse}) = 0.3$. This is a valid probability assignment since the three probability values 0.2, 0.5, and 0.3 are all between zero and one and they sum to one. Figure 1.6 shows a diagram of these probabilities by recording the respective probability value with each of the outcomes. These probability values indicate that mechanical failures are most likely, with misuse failures being more likely than electrical failures.

In addition, $P(\text{mechanical}) = 0.5$ indicates that about half of the failures will be attributable to mechanical causes. This does not mean that of the next two machine breakdowns, exactly one will be for mechanical reasons, or that in the next ten machine breakdowns, exactly five will be for mechanical reasons. However, it means that in the *long run*, the manager can reasonably expect that roughly half of the breakdowns will be for mechanical reasons. Similarly, in the long run, the manager will expect that about 20% of the breakdowns will be for electrical reasons, and that about 30% of the breakdowns will be attributable to operator misuse.

Example 3
Software Errors

Suppose that the number of errors in a software product has probabilities

$$P(0 \text{ errors}) = 0.05, \quad P(1 \text{ error}) = 0.08, \quad P(2 \text{ errors}) = 0.35,$$
$$P(3 \text{ errors}) = 0.20, \quad P(4 \text{ errors}) = 0.20, \quad P(5 \text{ errors}) = 0.12,$$
$$P(i \text{ errors}) = 0, \quad \text{for } i \geq 6$$

These probabilities show that there are at most five errors since the probability values are zero for six or more errors. In addition, it can be seen that the most likely number of errors is two and that three and four errors are equally likely.

It is reasonable to ask how anybody would ever know the probability assignments in the above two examples. In other words, how would the engineer know that there is a probability of 0.2 that a breakdown will be due to an electrical fault, or how would a computer programmer know that the probability of an error-free product is 0.05? In practice these probabilities would have to be estimated from a collection of data and prior experiences. Later in this book, in Chapters 7 and 10, it will be shown how statistical analysis techniques can be employed to help the engineer and programmer conduct studies to **estimate** probabilities of these kinds.

In some situations, notably games of chance, the experiments are conducted in such a way that all of the possible outcomes can be considered to be equally likely, so that they must be assigned identical probability values. If there are n outcomes in the sample space that are equally likely, then the condition that the probabilities sum to one requires that each probability value be $1/n$.

GAMES OF CHANCE

For a coin toss, the probabilities will in general be given by

$$P(\text{head}) = p, \qquad P(\text{tail}) = 1 - p$$

for some value of p with $0 \le p \le 1$. A fair coin will have $p = 0.5$ so that

$$P(\text{head}) = P(\text{tail}) = 0.5$$

with the two outcomes being equally likely. A biased coin will have $p \neq 0.5$. For example, if $p = 0.6$, then

$$P(\text{head}) = 0.6, \qquad P(\text{tail}) = 0.4$$

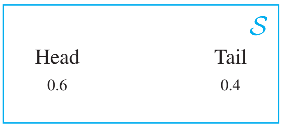


FIGURE 1.7

Probability values for a biased coin

as shown in Figure 1.7, and the coin toss is more likely to record a head.

A fair die will have each of the six outcomes equally likely, with each being assigned the same probability. Since the six probabilities must sum to one, this implies that each of the six outcomes must have a probability of $1/6$, so that

$$P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = \frac{1}{6}$$

This case is shown in Figure 1.8. An example of a biased die would be one for which

$$P(1) = 0.10, \quad P(2) = 0.15, \quad P(3) = 0.15,$$
$$P(4) = 0.15, \quad P(5) = 0.15, \quad P(6) = 0.30$$

as in Figure 1.9. In this case the die is most likely to score a 6, which will happen roughly three times out of ten as a long-run average. Scores of 2, 3, 4, and 5 are equally likely, and a score of 1 is the least likely event, happening only one time in ten on average.

If two dice are thrown and each of the 36 outcomes are equally likely (as will be the case with two fair dice that are shaken properly), the probability value of each outcome will necessarily be $1/36$. This is shown in Figure 1.10.

If a card is drawn at random from a pack of cards, then there are 52 possible outcomes in the sample space, and each one is equally likely so that each would be assigned a probability value of $1/52$. Thus, for example, $P(A\heartsuit) = 1/52$, as shown in Figure 1.11. If two cards are drawn with replacement, and if both the cards can be assumed to be chosen at random through suitable shuffling of the pack before and between the drawings, then each of the $52 \times 52 = 2704$ elements of the sample space will be equally likely and hence should each be assigned a probability value of $1/2704$. In this case $P(A\heartsuit, 2\clubsuit) = 1/2704$, for example, as shown in Figure 1.12. If the drawing is performed without replacement but again at random, then the sample space has only 2652 elements and each would have a probability of $1/2652$, as shown in Figure 1.13.

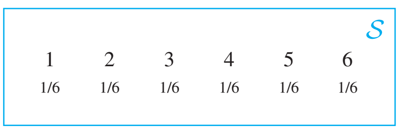


FIGURE 1.8

Probability values for a fair die

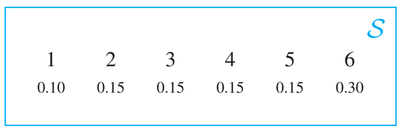


FIGURE 1.9

Probability values for a biased die

FIGURE 1.10

Probability values for rolling two dice

S

(1, 1) 1/36	(1, 2) 1/36	(1, 3) 1/36	(1, 4) 1/36	(1, 5) 1/36	(1, 6) 1/36
(2, 1) 1/36	(2, 2) 1/36	(2, 3) 1/36	(2, 4) 1/36	(2, 5) 1/36	(2, 6) 1/36
(3, 1) 1/36	(3, 2) 1/36	(3, 3) 1/36	(3, 4) 1/36	(3, 5) 1/36	(3, 6) 1/36
(4, 1) 1/36	(4, 2) 1/36	(4, 3) 1/36	(4, 4) 1/36	(4, 5) 1/36	(4, 6) 1/36
(5, 1) 1/36	(5, 2) 1/36	(5, 3) 1/36	(5, 4) 1/36	(5, 5) 1/36	(5, 6) 1/36
(6, 1) 1/36	(6, 2) 1/36	(6, 3) 1/36	(6, 4) 1/36	(6, 5) 1/36	(6, 6) 1/36

FIGURE 1.11

Probability values for choosing one card

S

A♡ 1/52	2♡ 1/52	3♡ 1/52	4♡ 1/52	5♡ 1/52	6♡ 1/52	7♡ 1/52	8♡ 1/52	9♡ 1/52	10♡ 1/52	J♡ 1/52	Q♡ 1/52	K♡ 1/52
A♣ 1/52	2♣ 1/52	3♣ 1/52	4♣ 1/52	5♣ 1/52	6♣ 1/52	7♣ 1/52	8♣ 1/52	9♣ 1/52	10♣ 1/52	J♣ 1/52	Q♣ 1/52	K♣ 1/52
A♢ 1/52	2♢ 1/52	3♢ 1/52	4♢ 1/52	5♢ 1/52	6♢ 1/52	7♢ 1/52	8♢ 1/52	9♢ 1/52	10♢ 1/52	J♢ 1/52	Q♢ 1/52	K♢ 1/52
A♠ 1/52	2♠ 1/52	3♠ 1/52	4♠ 1/52	5♠ 1/52	6♠ 1/52	7♠ 1/52	8♠ 1/52	9♠ 1/52	10♠ 1/52	J♠ 1/52	Q♠ 1/52	K♠ 1/52

FIGURE 1.12

Probability values for choosing two cards with replacement

S

(A♡, A♡) 1/2704	(A♡, 2♡) 1/2704	(A♡, 3♡) 1/2704	⋯ ⋯	(A♡, Q♠) 1/2704	(A♡, K♠) 1/2704
(2♡, A♡) 1/2704	(2♡, 2♡) 1/2704	(2♡, 3♡) 1/2704	⋯ ⋯	(2♡, Q♠) 1/2704	(2♡, K♠) 1/2704
(3♡, A♡) 1/2704	(3♡, 2♡) 1/2704	(3♡, 3♡) 1/2704	⋯ ⋯	(3♡, Q♠) 1/2704	(3♡, K♠) 1/2704
⋮	⋮	⋮		⋮	⋮
(Q♠, A♡) 1/2704	(Q♠, 2♡) 1/2704	(Q♠, 3♡) 1/2704	⋯ ⋯	(Q♠, Q♠) 1/2704	(Q♠, K♠) 1/2704
(K♠, A♡) 1/2704	(K♠, 2♡) 1/2704	(K♠, 3♡) 1/2704	⋯ ⋯	(K♠, Q♠) 1/2704	(K♠, K♠) 1/2704

FIGURE 1.13

Probability values for choosing two cards without replacement

S

	(A♡, 2♡) 1/2652	(A♡, 3♡) 1/2652	··· ···	(A♡, Q♠) 1/2652	(A♡, K♠) 1/2652
(2♡, A♡) 1/2652		(2♡, 3♡) 1/2652	··· ···	(2♡, Q♠) 1/2652	(2♡, K♠) 1/2652
(3♡, A♡) 1/2652	(3♡, 2♡) 1/2652		··· ···	(3♡, Q♠) 1/2652	(3♡, K♠) 1/2652
⋮	⋮	⋮		⋮	⋮
(Q♠, A♡) 1/2652	(Q♠, 2♡) 1/2652	(Q♠, 3♡) 1/2652	··· ···		(Q♠, K♠) 1/2652
(K♠, A♡) 1/2652	(K♠, 2♡) 1/2652	(K♠, 3♡) 1/2652	··· ···	(K♠, Q♠) 1/2652	

1.1.4 Problems

1.1.1 What is the sample space when a coin is tossed three times?

1.1.2 What is the sample space for counting the number of females in a group of n people?

1.1.3 What is the sample space for the number of Aces in a hand of 13 playing cards?

1.1.4 What is the sample space for a person's birthday?

1.1.5 A car repair is performed either on time or late and either satisfactorily or unsatisfactorily. What is the sample space for a car repair?

1.1.6 A bag contains balls that are either red or blue and either dull or shiny. What is the sample space when a ball is chosen from the bag?

1.1.7 A probability value p is often reported as an *odds ratio*, which is $p/(1-p)$. This is the ratio of the probability that the event happens to the probability that the event does not happen.

(a) If the odds ratio is 1, what is p?

(b) If the odds ratio is 2, what is p?

(c) If $p = 0.25$, what is the odds ratio?

1.1.8 An experiment has five outcomes, I, II, III, IV, and V. If $P(\text{I}) = 0.13$, $P(\text{II}) = 0.24$, $P(\text{III}) = 0.07$, and $P(\text{IV}) = 0.38$, what is $P(\text{V})$?

1.1.9 An experiment has five outcomes, I, II, III, IV, and V. If $P(\text{I}) = 0.08$, $P(\text{II}) = 0.20$, and $P(\text{III}) = 0.33$, what are the possible values for the probability of outcome V? If outcomes IV and V are equally likely, what are their probability values?

1.1.10 An experiment has three outcomes, I, II, and III. If outcome I is twice as likely as outcome II, and outcome II is three times as likely as outcome III, what are the probability values of the three outcomes?

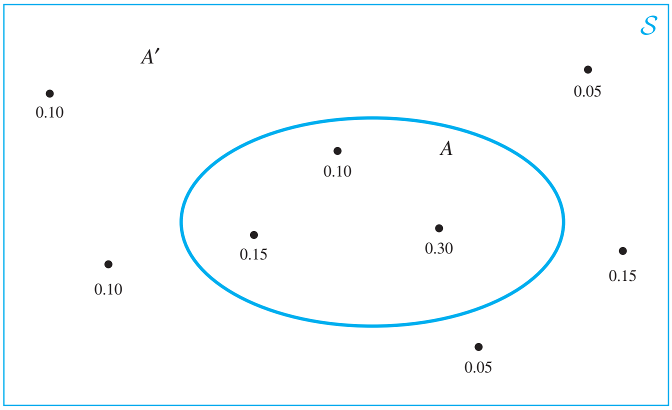
1.2 Events

1.2.1 Events and Complements

Interest is often centered not so much on the individual elements of a sample space, but rather on collections of individual outcomes. These collections of outcomes are called **events**.

FIGURE 1.14

$P(A) = 0.10 + 0.15 + 0.30 = 0.55$



Events

An **event** A is a subset of the sample space $\mathcal{S}$. It collects outcomes of particular interest. The probability of an event A, $P(A)$, is obtained by summing the probabilities of the outcomes contained within the event A.

An event is said to occur if one of the outcomes contained within the event occurs.

Figure 1.14 shows a sample space $\mathcal{S}$ consisting of eight outcomes, each of which is labeled with a probability value. Three of the outcomes are contained within the event A. The probability of the event A is calculated as the sum of the probabilities of these three events, so that

$$P(A) = 0.10 + 0.15 + 0.30 = 0.55$$

The **complement** of an event A is taken to mean the event consisting of everything in the sample space $\mathcal{S}$ that is not contained within the event A. The notation A' is used for the complement of A. In this example, the probability of the complement of A is obtained by summing the probabilities of the five outcomes not contained within A, so that

$$P(A') = 0.10 + 0.05 + 0.05 + 0.15 + 0.10 = 0.45$$

Notice that $P(A) + P(A') = 1$, which is a general rule.

Complements of Events

The event A', the **complement** of an event A, is the event consisting of everything in the sample space $\mathcal{S}$ that is not contained within the event A. In all cases

$$P(A) + P(A') = 1$$

It is useful to consider both individual outcomes and the whole sample space as also being events. Events that consist of an individual outcome are sometimes referred to as **elementary events** or **simple events**. If an event is defined to be a particular single outcome, then its probability is just the probability of that outcome. If an event is defined to be the whole sample space, then obviously its probability is one.

1.2.2 Examples of Events

Example 2
Defective Computer Chips

Consider the following probability values for the number of defective chips in a box of 500 chips:

$$P(0 \text{ defectives}) = 0.02, \quad P(1 \text{ defective}) = 0.11,$$
$$P(2 \text{ defectives}) = 0.16, \quad P(3 \text{ defectives}) = 0.21,$$
$$P(4 \text{ defectives}) = 0.13, \quad P(5 \text{ defectives}) = 0.08$$

and suppose that the probabilities of the additional elements of the sample space (6 defectives, 7 defectives, . . . , 500 defectives) are unknown. The company is thinking of claiming that each box has no more than 5 defective chips, and it wishes to calculate the probability that the claim is correct.

The event *correct* consists of the six outcomes listed above, so that

$$\begin{aligned}\text{correct} = \{&0 \text{ defectives},\ 1 \text{ defective},\ 2 \text{ defectives},\ 3 \text{ defectives},\\ &4 \text{ defectives},\ 5 \text{ defectives}\} \subset \mathcal{S}\end{aligned}$$

The probability of the claim being correct is then

$$\begin{aligned}P(\text{correct}) &= P(0 \text{ defectives}) + \cdots + P(5 \text{ defectives})\\ &= 0.02 + 0.11 + 0.16 + 0.21 + 0.13 + 0.08 = 0.71\end{aligned}$$

Consequently, on average, only about 71% of the boxes will meet the company's claim that there are no more than 5 defective chips. The complement of the event *correct* is that there will be at least 6 defective chips so that the company's claim will be incorrect. This has a probability of $1 - 0.71 = 0.29$.

Example 3
Software Errors

Consider the event A that there are no more than two errors in a software product. This event is given by

$$A = \{0 \text{ errors}, 1 \text{ error}, 2 \text{ errors}\} \subset \mathcal{S}$$

and its probability is

$$\begin{aligned}P(A) &= P(0 \text{ errors}) + P(1 \text{ error}) + P(2 \text{ errors})\\ &= 0.05 + 0.08 + 0.35 = 0.48\end{aligned}$$

The probability of the complement of the event A is

$$P(A') = 1 - P(A) = 1 - 0.48 = 0.52$$

which is the probability that a software product has three or more errors.

Example 4
Power Plant Operation

Consider the probability values given in Figure 1.15, where, for instance, the probability that all three plants are idle is $P((0, 0, 0)) = 0.07$, and the probability that only plant X is idle is $P((0, 1, 1)) = 0.18$. The event that plant X is idle is given by

$$A = \{(0, 0, 0), (0, 0, 1), (0, 1, 0), (0, 1, 1)\}$$

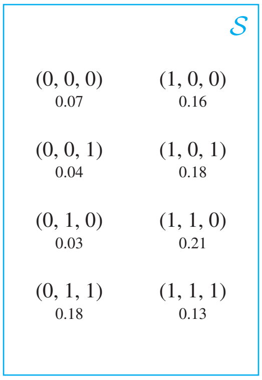


FIGURE 1.15

Probability values for power plant example

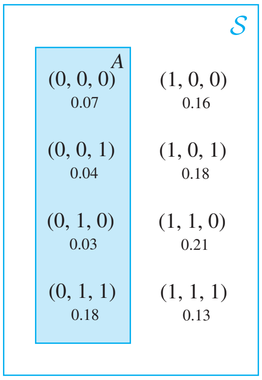


FIGURE 1.16

Event A: plant X idle

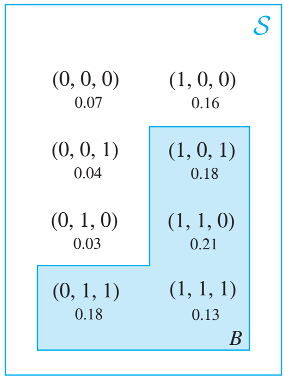


FIGURE 1.17

Event B: at least two plants generating electricity

as illustrated in Figure 1.16, and it has a probability of

$$\begin{aligned} P(A) &= P((0,0,0)) + P((0,0,1)) + P((0,1,0)) + P((0,1,1)) \\ &= 0.07 + 0.04 + 0.03 + 0.18 = 0.32 \end{aligned}$$

The complement of this event is

$$A' = \{(1,0,0), (1,0,1), (1,1,0), (1,1,1)\}$$

which corresponds to plant X generating electricity, and it has a probability of

$$P(A') = 1 - P(A) = 1 - 0.32 = 0.68$$

Suppose that the manager is interested in the proportion of the time that at least two out of the three plants are generating electricity. This event is given by

$$B = \{(0,1,1), (1,0,1), (1,1,0), (1,1,1)\}$$

as illustrated in Figure 1.17, with a probability of

$$\begin{aligned} P(B) &= P((0,1,1)) + P((1,0,1)) + P((1,1,0)) + P((1,1,1)) \\ &= 0.18 + 0.18 + 0.21 + 0.13 = 0.70 \end{aligned}$$

This result indicates that, on average, at least two of the plants will be generating electricity about 70% of the time. The complement of this event is

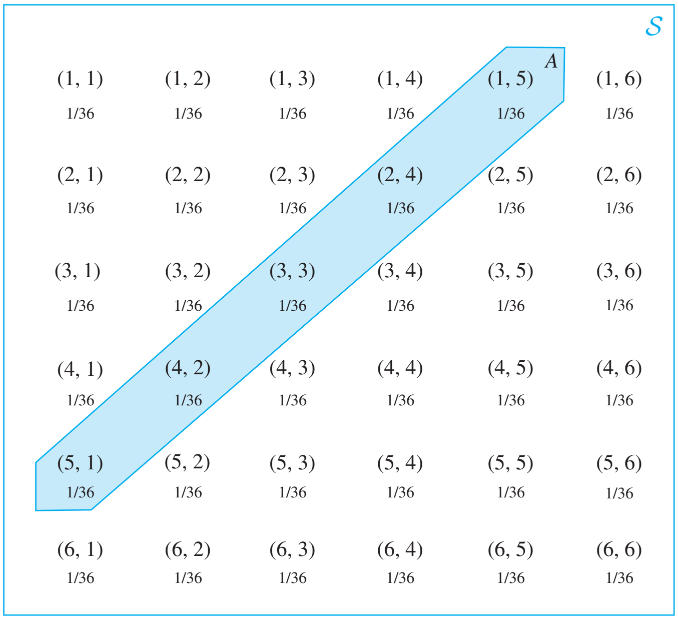
$$B' = \{(0,0,0), (0,0,1), (0,1,0), (1,0,0)\}$$

which corresponds to the situation in which at least two of the plants are idle. The probability of this is

$$P(B') = 1 - P(B) = 1 - 0.70 = 0.30$$

FIGURE 1.18

Event A: sum equal to 6



GAMES OF CHANCE

The event that an *even* score is recorded on the roll of a die is given by

$$\text{even} = \{2, 4, 6\}$$

For a fair die this event would have a probability of

$$P(\text{even}) = P(2) + P(4) + P(6) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}$$

Figure 1.18 shows the event that the sum of the scores of two dice is equal to 6. This event is given by

$$A = \{(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)\}$$

If each outcome is equally likely with a probability of $1/36$, then this event clearly has a probability of $5/36$. A sum of 6 will be obtained with two fair dice roughly 5 times out of 36 on average, that is, on about 14% of the throws. The probabilities of obtaining other sums can be obtained in a similar manner, and it is seen that 7 is the most likely score, with a probability of $6/36 = 1/6$. The least likely scores are 2 and 12, each with a probability of $1/36$.

Figure 1.19 shows the event that at least one of the two dice records a 6, which is seen to have a probability of $11/36$. The complement of this event is the event that neither die records a 6, with a probability of $1 - 11/36 = 25/36$.

Figure 1.20 illustrates the event that a card drawn from a pack of cards belongs to the heart suit. This event consists of the 13 outcomes corresponding to the 13 cards in the heart suit. If the drawing is done at random, with each of the 52 possible outcomes being equally likely with a probability of $1/52$, then the probability of drawing a heart is clearly $13/52 = 1/4$. This result makes sense since there are four suits that are equally likely. Figure 1.21 illustrates the event that a picture card (Jack, Queen, or King) is drawn, with a probability of $12/52 = 3/13$.

FIGURE 1.19

Event B: at least one 6 recorded

S

					B
(1, 1) 1/36	(1, 2) 1/36	(1, 3) 1/36	(1, 4) 1/36	(1, 5) 1/36	(1, 6) 1/36
(2, 1) 1/36	(2, 2) 1/36	(2, 3) 1/36	(2, 4) 1/36	(2, 5) 1/36	(2, 6) 1/36
(3, 1) 1/36	(3, 2) 1/36	(3, 3) 1/36	(3, 4) 1/36	(3, 5) 1/36	(3, 6) 1/36
(4, 1) 1/36	(4, 2) 1/36	(4, 3) 1/36	(4, 4) 1/36	(4, 5) 1/36	(4, 6) 1/36
(5, 1) 1/36	(5, 2) 1/36	(5, 3) 1/36	(5, 4) 1/36	(5, 5) 1/36	(5, 6) 1/36
(6, 1) 1/36	(6, 2) 1/36	(6, 3) 1/36	(6, 4) 1/36	(6, 5) 1/36	(6, 6) 1/36

FIGURE 1.20

Event A: card belongs to heart suit

S

A

A♡ 1/52	2♡ 1/52	3♡ 1/52	4♡ 1/52	5♡ 1/52	6♡ 1/52	7♡ 1/52	8♡ 1/52	9♡ 1/52	10♡ 1/52	J♡ 1/52	Q♡ 1/52	K♡ 1/52
A♣ 1/52	2♣ 1/52	3♣ 1/52	4♣ 1/52	5♣ 1/52	6♣ 1/52	7♣ 1/52	8♣ 1/52	9♣ 1/52	10♣ 1/52	J♣ 1/52	Q♣ 1/52	K♣ 1/52
A◇ 1/52	2◇ 1/52	3◇ 1/52	4◇ 1/52	5◇ 1/52	6◇ 1/52	7◇ 1/52	8◇ 1/52	9◇ 1/52	10◇ 1/52	J◇ 1/52	Q◇ 1/52	K◇ 1/52
A♠ 1/52	2♠ 1/52	3♠ 1/52	4♠ 1/52	5♠ 1/52	6♠ 1/52	7♠ 1/52	8♠ 1/52	9♠ 1/52	10♠ 1/52	J♠ 1/52	Q♠ 1/52	K♠ 1/52

FIGURE 1.21

Event B: picture card is chosen

S

A♡ 1/52	2♡ 1/52	3♡ 1/52	4♡ 1/52	5♡ 1/52	6♡ 1/52	7♡ 1/52	8♡ 1/52	9♡ 1/52	10♡ 1/52	J♡ 1/52	Q♡ 1/52	K♡ 1/52
A♣ 1/52	2♣ 1/52	3♣ 1/52	4♣ 1/52	5♣ 1/52	6♣ 1/52	7♣ 1/52	8♣ 1/52	9♣ 1/52	10♣ 1/52	J♣ 1/52	Q♣ 1/52	K♣ 1/52
A◇ 1/52	2◇ 1/52	3◇ 1/52	4◇ 1/52	5◇ 1/52	6◇ 1/52	7◇ 1/52	8◇ 1/52	9◇ 1/52	10◇ 1/52	J◇ 1/52	Q◇ 1/52	K◇ 1/52
A♠ 1/52	2♠ 1/52	3♠ 1/52	4♠ 1/52	5♠ 1/52	6♠ 1/52	7♠ 1/52	8♠ 1/52	9♠ 1/52	10♠ 1/52	J♠ 1/52	Q♠ 1/52	K♠ 1/52

B

1.2.3 Problems

1.2.1 Consider the sample space in Figure 1.22 with outcomes a, b, c, d, and e. Calculate:
(a) $P(b)$ **(b)** $P(A)$ **(c)** $P(A')$

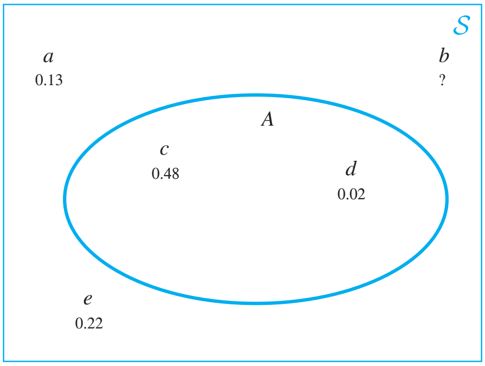


FIGURE 1.22

1.2.2 Consider the sample space in Figure 1.23 with outcomes a, b, c, d, e, and f. If $P(A) = 0.27$, calculate:
(a) $P(b)$ **(b)** $P(A')$ **(c)** $P(d)$

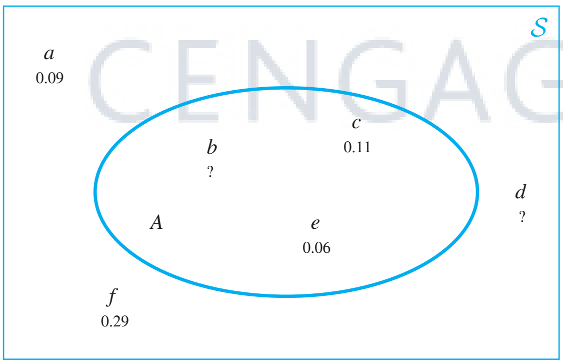


FIGURE 1.23

1.2.3 If birthdays are equally likely to fall on any day, what is the probability that a person chosen at random has a birthday in January? What about February?

1.2.4 If a fair die is thrown, what is the probability of scoring a *prime* number (suppose that the number 1 is considered to be a prime number)?

1.2.5 If two fair dice are thrown, what is the probability that at least one score is a prime number? What is the complement of this event? What is its probability?

1.2.6 Two fair dice are thrown, one red and one blue. What is the probability that the red die has a score that is *strictly greater* than the score of the blue die? Why is this probability less than 0.5? What is the complement of this event?

1.2.7 If a card is chosen at random from a pack of cards, what is the probability that the card is from one of the two black suits?

1.2.8 If a card is chosen at random from a pack of cards, what is the probability that it is an Ace?

1.2.9 A *winner* and a *runner-up* are decided in a tournament of four players, one of whom is Terica. If all the outcomes are equally likely, what is the probability that
(a) Terica is the winner?
(b) Terica is either the winner or the runner-up?

1.2.10 Three types of batteries are being tested, type I, type II, and type III. The outcome (I, II, III) denotes that the battery of type I fails first, the battery of type II next, and the battery of type III lasts the longest. The probabilities of the six outcomes are given in Figure 1.24. What is the probability that
(a) the type I battery lasts longest?
(b) the type I battery lasts shortest?
(c) the type I battery does not last longest?
(d) the type I battery lasts longer than the type II battery?
(This problem is continued in Problem 1.4.9.)

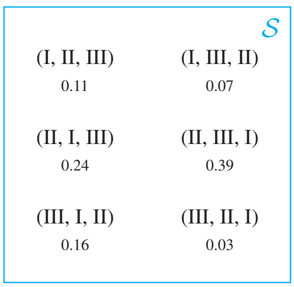


FIGURE 1.24

Probability values for battery lifetimes

1.2.11 A factory has two assembly lines, each of which is *shut down* (S), at *partial capacity* (P), or at *full capacity* (F). The sample space is given in Figure 1.25, where, for example, (S, P) denotes that the first assembly line is

$\mathcal{S}$

(S, S) 0.02	(S, P) 0.06	(S, F) 0.05
(P, S) 0.07	(P, P) 0.14	(P, F) 0.20
(F, S) 0.06	(F, P) 0.21	(F, F) 0.19

FIGURE 1.25

Probability values for assembly line operations

shut down and the second one is operating at partial capacity. What is the probability that

(a) both assembly lines are shut down?
(b) neither assembly line is shut down?
(c) at least one assembly line is at full capacity?
(d) exactly one assembly line is at full capacity?

What is the complement of the event in part (b)? What is the complement of the event in part (c)?
(This problem is continued in Problem 1.4.10.)

1.2.12 A fair coin is tossed three times. What is the probability that two heads will be obtained *in succession*?

1.3 Combinations of Events

In general, more than one event will be of interest for a particular experiment and sample space. For two events A and B, in addition to the consideration of the probability of event A occurring and the probability of event B occurring, it is often important to consider other probabilities such as the probability of **both** events occurring simultaneously. Other quantities of interest may be the probability that **neither** event A nor event B occurs, the probability that **at least one** of the two events occurs, or the probability that event A occurs, but event B does not.

1.3.1 Intersections of Events

Consider first the calculation of the probability that both events occur simultaneously. This can be done by defining a new event to consist of the outcomes that are in both event A and event B.

Intersections of Events

The event $A \cap B$ is the **intersection** of the events A and B and consists of the outcomes that are contained within both events A and B. The probability of this event, $P(A \cap B)$, is the probability that both events A and B occur simultaneously.

Figure 1.26 shows a sample space $\mathcal{S}$ that consists of nine outcomes. Event A consists of three outcomes, and its probability is given by

$$P(A) = 0.01 + 0.07 + 0.19 = 0.27$$

Event B consists of five outcomes, and its probability is given by

$$P(B) = 0.07 + 0.19 + 0.04 + 0.14 + 0.12 = 0.56$$

The intersection of these two events, shown in Figure 1.27, consists of the two outcomes that are contained within both events A and B. It has a probability of

$$P(A \cap B) = 0.07 + 0.19 = 0.26$$

which is the probability that both events A and B occur simultaneously.

FIGURE 1.26

Events A and B

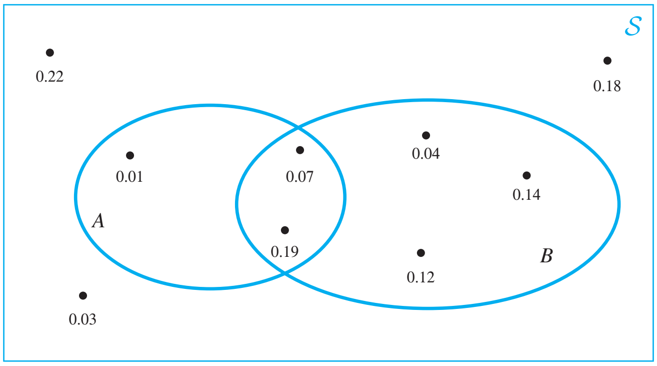
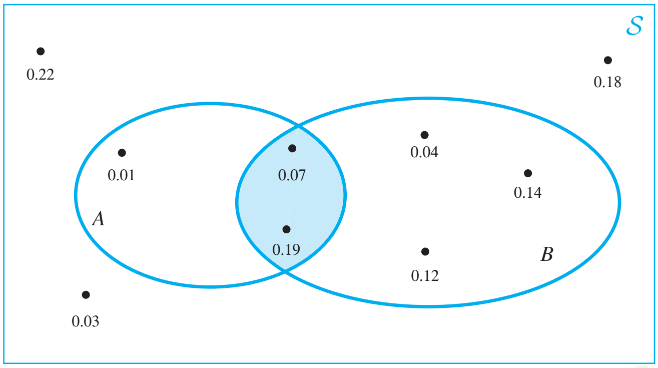


FIGURE 1.27

The event $A \cap B$



Event A', the complement of the event A, is the event consisting of the six outcomes that are not in event A. Notice that there are obviously no outcomes in $A \cap A'$, and this is written

$$A \cap A' = \emptyset$$

where $\emptyset$ is referred to as the "empty set," a set that does not contain anything. Consequently,

$$P(A \cap A') = P(\emptyset) = 0$$

and it is impossible for the event A to occur at the same time as its complement.

A more interesting event is the event $A' \cap B$ illustrated in Figure 1.28. This event consists of the three outcomes that are contained within event B but that are not contained within event A. It has a probability of

$$P(A' \cap B) = 0.04 + 0.14 + 0.12 = 0.30$$

which is the probability that event B occurs but event A does not occur. Similarly, Figure 1.29 shows the event $A \cap B'$, which has a probability of

$$P(A \cap B') = 0.01$$

This is the probability that event A occurs but event B does not.

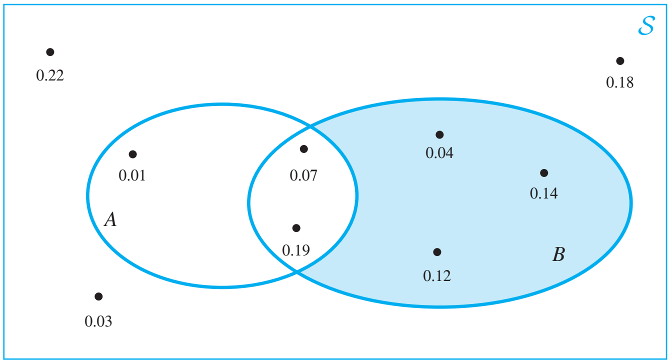


FIGURE 1.28

The event $A' \cap B$

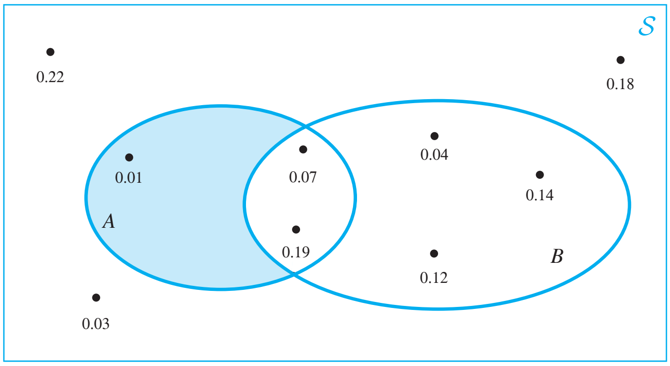


FIGURE 1.29

The event $A \cap B'$

Notice that

$$P(A \cap B) + P(A \cap B') = 0.26 + 0.01 = 0.27 = P(A)$$

and similarly that

$$P(A \cap B) + P(A' \cap B) = 0.26 + 0.30 = 0.56 = P(B)$$

The following two equalities hold in general for all events A and B:

$$P(A \cap B) + P(A \cap B') = P(A) \qquad P(A \cap B) + P(A' \cap B) = P(B)$$

Two events A and B that have *no* outcomes in common are said to be **mutually exclusive** events. In this case $A \cap B = \emptyset$ and $P(A \cap B) = 0$.

Mutually Exclusive Events

Two events A and B are said to be **mutually exclusive** if $A \cap B = \emptyset$ so that they have no outcomes in common.

Figure 1.30 illustrates a sample space $\mathcal{S}$ that consists of seven outcomes, three of which are contained within event A and two of which are contained within event B. Since no outcomes are contained within both events A and B, the two events are mutually exclusive.

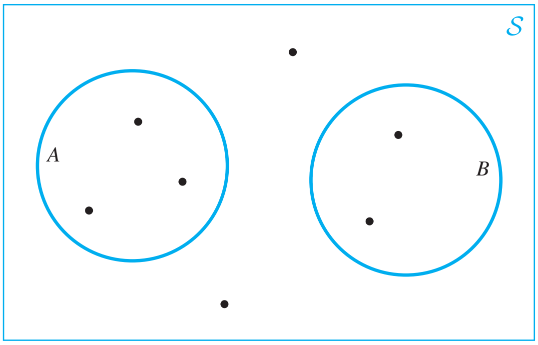


FIGURE 1.30

A and B are mutually exclusive events

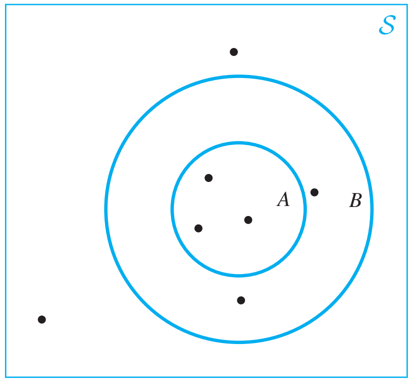


FIGURE 1.31

$A \subset B$

Finally, Figure 1.31 illustrates a situation where an event A is contained within an event B, that is, $A \subset B$. Each outcome in event A is also contained in event B. It is clear that in this case $A \cap B = A$.

Some other simple results concerning the intersections of events are as follows:

$$A \cap B = B \cap A \qquad A \cap A = A$$
$$A \cap \mathcal{S} = A \qquad A \cap \emptyset = \emptyset$$
$$A \cap A' = \emptyset \qquad A \cap (B \cap C) = (A \cap B) \cap C$$

1.3.2 Unions of Events

The event that *at least one* out of two events A and B occurs, shown in Figure 1.32, is denoted by $A \cup B$ and is referred to as the **union** of events A and B. The probability of this event, $P(A \cup B)$, is the sum of the probability values of the outcomes that are in either of events A or B (including those events that are in both events A and B).

Unions of Events

The event $A \cup B$ is the **union** of events A and B and consists of the outcomes that are contained within at least one of the events A and B. The probability of this event, $P(A \cup B)$, is the probability that at least one of the events A and B occurs.

Notice that the outcomes in the event $A \cup B$ can be classified into three kinds. They are

1. in event A, but not in event B
2. in event B, but not in event A
3. in both events A and B

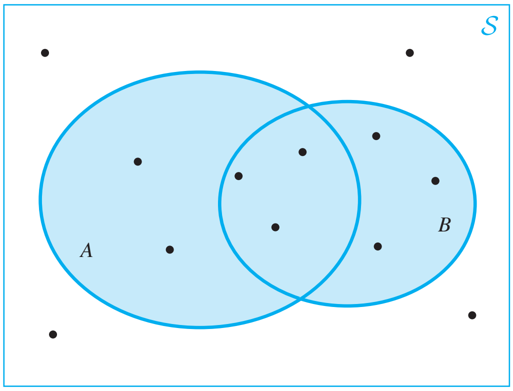


FIGURE 1.32

The event $A \cup B$

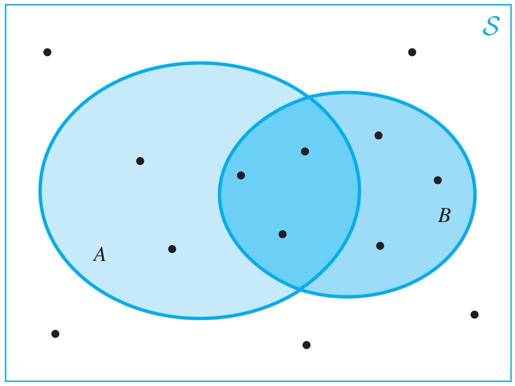


FIGURE 1.33

Decomposition of the event $A \cup B$

The outcomes of type 1 form the event $A \cap B'$, the outcomes of type 2 form the event $A' \cap B$, and the outcomes of type 3 form the event $A \cap B$, as shown in Figure 1.33. Since the probability of $A \cup B$ is obtained as the sum of the probability values of the outcomes within these three (mutually exclusive) events, the following result is obtained:

$$P(A \cup B) = P(A \cap B') + P(A' \cap B) + P(A \cap B)$$

This equality can be presented in another form using the relationships

$$P(A \cap B') = P(A) - P(A \cap B)$$

and

$$P(A' \cap B) = P(B) - P(A \cap B)$$

Substituting in these expressions for $P(A \cap B')$ and $P(A' \cap B)$ gives the following result:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

This equality has the intuitive interpretation that the probability of at least one of the events A and B occurring can be obtained by adding the probabilities of the two events A and B and then subtracting the probability that both the events occur simultaneously. The probability that both events occur, $P(A \cap B)$, needs to be subtracted since the probability values of the outcomes in the intersection $A \cap B$ have been counted *twice*, once in $P(A)$ and once in $P(B)$.

Notice that if events A and B are mutually exclusive, so that no outcomes are in $A \cap B$ and $P(A \cap B) = 0$ as in Figure 1.30, then $P(A \cup B)$ can just be obtained as the sum of the probabilities of events A and B.

If the events A and B are mutually exclusive so that $P(A \cap B) = 0$, then

$$P(A \cup B) = P(A) + P(B)$$

FIGURE 1.34

The event $A \cup B$

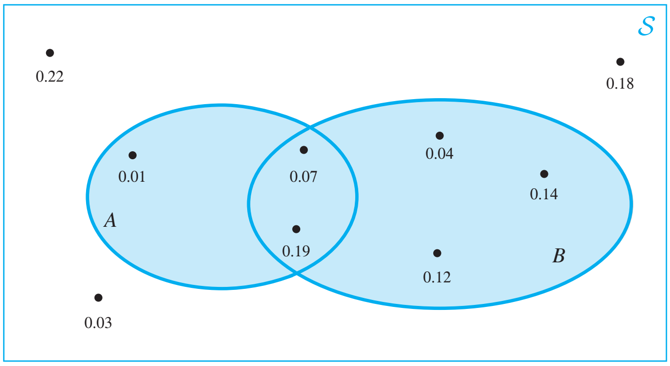
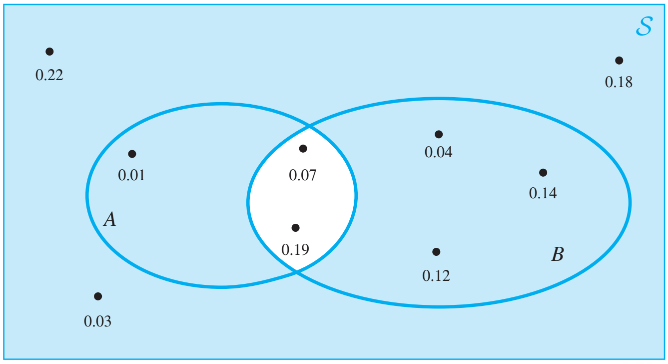


FIGURE 1.35

The event $A' \cup B'$



The sample space of nine outcomes illustrated in Figure 1.26 can be used to demonstrate some general relationships between unions and intersections of events. For this example, the event $A \cup B$ consists of the six outcomes illustrated in Figure 1.34, and it has a probability of

$$P(A \cup B) = 0.01 + 0.07 + 0.19 + 0.04 + 0.14 + 0.12 = 0.57$$

The event $(A \cup B)'$, which is the complement of the union of the events A and B, consists of the three outcomes that are neither in event A nor in event B. It has a probability of

$$P((A \cup B)') = 0.03 + 0.22 + 0.18 = 0.43 = 1 - P(A \cup B)$$

Notice that the event $(A \cup B)'$ can also be written as $A' \cap B'$ since it consists of those outcomes that are simultaneously neither in event A nor in event B. This is a general result:

$$(A \cup B)' = A' \cap B'$$

Furthermore, the event $A' \cup B'$ consists of the seven outcomes illustrated in Figure 1.35, and it has a probability of

$$P(A' \cup B') = 0.01 + 0.03 + 0.22 + 0.18 + 0.12 + 0.14 + 0.04 = 0.74$$

However, this event can also be written as $(A \cap B)'$ since it consists of the outcomes that are in the complement of the intersection of sets A and B. Hence, its probability could have been calculated by

$$P(A' \cup B') = P((A \cap B)') = 1 - P(A \cap B) = 1 - 0.26 = 0.74$$

Again, this is a general result:

$$(A \cap B)' = A' \cup B'$$

Finally, if event A is contained within event B, $A \subset B$, as shown in Figure 1.31, then clearly $A \cup B = B$.

Some other simple results concerning the unions of events are as follows:

$$A \cup B = B \cup A \qquad A \cup A = A \qquad A \cup \mathcal{S} = \mathcal{S}$$
$$A \cup \emptyset = A \qquad A \cup A' = \mathcal{S} \qquad A \cup (B \cup C) = (A \cup B) \cup C$$

1.3.3 Examples of Intersections and Unions

Example 4
Power Plant Operation

Consider again Figures 1.15, 1.16, and 1.17, and recall that event A, the event that plant X is idle, has a probability of 0.32, and that event B, the event that at least two out of the three plants are generating electricity, has a probability of 0.70.

The event $A \cap B$ consists of the outcomes for which plant X is idle *and* at least two out of the three plants are generating electricity. Clearly, the only outcome of this kind is the one where plant X is idle and both plants Y and Z are generating electricity, so that

$$A \cap B = \{(0, 1, 1)\}$$

as illustrated in Figure 1.36. Consequently,

$$P(A \cap B) = P((0, 1, 1)) = 0.18$$

The event $A \cup B$ consists of outcomes where *either* plant X is idle *or* at least two plants are generating electricity (or both). Seven out of the eight outcomes satisfy this condition, so that

$$A \cup B = \{(0, 0, 0), (0, 0, 1), (0, 1, 0), (0, 1, 1), (1, 0, 1), (1, 1, 0), (1, 1, 1)\}$$

as illustrated in Figure 1.37. The probability of the event $A \cup B$ is thus

$$\begin{aligned} P(A \cup B) &= P((0, 0, 0)) + P((0, 0, 1)) + P((0, 1, 0)) + P((0, 1, 1)) \\ &\quad + P((1, 0, 1)) + P((1, 1, 0)) + P((1, 1, 1)) \\ &= 0.07 + 0.04 + 0.03 + 0.18 + 0.18 + 0.21 + 0.13 = 0.84 \end{aligned}$$

Another way of calculating this probability is

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= 0.32 + 0.70 - 0.18 = 0.84 \end{aligned}$$

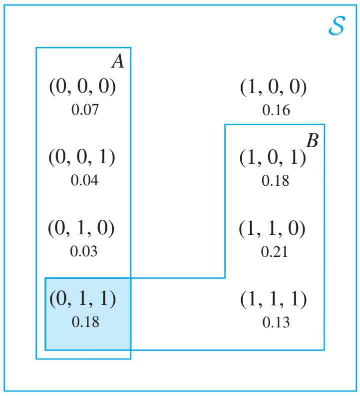


FIGURE 1.36

The event $A \cap B$

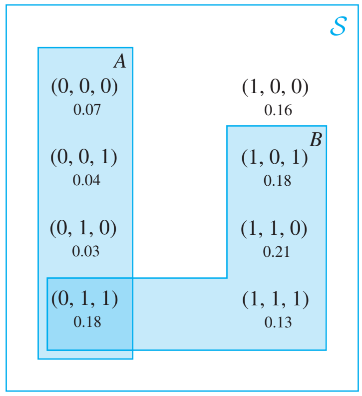


FIGURE 1.37

The event $A \cup B$

Still another way is to notice that the complement of the event $A \cup B$ consists of the single outcome (1, 0, 0), which has a probability value of 0.16, so that

$$P(A \cup B) = 1 - P((A \cup B)') = 1 - P((1,0,0)) = 1 - 0.16 = 0.84$$

Example 5

Television Set Quality

A company that manufactures television sets performs a final quality check on each appliance before packing and shipping it. The quality check has two components, the first being an evaluation of the quality of the *picture* obtained on the television set, and the second being an evaluation of the *appearance* of the television set, which looks for scratches or other visible deformities on the appliance. Each of the two evaluations is graded as *Perfect*, *Good*, *Satisfactory*, or *Fail*. The 16 outcomes are illustrated in Figure 1.38 together with a set of probability values, where the notation (P, G), for example, means that an appliance has a *Perfect* picture and a *Good* appearance.

The company has decided that an appliance that fails on either of the two evaluations will not be shipped. Furthermore, as an additional conservative measure to safeguard its reputation, it has decided that appliances that score an evaluation of *Satisfactory* on *both* accounts will also not be shipped.

An initial question of interest concerns the probability that an appliance cannot be shipped. This event A, say, consists of the outcomes

$$A = \{(F,P), (F,G), (F,S), (F,F), (P,F), (G,F), (S,F), (S,S)\}$$

as illustrated in Figure 1.39. The probability that an appliance cannot be shipped is then

$$\begin{aligned} P(A) &= P((F,P)) + P((F,G)) + P((F,S)) + P((F,F)) + P((P,F)) \\ &\quad + P((G,F)) + P((S,F)) + P((S,S)) \\ &= 0.004 + 0.011 + 0.009 + 0.008 + 0.007 + 0.012 + 0.010 + 0.013 \\ &= 0.074 \end{aligned}$$

In the long run about 7.4% of the television sets will fail the quality check.

From a technical point of view, the company is also interested in the probability that an appliance has a picture that is graded as either *Satisfactory* or *Fail*. This event B, say, is

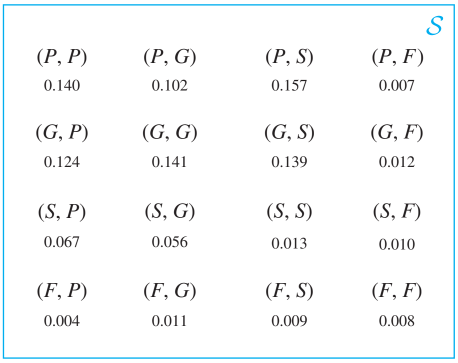


FIGURE 1.38

Probability values for television set example

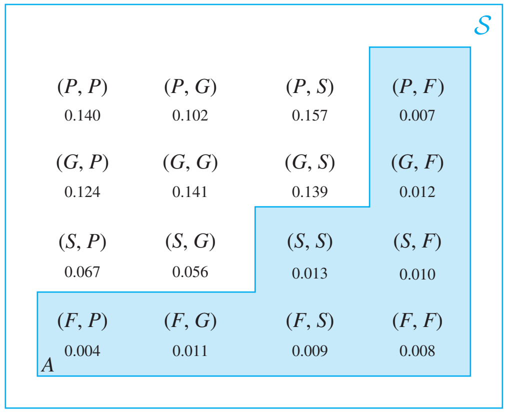


FIGURE 1.39

Event A: appliance not shipped

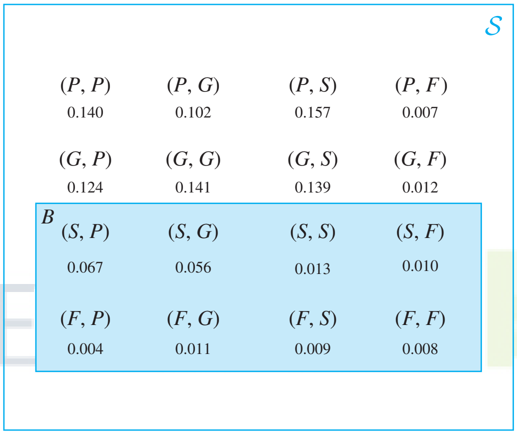


FIGURE 1.40

Event B: picture *Satisfactory* or *Fail*

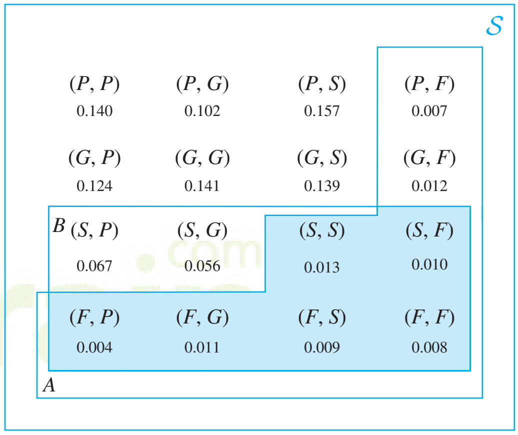


FIGURE 1.41

Event $A \cap B$

illustrated in Figure 1.40, and it has a probability of

$$\begin{aligned}
P(B) &= P((F, P)) + P((F, G)) + P((F, S)) + P((F, F)) + P((S, P)) \\
&\quad + P((S, G)) + P((S, S)) + P((S, F)) \\
&= 0.004 + 0.011 + 0.009 + 0.008 + 0.067 + 0.056 + 0.013 + 0.010 \\
&= 0.178
\end{aligned}$$

The event $A \cap B$ consists of outcomes where the appliance is not shipped *and* the picture is evaluated as being either *Satisfactory* or *Fail*. It contains the six outcomes illustrated in Figure 1.41, and it has a probability of

$$\begin{aligned}
P(A \cap B) &= P((F, P)) + P((F, G)) + P((F, S)) + P((F, F)) \\
&\quad + P((S, S)) + P((S, F)) \\
&= 0.004 + 0.011 + 0.009 + 0.008 + 0.013 + 0.010 = 0.055
\end{aligned}$$

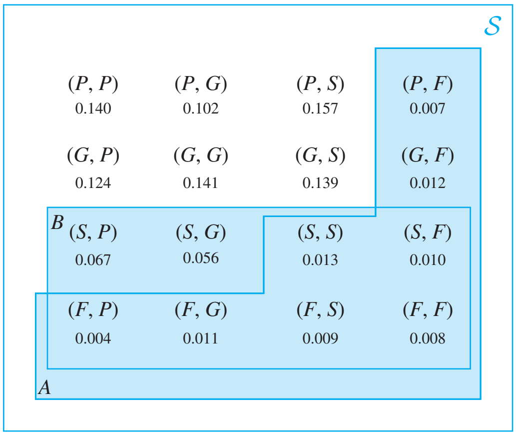


FIGURE 1.42

Event $A \cup B$

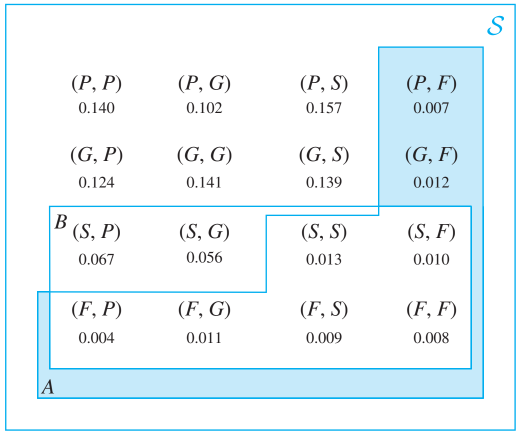


FIGURE 1.43

Event $A \cap B'$

The event $A \cup B$ consists of outcomes where the appliance was *either* not shipped *or* the picture was evaluated as being either *Satisfactory* or *Fail*. It contains the 10 outcomes illustrated in Figure 1.42, and its probability can be obtained either by summing the individual probability values of these ten outcomes or more simply as

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$= 0.074 + 0.178 - 0.055 = 0.197$$

Television sets that have a picture evaluation of either *Perfect* or *Good* but that cannot be shipped constitute the event $A \cap B'$. This event is illustrated in Figure 1.43 and consists of the outcomes

$$A \cap B' = \{(P, F), (G, F)\}$$

It has a probability of

$$P(A \cap B') = P((P, F)) + P((G, F)) = 0.007 + 0.012 = 0.019$$

Notice that

$$P(A \cap B) + P(A \cap B') = 0.055 + 0.019 = 0.074 = P(A)$$

as expected.

GAMES OF CHANCE

The event A that an even score is obtained from a roll of a die is

$$A = \{2, 4, 6\}$$

If the event B, a high score, is defined to be

$$B = \{4, 5, 6\}$$

then

$$A \cap B = \{4, 6\} \qquad \text{and} \qquad A \cup B = \{2, 4, 5, 6\}$$

If a fair die is used, then $P(A \cap B) = 2/6 = 1/3$, and $P(A \cup B) = 4/6 = 2/3$.

If two dice are thrown, recall that Figure 1.18 illustrates the event A, that the sum of the scores is equal to 6, and Figure 1.19 illustrates the event B, that at least one of the two dice records a 6. If all the outcomes are equally likely with a probability of $1/36$, then $P(A) = 5/36$ and $P(B) = 11/36$. Since there are no outcomes in both events A and B,

$$A \cap B = \emptyset$$

and $P(A \cap B) = 0$. Consequently, the events A and B are mutually exclusive.

The event $A \cup B$ consists of the five outcomes in event A together with the 11 outcomes in event B, and its probability is

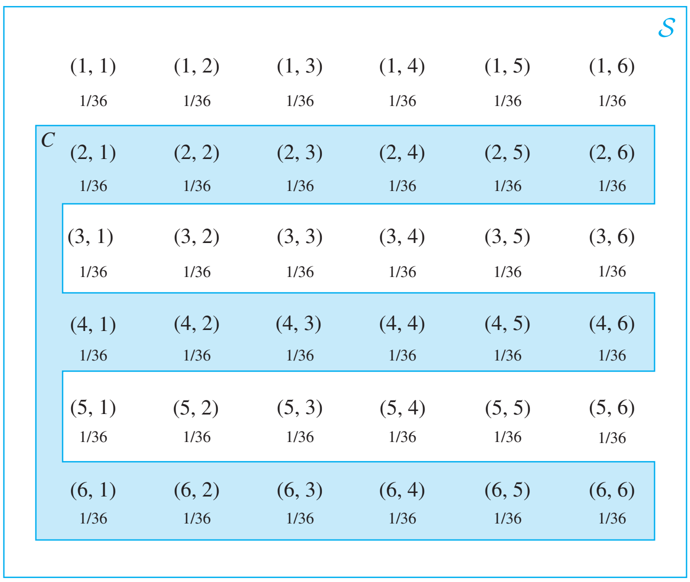
$$P(A \cup B) = \frac{16}{36} = \frac{4}{9} = P(A) + P(B)$$

If one die is red and the other is blue, then Figure 1.44 illustrates the event C, say, that an even score is obtained on the red die, and Figure 1.45 illustrates the event D, say, that an even score is obtained on the blue die. Figure 1.46 then illustrates the event $C \cap D$, which is the event that both dice have even scores. If all outcomes are equally likely, then this event has a probability of $9/36 = 1/4$. Figure 1.47 illustrates the event $C \cup D$, the event that at least one die has an even score. This event has a probability of $27/36 = 3/4$. Notice that $(C \cup D)'$, the complement of the event $C \cup D$, is just the event that both dice have odd scores.

Recall that Figure 1.20 illustrates the event A, that a card drawn from a pack of cards belongs to the heart suit, and Figure 1.21 illustrates the event B, that a picture card is drawn.

FIGURE 1.44

Event C: even score on red die



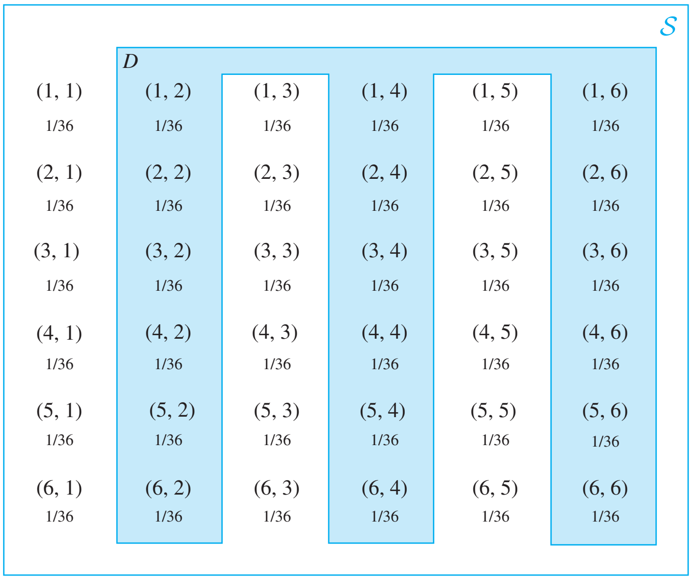


FIGURE 1.45

Event D: even score on blue die

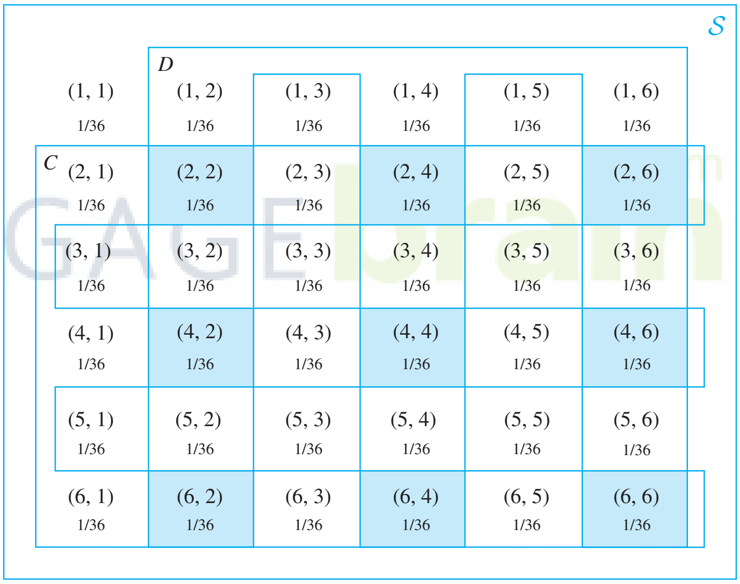


FIGURE 1.46

Event $C \cap D$

If all outcomes are equally likely, then $P(A) = 13/52 = 1/4$, and $P(B) = 12/52 = 3/13$. Figure 1.48 then illustrates the event $A \cap B$, which is the event that a picture card from the heart suit is drawn. This has a probability of $3/52$. Figure 1.49 illustrates the event $A \cup B$, the event that either a heart or a picture card (or both) is drawn, which has a probability of $22/52 = 11/26$. Notice that, as expected,

$$P(A) + P(B) - P(A \cap B) = \frac{13}{52} + \frac{12}{52} - \frac{3}{52} = \frac{22}{52} = P(A \cup B)$$

FIGURE 1.47

Event $C \cup D$

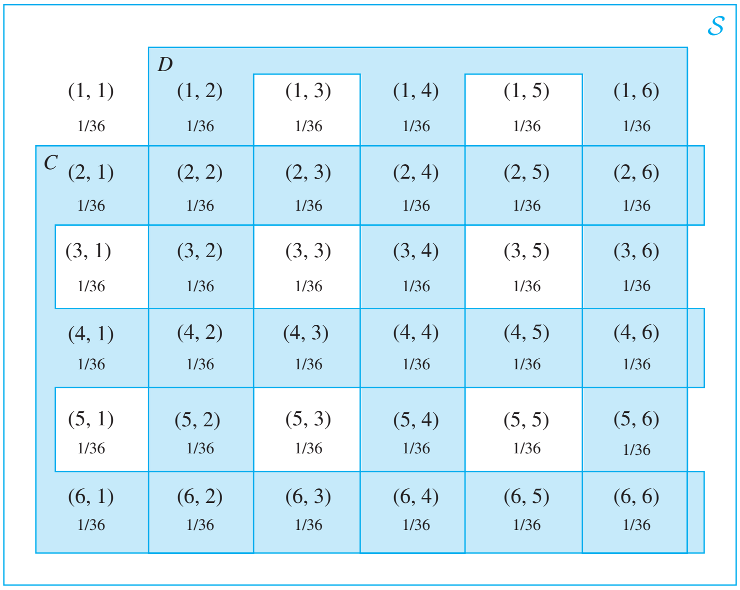


FIGURE 1.48

Event $A \cap B$

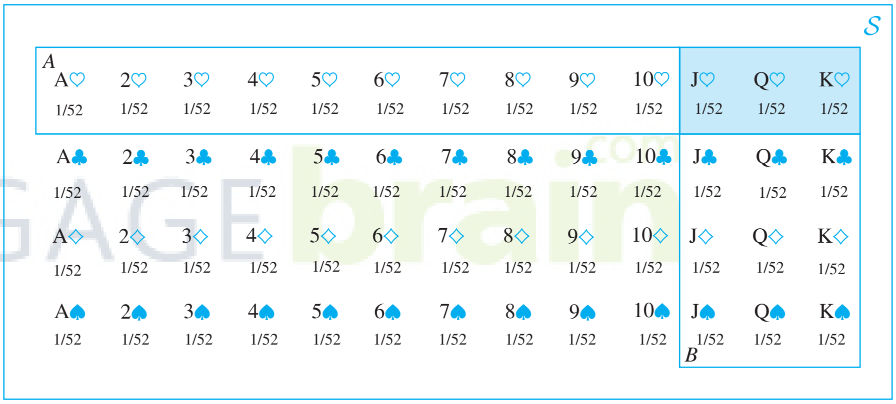


FIGURE 1.49

Event $A \cup B$

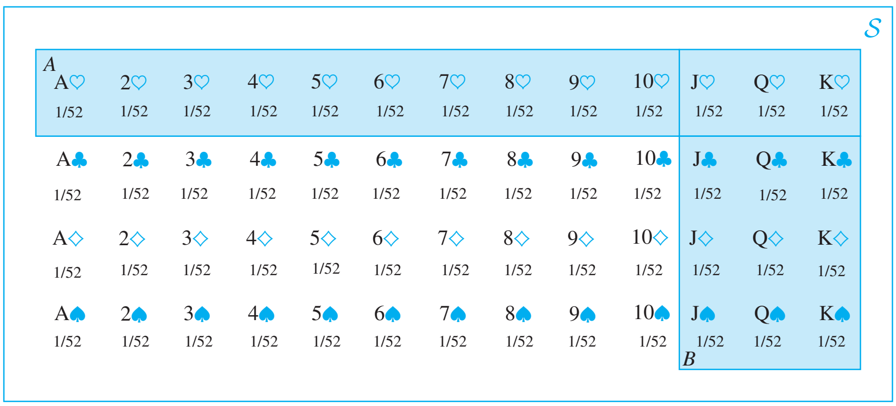


FIGURE 1.50

Event $A' \cap B$

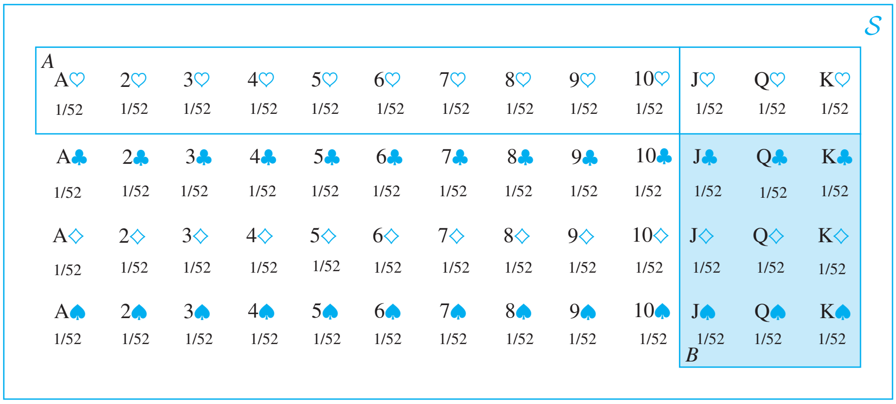
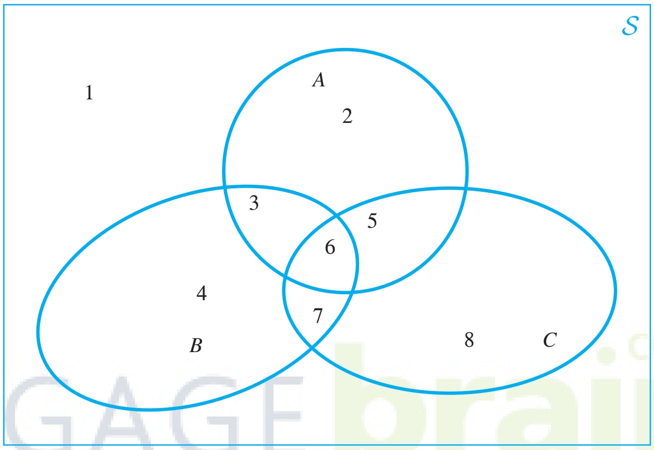


FIGURE 1.51

Three events decompose the sample space into eight regions



Finally, Figure 1.50 illustrates the event $A' \cap B$, which is the event that a picture card from a suit other than the heart suit is drawn. It has a probability of $9/52$. Again, notice that

$$P(A \cap B) + P(A' \cap B) = \frac{3}{52} + \frac{9}{52} = \frac{12}{52} = P(B)$$

as expected.

1.3.4 Combinations of Three or More Events

Intersections and unions can be extended in an obvious manner to three or more events. Figure 1.51 illustrates how three events A, B, and C can divide a sample space into eight distinct and separate regions. The event A, for example, is composed of the regions 2, 3, 5, and 6, and the event $A \cap B$ is composed of the regions 3 and 6.

The event $A \cap B \cap C$, the intersection of the events A, B, and C, consists of the outcomes that are simultaneously contained within all three events A, B, and C. In Figure 1.51 it corresponds to region 6. The event $A \cup B \cup C$, the union of the events A, B, and C, consists of the outcomes that are in at least one of the three events A, B, and C. In Figure 1.51 it corresponds to all of the regions except for region 1. Hence region 1 can be referred to as $(A \cup B \cup C)'$ since it is the complement of the event $A \cup B \cup C$.

In general, care must be taken to avoid ambiguities when specifying combinations of three or more events. For example, the expression

$$A \cup B \cap C$$

is ambiguous since the two events

$$A \cup (B \cap C) \qquad \text{and} \qquad (A \cup B) \cap C$$

are different. In Figure 1.51 the event $B \cap C$ is composed of regions 6 and 7, so $A \cup (B \cap C)$ is composed of regions 2, 3, 5, 6, and 7. In contrast, the event $A \cup B$ is composed of regions 2, 3, 4, 5, 6, and 7, so $(A \cup B) \cap C$ is composed of just regions 5, 6, and 7.

Figure 1.51 can also be used to justify the following general expression for the probability of the union of three events:

Union of Three Events

The probability of the **union of three events** A, B, and C is the sum of the probability values of the simple outcomes that are contained within at least one of the three events. It can also be calculated from the expression

$$P(A \cup B \cup C) = [P(A) + P(B) + P(C)] \\ - [P(A \cap B) + P(A \cap C) + P(B \cap C)] + P(A \cap B \cap C)$$

The expression for $P(A \cup B \cup C)$ can be checked by matching up the regions in Figure 1.51 with the various terms in the expression. The required probability, $P(A \cup B \cup C)$, is the sum of the probability values of the outcomes in regions 2, 3, 4, 5, 6, 7, and 8. However, the sum of the probabilities $P(A)$, $P(B)$, and $P(C)$ counts regions 3, 5, and 7 *twice*, and region 6 *three* times. Subtracting the probabilities $P(A \cap B)$, $P(A \cap C)$, and $P(B \cap C)$ removes the double counting of regions 3, 5, and 7 but also subtracts the probability of region 6 *three* times. The expression is then completed by adding back on $P(A \cap B \cap C)$, the probability of region 6.

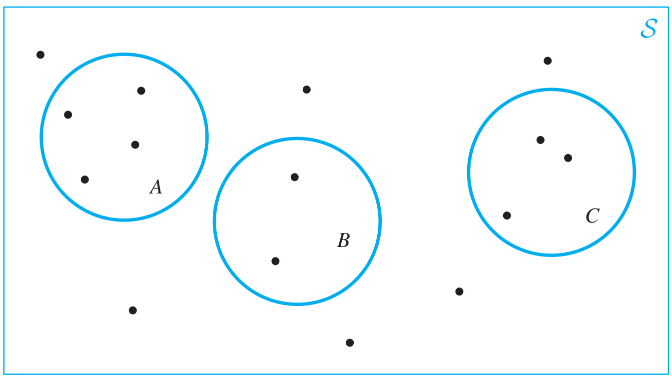
Figure 1.52 illustrates three events A, B, and C that are mutually exclusive because no two events have any outcomes in common. In this case,

$$P(A \cup B \cup C) = P(A) + P(B) + P(C)$$

because the event intersections all have probabilities of zero. More generally, for a sequence $A_1, A_2, \ldots, A_n$ of mutually exclusive events where no two of the events have any outcomes

FIGURE 1.52

Three mutually exclusive events



in common, the probability of the union of the events can be obtained by summing the probabilities of the individual events.

Union of Mutually Exclusive Events

For a sequence $A_1, A_2, \ldots, A_n$ of **mutually exclusive events**, the probability of the **union** of the events is given by

$$P(A_1 \cup \cdots \cup A_n) = P(A_1) + \cdots + P(A_n)$$

If a sequence $A_1, A_2, \ldots, A_n$ of mutually exclusive events has the additional property that their union consists of the whole sample space $\mathcal{S}$, then they are said to be an **exhaustive** sequence. They are also said to provide a **partition** of the sample space.

Sample Space Partitions

A **partition** of a sample space is a sequence $A_1, A_2, \ldots, A_n$ of *mutually exclusive* events for which

$$A_1 \cup \cdots \cup A_n = \mathcal{S}$$

Each outcome in the sample space is then contained within one and only one of the events A_i.

Figure 1.53 illustrates a partition of a sample space $\mathcal{S}$ into eight mutually exclusive events.

Example 5

Television Set Quality

In addition to the events A and B discussed before, consider also the event C that an appliance is of "mediocre quality." The event is defined to be appliances that score either *Satisfactory* or *Good* on each of the two evaluations, so that

$$C = \{(S, S), (S, G), (G, S), (G, G)\}$$

The three events A, B, and C are illustrated in Figure 1.54.

Notice that

$$A \cap C = \{(S, S)\} \qquad \text{and} \qquad B \cap C = \{(S, S), (S, G)\}$$

FIGURE 1.53

A partition of the sample space

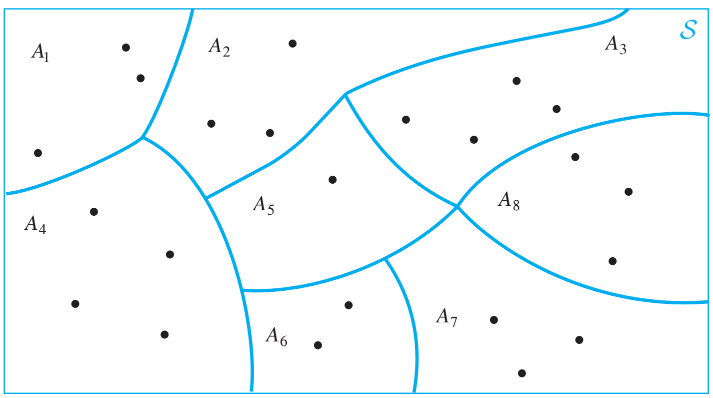
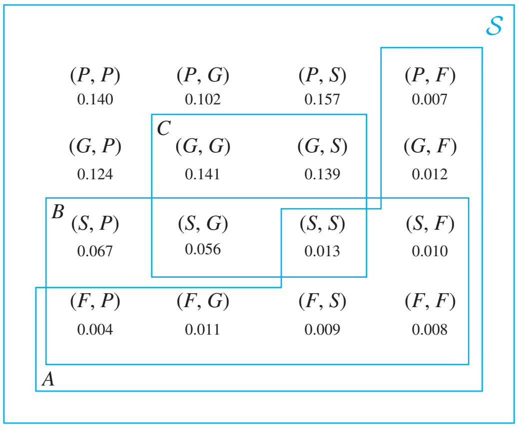


FIGURE 1.54

Events A, B, and C



Also, the intersection of the three events is

$$A \cap B \cap C = \{(S, S)\}$$

The fact that the events $A \cap C$ and $A \cap B \cap C$ are identical, both consisting of the single outcome (S, S), is a consequence of the fact that $A \cap B' \cap C = \emptyset$. There are no outcomes that are not shipped (in event A), have a picture rating of *Good* or *Perfect* (in event B'), and are of mediocre quality (in event C).

The company may be particularly interested in the event D, that an appliance is of "high quality," defined to be the complement of the union of the events A, B, and C:

$$D = (A \cup B \cup C)'$$

Notice that this event can also be written as

$$D = A' \cap B' \cap C'$$

since it consists of the outcomes that are shipped (in event A'), have a picture rating of *Good* or *Perfect* (in event B'), and are not of mediocre quality (in event C'). Specifically, the event D consists of the outcomes

$$D = \{(G, P), (P, P), (P, G), (P, S)\}$$

and it has a probability of

$$\begin{aligned} P(D) &= P((G, P)) + P((P, P)) + P((P, G)) + P((P, S)) \\ &= 0.124 + 0.140 + 0.102 + 0.157 = 0.523 \end{aligned}$$

1.3.5 Problems

1.3.1 Consider the sample space $\mathcal{S} = \{0, 1, 2\}$ and the event $A = \{0\}$. Explain why $A \neq \emptyset$.

1.3.2 Consider the sample space and events in Figure 1.55. Calculate the probabilities of the events:

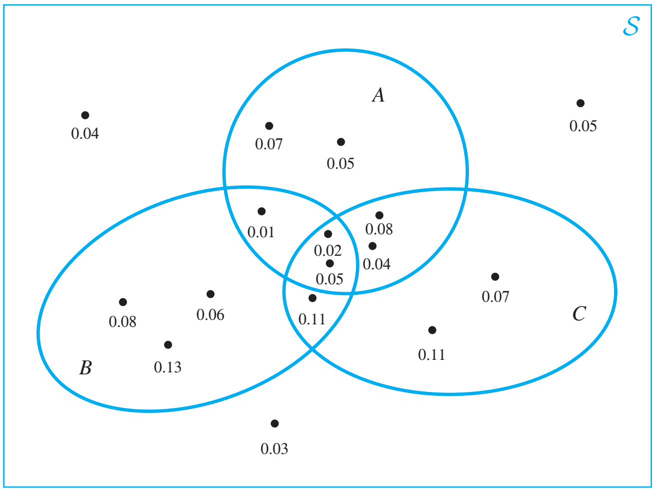
(a) B **(b)** $B \cap C$
(c) $A \cup C$ **(d)** $A \cap B \cap C$
(e) $A \cup B \cup C$ **(f)** $A' \cap B$
(g) $B' \cup C$ **(h)** $A \cup (B \cap C)$
(i) $(A \cup B) \cap C$ **(j)** $(A' \cup C)'$

(This problem is continued in Problem 1.4.1.)

1.3.3 Use Venn diagrams to illustrate the equations:

(a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

FIGURE 1.55



(b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
(c) $(A \cap B \cap C)' = A' \cup B' \cup C'$

1.3.4 Let A be the event that a person is *female*, let B be the event that a person has *black hair*, and let C be the event that a person has *brown eyes*. Describe the kinds of people in the following events:
(a) $A \cap B$ **(b)** $A \cup C'$
(c) $A' \cap B \cap C$ **(d)** $A \cap (B \cup C)$

1.3.5 A card is chosen from a pack of cards. Are the events that a card from one of the two red suits is chosen and that a card from one of the two black suits is chosen mutually exclusive? What about the events that an Ace is chosen and that a heart is chosen?

1.3.6 If $P(A) = 0.4$ and $P(A \cap B) = 0.3$, what are the possible values for $P(B)$?

1.3.7 If $P(A) = 0.5$, $P(A \cap B) = 0.1$, and $P(A \cup B) = 0.8$, what is $P(B)$?

1.3.8 A fair die is thrown. A is the event that an *even* score is obtained, and B is the event that a *prime* score is obtained. Give the probabilities:
(a) $A \cap B$ **(b)** $A \cup B$ **(c)** $A \cap B'$

1.3.9 A card is drawn at random from a pack of cards. A is the event that a heart is obtained, B is the event that a club is obtained, and C is the event that a diamond is obtained. Are these three events mutually exclusive? What is $P(A \cup B \cup C)$? Explain why $B \subset A'$.

1.3.10 A card is drawn from a pack of cards. A is the event that an Ace is obtained, B is the event that a card from one of the two red suits is obtained, and C is the event that a picture card is obtained. What cards do the following events consist of?
(a) $A \cap B$ **(b)** $A \cup C$
(c) $B \cap C'$ **(d)** $A \cup (B' \cap C)$

1.3.11 A car repair can be performed either on time or late and either satisfactorily or unsatisfactorily. The probability of a repair being on time and satisfactory is 0.26. The probability of a repair being on time is 0.74. The probability of a repair being satisfactory is 0.41. What is the probability of a repair being late and unsatisfactory?

1.3.12 A bag contains 200 balls that are either red or blue and either dull or shiny. There are 55 shiny red balls, 91 shiny balls, and 79 red balls. If a ball is chosen at random, what is the probability that it is either a shiny ball or a red ball? What is the probability that it is a dull blue ball?

1.3.13 In a study of patients arriving at a hospital emergency room, the gender of the patients is considered, together with whether the patients are younger or older than 30 years of age, and whether or not the patients are admitted to the hospital. It is found that 45% of the patients are male, 30% of the patients are younger than 30 years of age, 15% of the patients are females older than 30 years of age who are admitted to the hospital, and 21% of the patients are females younger than 30 years of age. What proportion of the patients are females older than 30 years of age who are not admitted to the hospital?

1.4 Conditional Probability

1.4.1 Definition of Conditional Probability

For experiments with two or more events of interest, attention is often directed not only at the probabilities of individual events, but also at the probability of an event occurring **conditional** on the knowledge that another event has occurred. Probabilities such as these are important and very useful since they provide appropriate **revisions** of a set of probabilities once a particular event is known to have occurred.

The probability that event A occurs conditional on event B having occurred is written

$$P(A|B)$$

Its interpretation is that if the outcome occurring is known to be contained within the event B, then this **conditional probability** measures the probability that the outcome is also contained within the event A. Conditional probabilities can easily be obtained using the following formula:

Conditional Probability

The **conditional probability** of event A conditional on event B is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

for $P(B) > 0$. It measures the probability that event A occurs when it is known that event B occurs.

One simple example of conditional probability concerns the situation in which two events A and B are mutually exclusive. Since in this case events A and B have no outcomes in common, it is clear that the occurrence of event B precludes the possibility of event A occurring, so that intuitively, the probability of event A conditional on event B must be zero. Since $A \cap B = \emptyset$ for mutually exclusive events, this intuitive reasoning is in agreement with the formula

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0}{P(B)} = 0$$

Another simple example of conditional probability concerns the situation in which an event B is contained within an event A, that is $B \subset A$. Then if event B occurs, it is clear that event A must also occur, so that intuitively, the probability of event A conditional on event B must be one. Again, since $A \cap B = B$ here, this intuitive reasoning is in agreement with the formula

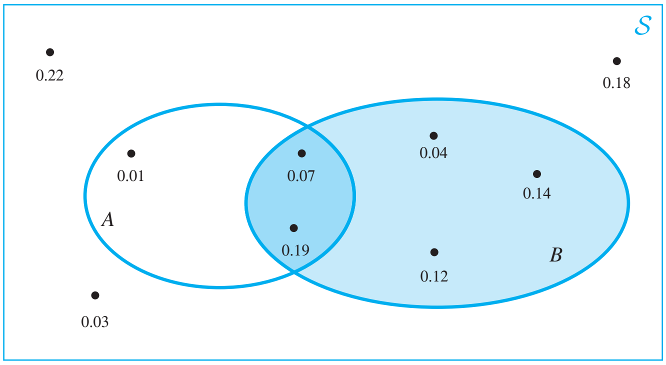
$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$$

For a less obvious example of conditional probability, consider again Figure 1.26 and the events A and B shown there. Suppose that event B is known to occur. In other words, suppose that it is known that the outcome occurring is one of the five outcomes contained within the event B. What then is the conditional probability of event A occurring?

Since two of the five outcomes in event B are also in event A (that is, there are two outcomes in $A \cap B$), the conditional probability is the probability that one of these two outcomes occurs

FIGURE 1.56

$P(A|B) = P(A \cap B)/P(B)$



rather than one of the other three outcomes (which are in $A' \cap B$). As Figure 1.56 shows, the conditional probability is calculated to be

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.26}{0.56} = 0.464$$

Notice that this conditional probability is different from $P(A) = 0.27$. If the event B is known *not* to occur, then the conditional probability of event A is

$$P(A|B') = \frac{P(A \cap B')}{P(B')} = \frac{P(A) - P(A \cap B)}{1 - P(B)} = \frac{0.27 - 0.26}{1 - 0.56} = 0.023$$

In the same way that $P(A) + P(A') = 1$, it is also true that

$$P(A|B) + P(A'|B) = 1$$

This is reasonable because if event B occurs, it is still the case that either event A occurs or it does not, and so the two conditional probabilities should sum to one. Formally, this result can be shown by noting that

$$P(A|B) + P(A'|B) = \frac{P(A \cap B)}{P(B)} + \frac{P(A' \cap B)}{P(B)}$$
$$= \frac{1}{P(B)}(P(A \cap B) + P(A' \cap B)) = \frac{1}{P(B)}P(B) = 1$$

However, there is no general relationship between $P(A|B)$ and $P(A|B')$.

Finally, the event conditioned on can be represented as a combination of events. For example,

$$P(A|B \cup C)$$

represents the probability of event A conditional on the event $B \cup C$, that is conditional on either event B or C occurring. It can be calculated from the formula

$$P(A|B \cup C) = \frac{P(A \cap (B \cup C))}{P(B \cup C)}$$

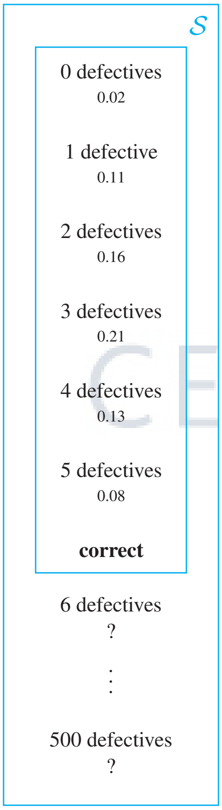


FIGURE 1.57

Sample space for computer chips example

1.4.2 Examples of Conditional Probabilities

Example 2

Defective Computer Chips

Consider Figure 1.57 that illustrates the sample space for the number of defective chips in a box of 500 chips, and recall that the event *correct*, with a probability of $P(\text{correct}) = 0.71$, consists of the six outcomes corresponding to no more than five defectives.

The probability that a box has no defective chips is

$$P(0 \text{ defectives}) = 0.02$$

so that if a box is chosen at random, it has a probability of only 0.02 of containing no defective chips. If the company guarantees that a box has no more than five defective chips, then customers can be classified as either satisfied or unsatisfied depending on whether the guarantee is met. Clearly, an unsatisfied customer did not purchase a box containing no defective chips. However, it is interesting to calculate the probability that a satisfied customer purchased a box that contained no defective chips. Intuitively, this conditional probability should be larger than the unconditional probability 0.02.

The required probability is the probability of no defectives conditional on there being no more than five defectives, which is calculated to be

$$P(0 \text{ defectives}|\text{correct}) = \frac{P(0 \text{ defectives} \cap \text{correct})}{P(\text{correct})} = \frac{P(0 \text{ defectives})}{P(\text{correct})}$$
$$= \frac{0.02}{0.71} = 0.028$$

This conditional probability indicates that whereas 2% of all the boxes contain no defectives, 2.8% of the satisfied customers purchased boxes that contained no defectives.

Example 4

Power Plant Operation

The probability that plant X is idle is $P(A) = 0.32$. However, suppose it is known that at least two out of the three plants are generating electricity (event B). How does this change the probability of plant X being idle?

The probability that plant X is idle (event A) conditional on at least two out of the three plants generating electricity (event B) is

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.18}{0.70} = 0.257$$

as shown in Figure 1.58. Therefore, whereas plant X is idle about 32% of the time, it is idle only about 25.7% of the time when at least two of the plants are generating electricity.

Example 5

Television Set Quality

Recall that the probability that an appliance has a picture graded as either *Satisfactory* or *Fail* is $P(B) = 0.178$. However, suppose that a technician takes a television set from a pile of sets that could not be shipped. What is the probability that the appliance taken by the technician has a picture graded as either *Satisfactory* or *Fail*?

The required probability is the probability that an appliance has a picture graded as either *Satisfactory* or *Fail* (event B) conditional on the appliance not being shipped (event A). As Figure 1.59 shows, this can be calculated to be

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{0.055}{0.074} = 0.743$$

so that whereas only about 17.8% of all the appliances manufactured have a picture graded as either *Satisfactory* or *Fail*, 74.3% of the appliances that cannot be shipped have a picture graded as either *Satisfactory* or *Fail*.

GAMES OF CHANCE

If a fair die is rolled the probability of scoring a 6 is $P(6) = 1/6$. If somebody rolls a die without showing you but announces that the result is *even*, then intuitively the chance that a 6 has been obtained is 1/3 since there are three equally likely even scores, one of which is a 6.

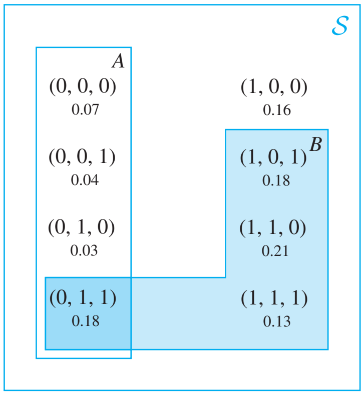


FIGURE 1.58

$P(A|B) = P(A \cap B)/P(B)$

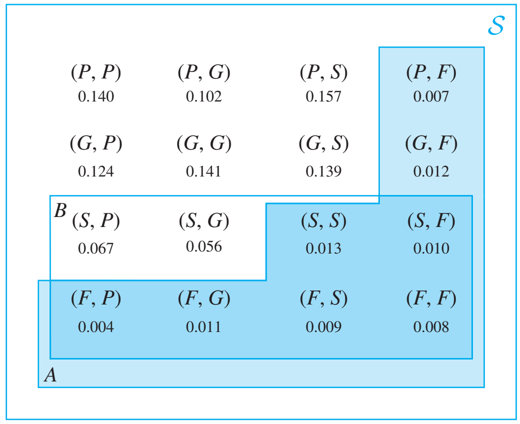


FIGURE 1.59

$P(B|A) = P(A \cap B)/P(A)$

Mathematically, this conditional probability is calculated to be

$$P(6|\text{even}) = \frac{P(6 \cap \text{even})}{P(\text{even})} = \frac{P(6)}{P(\text{even})}$$
$$= \frac{P(6)}{P(2) + P(4) + P(6)} = \frac{1/6}{1/6 + 1/6 + 1/6} = \frac{1}{3}$$

as expected.

If a red die and a blue die are thrown, with each of the 36 outcomes being equally likely, let A be the event that the red die scores a 6, so that

$$P(A) = \frac{6}{36} = \frac{1}{6}$$

Also, let B be the event that at least one 6 is obtained on the two dice (see Figure 1.19) with a probability of

$$P(B) = \frac{11}{36}$$

Suppose that somebody rolls the two dice without showing you but announces that at least one 6 has been scored. What then is the probability that the red die scored a 6? As Figure 1.60 shows, this conditional probability is calculated to be

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)} = \frac{1/6}{11/36} = \frac{6}{11}$$

As expected, this conditional probability is larger than $P(A) = 1/6$. It is also slightly larger than 0.5, which is accounted for by the outcome (6, 6) where both dice score a 6.

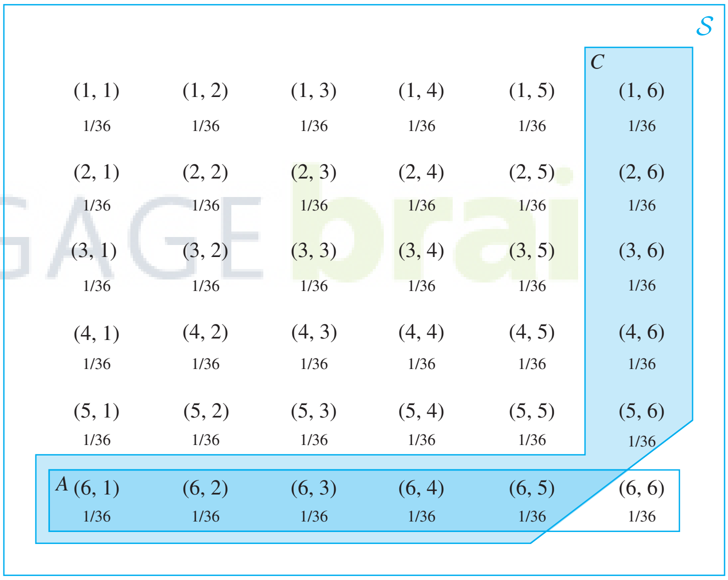
Contrast this problem with the situation where the announcement is that *exactly* one 6 has been scored, event C, say. In this case, it is intuitively clear that the 6 obtained is equally likely to have been scored on the red die or the blue die, so that the conditional probability $P(A|C)$

FIGURE 1.60

$P(A|B) = P(A \cap B)/P(B)$

FIGURE 1.61

$P(A|C) = P(A \cap C)/P(C)$



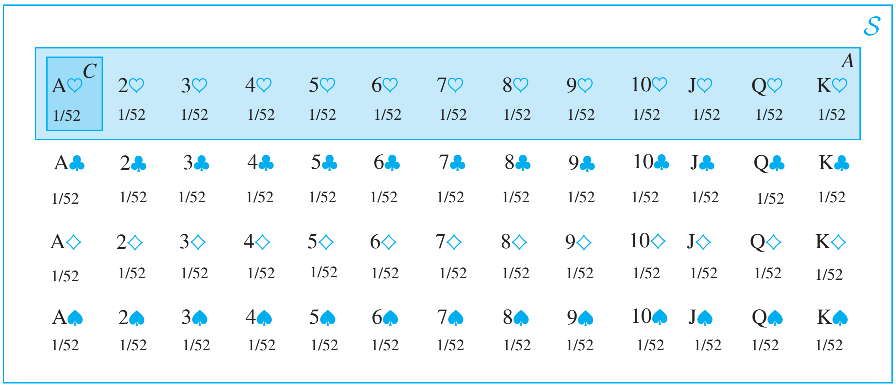
should be equal to $1/2$. As Figure 1.61 shows, this is correct since

$$P(A|C) = \frac{P(A \cap C)}{P(C)} = \frac{5/36}{10/36} = \frac{1}{2}$$

If a card is drawn from a pack of cards, let A be the event that a card from the heart suit is obtained, and let B be the event that a picture card is drawn. Recall that $P(A) = 13/52 = 1/4$ and $P(B) = 12/52 = 3/13$. Also, the event $A \cap B$, the event that a picture card from the heart suit is drawn, has a probability of $P(A \cap B) = 3/52$.

FIGURE 1.62

$P(C|A) = P(C \cap A)/P(A)$



Suppose that somebody draws a card and announces that it is from the heart suit. What then is the probability that it is a picture card? This conditional probability is

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{3/52}{1/4} = \frac{3}{13}$$

Notice that in this case, $P(B|A) = P(B)$ because the proportion of picture cards in the heart suit is identical to the proportion of picture cards in the whole pack. The events A and B are then said to be **independent** events, which are discussed more fully in Section 1.5.

Finally, let C be the event that the $A\heartsuit$ is chosen, with $P(C) = 1/52$. If it is known that a card from the heart suit is obtained, then intuitively the conditional probability of the card being $A\heartsuit$ is $1/13$ since there are 13 equally likely cards in the heart suit. As Figure 1.62 shows, this is correct because

$$P(C|A) = \frac{P(C \cap A)}{P(A)} = \frac{P(C)}{P(A)} = \frac{1/52}{1/4} = \frac{1}{13}$$

1.4.3 Problems

1.4.1 Consider again Figure 1.55 and calculate the probabilities:
(a) $P(A|B)$
(b) $P(C|A)$
(c) $P(B|A \cap B)$
(d) $P(B|A \cup B)$
(e) $P(A|A \cup B \cup C)$
(f) $P(A \cap B|A \cup B)$

1.4.2 Let A be the event that a *prime* number is obtained from the roll of a fair die. Calculate $P(5|A)$, $P(6|A)$, and $P(A|5)$.

1.4.3 A card is drawn at random from a pack of cards. Calculate:
(a) $P(A\heartsuit|\text{card from red suit})$
(b) $P(\text{heart}|\text{card from red suit})$
(c) $P(\text{card from red suit}|\text{heart})$
(d) $P(\text{heart}|\text{card from black suit})$
(e) $P(\text{King}|\text{card from red suit})$
(f) $P(\text{King}|\text{red picture card})$

1.4.4 If $A \subset B$ and $B' \neq \emptyset$, is $P(A)$ larger or smaller than $P(A|B)$? Provide some intuitive reasoning for your answer.

1.4.5 A ball is chosen at random from a bag containing 150 balls that are either red or blue and either dull or shiny. There are 36 red shiny balls and 54 blue balls. What is the probability of the chosen ball being shiny conditional on it being red? What is the probability of the chosen ball being dull conditional on it being red?

1.4.6 A car repair is either on time or late and either satisfactory or unsatisfactory. If a repair is made on time, then there is a probability of 0.85 that it is satisfactory.

There is a probability of 0.77 that a repair will be made on time. What is the probability that a repair is made on time and is satisfactory?

1.4.7 Assess whether the probabilities of the events (i) increase, decrease, or remain unchanged when they are conditioned on the events (ii).

(a) (i) It rains tomorrow, (ii) it is raining today.

(b) (i) A lottery winner has black hair, (ii) the lottery winner has brown eyes.

(c) (i) A lottery winner has black hair, (ii) the lottery winner owns a red car.

(d) (i) A lottery winner is more than 50 years old, (ii) the lottery winner is more than 30 years old.

1.4.8 Suppose that births are equally likely to be on any day. What is the probability that somebody chosen at random has a birthday on the *first day* of a month? How does this probability change conditional on the knowledge that the person's birthday is in March? In February?

1.4.9 Consider again Figure 1.24 and the battery lifetimes. Calculate the probabilities:

(a) A type I battery lasts longest conditional on it not failing first

(b) A type I battery lasts longest conditional on a type II battery failing first

(c) A type I battery lasts longest conditional on a type II battery lasting the longest

(d) A type I battery lasts longest conditional on a type II battery not failing first

1.4.10 Consider again Figure 1.25 and the two assembly lines. Calculate the probabilities:

(a) Both lines are at full capacity conditional on neither line being shut down

(b) At least one line is at full capacity conditional on neither line being shut down

(c) One line is at full capacity conditional on exactly one line being shut down

(d) Neither line is at full capacity conditional on at least one line operating at partial capacity

1.4.11 The length, width, and height of a manufactured part are classified as being either within or outside specified tolerance limits. In a quality inspection 86% of the parts are found to be within the specified tolerance limits for width, but only 80% of the parts are within the specified tolerance limits for all three dimensions. However, 2% of the parts are within the specified tolerance limits for width and length but not for height, and 3% of the parts are within the specified tolerance limits for width and height but not for length. Moreover, 92% of the parts are within the specified tolerance limits for either width or height, or both of these dimensions.

(a) If a part is within the specified tolerance limits for height, what is the probability that it will also be within the specified tolerance limits for width?

(b) If a part is within the specified tolerance limits for length and width, what is the probability that it will be within the specified tolerance limits for all three dimensions?

1.4.12 A gene can be either type A or type B, and it can be either dominant or recessive. If the gene is type B, then there is a probability of 0.31 that it is dominant. There is also a probability of 0.22 that a gene is type B and it is dominant. What is the probability that a gene is of type A?

1.4.13 A manufactured component has its quality graded on its performance, appearance, and cost. Each of these three characteristics is graded as either pass or fail. There is a probability of 0.40 that a component passes on both appearance and cost. There is a probability of 0.31 that a component passes on all three characteristics. There is a probability of 0.64 that a component passes on performance. There is a probability of 0.19 that a component fails on all three characteristics. There is a probability of 0.06 that a component passes on appearance but fails on both performance and cost.

(a) What is the probability that a component passes on cost but fails on both performance and appearance?

(b) If a component passes on both appearance and cost, what is the probability that it passes on all three characteristics?

1.4.14 An agricultural research establishment grows vegetables and grades each one as either good or bad for its taste, good or bad for its size, and good or bad for its appearance. Overall 78% of the vegetables have a good taste. However, only 69% of the vegetables have both a good taste and a good size. Also, 5% of the vegetables have both a good taste and a good appearance, but a bad size. Finally, 84% of the vegetables have either a good size or a good appearance.

(a) If a vegetable has a good taste, what is the probability that it also has a good size?

(b) If a vegetable has a bad size and a bad appearance, what is the probability that it has a good taste?

1.4.15 There is a 4% probability that the plane used for a commercial flight has technical problems, and this causes a delay in the flight. If there are no technical problems with the plane, then there is still a 33% probability that the flight is delayed due to all other reasons. What is the probability that the flight is delayed?

1.4.16 In a reliability test there is a 42% probability that a computer chip survives more than 500 temperature cycles. If a computer chip does not survive more than 500 temperature cycles, then there is a 73% probability that it was manufactured by company A. What is the probability that a computer chip is not manufactured by company A and does not survive more than 500 temperature cycles?

1.5 Probabilities of Event Intersections

1.5.1 General Multiplication Law

It follows from the definition of the conditional probability $P(A|B)$ that the probability of the intersection of two events $A \cap B$ can be calculated as

$$P(A \cap B) = P(B)\, P(A|B)$$

That is, the probability of events A and B both occurring can be obtained by multiplying the probability of event B by the probability of event A conditional on event B. It also follows from the definition of the conditional probability $P(B|A)$ that

$$P(A \cap B) = P(A)\, P(B|A)$$

so that the probability of events A and B both occurring can also be obtained by multiplying the probability of event A by the probability of event B conditional on event A. Therefore, it does not matter which of the two events A or B is conditioned upon.

More generally, since

$$P(C|A \cap B) = \frac{P(A \cap B \cap C)}{P(A \cap B)}$$

the probability of the intersection of three events can be calculated as

$$P(A \cap B \cap C) = P(A \cap B)\, P(C|A \cap B) = P(A)\, P(B|A)\, P(C|A \cap B)$$

Thus, the probability of all three events occurring can be obtained by multiplying together the probability of one event, the probability of a second event conditioned on the first event, and the probability of the third event conditioned on the intersection of the first and second events. This formula can be extended in an obvious way to the following **multiplication law** for the intersection of a series of events.

Probabilities of Event Intersections

The probability of the **intersection of a series of events** $A_1, \ldots, A_n$ can be calculated from the expression

$$P(A_1 \cap \cdots \cap A_n) = P(A_1) \times P(A_2|A_1) \times P(A_3|A_1 \cap A_2) \times \\ \cdots \times P(A_n|A_1 \cap \cdots \cap A_{n-1})$$

This expression for the probability of event intersections is particularly useful when the conditional probabilities $P(A_i|A_1 \cap \cdots \cap A_{i-1})$ are easily obtainable, as illustrated in the following example.

Suppose that two cards are drawn at random *without replacement* from a pack of cards. Let A be the event that the *first* card drawn is from the heart suit, and let B be the event that the *second* card drawn is from the heart suit. What is $P(A \cap B)$, the probability that both cards are from the heart suit?

Figure 1.13 shows the sample space for this problem, which consists of 2652 equally likely outcomes, each with a probability of $1/2652$. One way to calculate $P(A \cap B)$ is to count the number of outcomes in the sample space that are contained within the event $A \cap B$, that is, for which both cards are in the heart suit. In fact,

$$A \cap B = \{(A\heartsuit, 2\heartsuit), (A\heartsuit, 3\heartsuit), \ldots, (A\heartsuit, K\heartsuit), (2\heartsuit, A\heartsuit), (2\heartsuit, 3\heartsuit), \ldots,$$
$$(2\heartsuit, K\heartsuit), \ldots (K\heartsuit, A\heartsuit), (K\heartsuit, 2\heartsuit), \ldots, (K\heartsuit, Q\heartsuit)\}$$

which consists of $13 \times 12 = 156$ outcomes. Consequently, the required probability is

$$P(A \cap B) = \frac{156}{2652} = \frac{3}{51}$$

However, a more convenient way of calculating this probability is to note that it is the product of $P(A)$ and $P(B|A)$. When the first card is drawn, there are 13 heart cards out of a total of 52 cards, so

$$P(A) = \frac{13}{52} = \frac{1}{4}$$

Conditional on the first card being a heart (event A), when the second card is drawn there will be 12 heart cards remaining in the reduced pack of 51 cards, so that

$$P(B|A) = \frac{12}{51}$$

The required probability is then

$$P(A \cap B) = P(A)\,P(B|A) = \frac{1}{4} \times \frac{12}{51} = \frac{3}{51}$$

as before.

1.5.2 Independent Events

Two events A and B are said to be **independent** events if

$$P(B|A) = P(B)$$

so that the probability of event B remains the same whether or not the event A is conditioned upon. In other words, knowledge of the occurrence (or lack of occurrence) of event A does not affect the probability of event B. In this case

$$P(A \cap B) = P(A)\,P(B|A) = P(A)\,P(B)$$

and

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A)$$

Thus, in a similar way, the probability of event A remains the same whether or not event B is conditioned upon, and the probability of both events occurring, $P(A \cap B)$, is obtained simply by multiplying together the individual probabilities of the two events $P(A)$ and $P(B)$.

Independent Events

Two events A and B are said to be **independent** events if

$$P(A|B) = P(A), \quad P(B|A) = P(B), \quad \text{and} \quad P(A \cap B) = P(A)\,P(B)$$

Any one of these three conditions implies the other two. The interpretation of two events being independent is that knowledge about one event does not affect the probability of the other event.

The concept of independence is most easily understood from a practical standpoint, with two events being independent if they are "unrelated" to each other. For example, suppose that a person is chosen at random from a large group of people, as in a lottery, for instance. Let A be the event that the person is over 6 feet tall, and let B be the event that the person weighs more than 200 pounds. Intuitively, these two events are *not* independent because knowledge of one event influences our perception of the likelihood of the other event. For example, if the lottery winner is known to be over 6 feet tall, then this fact increases the likelihood that the person weighs more than 200 pounds.

On the other hand, if event C is that the person owns a red car, then intuitively the events A and C are independent, as are the events B and C. The knowledge that the lottery winner is over 6 feet tall does not change our perception of the probability that the person owns a red car. Conversely, the knowledge that the lottery winner owns a red car does not change our perception of the probability that the person is over 6 feet tall.

From a mathematical standpoint, two events A and B can be proven to be independent by establishing one of the conditions

$$P(A|B) = P(A), \quad P(B|A) = P(B), \quad \text{or} \quad P(A \cap B) = P(A)\,P(B)$$

In practice, however, an assessment of whether two events are independent or not is usually made by the practical consideration of whether the two events are "unrelated."

Events $A_1, A_2, \ldots, A_n$ are said to be independent if conditioning on combinations of some of the events does not affect the probabilities of the other events. In this case, the expression given earlier for the probability of the intersection of the events simplifies to the product of the probabilities of the individual events.

Intersections of Independent Events

The probability of the intersection of a series of **independent events** $A_1, \ldots, A_n$ is simply given by

$$P(A_1 \cap \cdots \cap A_n) = P(A_1)\,P(A_2) \cdots P(A_n)$$

Consider again the problem discussed above where two cards are drawn from a pack of cards, and where A is the event that the *first* card drawn is from the heart suit and B is the event that the *second* card drawn is from the heart suit. Suppose that now the drawings are made *with replacement*. What is $P(A \cap B)$ in this case?

Figure 1.12 shows the sample space for this problem, which consists of 2704 equally likely outcomes, each with a probability of $1/2704$. As before, one way to calculate $P(A \cap B)$ is to count the number of outcomes in the sample space that are contained within the event

$A \cap B$. This event is now

$$A \cap B = \{(A\heartsuit, A\heartsuit), (A\heartsuit, 2\heartsuit), \ldots, (A\heartsuit, K\heartsuit), (2\heartsuit, A\heartsuit), (2\heartsuit, 2\heartsuit), \ldots, (2\heartsuit, K\heartsuit), \ldots (K\heartsuit, A\heartsuit), (K\heartsuit, 2\heartsuit), \ldots, (K\heartsuit, K\heartsuit)\}$$

It consists of $13 \times 13 = 169$ outcomes, so that the required probability is

$$P(A \cap B) = \frac{169}{2704} = \frac{1}{16}$$

However, it is easier to notice that events A and B are independent with $P(A) = P(B) = 1/4$, so that

$$P(A \cap B) = P(A)\,P(B) = \frac{1}{4} \times \frac{1}{4} = \frac{1}{16}$$

The independence follows from the fact that with the replacement of the first card and with appropriate shuffling of the pack to ensure randomness, the outcome of the second drawing is not related to the outcome of the first drawing.

If the drawings are performed without replacement, then clearly events A and B are not independent. This can be confirmed mathematically by noting that

$$P(B|A) = \frac{12}{51} \quad \text{and} \quad P(B|A') = \frac{13}{51}$$

which are different from $P(B) = 1/4$.

1.5.3 Examples and Probability Trees

Example 2

Defective Computer Chips

Suppose that 9 out of the 500 chips in a particular box are defective, and suppose that 3 chips are *sampled* at random from the box without replacement. If each of the 3 chips sampled is tested to determine whether it is defective (1) or satisfactory (0), the sample space has eight outcomes. For example, the outcome (0, 1, 0) corresponds to the first and third chips being satisfactory and the second chip being defective.

The probability values of the eight outcomes can be calculated using a **probability tree** as illustrated in Figure 1.63. The events A, B, and C are, respectively, the events that the first, second, and third chips sampled are defective. These events are not independent since the sampling is conducted without replacement. The probability tree starts at the left with two branches corresponding to the events A and A'. The probabilities of these events

$$P(A) = \frac{9}{500} \qquad \text{and} \qquad P(A') = \frac{491}{500}$$

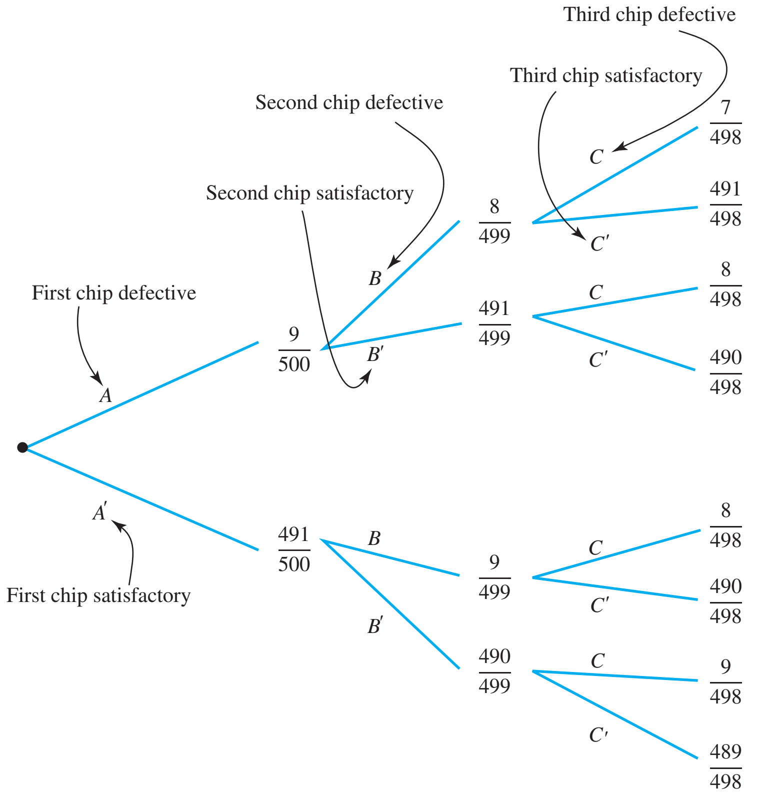
are recorded at the ends of the branches.

Each of these two branches then splits into two more branches corresponding to the events B and B', and the *conditional* probabilities of these events are recorded. These conditional probabilities are

$$P(B|A) = \frac{8}{499}, \qquad P(B'|A) = \frac{491}{499}, \qquad P(B|A') = \frac{9}{499}, \qquad P(B'|A') = \frac{490}{499}$$

which are constructed by considering how many of the 499 chips left in the box are defective when the second chip is chosen. For example, $P(B|A) = 8/499$ since if the first chip chosen is defective (event A), then 8 out of 499 chips in the box are defective when the second chip is chosen.

The probability tree is completed by adding additional branches for the events C and C', and by recording the probabilities of these events *conditional* on the outcomes of the first two



Outcome	Probability
(1,1,1)	$\frac{9}{500} \times \frac{8}{499} \times \frac{7}{498} \simeq 4.0 \times 10^{-6}$
(1,1,0)	$\frac{9}{500} \times \frac{8}{499} \times \frac{491}{498} \simeq 2.85 \times 10^{-4}$
(1,0,1)	$\frac{9}{500} \times \frac{491}{499} \times \frac{8}{498} \simeq 2.85 \times 10^{-4}$
(1,0,0)	$\frac{9}{500} \times \frac{491}{499} \times \frac{490}{498} \simeq 0.0174$
(0,1,1)	$\frac{491}{500} \times \frac{9}{499} \times \frac{8}{498} \simeq 2.85 \times 10^{-4}$
(0,1,0)	$\frac{491}{500} \times \frac{9}{499} \times \frac{490}{498} \simeq 0.0174$
(0,0,1)	$\frac{491}{500} \times \frac{490}{499} \times \frac{9}{498} \simeq 0.0174$
(0,0,0)	$\frac{491}{500} \times \frac{490}{499} \times \frac{489}{498} \simeq 0.9469$
	1

FIGURE 1.63
Probability tree for computer chip sampling

choices. For example,

$$P(C|A \cap B') = \frac{8}{498}$$

because conditional on the event $A \cap B'$ (the first choice is defective and the second is satisfactory), 8 out of the 498 chips in the box are defective when the third choice is made.

The probability values of the eight outcomes are found by *multiplying* the probabilities along the branches. Thus, the probability of choosing 3 defective chips is

$$\begin{aligned} P((1,1,1)) &= P(A \cap B \cap C) = P(A)P(B|A)P(C|A \cap B) \\ &= \frac{9}{500} \times \frac{8}{499} \times \frac{7}{498} = \frac{21}{5{,}177{,}125} \simeq 4.0 \times 10^{-6} \end{aligned}$$

The probability of choosing 2 satisfactory chips followed by a defective chip is

$$\begin{aligned} P((0,0,1)) &= P(A' \cap B' \cap C) = P(A')P(B'|A')P(C|A' \cap B') \\ &= \frac{491}{500} \times \frac{490}{499} \times \frac{9}{498} = \frac{72{,}177}{4{,}141{,}700} \simeq 0.0174 \end{aligned}$$

Notice that the probabilities of the outcomes (1, 0, 0), (0, 1, 0), and (0, 0, 1) are identical, although they are calculated in different ways. Similarly, the probabilities of the outcomes (1, 1, 0), (0, 1, 1), and (1, 0, 1) are identical. The probability of exactly 1 defective chip being found is

$$\begin{aligned} P(1 \text{ defective}) &= P((1,0,0)) + P((0,1,0)) + P((0,0,1)) \\ &\simeq 3 \times 0.0174 = 0.0522 \end{aligned}$$

In fact, if attention is focused solely on the number of defective chips in the sample, then the required probabilities can be found from the hypergeometric distribution which is discussed in Section 3.3. Finally, it is interesting to note that, in practice, the number of defective chips in a box will not usually be known, but probabilities of these kinds are useful in *estimating* the number of defective chips in the box. In later chapters of this book, statistical techniques will be employed to use the information provided by a random sample (here the number of defective chips found in the sample) to make inferences about the population that is sampled (here the box of chips).

Example 6

Satellite Launching

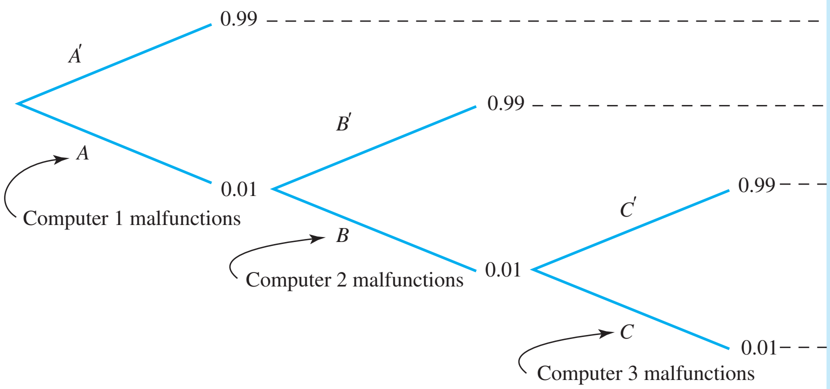
A satellite launch system is controlled by a computer (computer 1) that has two identical backup computers (computers 2 and 3). Normally, computer 1 controls the system, but if it has a malfunction then computer 2 automatically takes over. If computer 2 malfunctions then computer 3 automatically takes over, and if computer 3 malfunctions there is a general system shutdown.

The state space for this problem consists of the four situations

$$\mathcal{S} = \{\text{computer 1 in use, computer 2 in use, computer 3 in use, system failure}\}$$

Suppose that a computer malfunctions with a probability of 0.01 and that malfunctions of the three computers are *independent* of each other. Also, let the events A, B, and C be, respectively, the events that computers 1, 2, and 3 malfunction.

Figure 1.64 shows the probability tree for this problem, which starts at the left with two branches corresponding to the events A and A' with probabilities $P(A) = 0.01$ and



Outcome	Probability
Computer 1 in use	0.99
Computer 2 in use	$0.01 \times 0.99 = 0.0099$
Computer 3 in use	$0.01 \times 0.01 \times 0.99 = 0.000099$
System failure	$0.01 \times 0.01 \times 0.01 = 10^{-6}$
	1

FIGURE 1.64

Probability tree for computer backup system

$P(A') = 0.99$. The top branch (event A') corresponds to computer 1 being in use, and there is no need to extend it further. However, the bottom branch (event A) extends into two further branches for the events B and B'.

Since events A and B are independent, the probabilities of these second-stage branches (events B and B') do not need to be conditioned on the first-stage branch (event A), and so their probabilities are just $P(B) = 0.01$ and $P(B') = 0.99$. The probability tree is completed by adding branches for the events C and C' following on from the events A and B.

The probability values of the four situations are obtained by multiplying the probabilities along the branches, so that

$$P(\text{computer 1 in use}) = 0.99$$
$$P(\text{computer 2 in use}) = 0.01 \times 0.99 = 0.0099$$
$$P(\text{computer 3 in use}) = 0.01 \times 0.01 \times 0.99 = 0.000099$$
$$P(\text{system failure}) = 0.01 \times 0.01 \times 0.01 = 10^{-6}$$

The design of the system backup capabilities is obviously conducted with the aim of minimizing the probability of a system failure. Notice that a key issue in the determination of this probability is the assumption that the malfunctions of the three computers are *independent* events. In other words, a malfunction in computer 1 should not affect the probabilities of the other two computers malfunctioning. An essential part of such a backup system is ensuring that these events are as independent as it is possible to make them.

In particular, it is sensible to have three teams of programmers working independently to supply software to the three computers. If only one piece of software is written and then copied onto the three machines, then the computer malfunctions will not be independent since a malfunction due to a software error in computer 1 will be repeated in the other two computers.

Finally, it is worth noting that this system can be thought of as consisting of three computers connected in parallel, as discussed in Section 17.1.2, where system reliability is considered in more detail.

Example 7
Car Warranties

A company sells a certain type of car that it assembles in one of four possible locations. Plant I supplies 20% of the cars; plant II, 24%; plant III, 25%; and plant IV, 31%. A customer buying a car does not know where the car has been assembled, and so the probabilities of a purchased car being from each of the four plants can be thought of as being 0.20, 0.24, 0.25, and 0.31.

Each new car sold carries a one-year bumper-to-bumper warranty. The company has collected data that show

$$P(\text{claim}|\text{plant I}) = 0.05 \quad P(\text{claim}|\text{plant II}) = 0.11$$
$$P(\text{claim}|\text{plant III}) = 0.03 \quad P(\text{claim}|\text{plant IV}) = 0.08$$

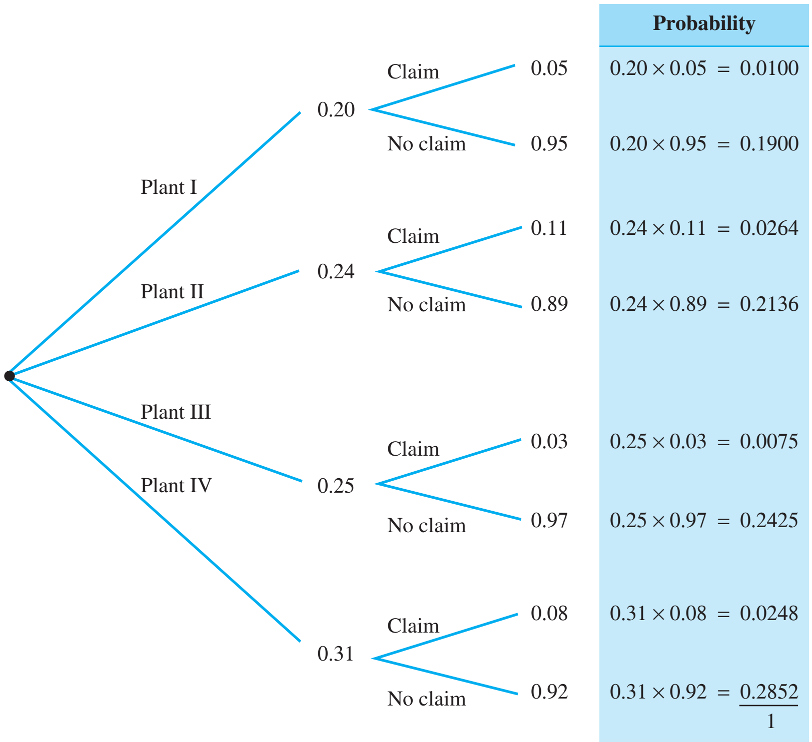
For example, a car assembled in plant I has a probability of 0.05 of receiving a claim on its warranty. This information, which is a closely guarded company secret, indicates which assembly plants do the best job. Plant III is seen to have the best record, and plant II the worst record. Notice that claims are clearly not independent of assembly location because these four conditional probabilities are unequal.

Figure 1.65 shows a probability tree for this problem. It is easily constructed because the probabilities of the second-stage branches are simply obtained from the conditional probabilities above. The probability that a customer purchases a car that was assembled in plant I and that does not require a claim on its warranty is seen to be

$$P(\text{plant I, no claim}) = 0.20 \times 0.95 = 0.19$$

FIGURE 1.65

Probability tree for car warranties example

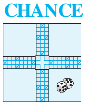


From a customer's point of view, the probability of interest is the probability that a claim on the warranty of the car will be required. This can be calculated as

$$\begin{aligned} P(\text{claim}) &= P(\text{plant I, claim}) + P(\text{plant II, claim}) + P(\text{plant III, claim}) \\ &\quad + P(\text{plant IV, claim}) \\ &= (0.20 \times 0.05) + (0.24 \times 0.11) + (0.25 \times 0.03) + (0.31 \times 0.08) \\ &= 0.0687 \end{aligned}$$

In other words, about 6.87% of the cars purchased will have a claim on their warranty. Notice that this overall claim rate is a *weighted* average of the four individual plant claim rates, with weights corresponding to the supply proportions of the four plants.

GAMES OF CHANCE



In the roll of a fair die, consider the events

$$\text{even} = \{2, 4, 6\} \qquad \text{and} \qquad \text{high score} = \{4, 5, 6\}$$

Intuitively, these two events are not independent since the knowledge that a high score is obtained increases the chances of the score being even, and vice versa, the knowledge that the score is even increases the chances of the score being high. Mathematically, this may be confirmed by noting that the probabilities

$$P(\text{even}) = \frac{1}{2} \qquad \text{and} \qquad P(\text{even}|\text{high score}) = \frac{2}{3}$$

are different.

If a red die and a blue die are rolled, consider the probability that both dice record even scores. In this case the scores on the two dice will be independent of each other since the score on one die does not affect the score that is obtained on the other die. If A is the event that the red die has an even score, and B is the event that the blue die has an even score, the required probability is

$$P(A \cap B) = P(A)\,P(B) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

A more tedious way of calculating this probability is to note that 9 out of the 36 outcomes in the sample space (see Figure 1.46) have both scores even, so that the required probability is $9/36 = 1/4$.

Suppose that two cards are drawn from a pack of cards *without replacement*. What is the probability that exactly one card from the heart suit is obtained? A very tedious way to solve this problem is to count the number of outcomes in the sample space (see Figure 1.13) that satisfy this condition. A better way is

$$\begin{aligned} P(\text{exactly one heart}) &= P(\text{first card heart, second card not heart}) \\ &\quad + P(\text{first card not heart, second card heart}) \\ &= \left(\frac{13}{52} \times \frac{39}{51}\right) + \left(\frac{39}{52} \times \frac{13}{51}\right) = \frac{13}{34} = 0.382 \end{aligned}$$

Since the second drawing is made without replacement, the events "first card heart" and "second card heart" are not independent. However, notice that if the second card is drawn *with replacement*, then the two events are independent, and the required probability is

$$\begin{aligned} P(\text{exactly one heart}) &= P(\text{first card heart, second card not heart}) \\ &\quad + P(\text{first card not heart, second card heart}) \\ &= \left(\frac{1}{4} \times \frac{3}{4}\right) + \left(\frac{3}{4} \times \frac{1}{4}\right) = \frac{3}{8} = 0.375 \end{aligned}$$

It is interesting that the probability is slightly higher when the second drawing is made without replacement.

1.5.4 Problems

1.5.1 Two cards are chosen from a pack of cards *without replacement*. Calculate the probabilities:
- **(a)** Both are picture cards.
- **(b)** Both are from red suits.
- **(c)** One card is from a red suit and one card is from a black suit.

1.5.2 Repeat Problem 1.5.1, except that the second drawing is made *with replacement*. Compare your answers with those from Problem 1.5.1.

1.5.3 Two cards are chosen from a pack of cards *without replacement*. Are the following events independent?
- **(a)** (i) The first card is a picture card, (ii) the second card is a picture card.
- **(b)** (i) The first card is a heart, (ii) the second card is a picture card.
- **(c)** (i) The first card is from a red suit, (ii) the second card is from a red suit.
- **(d)** (i) The first card is a picture card, (ii) the second card is from a red suit.
- **(e)** (i) The first card is a red picture card, (ii) the second card is a heart.

1.5.4 Four cards are chosen from a pack of cards *without replacement*. What is the probability that all four cards are hearts? What is the probability that all four cards are from red suits? What is the probability that all four cards are from different suits?

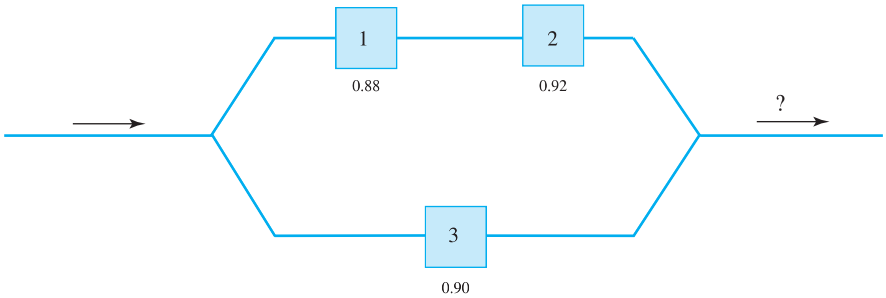


FIGURE 1.66

Switch diagram

1.5.5 Repeat Problem 1.5.4, except that the drawings are made *with replacement*. Compare your answers with those from Problem 1.5.4.

1.5.6 Show that if the events A and B are independent events, then so are the events
(a) A and B' **(b)** A' and B **(c)** A' and B'

1.5.7 Consider the network given in Figure 1.66 with three switches. Suppose that the switches operate independently of each other and that switch 1 allows a message through with probability 0.88, switch 2 allows a message through with probability 0.92, and switch 3 allows a message through with probability 0.90. What is the probability that a message will find a route through the network?

1.5.8 Suppose that birthdays are equally likely to be on any day of the year (ignore February 29 as a possibility). Show that the probability that two people chosen at random have different birthdays is 364/365. Show that the probability that three people chosen at random all have different birthdays is

$$\frac{364}{365} \times \frac{363}{365}$$

and extend this pattern to show that the probability that n people chosen at random all have different birthdays is

$$\frac{364}{365} \times \cdots \times \frac{366-n}{365}$$

What then is the probability that in a group of n people, at least two people will share the same birthday? Evaluate this probability for $n = 10$, $n = 15$, $n = 20$, $n = 25$, $n = 30$, and $n = 35$. What is the smallest value of n for which the probability is larger than a half? Do you think that birthdays are equally likely to be on any day of the year?

1.5.9 Suppose that 17 lightbulbs in a box of 100 lightbulbs are broken and that 3 are selected at random without replacement. Construct a probability tree for this problem. What is the probability that there will be no broken lightbulbs in the sample? What is the probability that there will be no more than 1 broken lightbulb in the sample? (This problem is continued in Problem 1.7.8.)

1.5.10 Repeat Problem 1.5.9, except that the drawings are made *with replacement*. Compare your answers with those from Problem 1.5.9.

1.5.11 Suppose that a bag contains 43 red balls, 54 blue balls, and 72 green balls, and that 2 balls are chosen at random without replacement. Construct a probability tree for this problem. What is the probability that 2 green balls will be chosen? What is the probability that the 2 balls chosen will have different colors?

1.5.12 Repeat Problem 1.5.11, except that the drawings are made *with replacement*. Compare your answers with those from Problem 1.5.11.

1.5.13 A biased coin has a probability p of resulting in a head. If the coin is tossed twice, what value of p minimizes the probability that the same result is obtained on both throws?

1.5.14 If a fair die is rolled six times, what is the probability that each score is obtained exactly once? If a fair die is rolled seven times, what is the probability that a 6 is not obtained at all?

1.5.15 **(a)** If a fair die is rolled five times, what is the probability that the numbers obtained are all even numbers?
(b) If a fair die is rolled three times, what is the probability that the three numbers obtained are all different?
(c) If three cards are taken at random from a pack of cards with replacement, what is the probability that there are two black cards and one red card?

(d) If three cards are taken at random from a pack of cards without replacement, what is the probability that there are two black cards and one red card?

1.5.16 Suppose that n components are available, and that each component has a probability of 0.90 of operating correctly, independent of the other components. What value of n is needed so that there is a probability of at least 0.995 that at least one component operates correctly?

1.5.17 Suppose that an insurance company insures its clients for flood damage to property. Can the company reasonably expect that the claims from its clients will be independent of each other?

1.5.18 A system has four computers. Computer 1 works with a probability of 0.88; computer 2 works with a probability of 0.78; computer 3 works with a probability of 0.92; computer 4 works with a probability of 0.85. Suppose that the operations of the computers are independent of each other.

(a) Suppose that the system works only when all four computers are working. What is the probability that the system works?

(b) Suppose that the system works only if at least one computer is working. What is the probability that the system works?

(c) Suppose that the system works only if at least three computers are working. What is the probability that the system works?

1.6 Posterior Probabilities

1.6.1 Law of Total Probability

Let $A_1, \ldots, A_n$ be a partition of a sample space $\mathcal{S}$ so that the events A_i are mutually exclusive with

$$\mathcal{S} = A_1 \cup \cdots \cup A_n$$

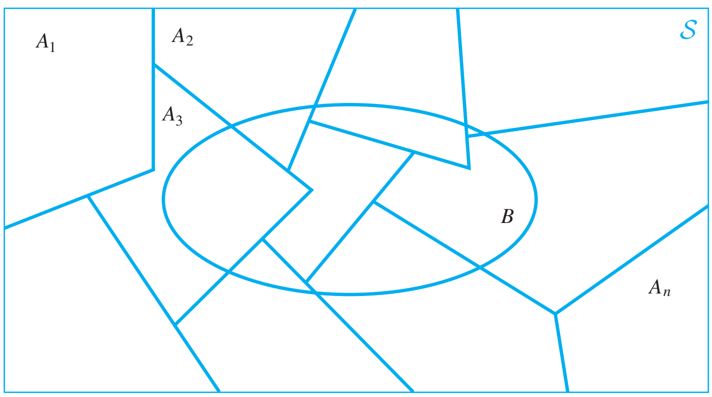
Suppose that the probabilities of these n events, $P(A_1), \ldots, P(A_n)$, are known. In addition, consider an event B as shown in Figure 1.67, and suppose that the conditional probabilities $P(B|A_1), \ldots, P(B|A_n)$ are also known.

An initial question of interest is how to use the probabilities $P(A_i)$ and $P(B|A_i)$ to calculate $P(B)$, the probability of the event B. In fact, this is easily achieved by noting that

$$B = (A_1 \cap B) \cup \cdots \cup (A_n \cap B)$$

FIGURE 1.67

A partition $A_1, \ldots, A_n$ and an event B



where the events $A_i \cap B$ are mutually exclusive, so that

$$\begin{aligned} P(B) &= P(A_1 \cap B) + \cdots + P(A_n \cap B) \\ &= P(A_1)\, P(B|A_1) + \cdots + P(A_n)\, P(B|A_n) \end{aligned}$$

This result, known as the **law of total probability**, has the interpretation that if it is known that one and only one of a series of events A_i can occur, then the probability of another event B can be obtained as the weighted average of the conditional probabilities $P(B|A_i)$, with weights equal to the probabilities $P(A_i)$.

Law of Total Probability

If $A_1, \ldots, A_n$ is a partition of a sample space, then the probability of an event B can be obtained from the probabilities $P(A_i)$ and $P(B|A_i)$ using the formula

$$P(B) = P(A_1)\, P(B|A_1) + \cdots + P(A_n)\, P(B|A_n)$$

Example 7

Car Warranties

The law of total probability was tacitly used in the previous section when the probability of a claim being made on a car warranty was calculated to be 0.0687. If A_1, A_2, A_3, and A_4 are, respectively, the events that a car is assembled in plants I, II, III, and IV, then they provide a partition of the sample space, and the probabilities $P(A_i)$ are the supply proportions of the four plants.

If B is the event that a claim is made, then the conditional probabilities $P(B|A_i)$ are the claim rates for the four individual plants, so that

$$\begin{aligned} P(B) &= P(A_1)\, P(B|A_1) + P(A_2)\, P(B|A_2) + P(A_3)\, P(B|A_3) + P(A_4)\, P(B|A_4) \\ &= (0.20 \times 0.05) + (0.24 \times 0.11) + (0.25 \times 0.03) + (0.31 \times 0.08) \\ &= 0.0687 \end{aligned}$$

as obtained before.

1.6.2 Calculation of Posterior Probabilities

An additional question of interest is how to use the probabilities $P(A_i)$ and $P(B|A_i)$ to calculate the probabilities $P(A_i|B)$, the revised probabilities of the events A_i conditional on the event B. The probabilities

$$P(A_1), \ldots, P(A_n)$$

can be thought of as being the *prior* probabilities of the events $A_1, \ldots, A_n$. However, the observation of the event B provides some additional information that allows the revision of these prior probabilities into a set of *posterior* probabilities

$$P(A_1|B), \ldots, P(A_n|B)$$

which are the probabilities of the events $A_1, \ldots, A_n$ conditional on the event B.

From the law of total probability, these posterior probabilities are calculated to be

$$P(A_i|B) = \frac{P(A_i \cap B)}{P(B)} = \frac{P(A_i)\, P(B|A_i)}{P(B)} = \frac{P(A_i)\, P(B|A_i)}{\sum_{j=1}^{n} P(A_j)\, P(B|A_j)}$$

which is known as **Bayes' theorem**.

HISTORICAL NOTE

Thomas Bayes was born in London, England, in 1702. He was ordained and ministered at a Presbyterian church in Tunbridge Wells, about 35 miles outside London. He was elected a Fellow of the Royal Society in 1742 and died on April 17, 1761. His work on posterior probabilities was discovered in his papers after his death.

Bayes' Theorem

If $A_1, \ldots, A_n$ is a partition of a sample space, then the **posterior probabilities** of the events A_i conditional on an event B can be obtained from the probabilities $P(A_i)$ and $P(B|A_i)$ using the formula

$$P(A_i|B) = \frac{P(A_i)\,P(B|A_i)}{\sum_{j=1}^{n} P(A_j)\,P(B|A_j)}$$

Bayes' theorem is an important result in probability theory because it shows how *new information* can properly be used to update or revise an existing set of probabilities. In some cases the prior probabilities $P(A_i)$ may have to be estimated based on very little information or on subjective feelings. It is then important to be able to improve these probabilities as more information becomes available, and Bayes' theorem provides the means to do this.

1.6.3 Examples of Posterior Probabilities

Example 7
Car Warranties

When a customer buys a car, the (prior) probabilities of it having been assembled in a particular plant are

$$P(\text{plant I}) = 0.20 \qquad P(\text{plant II}) = 0.24$$
$$P(\text{plant III}) = 0.25 \qquad P(\text{plant IV}) = 0.31$$

If a claim is made on the warranty of the car, how does this change these probabilities? From Bayes' theorem, the posterior probabilities are calculated to be

$$P(\text{plant I}|\text{claim}) = \frac{P(\text{plant I})\,P(\text{claim}|\text{plant I})}{P(\text{claim})} = \frac{0.20 \times 0.05}{0.0687} = 0.146$$
$$P(\text{plant II}|\text{claim}) = \frac{P(\text{plant II})\,P(\text{claim}|\text{plant II})}{P(\text{claim})} = \frac{0.24 \times 0.11}{0.0687} = 0.384$$
$$P(\text{plant III}|\text{claim}) = \frac{P(\text{plant III})\,P(\text{claim}|\text{plant III})}{P(\text{claim})} = \frac{0.25 \times 0.03}{0.0687} = 0.109$$
$$P(\text{plant IV}|\text{claim}) = \frac{P(\text{plant IV})\,P(\text{claim}|\text{plant IV})}{P(\text{claim})} = \frac{0.31 \times 0.08}{0.0687} = 0.361$$

which are tabulated in Figure 1.68. Notice that plant II has the largest claim rate (0.11), and its posterior probability 0.384 is much larger than its prior probability of 0.24. This is expected since the fact that a claim is made increases the likelihood that the car has been assembled in a plant that has a high claim rate. Similarly, plant III has the smallest claim rate (0.03), and its posterior probability 0.109 is much smaller than its prior probability of 0.25, as expected.

FIGURE 1.68

Prior and posterior probabilities for car warranties example

		Posterior Probabilities	
	Prior Probabilities	Claim	No claim
Plant I	0.200	0.146	0.204
Plant II	0.240	0.384	0.229
Plant III	0.250	0.109	0.261
Plant IV	0.310	0.361	0.306
	1.000	1.000	1.000

On the other hand, if *no* claim is made on the warranty, the posterior probabilities are calculated to be

$$P(\text{plant I}|\text{no claim}) = \frac{P(\text{plant I})\,P(\text{no claim}|\text{plant I})}{P(\text{no claim})} = \frac{0.20 \times 0.95}{0.9313} = 0.204$$

$$P(\text{plant II}|\text{no claim}) = \frac{P(\text{plant II})\,P(\text{no claim}|\text{plant II})}{P(\text{no claim})} = \frac{0.24 \times 0.89}{0.9313} = 0.229$$

$$P(\text{plant III}|\text{no claim}) = \frac{P(\text{plant III})\,P(\text{no claim}|\text{plant III})}{P(\text{no claim})} = \frac{0.25 \times 0.97}{0.9313} = 0.261$$

$$P(\text{plant IV}|\text{no claim}) = \frac{P(\text{plant IV})\,P(\text{no claim}|\text{plant IV})}{P(\text{no claim})} = \frac{0.31 \times 0.92}{0.9313} = 0.306$$

as tabulated in Figure 1.68. In this case when no claim is made, the probabilities decrease slightly for plant II and increase slightly for plant III.

Finally, it is interesting to notice that when a claim is made the probabilities are revised quite substantially, but when no claim is made the posterior probabilities are almost the same as the prior probabilities. Intuitively, this is because the claim rates are all rather small, and so a claim is an "unusual" occurrence, which requires a more radical revision of the probabilities.

Example 8

Chemical Impurity Levels

A chemical company has to pay particular attention to the impurity levels of the chemicals that it produces. Previous experience leads the company to estimate that about *one in a hundred* of its chemical batches has an impurity level that is too high.

To ensure better quality for its products, the company has invested in a new laser-based technology for measuring impurity levels. However, this technology is not foolproof, and its manufacturers warn that it will falsely give a reading of a high impurity level for about 5% of batches that actually have satisfactory impurity levels (these are "false-positive" results). On the other hand, it will falsely indicate a satisfactory impurity level for about 2% of batches that have high impurity levels (these are "false-negative" results). With this in mind, the chemical company is interested in questions such as these:

- If a high impurity reading is obtained, what is the probability that the impurity level really is high?
- If a satisfactory impurity reading is obtained, what is the probability that the impurity level really is satisfactory?

To answer these questions, let A be the event that the impurity level is too high. Event A and its complement A' form a partition of the sample space, and they have prior probability values of

$$P(A) = 0.01 \qquad \text{and} \qquad P(A') = 0.99$$

Let B be the event that a high impurity reading is obtained. The false-negative rate then indicates that

$$P(B|A) = 0.98 \qquad \text{and} \qquad P(B'|A) = 0.02$$

and the false-positive rate indicates that

$$P(B|A') = 0.05 \qquad \text{and} \qquad P(B'|A') = 0.95$$

If a high impurity reading is obtained, Bayes' theorem gives

$$\begin{aligned} P(A|B) &= \frac{P(A)\,P(B|A)}{P(A)\,P(B|A) + P(A')\,P(B|A')} \\ &= \frac{0.01 \times 0.98}{(0.01 \times 0.98) + (0.99 \times 0.05)} = 0.165 \end{aligned}$$

and

$$\begin{aligned} P(A'|B) &= \frac{P(A')\,P(B|A')}{P(A)\,P(B|A) + P(A')\,P(B|A')} \\ &= \frac{0.99 \times 0.05}{(0.01 \times 0.98) + (0.99 \times 0.05)} = 0.835 \end{aligned}$$

If a satisfactory impurity reading is obtained, Bayes' theorem gives

$$\begin{aligned} P(A|B') &= \frac{P(A)\,P(B'|A)}{P(A)\,P(B'|A) + P(A')\,P(B'|A')} \\ &= \frac{0.01 \times 0.02}{(0.01 \times 0.02) + (0.99 \times 0.95)} = 0.0002 \end{aligned}$$

and

$$\begin{aligned} P(A'|B') &= \frac{P(A')\,P(B'|A')}{P(A)\,P(B'|A) + P(A')\,P(B'|A')} \\ &= \frac{0.99 \times 0.95}{(0.01 \times 0.02) + (0.99 \times 0.95)} = 0.9998 \end{aligned}$$

These posterior probabilities are tabulated in Figure 1.69.

FIGURE 1.69

Prior and posterior probabilities for the chemical impurities example

		Posterior Probabilities	
	Prior Probabilities	**B: high reading**	**B': satisfactory reading**
A: impurity level too high	0.0100	0.1650	0.0002
A': impurity level satisfactory	0.9900	0.8350	0.9998
	1.0000	1.0000	1.0000

We can see that if a satisfactory impurity reading is obtained, then the probability of the impurity level actually being too high is only 0.0002, so that on average, only 1 in 5000 batches testing satisfactory is really not satisfactory. However, if a high impurity reading is obtained, there is a probability of only 0.165 that the impurity level really is high, and the probability is 0.835 that the batch is really satisfactory. In other words, only about 1 in 6 of the batches testing high actually has a high impurity level.

At first this may appear counterintuitive. Since the false-positive and false-negative error rates are so low, why is it that most of the batches testing high are really satisfactory? The answer lies in the fact that about 99% of the batches have satisfactory impurity levels, so that 99% of the time there is an "opportunity" for a false-positive result, and only about 1% of the time is there an "opportunity" for a genuine positive result.

In conclusion, the chemical company should realize that it is wasteful to disregard off-hand batches that are indicated to have high impurity levels. Further investigation of these batches should be undertaken to identify the large proportion of them that are in fact satisfactory products.

1.6.4 Problems

1.6.1 Suppose it is known that 1% of the population suffers from a particular disease. A blood test has a 97% chance of identifying the disease for diseased individuals, but also has a 6% chance of falsely indicating that a healthy person has the disease.
(a) What is the probability that a person will have a positive blood test?
(b) If your blood test is positive, what is the chance that you have the disease?
(c) If your blood test is negative, what is the chance that you do not have the disease?

1.6.2 Bag A contains 3 red balls and 7 blue balls. Bag B contains 8 red balls and 4 blue balls. Bag C contains 5 red balls and 11 blue balls. A bag is chosen at random, with each bag being equally likely to be chosen, and then a ball is chosen at random from that bag. Calculate the probabilities:
(a) A red ball is chosen.
(b) A blue ball is chosen.
(c) A red ball from bag B is chosen.
If it is known that a red ball is chosen, what is the probability that it comes from bag A? If it is known that a blue ball is chosen, what is the probability that it comes from bag B?

1.6.3 A class had two sections. Section I had 55 students of whom 10 received A grades. Section II had 45 students of whom 11 received A grades. Now 1 of the 100 students is chosen at random, with each being equally likely to be chosen.
(a) What is the probability that the student was in section I?
(b) What is the probability that the student received an A grade?
(c) What is the probability that the student received an A grade if the student is known to have been in section I?
(d) What is the probability that the student was in section I if the student is known to have received an A grade?

1.6.4 An island has three species of bird. Species 1 accounts for 45% of the birds, of which 10% have been tagged. Species 2 accounts for 38% of the birds, of which 15% have been tagged. Species 3 accounts for 17% of the birds, of which 50% have been tagged. If a tagged bird is observed, what are the probabilities that it is of species 1, of species 2, and of species 3?

1.6.5 After production, an electrical circuit is given a quality score of A, B, C, or D. Over a certain period of time, 77% of the circuits were given a quality score A, 11% were given a quality score B, 7% were given a quality score C, and 5% were given a quality score D. Furthermore, it was found that 2% of the circuits given a quality score A eventually failed, and the failure rate was 10% for circuits given a quality score B, 14% for circuits given a quality score C, and 25% for circuits given a quality score D.
(a) If a circuit failed, what is the probability that it had received a quality score either C or D?

(b) If a circuit did not fail, what is the probability that it had received a quality score A?

1.6.6 The weather on a particular day is classified as either cold, warm, or hot. There is a probability of 0.15 that it is cold and a probability of 0.25 that it is warm. In addition, on each day it may either rain or not rain. On cold days there is a probability of 0.30 that it will rain, on warm days there is a probability of 0.40 that it will rain, and on hot days there is a probability of 0.50 that it will rain. If it is not raining on a particular day, what is the probability that it is cold?

1.6.7 A valve can be used at four temperature levels. If the valve is used at a cold temperature, then there is a probability of 0.003 that it will leak. If the valve is used at a medium temperature, then there is a probability of 0.009 that it will leak. If the valve is used at a warm temperature, then there is a probability of 0.014 that it will leak. If the valve is used at a hot temperature, then there is a probability of 0.018 that it will leak. Under standard operating conditions, the valve is used at a cold temperature 12% of the time, at a medium temperature 55% of the time, at a warm temperature 20% of the time, and at a hot temperature 13% of the time.

(a) If the valve leaks, what is the probability that it is being used at the hot temperature?

(b) If the valve does not leak, what is the probability that it is being used at the medium temperature?

1.6.8 A company sells five types of wheelchairs, with type A being 12% of the sales, type B being 34% of the sales, type C being 7% of the sales, type D being 25% of the sales, and type E being 22% of the sales. In addition, 19% of the type A wheelchair sales are motorized, 50% of the type B wheelchair sales are motorized, 4% of the type C wheelchair sales are motorized, 32% of the type D wheelchair sales are motorized, and 76% of the type E wheelchair sales are motorized.

(a) If a motorized wheelchair is sold, what is the probability that it is of type C?

(b) If a nonmotorized wheelchair is sold, what is the probability that it is of type D?

1.7 Counting Techniques

In many situations the sample space $\mathcal{S}$ consists of a very large number of outcomes that the experimenter will not want to list in their entirety. However, if the outcomes are equally likely, then it suffices to know the *number* of outcomes in the sample space and the *number* of outcomes contained within an event of interest. In this section, various **counting techniques** are discussed that can be used to facilitate such computations. Remember that if a sample space $\mathcal{S}$ consists of N equally likely outcomes, of which n are contained within the event A, then the probability of the event A is

$$P(A) = \frac{n}{N}$$

1.7.1 Multiplication Rule

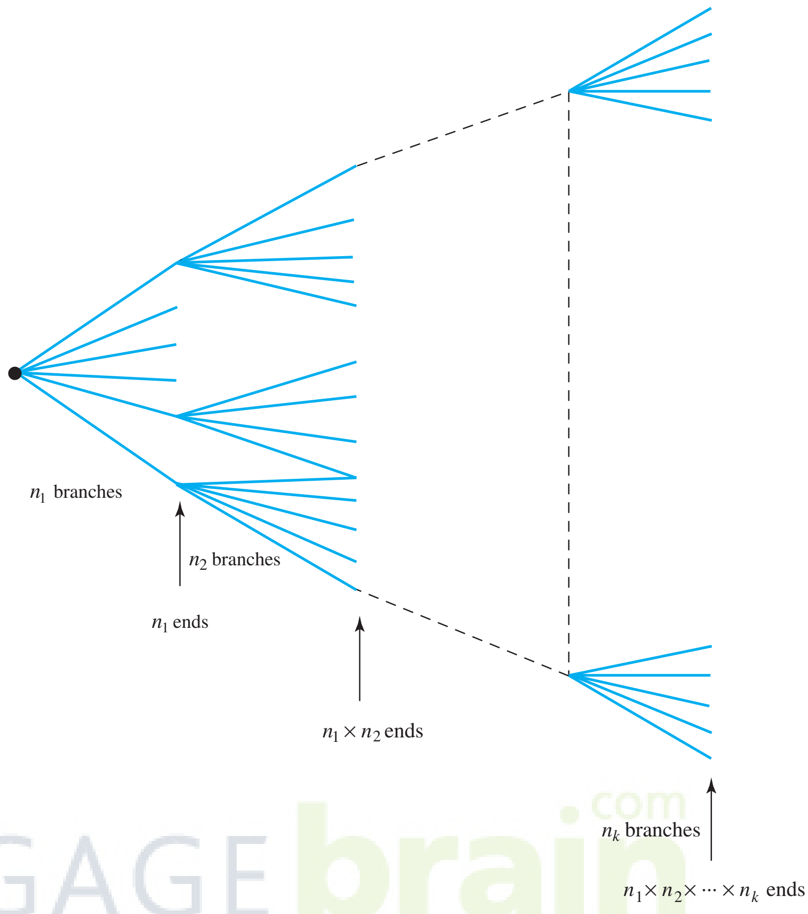
Suppose that an experiment consists of k "components" and that the ith component has n_i possible outcomes. The total number of experimental outcomes will then be equal to the product

$$n_1 \times n_2 \times \cdots \times n_k$$

This is known as the **multiplication rule** and can easily be seen by referring to the tree diagram in Figure 1.70. The n_1 outcomes of the first component are represented by the n_1 branches at the beginning of the tree, each of which splits into n_2 branches corresponding to the outcomes of the second component, and so on. The total number of experimental outcomes (the size of the sample space) is equal to the number of branch ends at the end of the tree, which is equal to the product of the n_i.

FIGURE 1.70

Probability tree illustrating the multiplication rule



Multiplication Rule

If an experiment consists of k components for which the number of possible outcomes are $n_1, \ldots, n_k$, then the total number of experimental outcomes (the size of the sample space) is equal to

$$n_1 \times n_2 \times \cdots \times n_k$$

Example 9

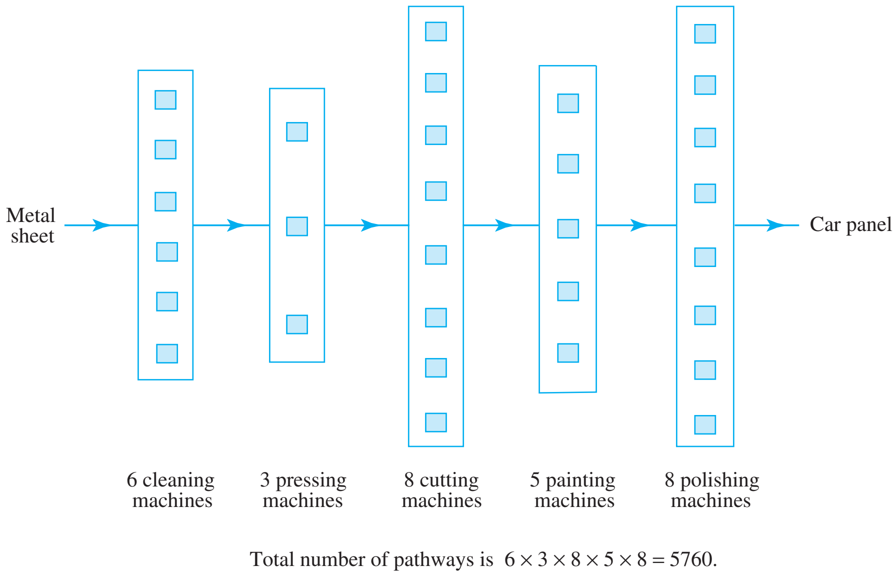
Car Body Assembly Line

A side panel for a car is made from a sheet of metal in the following way. The metal sheet is first sent to a cleaning machine, then to a pressing machine, and then to a cutting machine. The process is completed by a painting machine followed by a polishing machine. Each of the five tasks can be performed on one of several machines whose number and location within the factory are determined by the management so as to streamline the whole manufacturing process.

In particular, suppose that there are six cleaning machines, three pressing machines, eight cutting machines, five painting machines, and eight polishing machines, as illustrated in Figure 1.71. As a quality control procedure, the company attaches a bar code to each panel that

FIGURE 1.71

Manufacturing process for car side panels



identifies which of the machines have been used in its construction. The number of possible "pathways" through the manufacturing process is

$$6 \times 3 \times 8 \times 5 \times 8 = 5760$$

The number of pathways that include a particular pressing machine are

$$6 \times 8 \times 5 \times 8 = 1920$$

If the 5760 pathways can be considered to be equally likely, then a panel chosen at random will have a probability of $1/5760$ of having each of the pathways. However, notice that the pathways will probably not be equally likely, since, for example, the factory layout could cause a panel coming out of one pressing machine to be more likely to be passed on to a particular cutting machine than panels from another pressing machine.

Example 10

Fiber Coatings

Thin fibers are often coated by passing them through a cloud chamber containing the coating material. The fiber and the coating material are provided with opposite electrical charges to provide a means of attraction. Among other things, the quality of the coating will depend on the sizes of the *electrical charges* employed, the *density* of the coating material in the cloud chamber, the *temperature* of the cloud chamber, and the *speed* at which the fiber is passed through the chamber.

A chemical engineer wishes to conduct an experiment to determine how these four factors affect the quality of the coating. The engineer is interested in comparing three charge levels, five density levels, four temperature levels, and three speed levels, as illustrated in Figure 1.72. The total number of possible experimental conditions is then

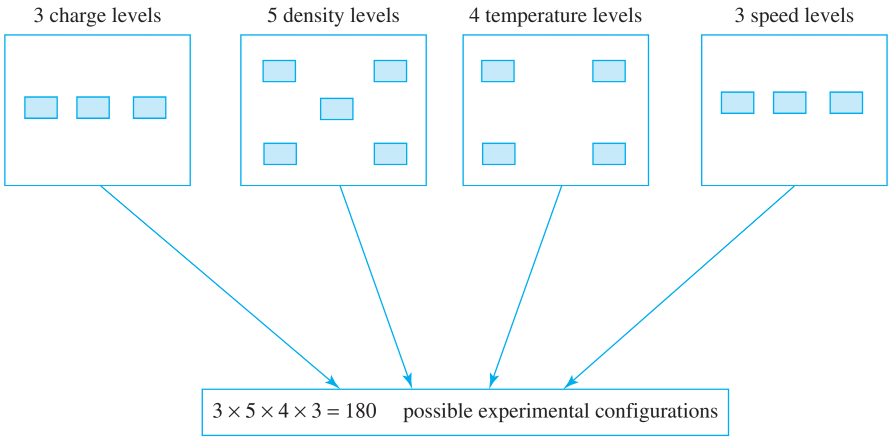
$$3 \times 5 \times 4 \times 3 = 180$$

In other words, there are 180 different combinations of the four factors that can be investigated.

However, the cost of running 180 experiments is likely to be prohibitive, and the engineer may have a budget sufficient to investigate, say, only 30 experimental conditions. Nevertheless,

FIGURE 1.72

Experimental configurations for fiber coatings



with an appropriate **experimental design** and statistical analysis, the engineer can carefully choose which experimental conditions to investigate in order to provide a maximum amount of information about the four factors and how they influence the quality of the coating. The analysis of experiments of this kind is discussed in Chapter 14.

Often the k components of an experiment are identical because they are replications of the same process. In such cases $n_1 = \cdots = n_k = n$, say, and the total number of experimental outcomes will be n^k. For example, if a die is rolled twice, there are $6 \times 6 = 36$ possible outcomes. If a die is rolled k times, there are 6^k possible outcomes.

Computer codes consist of a series of binary digits 0 and 1. The number of different strings consisting of k digits is then 2^k. For example, a string of 20 digits can have

$$2^{20} = 1{,}048{,}576$$

possible values. Calculations such as these indicate how much "information" can be carried by the strings.

Computer passwords typically consist of a string of eight characters, say, which are either 1 of the 26 letters or a numerical digit. The possible number of choices for a password is then

$$36^8 \simeq 2.82 \times 10^{12}$$

If a password is chosen at random, the chance of somebody "guessing" it is thus negligibly small. Nevertheless, a feeling of security could be an illusion for at least two reasons. First, if a computer can be programmed to search repeatedly through possible passwords quickly in an organized manner, it may not take it long to hit on the correct one. Second, few people choose passwords at random, since they themselves have to remember them.

1.7.2 Permutations and Combinations

Often it is important to be able to calculate how many ways a series of distinguishable k objects can be drawn from a pool of n objects. If the drawings are performed with replacement, then the k drawings are identical events, each with n possible outcomes, and the multiplication rule shows that there are n^k possible ways to draw the k objects.

If the drawings are made *without replacement*, then the outcome is said to be a **permutation** of k objects from the original n objects. If only one object is chosen, then clearly there are only n possible outcomes. If two objects are chosen, then there will be

$$n(n-1)$$

possible outcomes, since there are n possibilities for the first choice and then only $n-1$ possibilities for the second choice. More generally, if k objects are chosen, there will be

$$n(n-1)(n-2)\cdots(n-k+1)$$

possible outcomes, which is obtained by multiplying together the number of choices at each drawing.

For dealing with expressions such as these, it is convenient to use the following notation:

Factorials

If n is a positive integer, the quantity $n!$ called "n factorial" is defined to be

$$n! = n(n-1)(n-2)\cdots(1)$$

Also, the quantity $0!$ is taken to be equal to 1.

The number of permutations of k objects from n objects is given the notation P_k^n.

Permutations

A **permutation** of k objects from n objects ($n \geq k$) is an **ordered** sequence of k objects selected without replacement from the group of n objects. The number of possible permutations of k objects from n objects is

$$P_k^n = n(n-1)(n-2)\cdots(n-k+1) = \frac{n!}{(n-k)!}$$

Notice that if $k = n$, the number of permutations is

$$P_n^n = n(n-1)(n-2)\cdots 1 = n!$$

which is just the number of ways of ordering n objects.

Example 11
Taste Tests

A food company has four different recipes for a potential new product and wishes to compare them through consumer taste tests. In these tests, a participant is given the four types of food to taste in a random order and is asked to rank various aspects of their taste. This ranking procedure simply provides an ordering of the four products, and the number of possible ways in which it can be performed is

$$P_4^4 = 4! = 4 \times 3 \times 2 \times 1 = 24$$

In a different taste test, each participant samples eight products and is asked to pick the best, the second best, and the third best. The number of possible answers to the test is then

$$P_3^8 = 8 \times 7 \times 6 = 336$$

Notice that with permutations, the *order* of the sequence is important. For example, if the eight products in the taste test are labeled A–H, then the permutation ABC (in which product A is judged to be best, product B second best, and product C third best) is considered to be different from the permutation ACB, say. That is, each of the six orderings of the products A, B, and C is considered to be a different permutation.

Sometimes when k objects are chosen from a group of n objects, the ordering of the drawing of the k objects is not of importance. In other words, interest is focused on which k objects are chosen, but not on the order in which they are chosen. Such a collection of objects is called a *combination* of k objects from n objects. The notation C_k^n is used for the total possible number of such combinations, and it is calculated using the formula

$$C_k^n = \frac{n!}{(n-k)!\,k!}$$

This formula for the number of combinations follows from the fact that each combination of k objects can be associated with the $k!$ permutations that consist of those objects. Consequently,

$$P_k^n = k! \times C_k^n$$

so that

$$C_k^n = \frac{P_k^n}{k!} = \frac{n!}{(n-k)!\,k!}$$

A common alternative notation for the number of combinations is

$$C_k^n = \binom{n}{k}$$

Combinations

A **combination** of k objects from n objects ($n \geq k$) is an **unordered** collection of k objects selected without replacement from the group of n objects. The number of possible combinations of k objects from n objects is

$$C_k^n = \binom{n}{k} = \frac{n!}{(n-k)!\,k!}$$

Notice that

$$C_1^n = \binom{n}{1} = \frac{n!}{(n-1)!\,1!} = n$$

and

$$C_2^n = \binom{n}{2} = \frac{n!}{(n-2)!\,2!} = \frac{n(n-1)}{2}$$

so that there are n ways to choose one object from n objects, and $n(n-1)/2$ ways to choose two objects from n objects (without attention to order). Also,

$$C_{n-1}^n = \binom{n}{n-1} = \frac{n!}{1!\,(n-1)!} = n$$

and

$$C_n^n = \binom{n}{n} = \frac{n!}{0!\, n!} = 1$$

This last equation just indicates that there is only one way to choose all n objects. It is also useful to note that $C_k^n = C_{n-k}^n$.

Example 11
Taste Tests

Suppose that in the taste test, each participant samples eight products and is asked to select the three best products, but not in any particular order. The number of possible answers to the test is then

$$\binom{8}{3} = \frac{8!}{5!\, 3!} = \frac{8 \times 7 \times 6}{3 \times 2 \times 1} = 56$$

Example 2
Defective Computer Chips

Suppose again that 9 out of 500 chips in a particular box are defective, and that 3 chips are sampled at random from the box without replacement. The total number of possible samples is

$$\binom{500}{3} = \frac{500!}{497!\, 3!} = \frac{500 \times 499 \times 498}{3 \times 2 \times 1} = 20{,}708{,}500$$

which are all equally likely.

The probability of choosing 3 defective chips can be calculated by dividing the number of samples that contain 3 defective chips by the total number of samples. Since there are 9 defective chips, the number of samples that contain 3 defective chips is

$$\binom{9}{3} = \frac{9!}{6!\, 3!} = \frac{9 \times 8 \times 7}{3 \times 2 \times 1} = 84$$

so that the probability of choosing 3 defective chips is

$$\frac{\binom{9}{3}}{\binom{500}{3}} = \frac{\left(\frac{9!}{6!\, 3!}\right)}{\left(\frac{500!}{497!\, 3!}\right)} = \frac{9 \times 8 \times 7}{500 \times 499 \times 498} \simeq 4.0 \times 10^{-6}$$

as obtained before.

Also, the number of samples that contains exactly 1 defective chip is

$$9 \times \binom{491}{2}$$

since there are 9 ways to choose the defective chip and C_2^{491} ways to choose the 2 satisfactory chips. Consequently, the probability of obtaining exactly 1 defective chip is

$$\frac{9 \times \binom{491}{2}}{\binom{500}{3}} = \frac{\left(9 \times \frac{491!}{489!\, 2!}\right)}{\left(\frac{500!}{497!\, 3!}\right)} = \frac{9 \times 491 \times 490 \times 3}{500 \times 499 \times 498} = 0.0522$$

as obtained before. These calculations are examples of the hypergeometric distribution that is discussed in Section 3.3.

GAMES OF CHANCE

Suppose that four cards are taken at random without replacement from a pack of cards. What is the probability that two Kings and two Queens are chosen?

The number of ways to choose four cards is

$$\binom{52}{4} = \frac{52!}{48!\,4!} = \frac{52 \times 51 \times 50 \times 49}{4 \times 3 \times 2 \times 1} = 270{,}725$$

The number of ways of choosing two Kings from the four Kings in the pack as well as the number of ways of choosing two Queens from the four Queens in the pack is

$$\binom{4}{2} = \frac{4!}{2!\,2!} = \frac{4 \times 3}{2 \times 1} = 6$$

so that the number of hands consisting of two Kings and two Queens is

$$\binom{4}{2} \times \binom{4}{2} = 36$$

The required probability is thus

$$\frac{36}{270{,}725} \simeq 1.33 \times 10^{-4}$$

which is a chance of about 13 out of 100,000.

1.7.3 Problems

1.7.1 Evaluate:
(a) 7! **(b)** 8! **(c)** 4! **(d)** 13!

1.7.2 Evaluate:
(a) P_2^7 **(b)** P_5^9 **(c)** P_2^5 **(d)** P_4^{17}

1.7.3 Evaluate:
(a) C_2^6 **(b)** C_4^8 **(c)** C_2^5 **(d)** C_6^{14}

1.7.4 A menu has five appetizers, three soups, seven main courses, six salad dressings, and eight desserts. In how many ways can a full meal be chosen? In how many ways can a meal be chosen if either an appetizer or a soup is ordered, but not both?

1.7.5 In an experiment to test iron strengths, three different ores, four different furnace temperatures, and two different cooling methods are to be considered. Altogether, how many experimental configurations are possible?

1.7.6 Four players compete in a tournament and are ranked from 1 to 4. They then compete in another tournament and are again ranked from 1 to 4. Suppose that their performances in the second tournament are unrelated to their performances in the first tournament, so that the two sets of rankings are independent.

(a) What is the probability that each competitor receives an identical ranking in the two tournaments?

(b) What is the probability that nobody receives the same ranking twice?

1.7.7 Twenty players compete in a tournament. In how many ways can rankings be assigned to the top five competitors? In how many ways can the best five competitors be chosen (without being in any order)?

1.7.8 There are 17 broken lightbulbs in a box of 100 lightbulbs. A random sample of 3 lightbulbs is chosen without replacement.

(a) How many ways are there to choose the sample?

(b) How many samples contain no broken lightbulbs?

(c) What is the probability that the sample contains no broken lightbulbs?

(d) How many samples contain exactly 1 broken lightbulb?

(e) What is the probability that the sample contains no more than 1 broken lightbulb?

1.7.9 Show that $C_k^n = C_k^{n-1} + C_{k-1}^{n-1}$. Can you provide an interpretation of this equality?

1.7.10 A poker hand consists of five cards chosen at random from a pack of cards.

(a) How many different hands are there?

(b) How many hands consist of all hearts?

(c) How many hands consist of cards all from the same suit (a "flush")?
(d) What is the probability of being dealt a flush?
(e) How many hands contain all four Aces?
(f) How many hands contain four cards of the same number or picture?
(g) What is the probability of being dealt a hand containing four cards of the same number or picture?

1.7.11 In an arrangement of n objects in a circle, an object's neighbors are important, but an object's place in the circle is not important. Thus, rotations of a given arrangement are considered to be the same arrangement. Explain why the number of different arrangements is $(n-1)!$

1.7.12 In how many ways can six people sit in six seats in a line at a cinema? In how many ways can the six people sit around a dinner table eating pizza after the movie?

1.7.13 Repeat Problem 1.7.12 with the condition that one of the six people, Andrea, must sit next to Scott. In how many ways can the seating arrangements be made if Andrea refuses to sit next to Scott?

1.7.14 A total of n balls are to be put into k boxes with the conditions that there will be n_1 balls in box 1, n_2 balls in box 2, and so on, with n_k balls being placed in box k $(n_1 + \cdots + n_k = n)$. Explain why the number of ways of doing this is

$$\frac{n!}{n_1! \times \cdots \times n_k!}$$

Explain why this is just $C^n_{n_1} = C^n_{n_2}$ when $k = 2$.

1.7.15 Explain why the following two problems are identical and solve them.
(a) In how many ways can 12 balls be placed in 3 boxes, when the first box can hold 3 balls, the second box can hold 4 balls, and the third box can hold 5 balls.
(b) In how many ways can 3 red balls, 4 blue balls, and 5 green balls be placed in a straight line?
(See Problem 1.7.14.)

1.7.16 A garage employs 14 mechanics, of whom 3 are needed on one job and, at the same time, 4 are needed on another job. The remaining 7 are to be kept in reserve. In how many ways can the job assignments be made?
(See Problem 1.7.14.)

1.7.17 A company has 15 applicants to interview, and 3 are to be invited on each day of the working week. In how many ways can the applicants be scheduled?
(See Problem 1.7.14.)

1.7.18 A quality inspector selects a sample of 12 items at random from a collection of 60 items, of which 18 have excellent quality, 25 have good quality, 12 have poor quality, and 5 are defective.
(a) What is the probability that the sample only contains items that have either excellent or good quality?
(b) What is the probability that the sample contains three items of excellent quality, three items of good quality, three items of poor quality, and three defective items?

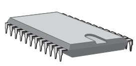
1.7.19 A salesman has to visit 10 different cities. In how many different ways can the ordering of the visits be made? If he decides that 5 of the visits will be made one week, and the other 5 visits will be made the following week, in how many different ways can the 10 cities be split into two groups of 5 cities?

1.7.20 Suppose that 5 cards are taken without replacement from a deck of 52 cards. How many ways are there to do this so that there are 2 red cards and 3 black cards?

1.7.21 A hand of 8 cards is chosen at random from an ordinary deck of 52 playing cards without replacement.
(a) What is the probability that the hand does not have any hearts?
(b) What is the probability that the hand consists of two hearts, two diamonds, two clubs, and two spades?

1.7.22 A box contains 40 batteries, 5 of which have low lifetimes, 30 of which have average lifetimes, and 5 of which have high lifetimes. A consumer requires 8 batteries to run an appliance and randomly selects them all from the box. What is the probability that among the 8 batteries fitted into the consumer's appliance, there are exactly 2 low, 4 average and 2 high lifetimes batteries?

1.8 Case Study: Microelectronic Solder Joints



Suppose that using a particular production method there is a probability of 0.85 that a solder joint has a barrel shape, there is a probability of 0.03 that a solder joint has a cylinder shape, and there is a probability of 0.12 that a solder joint has an hourglass shape. If it is known that a particular solder joint does not have a barrel shape, what is the probability that it has a cylinder shape? This is a conditional probability that can be calculated as

$$P(\text{cylinder}|\text{not barrel}) = \frac{P(\text{cylinder and not barrel})}{P(\text{not barrel})} = \frac{P(\text{cylinder})}{P(\text{cylinder}) + P(\text{hourglass})}$$
$$= \frac{0.03}{0.03 + 0.12} = 0.2$$

Furthermore, suppose that after a certain number of temperature cycles in an accelerated life test there is a probability of 0.002 that a solder joint is cracked if it has a barrel shape, there is a probability of 0.004 that a solder joint is cracked if it has a cylinder shape, and there is a probability of 0.005 that a solder joint is cracked if it has an hourglass shape. This information can be represented by the conditional probabilities shown in Figure 1.73.

If a solder joint is known to be cracked, Bayes' theorem can be used to calculate the probabilities of it having each of the three shapes. For example, the probability that it has a barrel shape is

$$P(\text{barrel}|\text{cracked})$$
$$= \frac{P(\text{barrel})\,P(\text{cracked}|\text{barrel})}{\left(\begin{array}{l} P(\text{barrel})\,P(\text{cracked}|\text{barrel}) + P(\text{cylinder})\,P(\text{cracked}|\text{cylinder}) \\ \quad + P(\text{hourglass})\,P(\text{cracked}|\text{hourglass}) \end{array}\right)}$$
$$= \frac{0.85 \times 0.002}{(0.85 \times 0.002) + (0.03 \times 0.004) + (0.12 \times 0.005)} = 0.70248$$

Similarly, if a solder joint is known not to be cracked, then Bayes' theorem can be used to calculate the probability that it has a cylinder shape, for example, as

$$P(\text{cylinder}|\text{not cracked})$$
$$= \frac{P(\text{cylinder})\,P(\text{not cracked}|\text{cylinder})}{\left(\begin{array}{l} P(\text{barrel})\,P(\text{not cracked}|\text{barrel}) + P(\text{cylinder})\,P(\text{not cracked}|\text{cylinder}) \\ \quad + P(\text{hourglass})\,P(\text{not cracked}|\text{hourglass}) \end{array}\right)}$$
$$= \frac{0.03 \times 0.996}{(0.85 \times 0.998) + (0.03 \times 0.996) + (0.12 \times 0.995)} = 0.02995$$

Figure 1.74 shows all of the shape probabilities conditional on whether the solder joint is known to be cracked or not cracked. Notice that the probabilities in each column sum to one, and that whereas the knowledge that the solder joint is not cracked has little effect on the shape probabilities, the knowledge that the solder joint is cracked (which is a considerably rarer event) has much more effect on the shape probabilities.

FIGURE 1.73

Conditional probabilities of cracking for solder joints

$P(\text{cracked}|\text{barrel}) = 0.002$	$P(\text{not cracked}|\text{barrel}) = 0.998$
$P(\text{cracked}|\text{cylinder}) = 0.004$	$P(\text{not cracked}|\text{cylinder}) = 0.996$
$P(\text{cracked}|\text{hourglass}) = 0.005$	$P(\text{not cracked}|\text{hourglass}) = 0.995$

FIGURE 1.74

Shape probabilities conditional on whether the solder joint is cracked or not

No information on whether the solder joint is cracked or not	Solder joint is known to be cracked	Solder joint is known not to be cracked
$P(\text{barrel}) = 0.85$	$P(\text{barrel}|\text{cracked}) = 0.70248$	$P(\text{barrel}|\text{not cracked}) = 0.85036$
$P(\text{cylinder}) = 0.03$	$P(\text{cylinder}|\text{cracked}) = 0.04959$	$P(\text{cylinder}|\text{not cracked}) = 0.02995$
$P(\text{hourglass}) = 0.12$	$P(\text{hourglass}|\text{cracked}) = 0.24793$	$P(\text{hourglass}|\text{not cracked}) = 0.11969$

Finally, suppose that an assembly consists of 16 solder joints and that unknown to the researcher 5 of these solder joints are cracked. If the researcher randomly chooses a sample of 4 of the solder joints for inspection, then the state space of the number of cracked joints in the sample is {0, 1, 2, 3, 4}. The total number of different samples that can be chosen is

$$\binom{16}{4} = \frac{16!}{12!4!} = 1820$$

and the probability that there will be exactly two cracked solder joints in the researcher's sample is

$$\frac{\binom{5}{2} \times \binom{11}{2}}{\binom{16}{4}} = \frac{10 \times 55}{1820} = 0.302$$

This hypergeometric distribution is discussed more comprehensively in Section 3.3.

1.9 Supplementary Problems

1.9.1 What is the sample space for the average score of two dice?

1.9.2 What is the sample space when a *winner* and a *runner-up* are chosen in a tournament with four contestants.

1.9.3 A *biased* coin is known to have a greater probability of recording a head than a tail. How can it be used to determine *fairly* which team in a football game has the choice of kick-off?

1.9.4 If two fair dice are thrown, what is the probability that their two scores differ by no more than one?

1.9.5 If a card is chosen at random from a pack of cards, what is the probability of choosing a diamond picture card?

1.9.6 Two cards are drawn from a pack of cards. Is it more likely that two hearts will be drawn when the drawing is with replacement or without replacement?

1.9.7 Two fair dice are thrown. A is the event that the sum of the scores is no larger than 4, and B is the event that the two scores are identical. Calculate the probabilities:
(a) $A \cap B$ **(b)** $A \cup B$ **(c)** $A' \cup B$

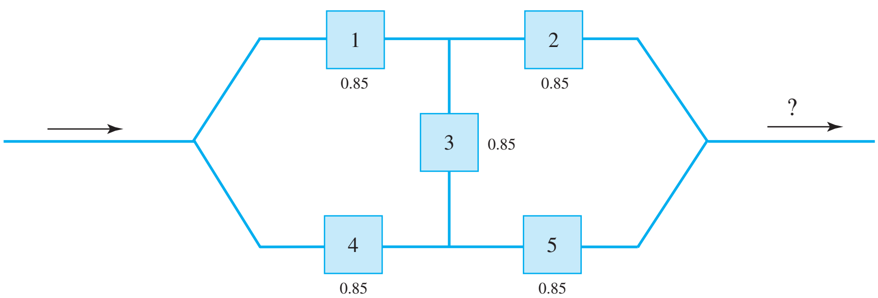
1.9.8 Two fair dice are thrown, one red and one blue. Calculate:
(a) P(red die is 5|sum of scores is 8)
(b) P(either die is 5|sum of scores is 8)
(c) P(sum of scores is 8|either die is 5)

1.9.9 Consider the network shown in Figure 1.75 with five switches. Suppose that the switches operate independently and that each switch allows a message through with a probability of 0.85. What is the probability that a message will find a route through the network?

1.9.10 Which is more likely: obtaining at least one head in two tosses of a fair coin, or at least two heads in four tosses of a fair coin?

FIGURE 1.75

Switch diagram



1.9.11 Bag 1 contains 6 red balls, 7 blue balls, and 3 green balls. Bag 2 contains 8 red balls, 8 blue balls, and 2 green balls. Bag 3 contains 2 red balls, 9 blue balls, and 8 green balls. Bag 4 contains 4 red balls, 7 blue balls, and no green balls. Bag 1 is chosen with a probability of 0.15, bag 2 with a probability of 0.20, bag 3 with a probability of 0.35, and bag 4 with a probability of 0.30, and then a ball is chosen at random from the bag. Calculate the probabilities:

(a) A blue ball is chosen
(b) Bag 4 was chosen if the ball is green
(c) Bag 1 was chosen if the ball is blue

1.9.12 A fair die is rolled. If an even number is obtained, then that is the recorded score. However, if an odd number is obtained, then a fair coin is tossed. If a head is obtained, then the recorded score is the number on the die, but if a tail is obtained, then the recorded score is *twice* the number on the die.

(a) Give the possible values of the recorded score.
(b) What is the probability that a score of 10 is recorded?
(c) What is the probability that a score of 3 is recorded?
(d) What is the probability that a score of 6 is recorded?
(e) What is the probability that a score of 4 is recorded if it is known that the coin is tossed?
(f) If a score of 6 is recorded, what is the probability that an odd number was obtained on the die?

1.9.13 How many sequences of length 4 can be made when each component of the sequence can take 5 different values? How many sequences of length 5 can be made when each component of the sequence can take 4 different values? In general, if $3 \leq n_1 < n_2$, are there more sequences of length n_1 with n_2 possible values for each component, or more sequences of length n_2 with n_1 possible values for each component?

1.9.14 Twenty copying jobs need to be done. If there are four copy machines, in how many ways can five jobs be assigned to each of the four machines? If an additional copier is used, in how many ways can four jobs be assigned to each of the five machines?

1.9.15 A bag contains two counters with each independently equally likely to be either black or white. What is the distribution of X, the number of white counters in the bag? Suppose that a white counter is added to the bag and then one of the three counters is selected at random and taken out of the bag. What is the distribution of X conditional on the counter taken out being white? What if the counter taken out of the bag is black?

1.9.16 It is found that 28% of orders received by a company are from first-time customers, with the other 72% coming from repeat customers. In addition, 75% of the orders from first-time customers are dispatched within one day, and overall 30% of the company's orders are from repeat customers whose orders are not dispatched within one day. If an order is dispatched within one day, what is the probability that it was for a first-time customer?

1.9.17 When asked to select their favorite opera work, 26% of the respondents selected a piece by Puccini, and 22% of the respondents selected a piece by Verdi. Moreover, 59% of the respondents who selected a piece by Puccini were female, and 45% of the respondents who selected a piece by Verdi were female. Altogether, 62% of the respondents were female.

(a) If a respondent selected a piece that is by neither Puccini nor Verdi, what is the probability that the respondent is female?
(b) What proportion of males selected a piece by Puccini?

1.9.18 A random sample of 10 fibers is taken from a collection of 92 fibers that consists of 43 fibers of polymer A, 17 fibers of polymer B, and 32 fibers of polymer C.

(a) What is the probability that the sample does not contain any fibers of polymer B?
(b) What is the probability that the sample contains exactly one fiber of polymer B?
(c) What is the probability that the sample contains three fibers of polymer A, three fibers of polymer B, and four fibers of polymer C?

1.9.19 A fair coin is tossed five times. What is the probability that there is not a sequence of three outcomes of the same kind?

1.9.20 Consider telephone calls made to a company's complaint line. Let A be the event that the call is answered within 10 seconds. Let B be the event that the call is answered by one of the company's experienced telephone operators. Let C be the event that the call lasts less than 5 minutes. Let D be the event that the complaint is handled successfully by the telephone operator. Describe the following events.

(a) $B \cap C'$ **(b)** $(A \cup B') \cap D$
(c) $A' \cap C' \cap D'$ **(d)** $(A \cap C) \cup (B \cap D)$

1.9.21 A manager has 20 different job orders, of which 7 must be assigned to production line I, 7 must be assigned to production line II, and 6 must be assigned to production line III.

(a) In how many ways can the assignments be made?
(b) If the first job and the second job must be assigned to the same production line, in how many ways can the assignments be made?
(c) If the first job and the second job cannot be assigned to the same production line, in how many ways can the assignments be made?

1.9.22 A hand of 3 cards (without replacement) is chosen at random from an ordinary deck of 52 playing cards.
(a) What is the probability that the hand contains only diamonds?
(b) What is the probability that the hand contains one Ace, one King, and one Queen?

1.9.23 A hand of 4 cards (without replacement) is chosen at random from an ordinary deck of 52 playing cards.
(a) What is the probability that the hand does not have any Aces?
(b) What is the probability that the hand has exactly one Ace?
Suppose now that the 4 cards are taken with replacement.
(c) What is the probability that the same card is obtained four times?

1.9.24 Are the following statements true or false?
(a) If a fair coin is tossed three times, the probability of obtaining two heads and one tail is the same as the probability of obtaining one head and two tails.
(b) If a card is drawn at random from a deck of cards, the probability that it is a heart increases if it is conditioned on the knowledge that it is an Ace.
(c) The number of ways of choosing five different letters from the alphabet is more than the number of seconds in a year.
(d) If two events are independent, then the probability that they both occur can be calculated by multiplying their individual probabilities.
(e) It is always true that $P(A|B) + P(A'|B) = 1$.
(f) It is always true that $P(A|B) + P(A|B') = 1$.
(g) It is always true that $P(A|B) \leq P(A)$.

1.9.25 There is a probability of 0.55 that a soccer team will win a game. There is also a probability of 0.85 that the soccer team will not have a player sent off in the game. However, if the soccer team does not have a player sent off, then there is a probability of 0.60 that the team will win the game. What is the probability that the team has a player sent off but still wins the game?

1.9.26 A warehouse contains 500 machines. Each machine is either new or used, and each machine has either good quality or bad quality. There are 120 new machines that have bad quality. There are 230 used machines. Suppose that a machine is chosen at random, with each machine being equally likely to be chosen.
(a) What is the probability that the chosen machine is a new machine with good quality?
(b) If the chosen machine is new, what is the probability that it has good quality?

1.9.27 A class has 250 students, 113 of whom are male, and 167 of whom are mechanical engineers. There are 52 female students who are not mechanical engineers. There are 19 female mechanical engineers who are seniors.
(a) If a randomly chosen student is not a mechanical engineer, what is the probability that the student is a male?
(b) If a randomly chosen student is a female mechanical engineer, what is the probability that the student is a senior?

1.9.28 A business tax form is either filed on time or late, is either from a small or a large business, and is either accurate or inaccurate. There is an 11% probability that a form is from a small business and is accurate and on time. There is a 13% probability that a form is from a small business and is accurate but is late. There is a 15% probability that a form is from a small business and is on time. There is a 21% probability that a form is from a small business and is inaccurate and is late.
(a) If a form is from a small business and is accurate, what is the probability that it was filed on time?
(b) What is the probability that a form is from a large business?

1.9.29 **(a)** If four cards are taken at random from a pack of cards without replacement, what is the probability of having exactly two hearts?
(b) If four cards are taken at random from a pack of cards without replacement, what is the probability of having exactly two hearts and exactly two clubs?
(c) If four cards are taken at random from a pack of cards without replacement and it is known that there are no clubs, what is the probability that there are exactly three hearts?

1.9.30 An applicant has a 0.26 probability of passing a test when they take it for the first time, and if they pass it they can move on to the next stage. However, if they fail the test the first time, they must take the test a second time, and when an applicant takes the test for the second time there is a 0.43 chance that they will pass and be allowed to move on to the next stage. The applicant is rejected if the test is failed on the second attempt.

(a) What is the probability that an applicant moves on to the next stage but needs two attempts at the test?

(b) What is the probability that an applicant moves on to the next stage?

(c) If an applicant moves on to the next stage, what is the probability that they passed the test on the first attempt?

1.9.31 A fair die is rolled five times. What is the probability that the first score is strictly larger than the second score which is strictly larger than the third score which is strictly larger than the fourth score which is strictly larger than the fifth score (i.e., the five scores are strictly decreasing).

1.9.32 A software engineer makes two backup copies of his file, one on a CD and another on a diskette. Suppose that there is a probability of 0.05% that the file is corrupted when it is backed-up onto the CD, and a probability of 0.1% that the file is corrupted when it is backed-up onto the diskette, and that these events are independent of each other. What is the probability that the engineer will have at least one uncorrupted copy of the file?

1.9.33 A warning light in the cockpit of a plane is supposed to indicate when a hydraulic pump is inoperative. If the pump is inoperative, then there is a probability of 0.992 that the warning light will come on. However, there is a probability of 0.003 that the warning light will come on even when the pump is operating correctly. Furthermore, there is a probability of 0.996 that the pump is operating correctly. If the warning light comes on, what is the probability that the pump really is inoperative?

1.9.34 A hand of 10 cards is chosen at random without replacement from a deck of 52 cards. What is the probability that the hand contains exactly 2 Aces, 2 Kings, 3 Queens, and 3 Jacks?

1.9.35 There are 11 items of a product on a shelf in a retail outlet, and unknown to the customers, 4 of the items are overage. Suppose that a customer takes 3 items at random.

(a) What is the probability that none of the overage products are selected by the customer?

(b) What is the probability that exactly 2 of the items taken by the customer are overage?

1.9.36 Among those people who are infected with a certain virus, 32% have strain A, 59% have strain B, and the remaining 9% have strain C. Furthermore, 21% of people infected with strain A of the virus exhibit symptoms, 16% of people infected with strain B of the virus exhibit symptoms, and 63% of people infected with strain C of the virus exhibit symptoms.

(a) If a person has the virus and exhibits symptoms of it, what is the probability that they have strain C?

(b) If a person has the virus but doesn't exhibit any symptoms of it, what is the probability that they have strain A?

(c) What is the probability that a person who has the virus does not exhibit any symptoms of it?

> "When solving mysteries like this one, it's always a question of prior probabilities and posterior probabilities." (From *Inspector Morimoto and the Two Umbrellas*, by Timothy Hemion)

ANSWERS TO ODD-NUMBERED PROBLEMS

Chapter 1 Probability Theory

1.1 Probabilities

1.1.1 $\mathcal{S}$ = {(head, head, head), (head, head, tail), (head, tail, head), (head, tail, tail), (tail, head, head), (tail, head, tail), (tail, tail, head), (tail, tail, tail)}

1.1.3 $\mathcal{S} = \{0,1,2,3,4\}$

1.1.5 $\mathcal{S}$ = {(on time, satisfactory), (on time, unsatisfactory), (late, satisfactory), (late, unsatisfactory)}

1.1.7 (a) $p = 0.5$ (b) $p = \frac{2}{3}$ (c) $\frac{1}{3}$

1.1.9 $0 \leq P(V) \leq 0.39$ $P(IV) = P(V) = 0.195$

1.2 Events

1.2.1 (a) $P(b) = 0.15$ (b) $P(A) = 0.50$ (c) $P(A') = 0.50$

1.2.3 $\frac{124}{1461}$ and $\frac{113}{1461}$

1.2.5 P(at least one score is a prime number) $= \frac{8}{9}$

P(neither score prime) $= \frac{1}{9}$

1.2.7 P(♠ or ♣) $= \frac{1}{2}$

1.2.9 (a) P(Terica is winner) $= \frac{1}{4}$

(b) P(Terica is winner or runner up) $= \frac{1}{2}$

1.2.11 (a) P(both assembly lines are shut down) $= 0.02$

(b) P(neither assembly line is shut down) $= 0.74$

(c) P(at least one assembly line is at full capacity) $= 0.71$

(d) P(exactly one assembly line at full capacity) $= 0.52$

The complement of *"neither assembly line is shut down"* consists of the outcomes

{(S, S), (S, P), (S, F), (P, S), (F, S)}.

The complement of *"at least one assembly line is at full capacity"* consists of the outcomes

{(S, S), (S, P), (P, S), (P, P)}.

1.3 Combinations of Events

1.3.1 The event A contains the outcome 0 while the empty set does not contain any outcomes.

1.3.5 Yes No

1.3.7 $P(B) = 0.4$

1.3.9 Yes

$P(A \cup B \cup C) = \frac{3}{4}$

1.3.11 $P(O' \cap S') = 0.11$

1.3.13 0.19

1.4 Conditional Probability

1.4.1 (a) $P(A \mid B) = 0.1739$ (b) $P(C \mid A) = 0.59375$

(c) $P(B \mid A \cap B) = 1$ (d) $P(B \mid A \cup B) = 0.657$

(e) $P(A \mid A \cup B \cup C) = 0.3636$

(f) $P(A \cap B \mid A \cup B) = 0.1143$

1.4.3 (a) $P(A\heartsuit \mid \text{red suit}) = \frac{1}{26}$ (b) P(heart | red suit) $= \frac{1}{2}$

(c) P(red suit | heart) $= 1$ (d) P(heart | black suit) $= 0$

(e) P(King | red suit) $= \frac{1}{13}$

(f) P(King | red picture card) $= \frac{1}{3}$

1.4.5 P(shiny | red) $= \frac{3}{8}$

P(dull | red) $= \frac{5}{8}$

1.4.9 (a) 0.512 (b) 0.619

(c) 0 (d) 0.081

1.4.11 (a) 0.9326 (b) 0.9756

1.4.13 (a) 0.02 (b) 0.775

1.4.15 P(delay) $= 0.3568$

1.5 Probabilities of Event Intersections

1.5.1 (a) $P(\text{both cards are picture cards}) = \frac{132}{2652}$
(b) $P(\text{both cards are from red suits}) = \frac{650}{2652}$
(c) $P(\text{one card is from a red suit and one is from black suit}) = \frac{26}{51}$

1.5.3 (a) No, they are not independent.
(b) Yes, they are independent.
(c) No, they are not independent.
(d) Yes, they are independent.
(e) No, they are not independent.

1.5.5 $P(\text{all 4 cards are hearts}) = \frac{1}{256}$
$P(\text{all 4 cards are from red suits}) = \frac{1}{16}$
$P(\text{all 4 cards from different suits}) = \frac{3}{32}$

1.5.7 $P(\text{message gets through the network}) = 0.98096$

1.5.9 $P(\text{no broken bulbs}) = 0.5682$
$P(\text{no more than one broken bulb in the sample}) = 0.9260$

1.5.11 $P(\text{drawing 2 green balls}) = 0.180$
$P(\text{two balls different colors}) = 0.656$

1.5.13 $p = 0.5$ (a fair coin)

1.5.15 (a) $\frac{1}{32}$ (b) $\frac{5}{9}$
(c) $\frac{3}{8}$ (d) $\frac{13}{34}$

1.6 Posterior Probabilities

1.6.1 (a) $P(\text{positive blood test}) = 0.0691$
(b) $P(\text{disease} \mid \text{positive blood test}) = 0.1404$
(c) $P(\text{no disease} \mid \text{negative blood test}) = 0.9997$

1.6.3 (a) $P(\text{Section I}) = \frac{55}{100}$ (b) $P(\text{grade is A}) = \frac{21}{100}$
(c) $P(\text{A} \mid \text{Section I}) = \frac{10}{55}$ (d) $P(\text{Section I} \mid \text{A}) = \frac{10}{21}$

1.6.5 (a) 0.4579
(b) $P(A \mid \text{did not fail}) = 0.7932$

1.6.7 (a) $P(H \mid L) == 0.224$
(b) $P(M \mid L') = 0.551$

1.7 Counting Techniques

1.7.1 (a) $7! = 5040$
(b) $8! = 40320$
(c) $4! = 24$
(d) $13! = 6,227,020,800$

1.7.3 (a) $C_2^6 == 15$ (b) $C_4^8 = 70$
(c) $C_2^5 = 10$ (d) $C_6^{14} = 3003$

1.7.5 24

1.7.7 1,860,480
15504

1.7.13 48
72

1.7.15 (a) 27720

1.7.17 168,168,000

1.7.19 3,628,800
252

1.7.21 (a) 0.082
(b) 0.049

This page contains answers for this chapter only.