OPERATION RESEARCH

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MS (DATA SCIENCE)

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Operations Research

UNIT – I

Operations Research (often referred to as management science) is simply a scientific approach to decision making that seeks to best design and operate a system, usually under conditions requiring the allocation of scarce resources. By system we mean an organization of independent components that work together to accomplish the goal of the system.

G.B.Dantzig, the Father of LPP, was making calculations for the Air Force in 1947 developed Simplex Method for programming (military jargon for planning and not a computer programming as usually understood) in a linear structure.
The process which help us to choose from several possible decisions only the best decision subject to satisfying the limitations called constraints of the problem may be termed as Operations Research

Applications of Operations Research

- *Supply Chain Management*
- *Marketing and Revenue Management System*
- *Manufacturing Plants*
- *Financial Engineering*
- *Telecommunication and the Environment Networks*
- *Healthcare Management*
- *Transportation Network*

Why Optimization ?

- About 68,000 Ideas come to the minds of a healthy person every day.
- Many of these ideas are converted into decisions
- We do not know whether these decisions are good or bad
- We try to take only good decisions in our life to make us happy, else we would be sad!

Formulation of a LP Model

1. Identify the decision variables and express them in algebraic symbols
2. Identify all the constraints or limitations and express as equations
3. Identify the Objective Function and express it as a linear function.
4. Add Non-negativity Condition

General Mathematical Formulation of LPP

Optimize (Maximize or Minimize)

$Z = c_1 x_1 + c_2 x_2 + \ldots\ldots + c_n x_n$

Subject to:

$a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n \ (<=, =, >=)\ b_1$
$a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n \ (<=, =, >=)\ b_2$

.
.
$a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n \ (<=, =, >=)\ b_m$

and $x_1, x_2, \ldots, x_n >= 0$

The above formulation may also be expressed with the following notations:

Optimize (Max. or Min.) $Z = \Sigma\, c_j x_j$ for $j = 1..n$, (Objective Function)

Subject to:

$\Sigma\, a_{ij} x_j \ (<=, =, >=)\ b_i$; for $j = 1 ..n$, $i = 1,2, \ldots m$
(Constraints)

and $x_j >= 0$; $j= 1, 2, \ldots, n$
(Non negativity restrictions)

Some Definitions

1. **Basic Solution :** For a set of m simultaneous equations in n unknowns (n>m), a solution obtained by setting n- m of the variables equal to zero and solving the m equation in n unknowns is called basic solution.
2. **Basic Feasible Solution :** A basic solution which satisfy non-negativity condition also is basic feasible solution.

3. **Optimum Feasible Solution :** Any basic feasible solution which optimizes the objective function.
4. **Degenerate Solution :** If one or more basic variable becomes equal to zero.
5. **Non-degenerate Solution :** A solution with no basic variables becoming equal to zero. We prefer non-degenerate solution.

Steps for Graphical Solution

A. Corner Point Method

1. Define the problem mathematically
2. Graph by constraints by treating each inequality as equality.
3. Locate the feasible region and the corner points.
4. Find out the value of objective function at these points.
5. Find out the optimal solution and the optimal value of Objective function.

B. Iso-Profit or Iso-Cost Line Method

1. Define the problem mathematically
2. Graph by constraints by treating each inequality as equality.
3. Locate the feasible region and the corner points.
4. Draw out a line having the slope of Objective Function Equation (this is called Iso-Cost / Profit Line in Minimization and Maximization problems respectively) somewhere in the middle of the feasible region
5. Move this line away from origin (in case of Minimization) or towards Origin (in case of Maximization) until it touches the extreme point of the feasible region.
6. If a single point is encountered, that reflects optimality and its coordination the solution. If Iso-Profit/ Cost line coincides with any constraint line at the extreme, then this is the case of multiple optimum solutions.

An Electronics Company manufactures two types of LEDs (model 1 and model 2). The daily capacity of the Company is 60 LEDs of model 1 and 75 LEDs of Model 2. Each model of the first type uses 10 pieces of an electronic component where as a model of second type uses 8 pieces of this component. The maximum availability of this component is 800 pieces. The Company is committed to manufacture at least 20 pieces of Model 2 per day. If the net profit on the sale of these two models is Rs.40 and Rs.30 resp. Determine the production schedule which maximizes the profit.

Maximize $Z = 40x_1 + 30x_2$

Sub to $10x_1 + 8x_2 \leq 800$

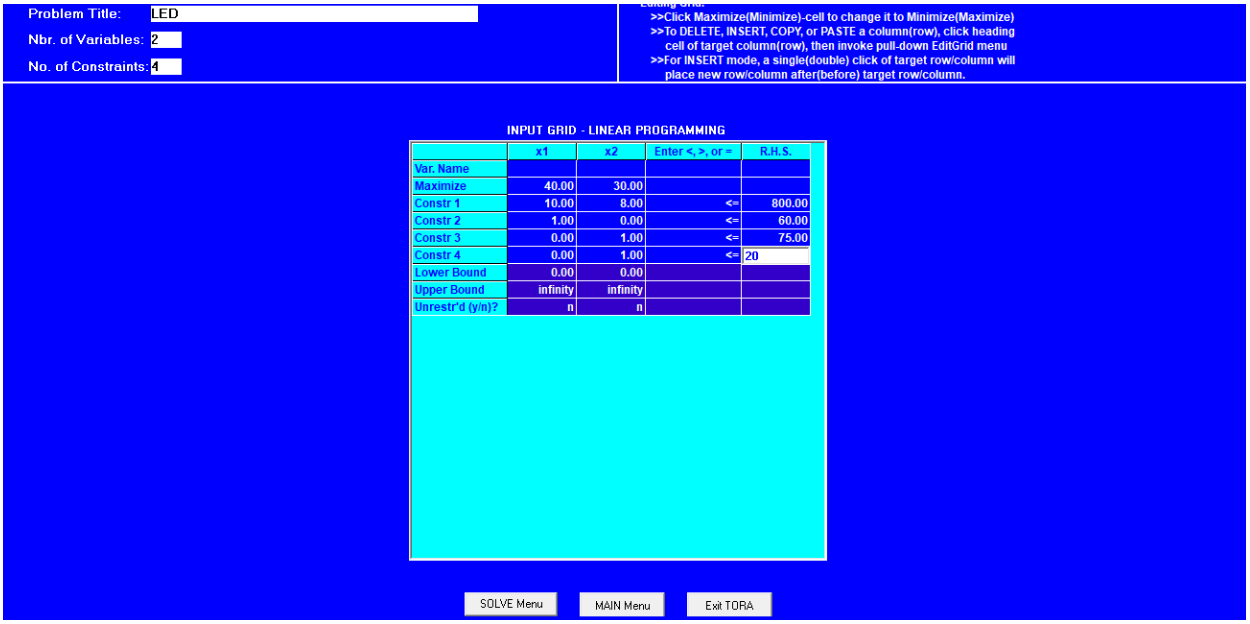
$x_1 \leq 60$

$x_2 \leq 75$

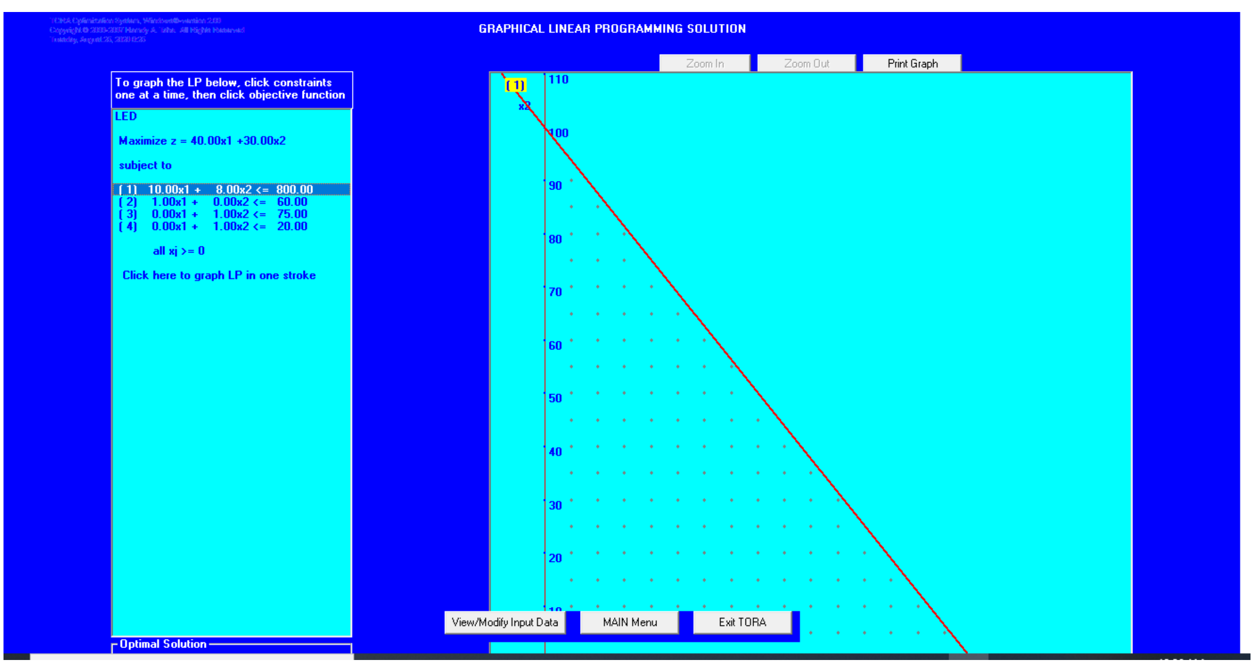
$x_2 \geq 20$

$x_1, x_2 \geq 0$

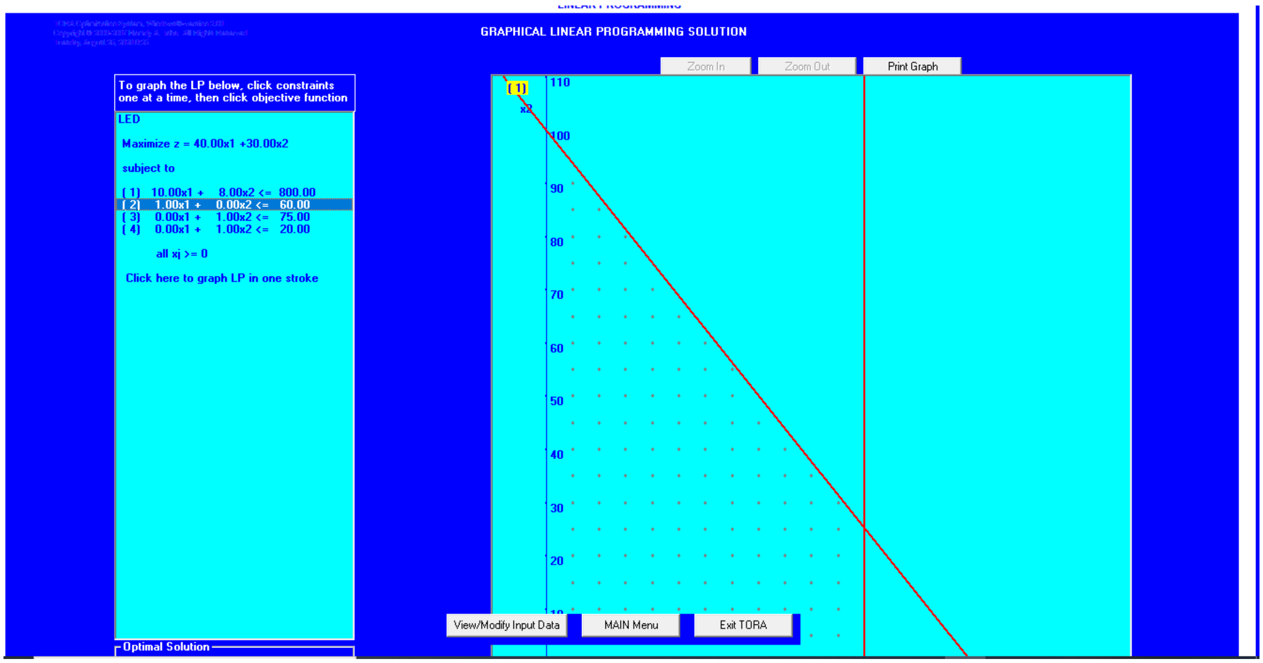
We plot 10x1+8x2 ≤800



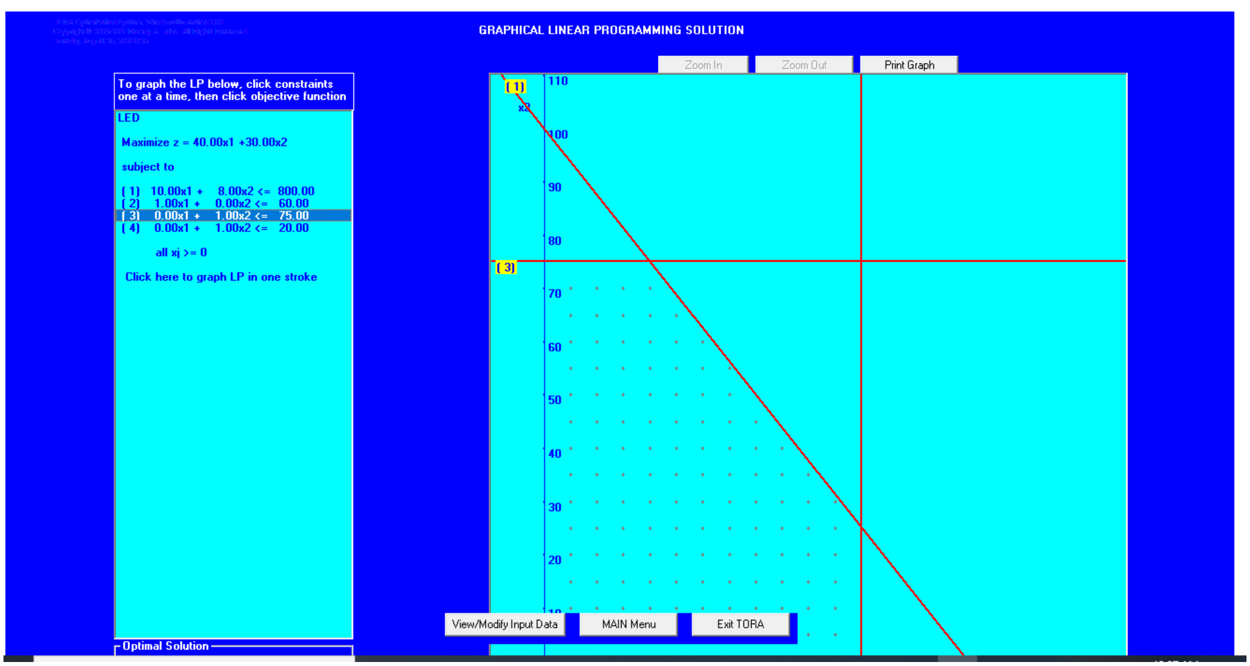
We plot 10x1+8x2 ≤800



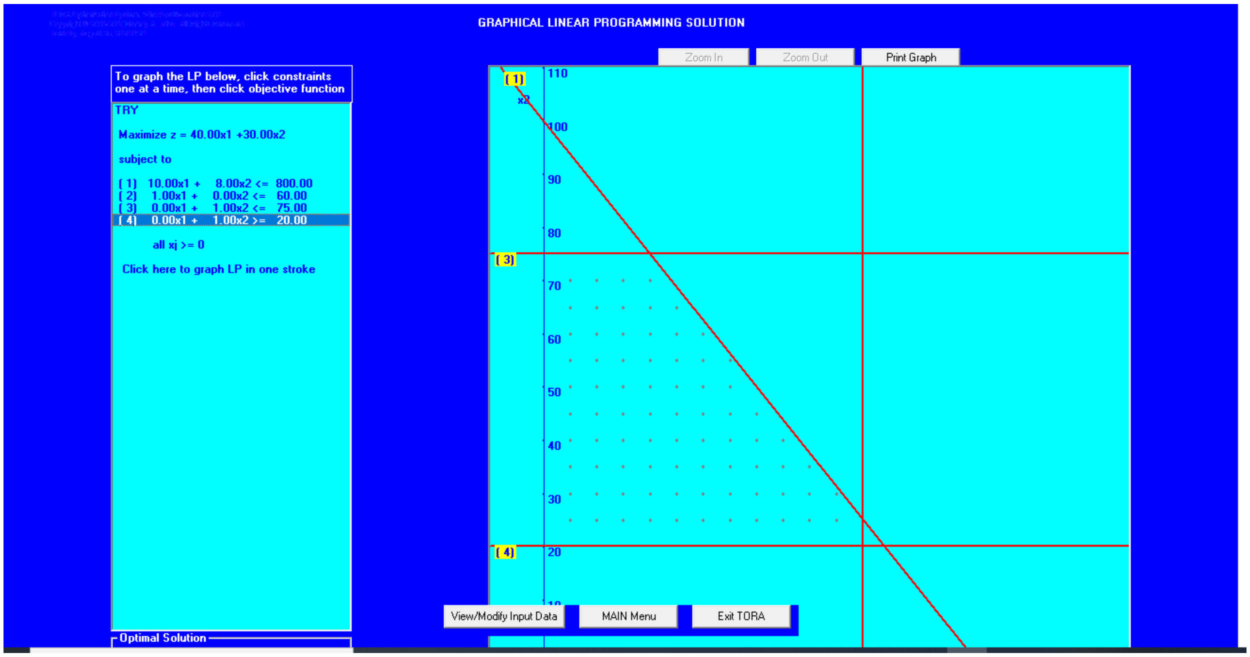
We plot x1≤60



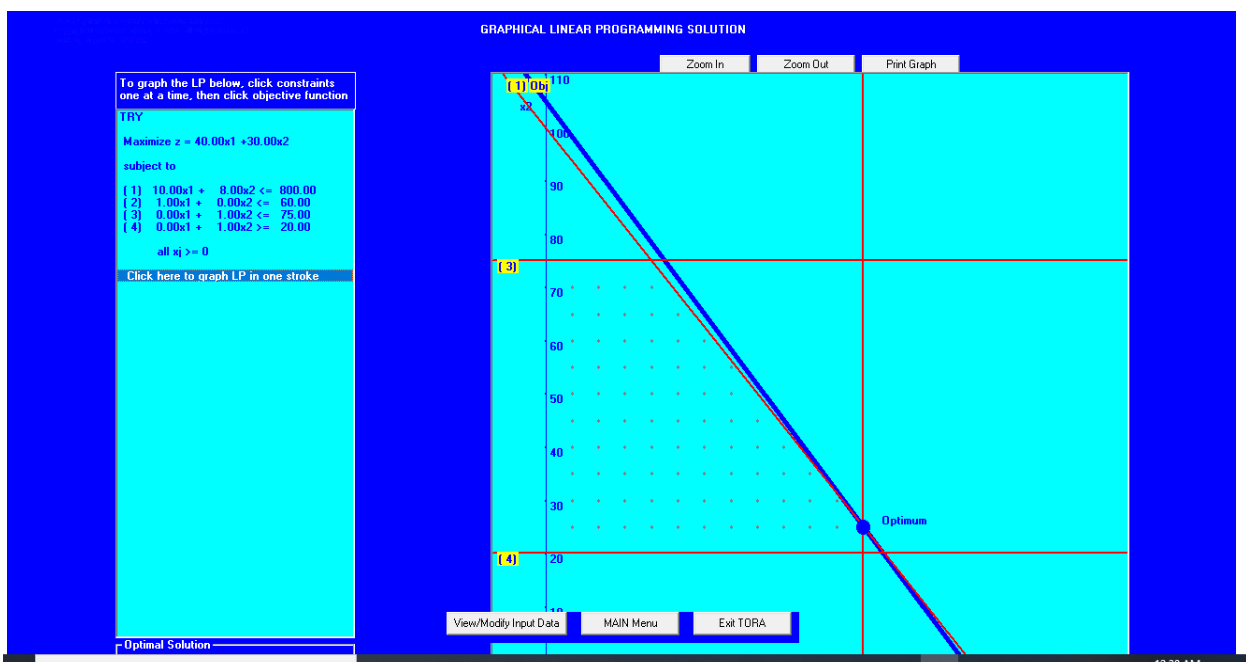
We plot x2≤75



We plot x2≥20



We find the Opimal solution by drawing contours of the objective function



READY MIX PROBLEM (GRAPHICAL SOLUTION)

Ready Mix Produces interior and exterior paints from two raw materials M1 & M2.

The requirement of raw material for these paints as well as their availability along with the profit per tons is being expressed in the following table:

	Tons of Raw Material / Tons of		**Maximum Availability /day (tons)**
	Exterior Paint	Interior Paint	
Raw Material M1	6	4	24
Raw Material M2	1	2	6
Profit/Ton (‘000$)	**5**	**4**	

Market survey indicates that the daily demand for Interior Paint can not exceed that of Exterior Paint by more than 1 ton. Also, maximum daily demand of interior paint is 2 tons.

Determine the optimal product mix that maximizes total daily profit.

Formulation:

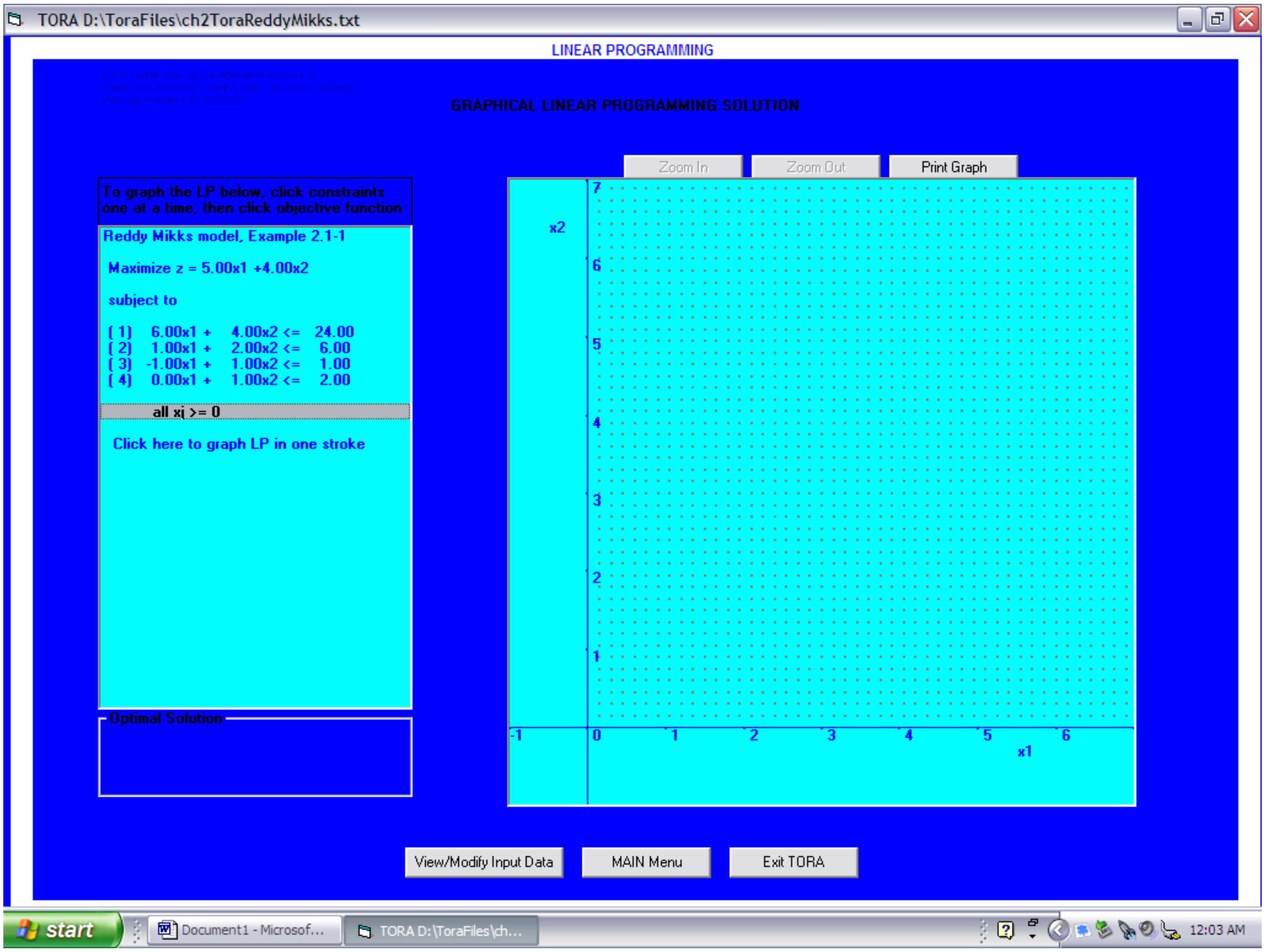
Objective Function:

Maximize $Z = 5\,x_1 + 4\,x_2$

Subject to:

Constraints:

1. $6x_1 + 4x_2 <= 24$
2. $x_1 + 2x_2 <= 6$
3. $-x_1 + x_2 <= 1$
4. $x_2 <= 2$
5. $x_1 >= 0$
6. $x_2 >= 0$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LINEAR PROGRAMMING
GRAPHICAL LINEAR PROGRAMMING SOLUTION
Zoom In
Zoom Out
Print Graph
To graph the LP below, click constraints one at a time, then click objective function
Reddy Mikks model, Example 2.1-1
Maximize z = 5.00x1 +4.00x2
subject to
(1) 6.00x1 + 4.00x2 <= 24.00
(2) 1.00x1 + 2.00x2 <= 6.00
(3) -1.00x1 + 1.00x2 <= 1.00
(4) 0.00x1 + 1.00x2 <= 2.00
all xj >= 0
Click here to graph LP in one stroke
Optimal Solution
x2
7
6
5
4
3
2
1
-1
0
1
2
3
4
5
6
x1
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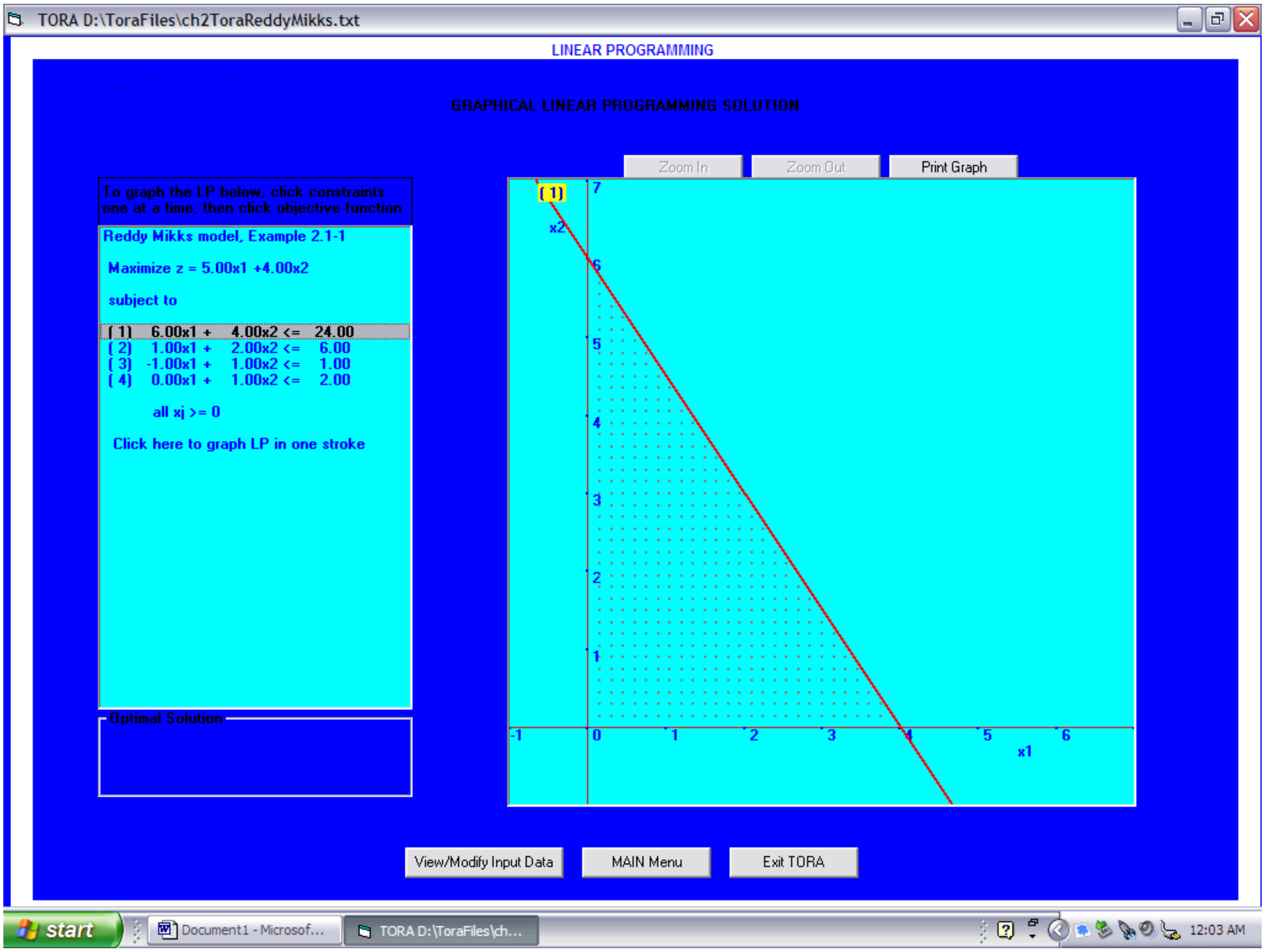
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(3) -1.00x1 + 1.00x2 <= 1.00
(4) 0.00x1 + 1.00x2 <= 2.00
all xj >= 0
Click here to graph LP in one stroke
Optimal Solution
(1)
x2
x1
View/Modify Input Data
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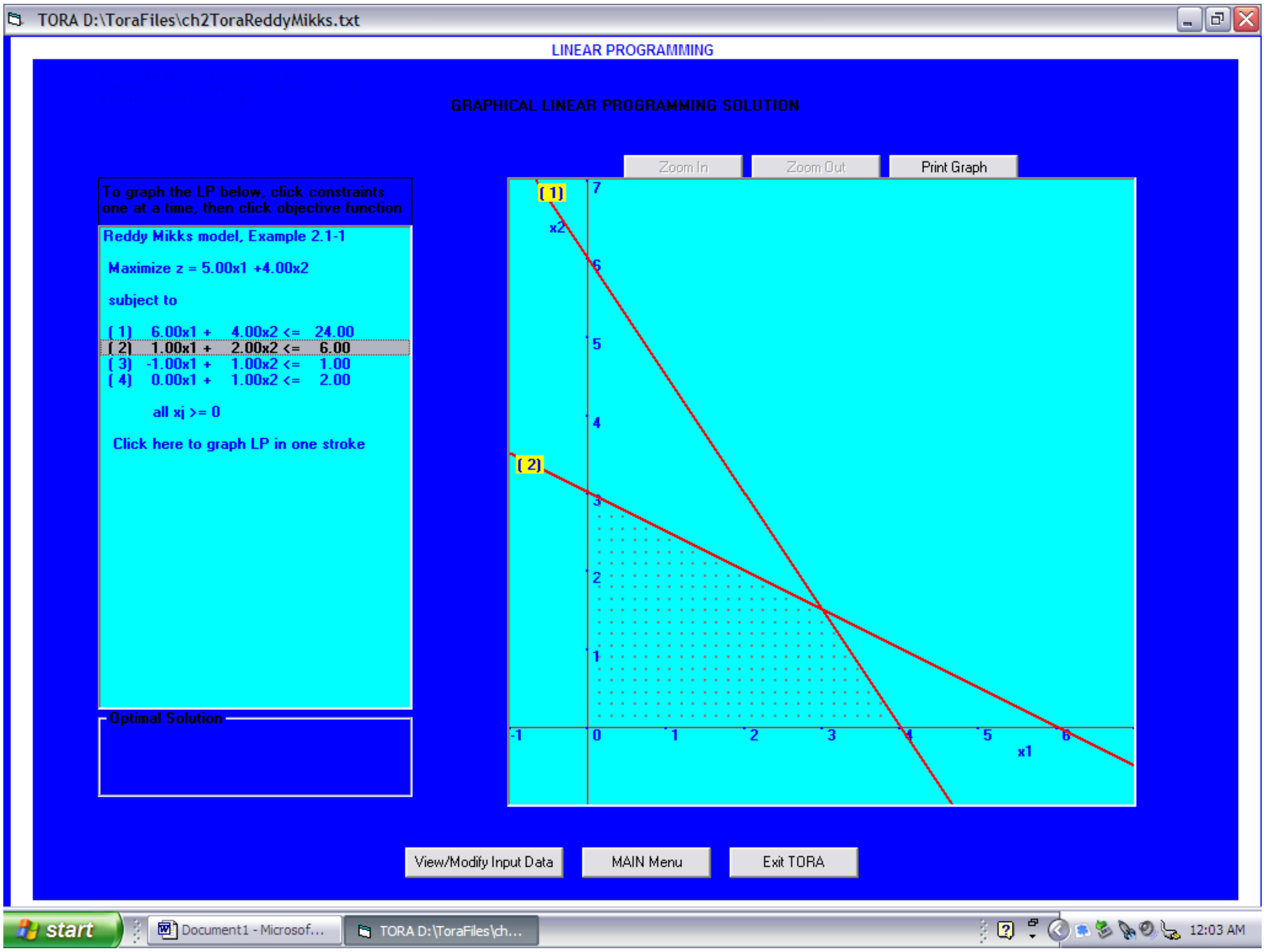
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(3) -1.00x1 + 1.00x2 <= 1.00
(4) 0.00x1 + 1.00x2 <= 2.00
all xj >= 0
Click here to graph LP in one stroke
Optimal Solution
(1)
(2)
x2
x1
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Document1 - Microsof...
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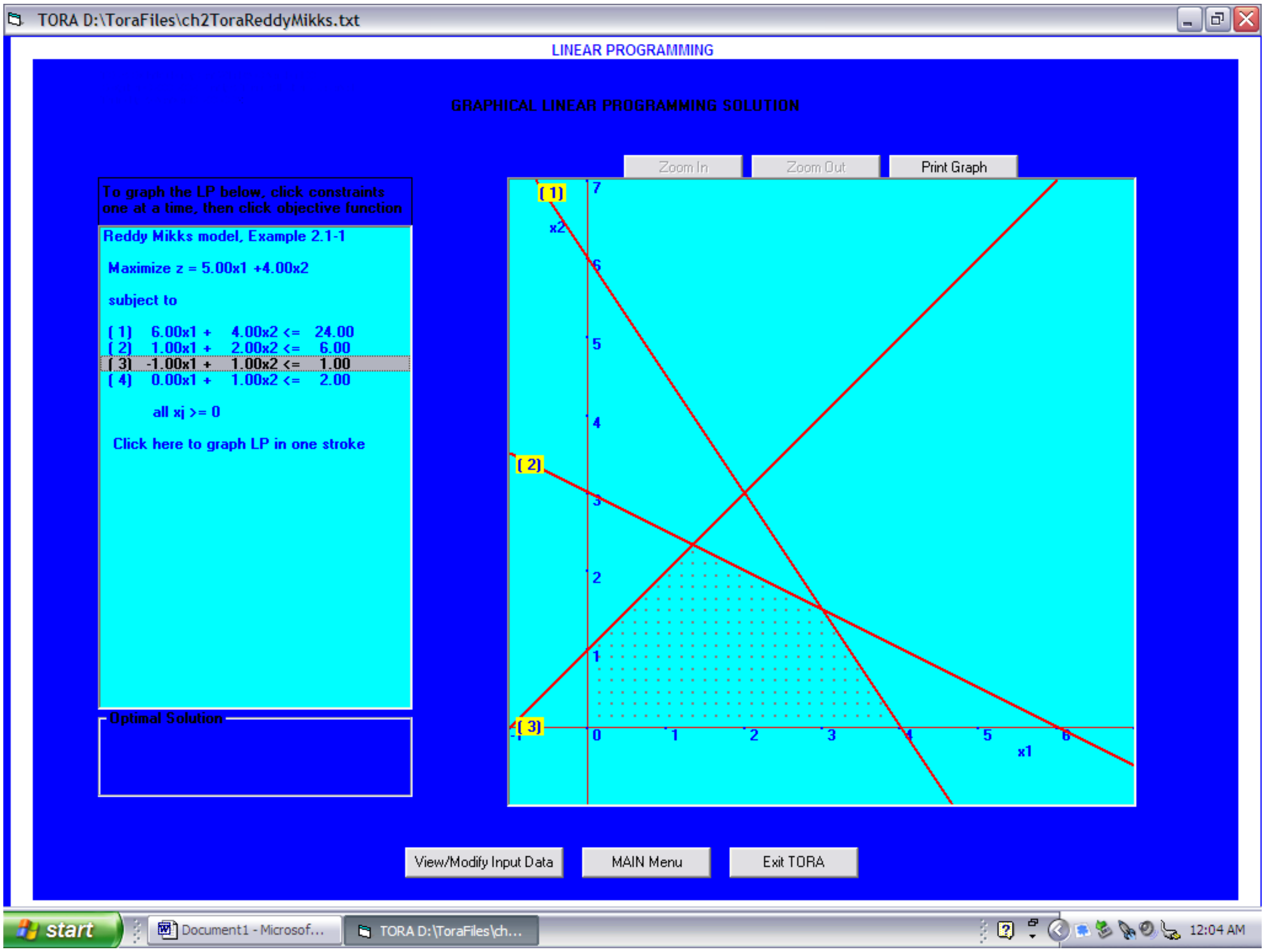
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(3) -1.00x1 + 1.00x2 <= 1.00
(4) 0.00x1 + 1.00x2 <= 2.00
all xj >= 0
Click here to graph LP in one stroke
Optimal Solution
(1)
(2)
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x2
x1
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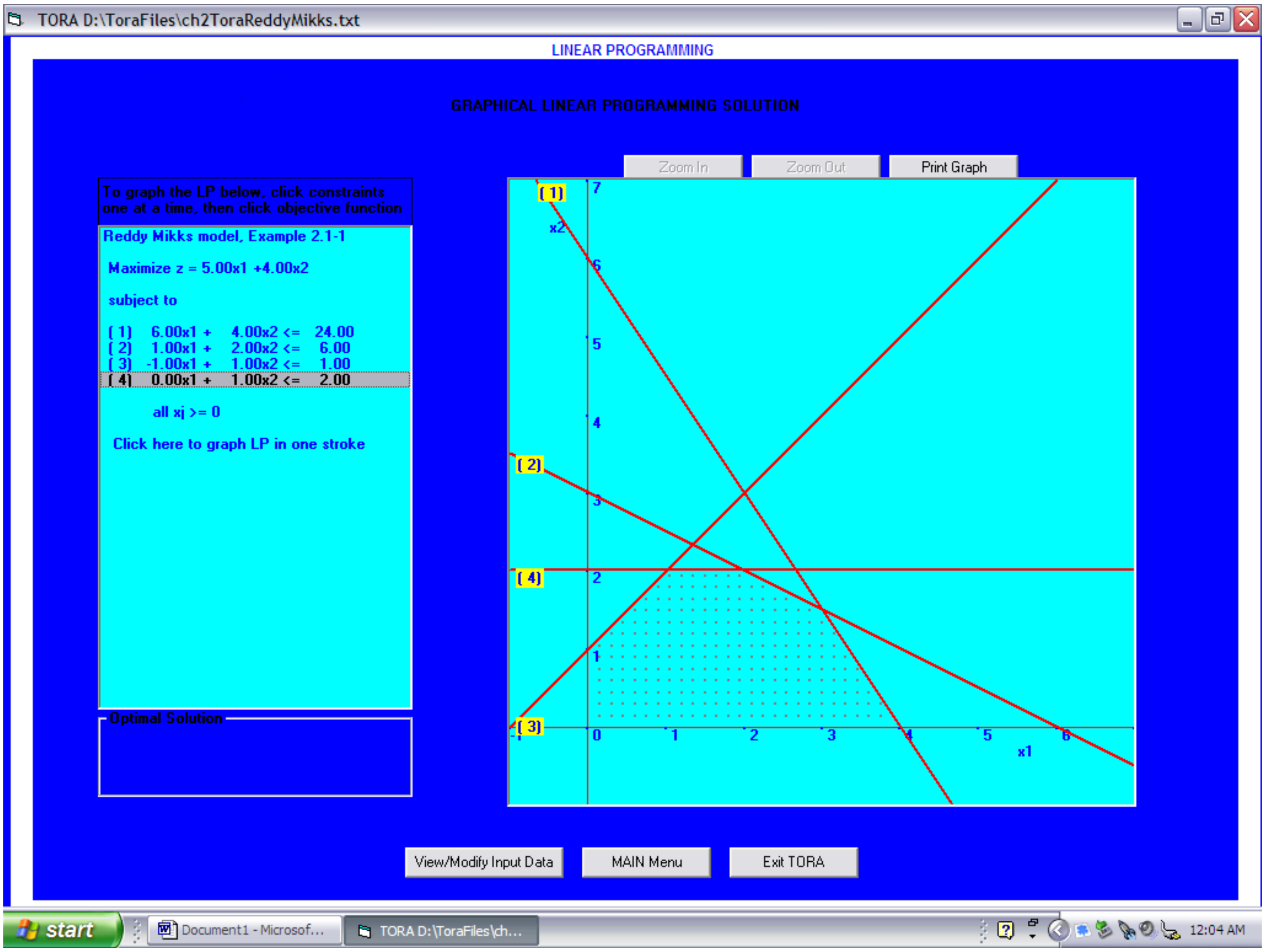
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GRAPHICAL LINEAR PROGRAMMING SOLUTION
Zoom In
Zoom Out
Print Graph
To graph the LP below, click constraints one at a time, then click objective function
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Maximize z = 5.00x1 +4.00x2
subject to
(1) 6.00x1 + 4.00x2 <= 24.00
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(3) -1.00x1 + 1.00x2 <= 1.00
(4) 0.00x1 + 1.00x2 <= 2.00
all xj >= 0
Click here to graph LP in one stroke
Optimal Solution
x2
x1
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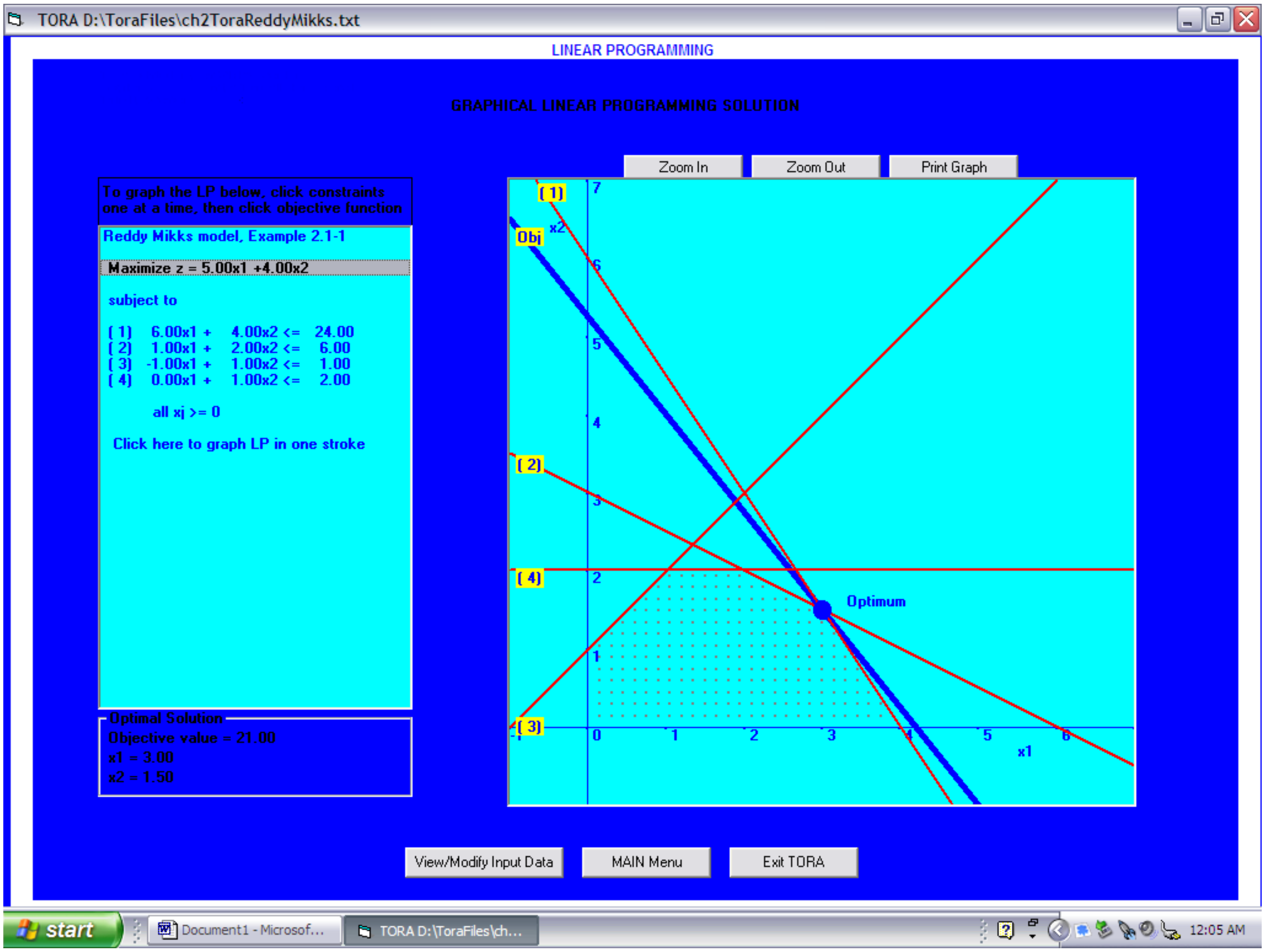
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(3) -1.00x1 + 1.00x2 <= 1.00
(4) 0.00x1 + 1.00x2 <= 2.00
all xj >= 0
Click here to graph LP in one stroke
Optimal Solution
Objective value = 21.00
x1 = 3.00
x2 = 1.50
Optimum
View/Modify Input Data
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MINIMIZATION PROBLEM (GRAPHICAL METHOD)

A wash Farm uses 800 kg of mixed feed daily having the following composition:

Feedstuff	Kg per Kg of Feedstuff		
	Protein	**Fiber**	**Cost Birr/ Kg**
Corn	0.09	0.02	0.30
Soybean Meal	0.60	0.06	0.90

The diet must contain at least 30% protein and at the most 5% fiber.

Determine the daily minimum cost feed mix.

Formulation:

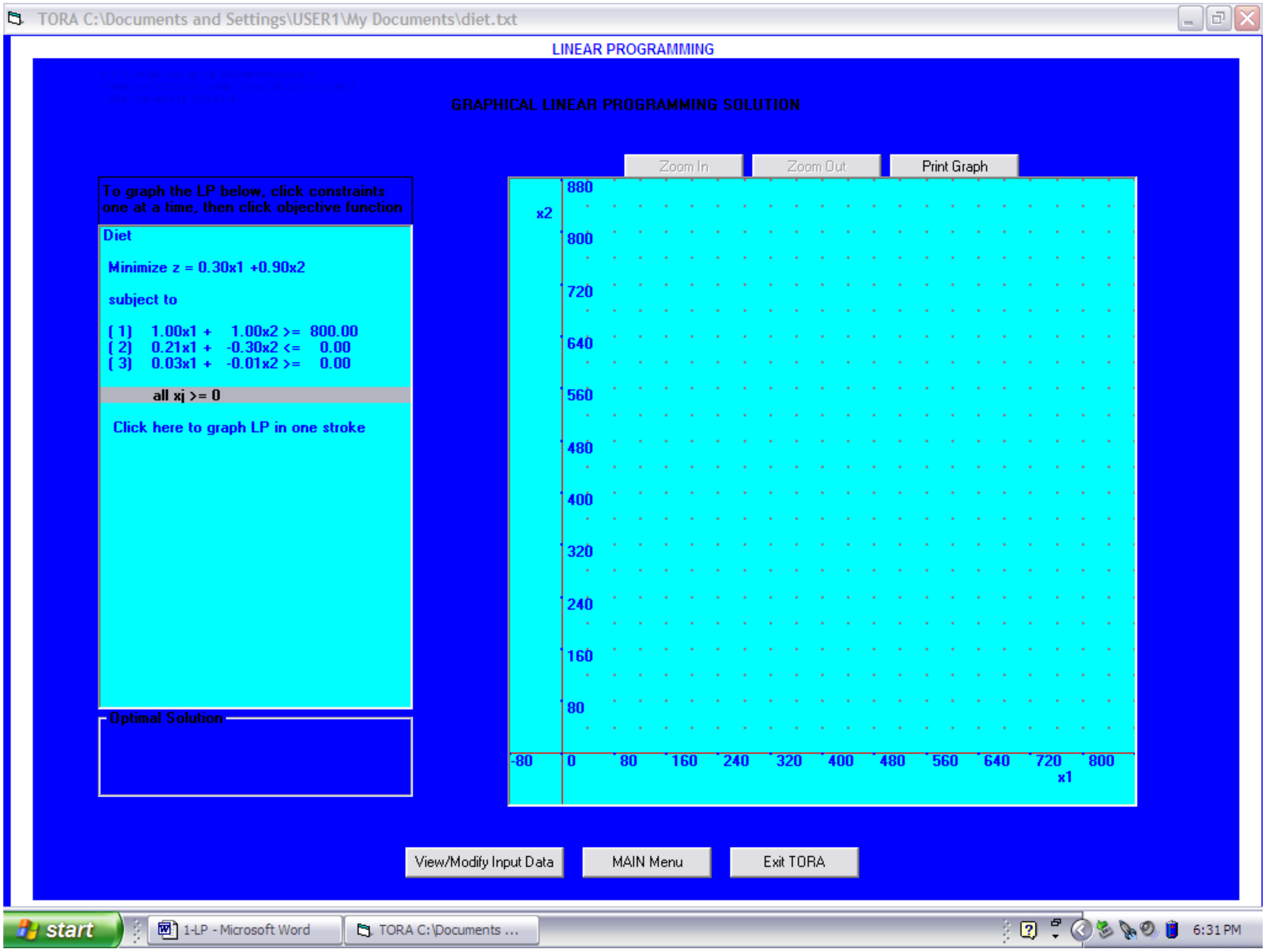
Min Z = $0.30\ x_1 + 0.90\ x_2$

Subject to :

$x_1 + x_2 >= 800$
$0.09\ x_1 + 0.60\ x_2 >= 0.30(x_1 + x_2)$
$0.02\ x_1 + 0.06\ x_2 <= 0.05(x_1 + x_2)$
and $x_1, x_2 >= 0$

Or

Min Z = $0.30\ x_1 + 0.90\ x_2$
S/t
$x_1 + x_2 >= 800$
$0.21\ x_1 - 0.30\ x_2 <= 0$
$0.03\ x_1 - 0.01\ x_2 >= 0$
and $x_1, x_2 >= 0$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Diet
Minimize z = 0.30x1 +0.90x2
subject to
(1) 1.00x1 + 1.00x2 >= 800.00
(2) 0.21x1 + -0.30x2 <= 0.00
(3) 0.03x1 + -0.01x2 >= 0.00
all xj >= 0
Click here to graph LP in one stroke
Optimal Solution
Zoom In
Zoom Out
Print Graph
x2
880
800
720
640
560
480
400
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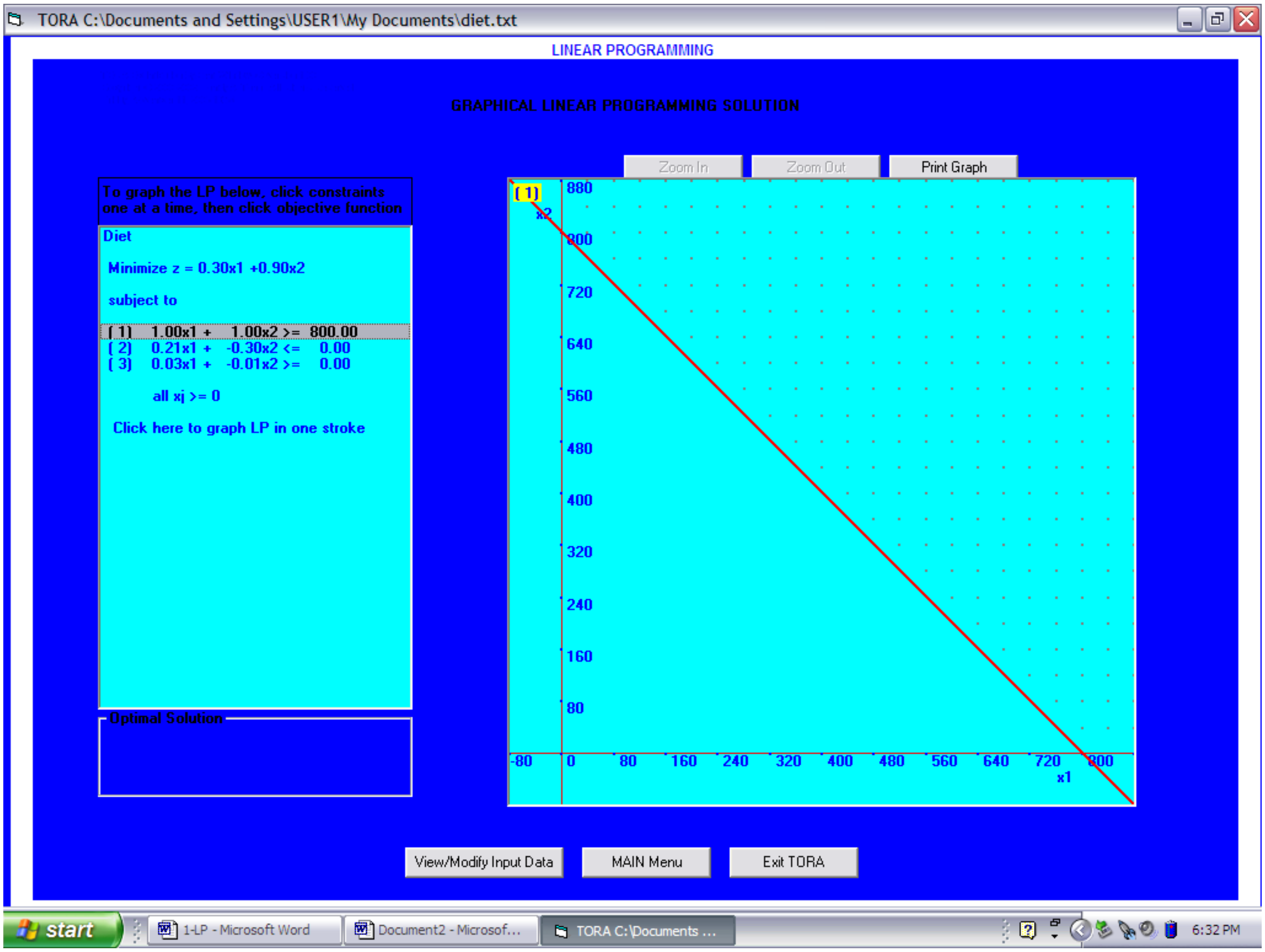
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Zoom In
Zoom Out
Print Graph
To graph the LP below, click constraints one at a time, then click objective function
Diet
Minimize z = 0.30x1 +0.90x2
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(1) 1.00x1 + 1.00x2 >= 800.00
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all xj >= 0
Click here to graph LP in one stroke
Optimal Solution
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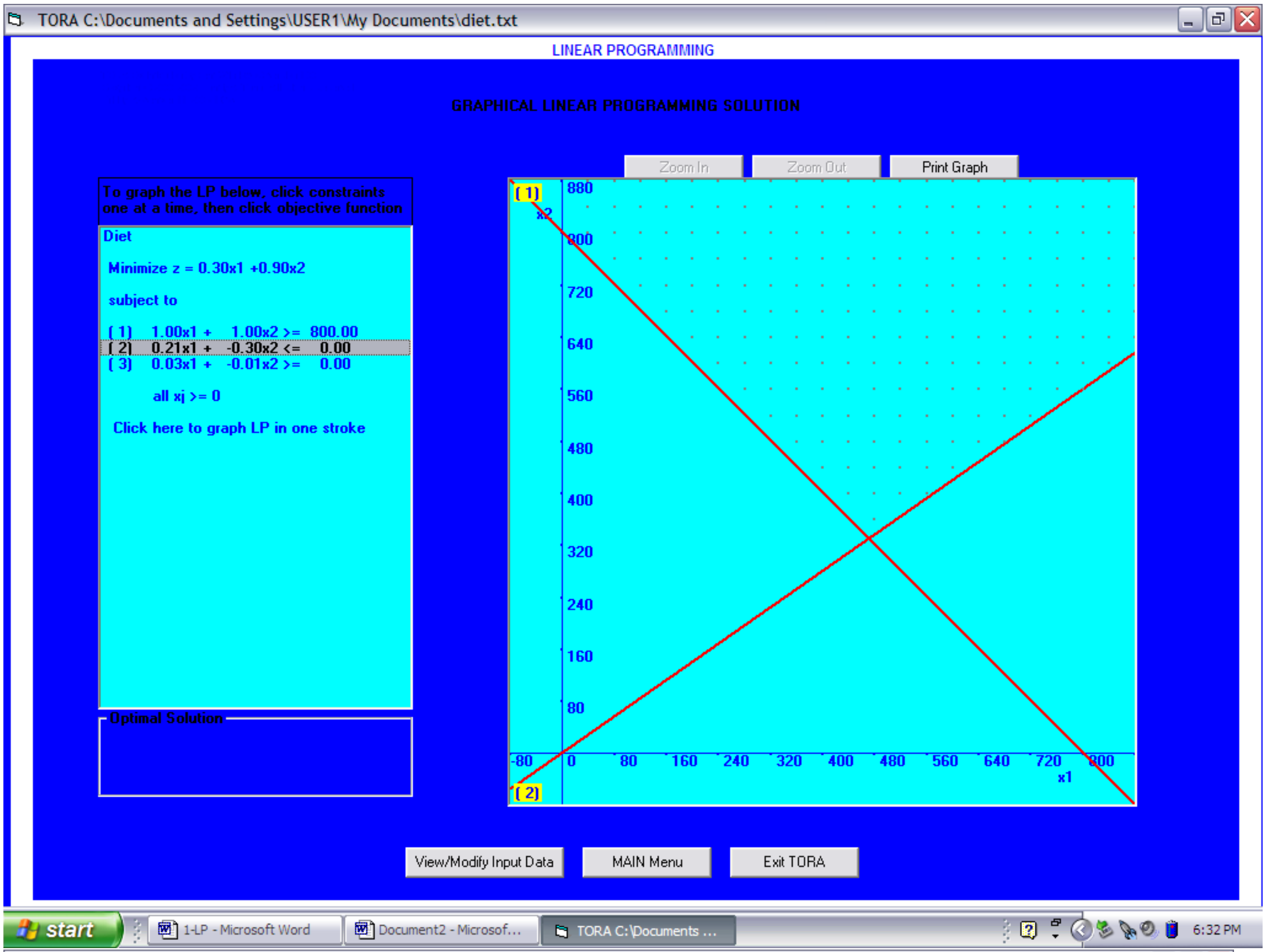
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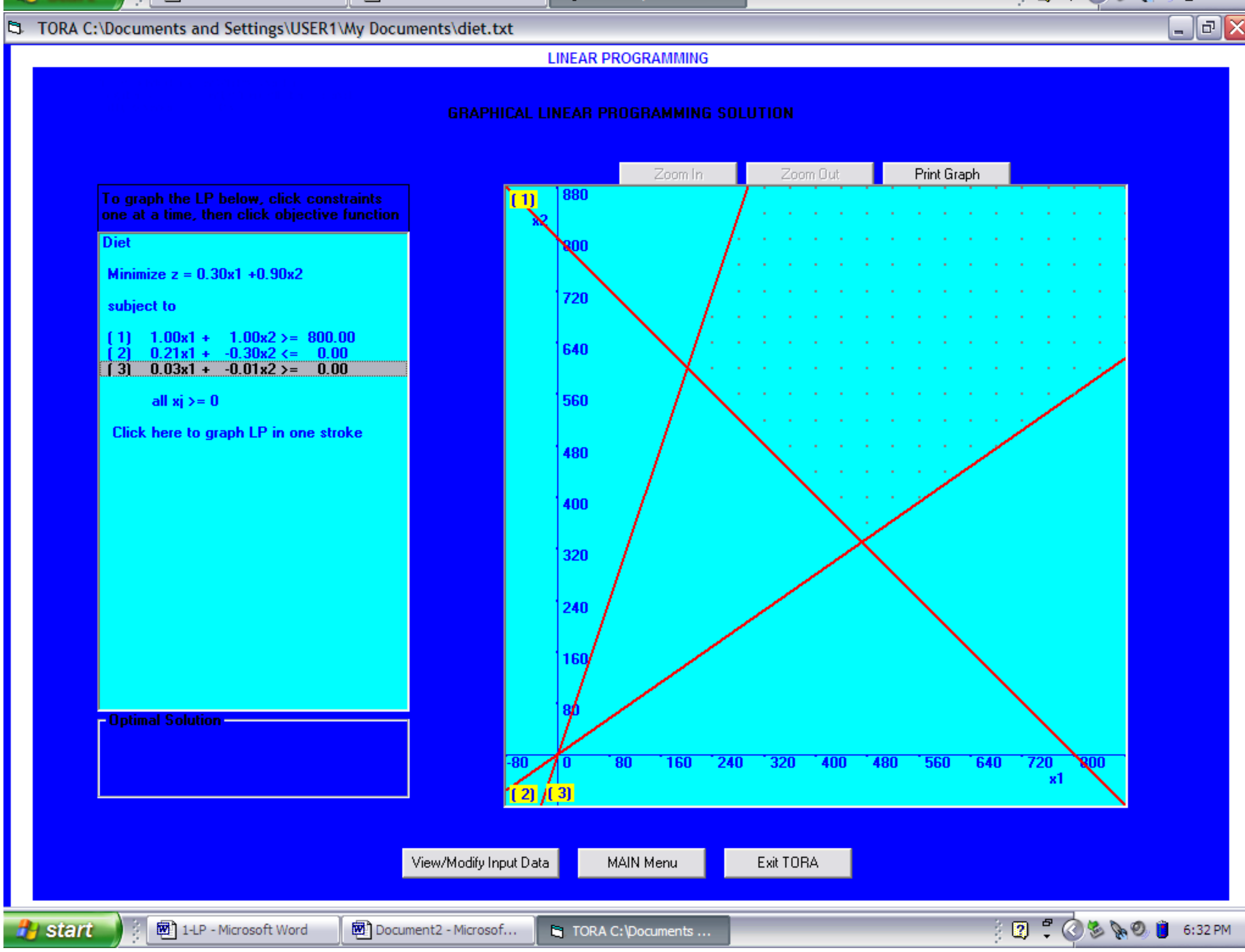
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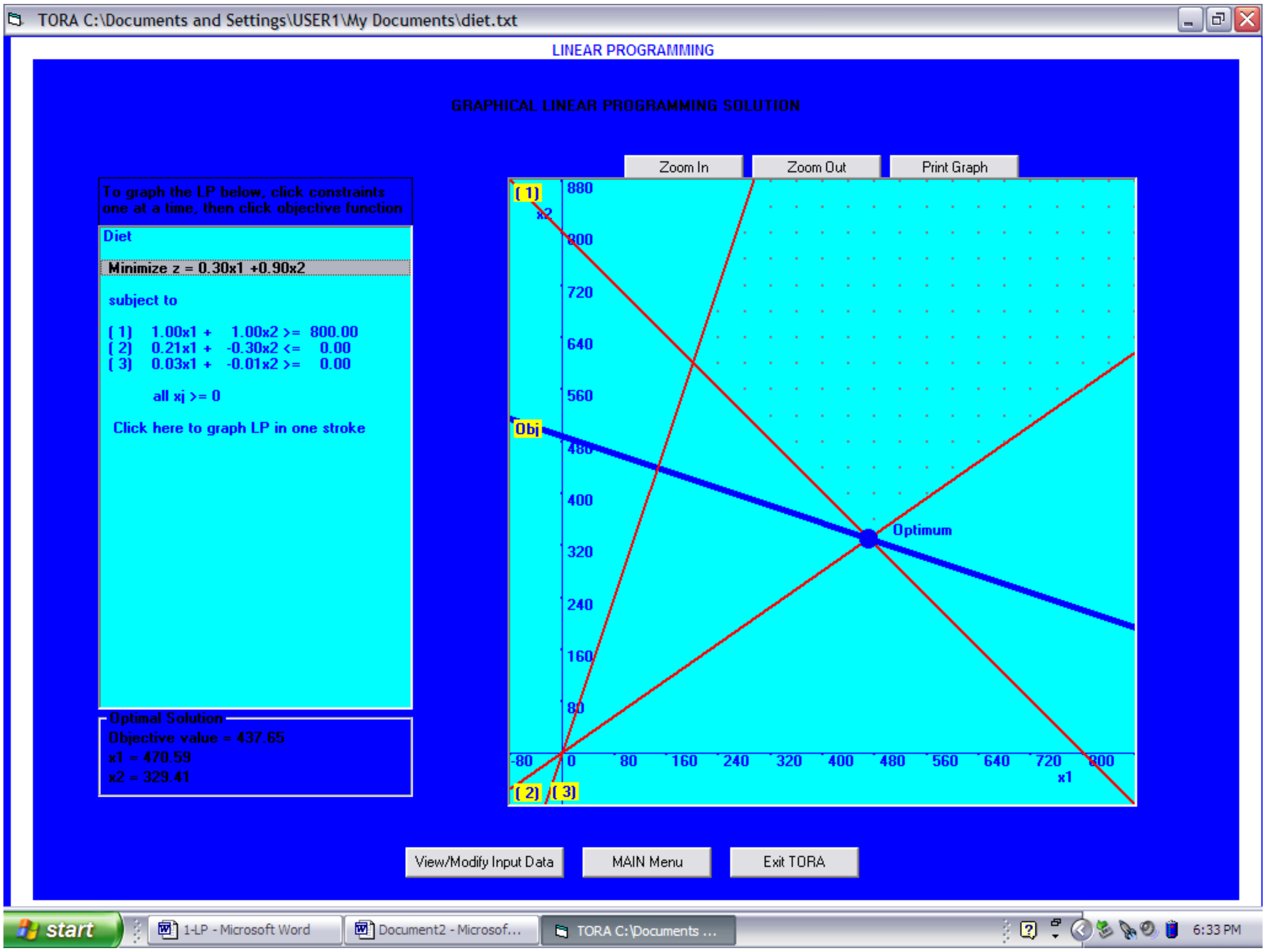


Some special cases:

1. Infeasible Solution
2. Multiple optimum solutions
3. Unbounded solution

Min $z = 40x_1 + 36x_2$

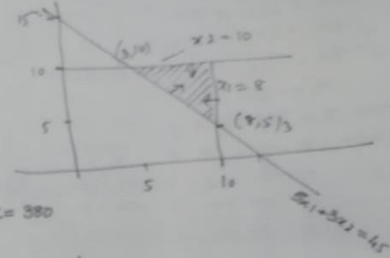
Sub to

$x_1 \leq 8$

$x_2 \leq 10$

$5x_1 + 3x_2 \geq 45$

$x_1 \geq 0,\ x_2 \geq 0$



$Z = 380$

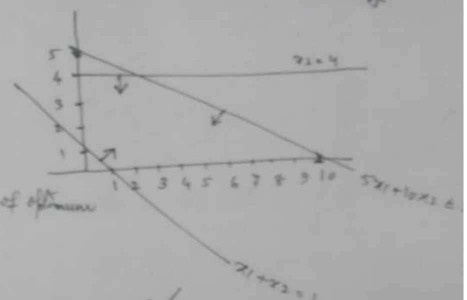
(B) Min $z = x_1 + x_2$

st $5x_1 + 10x_2 \leq 50$

$x_1 + x_2 \geq 1$

$x_2 \leq 4$

$x_1, x_2 \geq 0$



Infinite no of optimum sol^n

(C) Max $z = x_2 + 0.75x_1$

Sub to

$x_1 - x_2 \geq 0$

$-5x_1 + x_2 \leq 1$

$x_1, x_2 \geq 0$

$x_1 = 0, x_2 = 1$

$x_2 = 0, x_1 = -1/5$

$x_1 - x_2 \geq 2$

$x_1 = 0, x_2 = 0.$

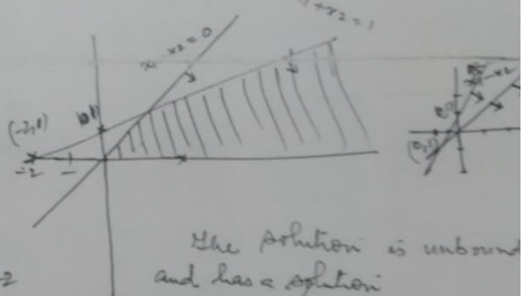
$x_1 - x_2 = 2$

$x_1 = 2, x_2 = -2$

$x_2 = 0,\ x_1 = 0$

$x_1 = 0,\ x_2 = 1$

$x_2 = 0,\ x_1 = -2$



The solution is unbound and has a solution

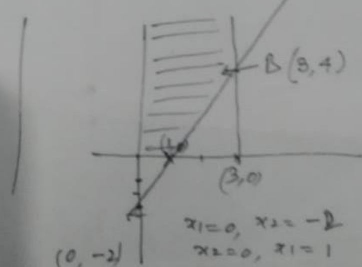
(D) Max $z = 6x_1 - 2x_2$

Sub to

$2x_1 - x_2 \leq 2$

$x_1 \leq 3,$

$x_1, x_2 \geq 0$



$z =$

$x_1 = 0,\ x_2 = -2$

$x_2 = 0,\ x_1 = 1$

Feasible Region is unbounded but optimal sol^n exists

Basic Solution (BS) : We consider the following system of equations :

$$2x_1 + x_2 - x_3 = 2$$
$$3x_1 + 2x_2 + x_3 = 3$$

And find the set of basic solutions by setting **n-m** variables equal to zero and then solve for the resulting equations. Since n = 3 and m = 2 and n-m= 1 therefore, we set one variable at a time equal to zero. First setting $x_1 = 0$, we have

$$x_2 - x_3 = 2$$
$$2x_2 + x_3 = 3$$

Solving the above equations, yield $x_2 = 5/3$ and $x_3 = 2/3$
Thus, we have $x_1 = 0$, $x_2 = 5/3$ and $x_3 = 2/3$ ----(I)
Similarly, setting $x_2 = 0$, we get $x_1 = 1$ and $x_3 = 0$ --------(II) and setting, $x_3 = 0$, we get, $x_1 = 1$ and $x_2 = 0$ -------------(III) The variables which are put equal to zero are called non-basic variables and variables for which we solve the remaining system of equations are called basic variables. They will be equal to m in number. In solution (I), variable $x_1$is a non basic variable and x_2 and x_3 are basic variables.

Degenerate Solution : If at least one of the basic variables is/are equal to zero, then we call the solution as a degenerate solution. Normally, we do not prefer degenerate solution. In the above example, we have solutions (II) and (III) as Degenerate solutions.

Non-Degenerate Solution : If none of the basic variables is zero, then we call the solution as a Non-Degenerate solution. We prefer Non-Degenerate solution. In the above example, we have solution (I) as Non-Degenerate solution.

SIMPLEX METHOD

We have already used Graphical Method to find an optimal solution to a linear Programming problem (LPP). However, we use Graphical Method to solve an LPP when we have LPP in two variables only. Thus for more than two variables, we always prefer the Simplex Method to the Graphical Method as it becomes difficult to graph more than two variables. We call it a Simplex Method because it attains it optimum at one of the simplices of the polyhedron. The Method was developed by **G.B.Dantzig** in 1947.

Example :-

$$\text{Max. } Z = 5x_1 + 3x_2$$

Subject to constraints:

$$x_1 + x_2 < 2$$
$$5x_1 + 2x_2 < 10$$
$$3x_1 + 8x_2 < 12$$
$$x_1, x_2 > 0$$

Solution :-

Step 1 : Bring the given LPP in to the standard form.

(i) No right hand side constants must be negative.

(ii) Convert all the inequality constraints into equations by the use of slack variables and check whether the LPP is in canonical form.

i.e. it has the number of linearly independent columns equal to m (number of constraints) or not? If yes, then go to step II else create an artificial basis and only then go to step II.

Introducing x_1, x_2, x_3 three slack variables into the inequality constraints and rewriting the objective function with slack variables having zero coefficients. Then the LPP Model can be re-written as :-

$$\text{Maximize } Z = 5x_1 + 3x_2 + 0\cdot x_3 + 0\cdot x_4 + 0\cdot x_5$$

Subject to constraints:

$$x_1 + x_2 + x_3 = 2$$
$$5x_1 + 2x_2 + x_4 = 10$$
$$3x_1 + 8x_2 + x_5 = 12$$
$$x_1, x_2, x_3, x_4, x_5 > 0$$

Step II: We set up an initial table as:

Iteration 0.

C_b	$C_j \rightarrow$ $X_b \downarrow$	5 y1	3 y2	0 y3	0 y4	0 y5	b	Ratio
0	x_3	**1**	1	1	0	0	2	**2/1=2 →**
0	x_4	5	2	0	1	0	10	**10/5=2**
0	x_5	3	8	0	0	1	12	**12/3=4**
Z_j - C_j		**-5 ↑**	**-3**	**0**	**0**	**0**		**Z = 0**

From the above table, we find the initial basic feasible solution (BFS) as
$x_3 = 2$, $x_4 = 10$, $x_5 = 12$, $x_1 = 0$, $x_2 = 0$,

And the value of the objective function is

$$\text{Min } Z = 5x_1 + 3x_2 + 0\cdot x_3 + 0\cdot x_4 + 0\cdot x_5 = 0 \text{ as}$$

$$Z = 5\cdot 0 + 3\cdot 0 + 0\cdot 2 + 0\cdot 10 + 0\cdot 12 = 0$$

Thus we have x_3, x_4 and x_5 as basic variables and x_1 and x_2 are non basic variables (or free variables).

***<u>Step III</u>*:-**

We find net evaluations also called relative profits as

$$Z_j - C_j = C_b^T y_j - C_j, \qquad j = 1,2,,\ldots,5$$

If all **Z_j - $C_j \geq 0$** for a maximization problem, then stop, and the current basic feasible solution is also optimal. If at least one Z_j - C_j is negative, then go to Step IV.

<u>Step IV:</u>

(a) Choose the entering variable with a negative value in Z_j - C_j row (most negative in case of more than one Z_j - C_j are negative) .
In our case, it is **5** corresponding to **y_1** column, therefore, **x_1** is an **entering variable** and the corresponding column is known as **pivotal column**.

(b) After finding the pivot column corresponding the most negative value in last row (Z_j - C_j) say j[th] column. We choose the leaving variable with the minimum ratio, calculated as min (b_i / y_{jr}, r=1,2,3 i.e. solution col./$y_j$col., y_j > 0). In our case, we have min (2/1=2, 10/5=2, 12/3=4) =2 which corresponds to x_3**.** In case of a tie, we break the ties arbitrarily. Thus, corresponding to the minimum ratio **2** ,x_3 is chosen as the **leaving variable** and the corresponding row is known as **pivotal row** .
The common element between the pivotal column and pivotal row is the **pivotal element**. Thus, we replace $\mathbf{x_3}$ by $\mathbf{x_1}$ in the basis and x_1 becomes the basic variable whereas x3 becomes a non-basic variable. Then, we go to StepV.

Step V:

We update the new table by making pivotal element **1** and the remaining elements in that column zeroes by using row operations.

Iteration 1

C_b	$C_j\rightarrow$ $X_b\downarrow$	5 y1	3 y2	0 y3	0 y4	0 y5	b
5	x_1	1	1	1	0	0	2
0	x_4	0	-3	-5	1	0	0
0	x_5	0	5	-3	0	1	6
Zj-Cj		0	2	5	0	0	**Z = 10**

Since all $\mathbf{Z_j - C_j \geq 0}$, the given table is optimal and the optimal solution is x_1= 5, x_2 = 0, x_3 = 0, x_4 = 0 and x_5 = 0 and the optimal value of the objective function is

$$\boldsymbol{Max\ Z = 5x_1 + 3x_2 + 0.x_3 + 0.x_4 + 0.x_5 = 10}$$

We can solve the given LPP as a minimization LPP also.

Max. $\mathbf{Z = 5x_1 + 3x_2}$

Subject to constraints :

$$x_1 + x_2 < 2$$
$$5x_1 + 2x_2 < 10$$
$$3x_1 + 8x_2 < 12$$
$$x_1,\ x_2 \geq 0$$

Maximize $Z = 5x_1 + 3x_2 + 0{\cdot}x_3 + 0{\cdot}x_4 + 0{\cdot}x_5$

or **Max Z = -{Min(-Z)}**

We take Min(-Z) = $\{-5x_1-3x_2+0{\cdot}x_3+0{\cdot}x_4+0{\cdot}x_5\}$

We will use only $\{-5x_1-3x_2+0{\cdot}x_3+0{\cdot}x_4+0{\cdot}x_5\}$ and will multiply the final value by -1 once again.

Subject to constraints:

$$x_1+x_2+x_3=2$$
$$5x_1+2x_2+x_4=10$$
$$3x_1+8x_2+x_5=12$$
$$x_1,\ x_2,\ x_3,\ x_4,\ x_5>0$$

Step II: We set up an initial table as :

Iteration 0

C_b	$C_j \rightarrow$ $X_b \downarrow$	5 y_1	3 y_2	0 y_3	0 y_4	0 y_5	b	Ratio
0	x_1	**1**	1	1	0	0	2	2/1=2 →
0	x_4	0	-3	-5	1	0	0	10/5=2
0	x_5	0	5	-3	0	1	6	12/3=4
$Z_j - C_j$		5 ↑	3	0	0	0	**Z = 10**	

From the above table, we find the initial basic feasible solution (BFS) as
$\mathbf{x_3 = 2, x_4 = 10, x_5 = 12, x_1 = 0, x_2 = 0,}$

And the value of the objective function is

$$\mathbf{Min\ Z = -5x_1 - 3x_2 + 0 \cdot x_3 + 0 \cdot x_4 + 0 \cdot x_5 = 0}$$

$$\mathbf{as\ Z = -5 \cdot 0 - 3 \cdot 0 + 0 \cdot 2 + 0 \cdot 10 + 0 \cdot 12 = 0}$$

Thus we have x_3, x_4 and x_5 as basic variables and x1 and x_2 are non basic variables (or free variables).

Step III:

We find net evaluations also called relative profits as $\mathbf{Z_j - C_j} = C^T y_j - C_j$, j=1,2,…,5. If all $\mathbf{Z_j - C_j \leq 0}$ for a minimization problem, then stop, and the current basic feasible solution is also optimal. If at least one $Z_j - C_j$ is positive, then go to Step IV.

Step IV :

(a) Choose the entering variable with a negative value in $Z_j - C_j$ row (most negative in case of more than one $Z_j - C_j$ are negative) .
In our case, it is **5** corresponding to y_1 column, therefore, x_1 is an **entering variable** and the corresponding column is known as **pivotal column**.

(b) After finding the pivot column corresponding the most negative value in last row ($Z_j - C_j$) say jth column. We choose the leaving variable with the minimum ratio, calculated as min (bi/yjr, r=1,2,3 i.e. solution col./yjcol. ,yj > 0). In our case, we have min (2/1=2, 10/5=2, 12/3=4) =2 which corresponds to x_3**.** In case of a tie, we break the ties arbitrarily. Thus, corresponding to the minimum ratio **2** ,x_3 is chosen as the **leaving variable** and the corresponding row is known as **pivotal row** . The common element between the pivotal column and pivotal row is the **pivotal element**. Thus, we replace $\mathbf{x_3}$ by $\mathbf{x_1}$ in the basis and x1 becomes the basic variable whereas x_3 becomes a non-basic variable. Then, we go to Step V.

Step V: *We update the new table by making pivotal element 1 and the remaining elements in that column zeroes by using row operations.*

Iteration 1

C_b	$C_j\rightarrow$ $X_b\downarrow$	-5 y1	-3 y2	0 y3	0 y4	0 y5	b
-5	x_1	1	1	1	0	0	2
0	x_4	0	-3	-5	1	0	0
0	x_5	0	5	-3	0	1	6
Zj-Cj		0	-2	-5	0	0	**Z = - 10**

Since all *Zj-Cj are less or equal to zero, the given table is optimal and the optimal solution is* ***x_1= 5, x_2=0, x_3=0, x_4=0 and x_5 = 0*** *and optimal value of the objective function is*

Min Z = $-5x_1 - 3x_2 + 0.x_3 + 0.x_4 + 0.x_5 = -10$
Therefore , ***Max Z = -(-10) = 10***
We have to multiply by -1 as we had used the relation
Max Z = -(Min(-Z)) .

Thus, we would prefer solving minimization LPP to maximization LPP so that there is no confusion in remembering the optimality conditions for the two problems.

Artificial Variable Technique

If the number of basic variables in an LPP are not equal to the number of the constraints (m) i.e., an LPP, which is not in a canonical form even after introducing slack variables (constraints having $\geq$ and = signs) , we introduce a new type of variable/s , called the ***artificial variable/s***. These variables are fictitious and cannot have any physical meaning.
Big M-Method (also known as **Penalty Method**) and **Two Phase Simplex Method are** used to solve such problems.

Big M-Method (Penalty Method)

In this method, we assign very large costs in the objective function. The general practice is to assign the letter -M for a maximization problem and M for a minimization problem

Maximize $Z = 6x_1 + 4x_2$
Sub to

$$2x_1 + 3x_2 \leq 30$$
$$3x_1 + 2x_2 \leq 24$$
$$x_1 + x_2 \geq 3$$
$$x_1, x_2 \geq 0$$

We introduce slack and artificial variables into the problem
Maximize $Z = 6x_1 + 4x_2 + 0 \cdot s_1 + 0 \cdot s_2 + 0 \cdot s_3 – Ma_1$
Sub to

$$2x_1 + 3x_2 + s_1 = 30$$
$$3x_1 + 2x_2 + s_2 = 24$$
$$x_1 + x_2 - s_3 + a_1 = 3$$

$x_1, x_2, s_1, s_2, s_3 \geq 0$, where s_1, s_2, s_3 are slack variables and a_1 is an artificial variable.

Iteration 0.

C_b	X_b	6	4	0	0	0	-M	b	Ratio
		y_1	y_2	y_3	y_4	y_5	y_6		
0	s_1	2	3	1	0	0	0	30	30/2=15
0	s_2	3	2	0	1	0	0	24	24/3=8
-M	a_1	1	1	0	0	-1	1	3	3/1=3 →
	Z_j-C_j	-M -6 ↑	-M-4	0	0	M	0	Z=-3M	

Iteration 1

C_b	X_b	6	4	0	0	0	b	Ratio
		y_1	y_2	y_3	y_4	y_5		
0	s_1	0	1	1	0	2	24	24/2=12
0	s_2	0	-1	0	1	**3**	15	15/3=5 →
6	x_1	1	1	0	0	-1	3	-
	Z_j-C_j	0	2	0	0	-6 ↑	**Z=18**	

We have deleted the information pertaining to the artificial variable from the Simplex Table in the following Iteration 2 as it is no more required.

Iteration 2

C_b	X_b	6	4	0	0	0	b
		y_1	y_2	y_3	y_4	y_5	
0	s_1	0	-5/3	1	-2/3	0	14
0	s_3	0	-1/3	0	1/3	1	5
6	x_1	1	2/3	0	1/3	0	8
	Z_j-C_j	0	0	0	2	0	**48**

Since all $\mathbf{Z_j\text{-}C_j \geq 0}$, therefore, the given solution is optimal. The optimal solution **is $x_1 = 8$, $x_2 = 0$ $s_1 = 14$, $s_2 = 0$ and $s_3 = 5$.**
The value of the objective function is **48.**

Example 2. Big M-Method Method:

Minimize $Z=-3x_1+x_2+x_3$

Sub to
$$x_1 - 2x_2 + x_3 \leq 11$$
$$-4x_1 + x_2 + 2x_3 \geq 3$$
$$2x_1 \quad - x_3 = -1$$
$$x_1, x_2, x_3 \geq 0$$

<u>Standarazing the LPP</u>

Minimize $Z = -3x_1 + x_2 + x_3 + 0 \cdot s_1 - 0 \cdot s_2 + Ma_1 + Ma_2$

Sub to
$$x_1 - 2x_2 + x_3 + s_1 = 11$$
$$-4x_1 + x_2 + 2x_3 - s_2 + a_1 = 3$$
$$-2x_1 \quad + x_3 \quad + a_2 = 1$$

$x_1, x_2, x_3, +s_1, s_2 \geq 0$ and a_1 and a_2 are artificial variables.

Iteration 0

C_b	X_b	-3	1	1	0	0	M	M	b	Ratio
		y_1	y_2	y_3	y_4	y_5	y_6	y_7		
0	S_1	1	-2	1	1	0	0	0	11	11/1=11
M	a_1	-4	1	2	0	-1	1	0	3	3/2=1.5
M	a_2	-2	0	**1**	0	0	0	1	1	1/1=1→
	Z_j-C_j	-6M+3	M-1	3M-1↑	0	-M	0	0	**Z=4M**	

Iteration 1

C_b	**X_b**	**-3**	**1**	**1**	**0**	**0**	**M**	**M**	**b**	**Ratio**
		y_1	**y_2**	**y_3**	**y_4**	**y_5**	**y_6**	**y_7**		
0	S_1	3	-2	0	1	0	0	-1	10	-
M	a_1	0	**1**	0	0	-1	1	-2	1	1/1=1→
1	x_3	-2	0	1	0	0	0	1	1	-
	Z_j-C_j	-1	M+1 ↑	0	0	-M	0	-3M+1	**Z=M+1**	

Iteration 2

C_b	X_b	-3	1	1	0	0	M	M	b	Ratio
		y_1	y_2	y_3	y_4	y_5	y_6	y_7		
0	s_1	***3***	0	0	1	-2	2	-5	12	**12/3=4→**
1	x_2	0	1	0	0	-1	1	-2	1	**-**
1	x_3	-2	0	1	0	0	0	1	1	**-**
	Z_j-C_j	1 ↑	0	0	0	-1	1-M	-1-M	**Z=2**	

Iteration 3

C_b	X_b	-3	1	1	0	0	M	M	b
		y_1	y_2	y_3	y_4	y_5	y_6	y_7	
-3	x_1	1	0	0	1/3	-2/3	2/3	-5/3	4
1	x_2	0	1	0	0	-1	1	-2	1
1	x_3	0	0	1	2/3	-4/3	4/3	-7/3	9
	Z_j-C_j	0	0	0	-1/3	-1/3	1/3-M	2/3+M	**Z=-2**

Since all $Z_j - C_j \geq 0$, therefore, the given solution is optimal. The optimal solution **is $x_1 = 4$, $x_2 = 1$, $x_3 = 9$, $s_1 = 0$, s2=0.** The value of the objective function is **-2.**

Remark 1: An artificial variable is added merely as a basic variable in a particular equation. Once, it is replaced by a real (decision) variable, there is no need to retain the artificial variable in the Simplex Table.
Remark 2: If an artificial variable remains in the optimal table, then the problem is infeasible.
Remark 3: For Computer solutions M, has to be given a specific value, every time, the problem is solved, this may not be convenient. Thus, if the computer solution is required, then we use Two-Phase Method

Two-Phase Method:

Phase 1: In Phase 1, we create an auxiliary objective function as the sum of only the artificial variables and minimize the sum to zero. Once the value of the objective function is zero, Phase 1 terminates
Phase 2: In Phase 2, we assign the coefficients back to the original variables and use the Simplex method in the usual manner.

Minimize $Z = -3x_1 + x_2 + x_3$

Sub to

$$x_1 - 2x_2 + x_3 \leq 11$$
$$-4x_1 + x_2 + 2x_3 \geq 3$$
$$2x_1 \quad -x_3 = -1$$
$$x_1, x_2, x_3 \geq 0$$

<u>Standarizing the problem</u>

Minimize $Z= -3x_1 + x_2 + x_3 + 0 \cdot s_1 - 0 \cdot s_2 + Ma_1 + Ma_2$

Sub to

$$x_1 - 2x_2 + x_3 + s_1 = 11$$
$$-4x_1 + x_2 + 2x_3 - s_2 + a_1 = 3$$
$$-2x_1 \quad + x_3 \quad + a_2 = 1$$

$x_1, x_2, x_3, \; s_1, s_2 \geq 0$ and a_1 and a_2 are artificial variables.

Iteration 0.

Min W = $a_1 + a_2$

C_b	X_b	0	0	0	0	0	1	1	b	Ratio
		y_1	y_2	y_3	y_4	y_5	y_6	y_7		
0	S_1	1	-2	1	1	0	0	0	11	11/1=11
1	a_1	-4	1	2	0	-1	1	0	3	3/2=1.5
1	a_2	-2	0	1	0	0	0	1	1	1/1=1
	Z_j-C_j	-6	1	3	0	-1	0	0		

Iteration 1

C_b	**X_b**	**0**	**0**	**0**	**0**	**0**	**1**	**1**	**b**	**Ratio**
		y_1	**y_2**	**y_3**	**y_4**	**y_5**	**y_6**	**y_7**		
0	S_1	3	-2	0	1	0	0	-1	10	-
1	a_1	0	**<u>1</u>**	0	0	-1	1	-2	1	1/1=1→
0	x_3	-2	0	1	0	0	0	1	1	-
	Z_j-C_j	0	1 ↑	0	0	-1	0	-3	**W=1**	

Iteration 2

C_b	X_b	-3	1	1	0	0	M	M	b
		y_1	y_2	y_3	y_4	y_5	y_6	y_7	
0	S_1	3	0	0	1	-2	2	-5	12
0	x_2	0	1	0	0	-1	1	-2	1
0	x_3	-2	0	1	0	0	0	1	1
	Z_j-C_j	0	0	0	0		-1	-1	**W=0**

Iteration 3

C_b	X_b	-3	1	1	0	0	b	Ratio
		y_1	y_2	y_3	y_4	y_5		
0	S_1	$\underline{3}$	0	0	1	-2	12	12/3=4→
1	x_2	0	1	0	0	-1	1	-
1	x_3	-2	0	1	0	0	1	-
	Z_j-C_j	1 ↑	0	0	0	-1	**Z=2**	

Iteration 4

C_b	X_b	-3	1	1	0	0	b
		y_1	y_2	y_3	y_4	y_5	
-3	x_1	1	0	0	1/3	-2/3	4
1	x_2	0	1	0	0	-1	1
1	x_3	0	0	1	2/3	-4/3	9
	Z_j-C_j	0	0	0	-1/3	-1/3	**Z=-2**

Dual Simplex Method

We use Simplex Method with the condition that the right side constants must be non-negative. The Simplex method starts with a feasible solution but is non optimal solution. This improves the feasible solution to optimal solution. However, in Dual Simplex Method, it has no problem with negative right hand side constants and the Method starts with an optimal solution but with an infeasible solution and it improves the feasibility to optimality.

Dual Simplex Method : Iterative steps involved in the dual simplex method are:

1. All the constraints (except those with equality (=) sign) are modified to 'less-than-equal-to' sign. Constraints with greater-than-equal-to' sign are multiplied by -1 through out so that inequality sign gets reversed. Finally, all these constraints are transformed to equality sign by introducing required slack variables.
2. Modified problems, as in step 1, is expressed in the form of a simplex table. If all the cost coefficients are positive (i.e. optimality condition is satisfied) and one or more basic variables have negative values (i.e. non-feasible solution), then dual simplex method is applicable.
3. **Selection of exiting variable :** The basic variable with the highest negative value is the exiting variable. If there are two candidates for exiting variable, any one is selected. The row of the selected exiting variable is marked as pivotal row.
4. **Selection of entering variable :** Cost coefficients, corresponding to all the negative elements of the pivotal row, are identified and their ratios are calculated using the coefficients of Z-equation as the denominators except zeroes of pivotal row, i.e., The column corresponding to minimum ratio is identified as the pivotal column and associated decision variable is the entering variable.
5. **Pivotal operation :** Pivotal operation is exactly the same as in the case of simplex method, considering the pivotal element as the element at the intersection of pivotal row and pivotal column.
6. **Check for optimality :** If all the basic variables have nonnegative values then the optimum solution is reached. Otherwise, Steps 3 to 5 are repeated until the optimum is reached.

Example:

$$\text{Minimize } \mathbf{Z = 2x_1 + x_2}$$

Sub to

$$\mathbf{3x_1 + 3x_2 \geq 3}$$
$$\mathbf{4x_1 + 3x_2 \geq 6}$$
$$\mathbf{x_1 + 2x_2 \leq 3}$$
$$\mathbf{x_1, x_2 \geq 0}$$

We multiply constraints I and II by -1 to get the following LPP:

$$\mathbf{Min\ Z = 2x_1 + x_2}$$

Sub to $\mathbf{-3x_1 - x_2 \leq -3}$

$$\mathbf{-4x_1 - 3x_2 \leq -6}$$
$$\mathbf{x_1 + 2x_2 \leq 3}$$
$$\mathbf{x_1, x_2 \geq 0}$$

Standardizing the LPP :

$$\mathbf{Min\ Z = 2x_1 + x_2 + 0 \cdot s_1 + 0 \cdot s_2 + 0 \cdot s_3}$$

Sub to

$$\mathbf{-3x_1 - x_2 + s_1 = -3}$$
$$\mathbf{-4x_1 - 3x_2 + s_2 = -6}$$
$$\mathbf{x_1 + 2\,x_2 + s_3 = 3}$$
$$\mathbf{x_1, x_2, s_1, s_2, s_3 \geq 0,}$$

where s_1, s_2 and s_3 are slack variables.

We prepare the usual Simplex table.

Iteration 0

C_b	X_b	2	1	0	0	0	b
		y_1	y_2	y_3	y_4	y_5	
0	s_1	2	3	1	0	0	**-3**
0	s_2	-4	-3	0	1	o	**-6**
0	s_3	1	1	0	0	-1	**3**
	Z_j-C_j	**-2**	**-1**	**0**	**0**	**0**	**Z=0**

S_2 equation	:	**-4**	**- 3**	**0**	**1**	**0**
z_j-c_j	:	**-2**	**-1**	**0**	**0**	**0**
Ratios	:	**1/2**	**1/3**	**-**	**-**	**-**

Since smallest **ratio is 1/3**, therefore entering variable is **x_2**. We update the simplex table as usual.

Iteration 1

C_b	X_b	2	1	0	0	0	b
		y_1	y_2	y_3	y_4	y_5	
0	s_1	-5/3	0	1	-1/3	0	**-1**
1	x_2	4/3	1	0	-1/3	0	**2**
0	s_3	-5/3	0	0	2/3	1	**-1**
	Z_j-C_j	**-2/3**	**0**	**0**	**-1/3**	**0**	**Z=2**

Since there is a tie in **b_i, s_1** is arbitrarily selected to leave. The ratios are

S_2	-5/3	0	0	-1/3	0
Z_j - C_j	-2/3	0	0	-1/3	1
Ratio	2/5	0	0	1	0

Since smallest ratio is **2/5,** therefore entering variable is **x_1**. We update the simplex table as usual.

Iteration 2

C_b	X_b	2	1	0	0	0	b
		y_1	y_2	y_3	y_4	y_5	
2	s_1	1	0	-3/5	1/5	0	**3/5**
1	x_2	0	1	4/5	-3/5	0	**6/5**
0	s_3	0	0	-1	1	1	**0**
	Z_j - C_j	**0**	**0**	**-2/5**	**-1/5**	**0**	**Z=12/5**

Since all **z_j-$c_j \leq$ 0,** therefore, the given solution is optimal. The optimal solution is **x_1= 3/5, x_2 = 6/5 s_1=0, s2=0** and **s3=0.** The value of the objective function is **Z=12/5.**

Revised Simplex Method

We use Revised Simplex Method for large scale Linear programming problem (LPP). Revised simplex Method uses only small portion of the Simplex Table and saves lot of unnecessary computations. We consider the following example for the sake of illustrations as the given problem is not a large LPP.

Example 1 :

$$\textbf{Max. } Z = 40x_1 + 30x_2$$

Subject to constraints:

$$5x_1 + 4x_2 \le 400$$

$$x_1 \le 60$$

$$x_2 \le 75$$

$$x_1, x_2 \ge 0$$

Step I. Standardizing the given LPP:

$$\text{Max. } Z = 40x_1 + 30x_2 + 0\cdot x_3 + 0.x_4 + 0\cdot x_5$$

Subject to constraints:

$$5x_1 + 4x_2 + x_3 = 400$$

$$x_1 + x_4 = 60$$

$$x_2 + x_5 = 75$$

$$x_1, x_2, x_3, x_4, x_5 \ge 0$$

Step I : We express our set of constraints in the following form $\mathbf{x_1P_1 + x_2P_2 + \ldots + x_nP_n = b}$ where $\mathbf{P_1, P_2, \ldots, P_n}$ are the columns of our coefficients of decision and slack variables and b is a column vector of right hand side constants. In our example, we have

$$P1 = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}, P2 = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix} P3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} P4 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} P5 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, b = \begin{bmatrix} 400 \\ 60 \\ 75 \end{bmatrix}$$

In our case, we have $\mathbf{x_1P_1 + x_2P_2 + x_3P_3 + x_4P_4 + x_5P_5 = b}$

We prepare the following table:

C_b	X_b	B^{-1}			b
0	x_3	1	0	0	400
0	x_4	0	1	0	60
0	x_5	0	0	1	75

Step II : We calculate Simplex Multipliers as given below:

$$\pi = (\pi_1, \pi_2, \pi_3) = C_b^T B^{-1} = (0,0,0)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = (0,0,0)$$

$$where\ C_b^T = (0,0,0)\ \ and\ B^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Step III :*We* $calculate\ \overline{C_j} = \pi P_j - C_j,\ for\ j = 1,2\ (Non - basic\ \text{var}\ iables)$

$$\overline{C_1} = \pi P_1 - C_1 = (0,0,0)\begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix} - 40 = -40$$

$$and\ \overline{C_2} = \pi P_2 - C_2$$

$$\overline{C_2} = \pi P_2 - C_2 = (0,0,0)\begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix} - 30 = -30$$

If all C_j non-negative, then we terminate the process and the optimal solution has been obtained. In case, any of the is negative then we go to step IV.

Step IV. We select an entering variable corresponding to the negative $\bar{C}_j$ (most negative in case of more than one $\bar{C}_j$ being negative). We break the ties arbitrarily. Since more than one $\bar{C}_j$ are negative (-40 and -30), we choose -40 i.e., $\bar{C}_1$ .Thus x_1 is the entering variable and we go to step V.

Step V: For selecting the leaving variable, we first calculate the Pivotal Column $\bar{P}_1 = B^{-1} P_1$ corresponding to $\bar{C}_1$

$$\bar{P}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix} \textit{and calculate the ratios as}$$

$$\frac{b}{\bar{P}_1} \textit{as given in the following table.}$$

and choose the leaving variable corresponding to the smallest ratio . Thus x_4 becomes the leaving variable. We update the Simplex Table as usual

We prepare the following table. We choose the variable with minimum ratio as the leaving variable

C_b	X_b	B^{-1}			b	Pivotal Column	Ratio ($\frac{b}{\bar{P}_1}$)	
0	x_3	1	0	0	400	5	400/5=80	0
0	x_4	0	1	0	60	1	60/1=60	1
0	x_5	0	0	1	75	0	-	0

Thus, x_4 is the leaving variable and the new updated table is produced below;

C_b	X_b	B^{-1}			b
0	x_3	1	5	0	100
40	x_1	0	1	0	60
0	x_5	0	0	1	75

We calculate Simplex Multipliers again

We calculate again

$$\pi = (\pi_1, \pi_2, \pi_3) = C_b^T B^{-1} = (0,40,0)\begin{bmatrix} 1 & -5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = (0,40,0) \textit{ and}$$

$$\overline{C}_j = \pi\overline{P}_1 = B^{-1}P_1 - C_j, \ \textit{for } j = 2,4 (\textit{Non} - \textit{basic variables})$$

$$\overline{C}_2 = \pi P_2 - C_2 = (0,40,0)\begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix} - 30 = -30$$

$$\textit{and } \overline{C}_4 = \pi P_4 - C_4$$

$$\overline{C}_4 = \pi P_4 - C_4 = (0,40,0)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - 0 = 40$$

Since $\overline{C}_2 < 0$, thus, variable x_2 enters the basis. We calculate the ratios to identify the leaving variable

C_b	X_b	B^{-1}			b	Pivot Column $\overline{P}_2$	Ratios $\frac{b}{\overline{P}_2}$	
0	x_3	1	-5	0	100	4	100/4=25	1
40	x_1	0	1	0	60	0	-	0
0	x_5	0	0	1	75	1	75/1=75	0

Thus x_3 will leave the basis and we update the table

C_b	X_b	B^{-1}			b
30	x_2	1/4	-5/4	0	25
40	x_1	0	1	0	60
0	x_5	-1/4	5/4	1	50

We calculate again

$$\pi = (\pi_1, \pi_2, \pi_3) = C_b^T B^{-1} = (30,40,0)\begin{bmatrix} \frac{1}{4} & -\frac{5}{4} & 0 \\ 0 & 1 & 0 \\ -\frac{1}{4} & \frac{5}{4} & 1 \end{bmatrix} = \left(\frac{15}{2}, \frac{5}{2}, 0\right)$$

$$\overline{C}_j = \pi P_j - C_j, \quad for\ j = 3,4 (Non-basic\ variables)$$

$$\overline{C}_3 = \pi P_3 - C_3$$

$$\overline{C}_3 = \left(\frac{15}{2}, \frac{5}{2}, 0\right)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - 0 = \frac{15}{2}$$

$$and\ \overline{C}_4 = \pi P_4 - C_4$$

$$\overline{C}_4 = \pi P_4 - C_4 = \left(\frac{15}{2}, \frac{5}{2}, 0\right)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - 0 = \frac{5}{2}$$

Since $\overline{C}_3, \overline{C}_4 > 0$, an optimal solution has been reached and the optimal solution is $x_1 = 60$, $x_2 = 25$ $x_3 = 0$, $x_4 = 4$ and $x_5 = 50$ and the optimal value is **Rs 3150.**

DEGENERACY

Introduction :- At the stage of improving the solution during Simplex procedure, if a **tie for the minimum ratio occurs** at least one basic variable becomes equal to zero in the next iteration and the new solution is said to be **Degenerate.**

C_b	X_b	3	9	0	0	b	Ratio
		x_1	x_2	s_1	s_2		
0	s_1	1	**4**	1	0	8	2 →
0	s_2	1	2	0	1	4	2
	Z_j-C_j	-3	-9 ↑	0	0	0	

In this table s_1 and s_2 tie for the leaving variable. So any one can be considered as leaving variable. Therefore, x_2 is the entering variable and s_1 is the departing variable.

C_b	X_b	3	9	0	0	b	Ratio
		x_1	x_2	s_1	s_2		
0	x_2	1/4	1	1/4	0	2	8
0	s_2	**<u>1/2</u>**	0	-1/2	1	0	0 →
	Z_j-C_j	-3/4 ↑	0	9/4	0	**Z = 18**	

Therefore, x_1 is the entering variable and s_2 is the departing variable.

C_b	Basic Variable	3	9	0	0	b
		x_1	x_2	s_1	s_2	
0	x_2	0	1	1/2	-1/2	2
0	x_1	1	0	-1	2	0
	Z	0	0	3/2	3/2	**18**

Optimal Solution is : **x_1 = 0, x_2 = 2**

Z = 18

It results in a **Degenerate Basic Solution.**

- -

Convex Set : A set in n-dimensional space is said to be a convex set if

(i) For any two points $X_1, X_2 \in S$, then the line segment joining these two points is also in the set

(ii) Mathematically, if $X_1, X_2 \in S$, then $(\lambda X_1 + (1-\lambda)X_2) \in S, 0 \le \lambda \le 1$. It may be noted that the set S containing a single point is a convex set.

A linear functional f(X) is a real valued defined on an n-dimensional vector space such that for every vector $X = \alpha U + \beta V$

$f(X) = f(\alpha U + \beta V) = \alpha f(U) + \beta f(V)$ for all vectors (n-dimensional) U and V and all scalars α and β.

Example : **f(x) = 2x₁ + x₂, U = (3,4), V = (4,6),** $\alpha = \frac{1}{3}, \beta = \frac{2}{3}$

$X = \alpha U + \beta V = \frac{1}{3}(3,4) + \frac{2}{3}(4,6)$

$= (1, \frac{4}{3}) + (\frac{8}{3}, 4)$

$= (1 + \frac{8}{3}, \frac{4}{3} + 4)$

$= (\frac{11}{3}, \frac{16}{3})$

$f(x) = f(\alpha U + \beta V) = f(x)\big|_{(11/3,16/3)} = (2.\frac{11}{3} + 1.\frac{16}{3}) = \frac{38}{3}$

$\alpha f(U) + \beta f(V) = \frac{1}{3} f(x)\big|_{(3,4)} + \frac{2}{3} f(x)\big|_{(4,6)}$

$= \frac{1}{3}(2.3 + 1.4) + \frac{2}{3}(2.4 + 1.6)$

$= \frac{10}{3} + \frac{28}{3}$

$= \frac{38}{3}$

This implies that $f(\alpha U + \beta V) = \alpha f(U) + \beta f(V)$

Linear Convex Combination: The line segment between two points X1 and X2 is the set of all points

$X = \lambda X_1 + (1-\lambda)X_2$ for $0 \le \lambda \le 1$. This can be extended to a more general form

$X = \lambda_1 X_1 + \lambda_2 X_2 + \cdots + \lambda_n X_n$, for $0 \le \lambda_i \le 1, \sum \lambda_i = 1$ is called a convex combination.

Convex Hull : The convex hull of p-points $X_1, X_2, \ldots, X_n$ in E^n is the set of all points $b = \lambda_1 X_1 + \lambda_2 X_2 + \cdots + \lambda_n X_p$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_p = 1$

Example : For $X_1 = (6,6)$, $X_2 = (9,12)$, $X_3 = (3,9)$, The Convex hull of these points is

$X = \frac{1}{3}(6,6) + \frac{1}{3}(9,12) + \frac{1}{3}(3,9) = (2+3+1, 2+4+3) = (6,9)$.

Linear Independence : A set of vectors $P_1,P_2,\ldots,P_n$ of the same dimension is said to be linearly dependent if a set of scalars $\alpha_1,\alpha_2,\ldots,\alpha_n$ not all zero, can be found such that $\boldsymbol{\alpha_1 P_1 + \alpha_2 P_2 + \ldots + \alpha_n P_n = 0}$, where 0 represents a zero vector.

If the above relationship holds only when all the scalars $\alpha_1,\alpha_2,\ldots,\alpha_n$ are zero, then the set of vectors $P_1,P_2,\ldots,P_n$ is said to be linearly independent.

Ex.1. $P_1=\begin{bmatrix}3\\2\end{bmatrix}, P_2=\begin{bmatrix}6\\4\end{bmatrix}$,

Let $\alpha_1 P_1+\alpha_2 P_2=0$

or $3\alpha_1+6\alpha_2=0$ $2\alpha_1+4\alpha_2=0$

Solving these two equations, we get $\alpha_1=1$ and $\alpha_2=-1/2$. Thus the vectors P1 and P2 are linearly dependent. Also, for linearly dependent vectors determinant of P_1 and P_2 equals zero. In our Example 1, we have $\begin{vmatrix}3 & 6\\2 & 4\end{vmatrix}=0$.

Ex.2. $P_1=\begin{bmatrix}2\\3\end{bmatrix}, P_2=\begin{bmatrix}4\\2\end{bmatrix}$,

Let $\alpha_1 P_1+\alpha_2 P_2=0$

or $2\alpha_1+4\alpha_2=0$ $3\alpha_1+2\alpha_2=0$

Solving these two equations, we get $\alpha_1=\alpha_2=0$. Thus the vectors P_1 and P2 are linearly independent.

Basis for a vector space: A basis for vector space is a set of linearly independent vectors such that any vector in the vector space can be expressed as a linear combination of this set. The vectors in such a set are called the basis vectors.

Ex3. $P_1=\begin{bmatrix}1\\0\end{bmatrix}$, $P_2=\begin{bmatrix}0\\1\end{bmatrix}$, $V_3=\begin{bmatrix}200\\100\end{bmatrix}$ can be expressed as

V_3=200P1+100P2 and hence $P_1=\begin{bmatrix}1\\0\end{bmatrix}$ and $P_2=\begin{bmatrix}0\\1\end{bmatrix}$ are basis vectors for two dimensional vector space. The basis is $B=[P_1,P_2]=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$.

An n-dimensional Euclidean space E_n is a set of vectors with the property that there exists n linearly independent vectors while every set of n+1 vector is linearly dependent.

Extreme Point Theorems:

Theorem 1. The set of all feasible solutions of an LPP is a closed convex set.

Proof : We consider the following LPP:

$$\text{Min } Z = CX$$
$$\text{Sub to } AX = b$$
$$X \geq 0.$$

By definition $S=\{X : AX = b, X \geq 0\}$

Let us consider X_1 and X_2 as two feasible solutions to the above problem so that

$$AX_1=b,\ X_1\geq 0 \text{ and } AX_2=b,\ X_2\geq 0.$$

We consider convex combination of X_1 and X_2 as

$$X= \lambda X_1+(1-\lambda)X_2,\ 0 \leq \lambda \leq 1, \qquad \lambda + (1 - \lambda)=1$$

Thus we have, $AX=A[\lambda X_1+(1-\lambda)X_2]$

$$= \lambda(AX_1)+AX_2- \lambda(AX_2)$$
$$=\lambda b+b-\lambda b=b$$

Thus $AX=b$, since X_1, X_2, λ and $(1-\lambda)\geq 0$.

We have $X=\lambda AX_1+(1-\lambda)AX_2\geq 0$. Hence X is also a feasible solution to the problem and S is a convex set (A point is convex).

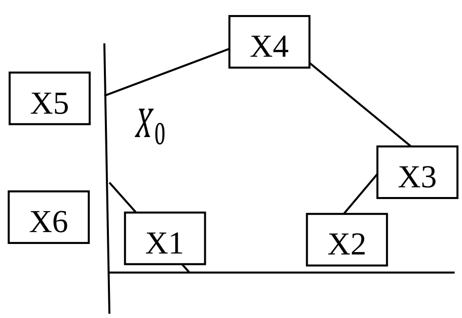
Theorem 2. (Fundamental Theorem of LPP).

Objective function assumes its minimum (optimal) solution at one of the extreme points of the bounded feasible region (S). If it assumes its minimum at more than one extreme points, then every convex combination of these particular points will also be minimum.

Proof : Since the feasible region is bounded convex polyhederon, then it will have finite number of extreme points say $X_1, X_2, \ldots, X_p$. We define X_0 to be the minimum extreme point if it satisfies

$f(X_0) \leq f(X_i)$, $i=1,2,\ldots,p$ ----------(1)

Let X_0 be any feasible point in S.



If it is an extreme point of S, then we have nothing to prove. Suppose that it is not an extreme point of S, then it can be expressed as linear convex combination of $X_1, X_2, \ldots, Xp$ of S, i.e.,

$X_0 = \lambda_1 X_1 + \lambda_2 X_2 + \ldots + \lambda_p X_p$ such that $0 \leq \lambda_i \leq 1$, $\sum \lambda_i = 1$

Since f(x) is a linear functional, then we can write

$$f(X_0) = f(\lambda_1 X_1 + \lambda_2 X_2 + \ldots + \lambda_p X_p)$$
$$= \lambda_1 f(X_1) + \lambda_2 f(X_2) + \ldots + \lambda_p f(X_p) \text{ ------------------ } (2)$$

Let min $\{f(X_1), f(X_2), \ldots, f(X_n)\} = f(X_m) = m$(say)

Replacing each $f(X_i)$, i=1,2,…,n by $f(X_m)$ in equation (2)

We can write $f(X_0) \geq \lambda_1 f(X_m) + \lambda_2 f(X_m) + \ldots + \lambda_p f(X_m)$

$$= (\lambda_1 + \lambda_2 + \ldots + \lambda_p) f(X_m)$$
$$= f(X_m) \text{ since } \lambda_1 + \lambda_2 + \ldots + \lambda_p = 1$$

Thus, we have $f(X_0) \geq f(X_m)$ for $X_i \in S$. ---------------------(3)

However, we had assumed that $f(X_0) \leq f(X_i)$, for $X_i \in S$

Thus from (1) and (3), we can only have $f(X_0) = f(X_m)$. As X_m is one of the extreme points, thus we conclude that X_m is an extreme point of S. This proves the first part of the theorem. To prove the second part of the theorem, we assume that the minimum occurs at more than one extreme points say at say $X_1, X_2, \ldots, X_q$. Thus, We consider any convex combination defined below:

$X = \lambda_1 f(X_1) + \lambda_2 f(X_2) + \ldots + \lambda_q f(X_q)$,

$= \lambda_1 f(X_m) + \lambda_2 f(X_m) + \ldots + \lambda_q f(X_m)$ since it assumes its minimum at more than one extreme points say $X_1, X_2, \ldots, X_q$.

Thus $f(X_1) = f(X_2) =, \ldots, = f(X_q) = f(X_m)$

Therefore $= \lambda_1 f(X_m) + \lambda_2 f(X_m) + \ldots + \lambda_q f(X_m)$

$$= (\lambda_1 + \lambda_2 + \ldots + \lambda_q) f(X_m) = f(X_m) \text{ as}$$

$\lambda_1 + \lambda_2 + \ldots + \lambda_q = 1$

which shows that it is also optimum.

Theorem 3. A basic feasible solution of the LPP is a vertex of the convex set of feasible solutions.

Proof. A point $X \in S$ is obvious and we can write

$X_1P_1+X_2P_2+\ldots+X_mP_m=B$------- (1)

and $X_j \geq 0$, $j=1,2,\ldots,m$

where $P_1,P_2,\ldots, P_m$ are linearly independent vectors.

Suppose that X is not an extreme point. Then there exists two different points X_1 and X_2 (say) different from X belonging to S such that

$X= \lambda X_1 + (1-\lambda)X_2$, $0 <\lambda<1$,

Since all the elements of X1 and X2 are non-negative and $0<\lambda<1$, the last elements of X_1 and X_2 are zero that is

$X_1=(x_{11},x_{21},\ldots,x_{m1},0,0,\ldots,0)$ and

$X_2=(x_{12},x_{22},\ldots,x_{m2},0,0,\ldots,0)$

Since X_1 and X_2 are feasible solutions, we have

$AX_1=B$, $X_1\geq 0$ and $AX_2=B$, $X_2\geq 0$ ---------------------(2)

Rewriting the above equations in (1), we have

$x_{11}P_1+ x_{21}P_2+,\ldots,+x_{m1}P_m=B$ ----------------------------(3)

$x_{12}P_1+ x_{22}P_2+,\ldots,+x_{m2}P_m=B$ ----------------------------(4)

From (1) and (3), we have

$(x_1-x_{11})P_1+ (x_2-x_{21})P_2+\ldots,+(x_m-x_{m1})P_m=0$

Since $P_{1,}P_2,\ldots,P_m$ are linearly independent vectors, we have

$(x_1-x_{11})=0$, $(x_2-x_{21})=0,\ldots, (x_m-x_{m1})= 0$

which implies that $x_1=x_{11},x_2=x_{21},\ldots,x_m=x_{m1}$.Thus $X=X_1$

Similarly from (1) and (4), we can show that

$x_1=x_{12},x_2=x_{22},\ldots,x_m=x_{m2}$. Thus $X=X_2$

Therefore, $X= X_1=X_2$ which contradicts the assumption that X cannot be expressed as a convex combination of X_1 and X_2.Thus X has to be a vertex of S.

Theorem 4. A vertex of S is a basic feasible solution.

Proof. Let X = $\{x_1, x_2, \ldots, x_n\}$ be a vertex of S, then we have $X \geq 0$ as $X \in S$.
Let r of x_j's, j=1,2,…,n be non zero where $r \leq n$.
Since $m \leq n$, either $r \leq m$ or $r > m$.
If $r \leq m$, X is obviously a basic feasible solution and so the theorem holds.
If $r > m$, then we may put
$X = \{x_1, x_2, \ldots, x_r, 0, 0, 0, \ldots 0\}$, where $x_j \geq 0$, for j=1,2,…,r.
Since $X \in S$, X is a solution of Ax=B
We have

$$x_1P_1 + x_2P_2 + \ldots + x_mP_m = B \text{ ------------------ } (1)$$

As r>m, the vectors $P_1, P_2, \ldots, P_r$ are not linearly independent vectors. Hence, there exist scalars
$\lambda_1, \lambda_2, \ldots, \lambda_r$ not all zero such that
$\lambda_1 P_1 + \lambda_2 P_2 + \ldots + \lambda_r P_r = 0$
Multiplying the above equation by C>0, we get
$C\lambda_1 P_1 + C\lambda_2 P_2 + \ldots + C\lambda_r P_r = 0$ ------------ (2)
From (1) and (2), we have (on adding both the equations)
$(x_1 + C\lambda_1)P_1 + (x_2 + C\lambda_2)P_2 + \ldots + (x_r + C\lambda_r)P_r = B$ --------------------- (3)
And subtracting (2) from (1),
$(x_1 - C\lambda_1) P_1 + (x_2 - C\lambda_2)P_2 + \ldots + (x_r - C\lambda_r)P_r = B$ ------------------- (4)
We choose C>0, sufficiently small to make,
$(x_j \pm C\lambda_j) > 0$ for j=1,2,…,r
Thus, we conclude from (3) and (4) that
$X_1 = \{(x_1 + C\lambda_1), (x_2 + C\lambda_2), \ldots, (x_r + C\lambda_r), 0, 0, \ldots 0\}$ and
$X_2 = \{(x_1 - C\lambda_1), (x_2 - C\lambda_2), \ldots, (x_r - C\lambda_r), 0, 0, \ldots 0\}$
are feasible solutions.
We now have three feasible solutions X, X_1 and X_2 which are connected through
$X = (1/2)X_1 + (1/2)X_2$
Hence, X is a convex combination of X_1 and X_2.. This means that X is not an extreme point which contradicts our initial assumption. Hence $r \ngtr m$ which implies that X is a basic feasible solution.

Theorem 6. If for any basic feasible solution X=$\{x_1, x_2, \ldots, x_m\}$, the conditions Z_j-C_j= $C_b^T y_j - C_j \leq 0$, hold for all j=1,2,…,n, then
$x_1P_1 + x_2P_2 + \ldots + x_mP_m = P_0$ --------(1)
with objective function value
$C_1x_1 + C_2x_2 + \ldots + C_mx_m = Z_0$ ------------------(2)
constitutes a minimum basic feasible (optimal) solution.

Proof : Let $y_1P_1+y_2P_2+\ldots+y_nP_n=P_0$--------------(3)

and $C_1y_1+C_2y_2+\ldots+C_ny_n=Z^*$ ---------------------(4)

be any other feasible solution with Z^* the corresponding value of the objective function. In order to prove the theorem, we must show that $Z_0 \leq Z^*$.

By the hypothesis $Z_j-C_j\leq 0 \Rightarrow Z_j \leq C_j$ for all j=1,2,…,n so that replacing C_j by Z_j in (4), we get

$y_1Z_1+y_2Z_2+\ldots+y_nZ_n \leq Z^*$ ---------------------(5)

Since $P_1,P_2,\ldots,P_m$ are linearly independent vectors, we can express any vector $P_1,P_2,\ldots,P_n$ in terms of $P_1,P_2,\ldots,P_m$.

Let P_j be given by

$x_{1j}P_1+x_{2j}P_2+\ldots+x_{mj}P_m = P_j$, j=1,2,…,n--------------(6)

and define

$x_{1j}C_1+x_{2j}C_2+\ldots+x_{mj}C_m=Z_j$, j=1,2,…,n------------------(7)

Replacing P_j in (3) by the expression of P_j in (6), we have

$$y_1(x_{11}P_1+x_{21}P_2+\ldots+x_{m1}P_m)+y_2(x_{12}P_1+x_{22}P_2+\ldots+x_{m2}P_m)$$
$$+\ldots+y_n(x_{1n}P_1+x_{2n}P_2+\ldots+x_{mn}P_m)=P_0$$

Or by regrouping like terms

We have

$$(y_1x_{11}P_1+y_2x_{12}P_1+\ldots+y_nx_{1n}P_1)+(y_1x_{21}P_2+y_2x_{22}P_2+\ldots+y_nx_{2n}P_2)+\ldots+(y_1x_{m1}P_m+$$
$$y_2x_{m2}P_m+\ldots+y_mx_{mn}P_m)=P_0$$

or

$$(\textstyle\sum_{j=1}^{n} y_jx_{1j})P_1+(\sum_{j=1}^{n} y_jx_{2j})P_2+\ldots+(\sum_{j=1}^{n} y_jx_{mj})P_m= P_0 \text{---(8)}$$

Similarly for each j, we substitute the expression for Z_j from (7) into (5) to obtain,

$$(\textstyle\sum_{j=1}^{n} y_jx_{1j})C_1+(\sum_{j=1}^{n} y_jx_{2j})C_2+\ldots+(\sum_{j=1}^{n} y_jx_{mj})C_m\leq Z^* \text{---------(9)}$$

Since the vectors $P_1,P_2,\ldots,P_m$ are linearly independent, the coefficients of the corresponding vectors in (1) and (8) must be equal and hence from (9)

$$x_1C_1+x_2C_2x_{m}+\ldots+C_mx_m\leq Z^*$$

or $C_1x_1+C_2x_2+\ldots+C_mx_m\leq Z^*$

or

$Z_0 \leq Z^*$ Proved.

Unrestricted Variables :

In an LPP we may have a situation where variable/s is/are unrestricted i.e., the unrestricted variable may have even negative value.

Example 1:

Max $Z=2x_1+5x_2$

Sub $6x_1+8x_2 \le 14$

$x_1+5x_2 \le 6$

$x_1 \ge 0$, x_2 unrestricted

We will replace this unrestricted variable by the difference of two variables (non-negative) as $x_2=x_2'-x_2''$. Thus,

Max $Z=2x_1+5(x_2'-x_2'')$

Sub to $6x_1+8(x_2'-x_2'') \le 14$

$x_1+5(x_2'-x_2'') \le 6$

$x_1, x_2', x_2'' \ge 0$

or

Max $Z = 2x_1+5x_2'-5x_2''$

Sub to $6x_1+8x_2'-8x_2'' \le 14$

$x_1+5x_2'-5x_2'' \le 6$

$x_1, x_2', x_2'' \ge 0$

We will solve this problem by Simplex Method as usual. However, in the final solution, we will calculate to find the value as $x_2=x_2'-x_2''$.

UNIT – II

Duality :

Duality theory plays an important role in linear programming problem as well as non linear programming. When we solve a linear programming problem, unknowingly, we are solving two linear programs, called as Primal and Dual problems. Interestingly, there exist very good relationship between the two programs. In fact, from the solution one , we can find the solution another and vice versa, we will investigate the relation between the to programs.

We consider the following LPP as Primal:

Primal : **Max Z = CX,**

Sub to

$$AX \leq b$$
$$X \geq 0$$

$C=(c_1, c_2, \ldots, c_n)$, $X=\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, A is a **m** x **n** matrix and $b=\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$

Dual : **Min W = b^TY,**

Sub to

$$A^T Y \geq C^T$$
$$Y \geq 0$$

where $Y=\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}$

The above problems are in Symmetric form.

Applications of Duality :

1. We can reduce the number of constraints by taking the dual of the primal.
2. The dual may be easier to solve
3. Any feasible solution to the dual problem gives a bound on the optimal objective function value in the primal problem.
4. The dual can be helpful in performing the sensitivity analysis.
5. The dual variables give the shadow prices for the primal constraints.

Rules of forming the dual :

1. We must bring the given LPP into the symmetric form.
2. Change the objective function from maximization to Minimization and vice versa.
3. Change the less than or equal to inequalities into more than or equal to inequalities and vice versa.
4. The dual variables will be greater than or equal to zero.
5. For every primal constraint equation, the corresponding dual variable will be unrestricted.

Ex. Max $Z = x_1 + 3x_2$,

Sub to

$$x_1 + 5x_2 \leq 6$$
$$4x_1 + 7x_2 \leq 11$$
$$4x_2 + 4x_2 \leq 8$$
$$x_1, x_2 \geq 0$$

Here, we have $c = (1,3)$, $b = \begin{pmatrix} 6 \\ 11 \\ 8 \end{pmatrix}$

$A = \begin{bmatrix} 1 & 5 \\ 4 & 7 \\ 4 & 4 \end{bmatrix}$, $A^T = \begin{bmatrix} 1 & 4 & 4 \\ 5 & 7 & 4 \end{bmatrix}$

$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$,

Dual: Minimize $W = b^TY = (6, 11, 8) \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = 6y_1 + 11y_2 + 8y_3$

Minimize $\mathbf{W = 6y_1 + 11y_2 + 8y_3}$

Sub to

$$A^TY \geq C^T$$

$$\begin{bmatrix} 1 & 4 & 4 \\ 5 & 7 & 4 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \geq \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \geq 0$$

or

$$y_1 + 4y_2 + 4y_3 \geq 1$$
$$5y_1 + 7y_2 + 4y_3 \geq 3$$
$$y_1, y_2, y_3 \geq 0.$$

Example : We consider the following Primal and dual problems:

Primal : **Max $Z = 5x_1 + 12x_2 + 4x_3$**

Sub to $x_1 + 2x_2 + x_3 \leq 10$

$$2x_1 - x_2 + 3x_3 = 8$$
$$x_1, x_2, x_3 \geq 0$$

We can write the second equality constraint of the primal in the form of two inequalities i.e., second and third constraint of the Dual.

Max $Z = 5x_1 + 12x_2 + 4x_3$

Sub to $x_1 + 2x_2 + x_3 \leq 10$

$$2x_1 - x_2 + 3x_3 \leq 8$$
$$2x_1 - x_2 + 3x \geq 8$$
$$x_1, x_2, x_3 \geq 0$$

Now, we write the LPP in the symmetric form. We convert inequality of third constraint into $\leq$ type multiplying by (-1) on both sides.

Max $Z = 5x_1 + 12x_2 + 4x_3$

Sub to $x_1 + 2x_2 + x_3 \leq 10$

$$2x_1 - x_2 + 3x_3 \leq 8$$
$$-2x_1 + x_2 - 3x_3 \leq -8$$
$$x_1, x_2, x_3 \geq 0$$

We write the dual of the above problem as

Min $W = 10y_1 + 8y_2 - 8y_3$

Sub to

$$y_1 + 2y_2 - 2y_3 \geq 5$$
$$2y_1 - y_2 + y_3 \geq 12$$
$$y_1 + 3y_2 - 3y_3 \geq 4$$
$$y_1, y_2, y_3 \geq 0$$

we can replace $y_2 - y_3$ by y_2' to get

Min $W = 10y_1 + 8y_2'$

Sub to

$$y_1 + 2y_2' \geq 5$$
$$2y_1 - y_2' \geq 12$$
$$y_1 + 3y_2' \geq 4$$

$y_1 \geq 0$ and $y_{2'}$ becomes unrestricted variable.
Therefore, for every equation in the Primal problem, the corresponding variable will be unrestricted in sign.

Now, we standardize the Primal, bring it into the canonical form and solve it by using a Big M Method.

Max $Z = 5x_1 + 12x_2 + 4x_3 + 0 \cdot x_4 - Ma_1$

Sub to $\quad x_1 + 2x_2 + x_3 + x_4 = 10$

$\quad 2x_1 - x_2 + 3x_3 + a_1 = 8$

$x_1, x_2, x_3, x_4 \geq 0$ and a_1 is an artificial variable.

We set up an initial table as :

C_b	$C_{j\rightarrow}$ $X_{b\downarrow}$	5	12	4	0	-M	b
		y_1	y_2	y_3	y_4	y_5	
0	x_4	1	2	1	1	0	10
-M	a_1	2	-1	3	0	1	8
	$Z_j - C_j$	2M-5	-M-12	0	0	0	Z = 0

The Final Optimal Simplex Table is obtained as

C_b	$C_{j\rightarrow}$ $X_{b\downarrow}$	5	12	4	0	-M	b
		y_1	y_2	y_3	y_4	y_5	
12	x_2	0	0	-1/5	2/5	-1/5	12/5
5	x_1	1	1	7/5	1/5	2/5	26/5
	$Z_j - C_j$	**0**	**0**	**3/5**	**29/5**	**-2/5+M**	**Z = 274/5**

In the above table we use the Rule :
The Optimal Z-equation coefficients of a starting variables in the Primal = Difference between the left and right sides of the dual constraint associated with the starting variables i.e.,
Starting Variables are

	x_4	a_1
Optimal Z - equation coefficients are	**29/5**	**M-2/5**

The dual constraints associated with x_4 and a_1 are $y_1 \geq 0$ and $y_2 \geq -M$

Applying the Rule :

$29/5 = y_1 - 0$ or $y_1 = 29/5$

and $M-2/5 = y_2 - M$ or $y_2 = -2/5$

The value of $\mathbf{W = 10y_1 + 8y_2 = 10(29/5) + 8(-2/5)}$

$= 290/5 - 16/5 = \mathbf{274/5}$

$\mathbf{Min\ W = 10y_1 + 8y_2' - 8y_2''}$

Sub to $y_1 + 2y_2' - 2y_2'' \geq 5$

$2y_1 - y_2' + y_2'' \geq 12$

$y_1 + 3y_2' - 3y_2'' \geq 4$

$y_1, y_2', y_2'' \geq 0.$

We would like to solve the dual.

Introducing surplus and artificial variables, we have

$$\mathbf{Min\ W = 10y_1 + 8y_2' - 8y_2'' + Ma_1 + Ma_2 + Ma_3}$$

Sub to $y_1 + 2y_2' - 2y_2'' - y_3 + a_1 = 5$

$2y_1 - y_2' - y_2'' - y_4 + a_2 = 12$

$y_1 + 3y_2' - 3y_2'' - y_5 + a_3 = 4$

$y_1, y_2', y_2'', y_3, y_4, y_5 \geq 0$ and a_1 , a_2 and a_3 are artificial variables.

Initial Simplex table is produced below.

C_b	$C_j\rightarrow$ $X_b\downarrow$	5	12	4	0	0	0	M	M	M	b
		y_1	y'_2	y''_2	y_3	y_4	y_5	a_1	a_2	a_3	
M	a_1	1	2	-2	-1	0	0	1	0	0	5
M	a_2	2	-1	1	0	-1	0	0	1	0	12
M	a_3	1	3	-3	0	0	-1	0	0	1	4
	$Z_j - C_j$	4m-10	4M+8	-4M+8	-M	-M	-M	0	0	0	Z=21M

The Final Simple Table in fifth iteration

C_b	$C_j \rightarrow$ / $X_b \downarrow$	5	12	4	0	0	0	M	M	M	b
		y_1	y'_2	y''_2	y_3	y_4	y_5	a_1	a_2	a_3	
M	y_5	0	0	0	-7/5	1/5	0	1	0	0	3/5
M	y''_2	0	-1	1	2/5	-1/5	0	0	1	0	2/5
M	y_1	1	0	0	-1/5	-2/5	-1	0	0	1	29/5
	$Z_j - C_j$	0	0	0	-26/5	-12/5	0	-M+26/5	-M+12/5	-M	Z=274/5

Starting Variables are in the dual are

a_1 a_2 a_3

The Optimal Z-equation coefficients of a starting variables in the dual

Optimal Z-equation coefficients are **-M+26/5 -M+12/5 -M**

Primal constraints associated with a_1, a_2 and a_3 are

$x_1 \geq M$, $x_2 \geq M$, $x_3 \geq M$

and difference between the left and right sides of the primal constraint associated with the starting dual variables are $x_1 - M$, $x_2 - M$, $x_3 - M$

Thus equating the difference between the left and right sides of the primal constraint associated with the starting dual variables to optimal Z-equation coefficients

$x_1 - M = -M + 26/5$ or $x_1 = 26/5$

$x_2 - M = -M + 12/5$ or $x_2 = 12/5$

and $x_3 - M = -M$ or $x_3 = 0$ and the optimal value of **Z** is **274/5.**

Sensitivity Analysis :

Sensitivity analysis refers to the changes in the optimal solution and the optimal value of the objective function due to the changes in the input data coefficients. These changes may occur due to many factors namely price rise, scarcity of raw material, rise or fall in the wages of the labor etc. We will confine ourselves to the following changes in the input data:

A. **Changes in the cost coefficients of the objective function.**
B. **Changes in the right hand the side constants .**
C. **Changes in the constraints (Coefficients of Matrix A)**

Example: A Dependable company plans production on three products-A, B and C. The unit profits on these products are Rs.2, Rs.3 and Rs.1 respectively and they require two-resources-labour and raw material. The Company's O.R. Department formulates the following LPP for determining the optimal product mix.

Max Z = $2x_1 + 3x_2 + x_3$

Sub to

$$\frac{1}{3}x_1 + \frac{1}{3}x_2 + \frac{1}{3}x_3 \leq 1\ (Labour)$$

$$\frac{1}{3}x_1 + \frac{4}{3}x_2 + \frac{7}{3}x_3 \leq 3(Material)$$

$x_1, x_2, x_3 \geq 0$

where x_1, x_2, x_3 denote the numbers of the products produced.

Now, we standardize the

Max Z = $2x_1 + 3x_2 + x_3$

Sub to

$$\frac{1}{3}x_1 + \frac{1}{3}x_2 + \frac{1}{3}x_3 + x_4 = 1(Labour)$$

$$\frac{1}{3}x_1 + \frac{4}{3}x_2 + \frac{7}{3}x_3 + x_5 = 3\ (Material)$$

$x_1, x_2, x_3 \geq 0$

Iteration 0 The initial Simplex table is

C_b	$C_j \rightarrow$ / $X_b \downarrow$	2	3	1	0	0	b
		y_1	y_2	y_3	y_4	y_5	
0	x_4	1/3	1/3	1/3	1	0	1
0	x_5	1	1	7/5	0	1	3
	Z_j-C_j	-2	-3	-1	0	0	Z=0

Iteration 1

C_b	**$C_j \rightarrow$ / $X_b \downarrow$**	**2**	**3**	**1**	**0**	**0**	**b**
		y_1	**y_2**	**y_3**	**y_4**	**y_5**	
0	**x_4**	**1/4**	**0**	**-1/4**	**1**	**-1/4**	**1/4**
3	**x_2**	**1/4**	**1**	**7/4**	**0**	**3/4**	**9/4**
	Z_j-C_j	**-5/4**	**0**	**17/4**	**0**	**9/4**	**Z=27/4**

The Final Optimal Simplex Table (F) is

C_b	**$C_j \rightarrow$ / $X_b \downarrow$**	**2**	**3**	**1**	**0**	**0**	**b**
		y_1	**y_2**	**y_3**	**y_4**	**y_5**	
2	**x_1**	**1**	**0**	**-1**	**4**	**-1**	**1**
3	**x_2**	**0**	**1**	**2**	**-1**	**1**	**2**
	Z_j-C_j	**0**	**0**	**3**	**5**	**1**	**Z=8**

A : Case (i) : *<u>Changes in cost coefficients of a non basic variable of the objective function.</u>*

In the optimal product mix, product C is not produced because of its low profit of Rs.1 per unit. We are interested in finding the range on the values of C_3 such that the present solution remains optimal. It is clear that when the profit on Product C decreases, it has no effect on the present solution.

However, when, the profit increases beyond a certain limit, product C may become profitable to be produced. The table is optimal as long as $Z_3 - C_3 \geq 0$.

Thus, we take C_3 as a variable i.e.,

$$\mathbf{Z_3 - C_3 = C_b\, y_3 - C_3 \geq 0}$$

or $(2, 3)\begin{bmatrix}-1\\ 2\end{bmatrix} - C_3 \geq 0$

$-2 + 6 - C_3 \geq 0$ or $-C_3 \geq -4$ or $C_3 \leq 4$.

Thus, as long as the profit on Product C is up to Rs. 4, it is not economical to produce product C. Suppose, the profit on product C increases to Rs.6, then $Z_3 - C_3 = 2$ and the current product mix is no more optimal

New table with $C_3 = 6$

C_b	$C_j \rightarrow$	2	3	6	0	0	b
	$X_b \downarrow$	y_1	y_2	y_3	y_4	y_5	
2	x_1	1	0	-1	4	-1	1
3	x_2	0	1	<u>2</u>	-1	1	2
	$Z_j - C_j$	0	0	-2	5	1	**Z=8**

The new final table is

C_b	$C_j \rightarrow$	2	3	6	0	0	b
	$X_b \downarrow$	y_1	y_2	y_3	y_4	y_5	
2	x_1	1	1/2	0	7/2	-1/2	1
6	x_3	0	1/2	1	-1/2	1/2	2
	$Z_j - C_j$	0	1	0	4	2	**Z=10**

Thus, the optimal product mix is to produce 2 units of product A and 1 unit of product C with a maximum profit of Rs.10. Generally, it takes one more iteration to reach the optimal product mix.

Case (ii) : <u>Changes in the coefficient of the basic variable</u>

Changing the coefficient of a basic variable: We wish to determine the effect of changes on the unit profit of product A. If the profit on A decreases, then it may not be profitable to produce any more. Even, when the profit increases to a certain level, this product may even become the only product to be produced.

To determine the range on C_1, we observe that change in C_1 changes the profit vector C_b . Since $C_b = (c_1, c_2)$, it can be verified that the relative profit coefficients of the basic variables namely $Z_3 - C_3$, $Z_4 - C_4$ and $Z_5 - C_5$ will change. But as long as these $Z_j - C_j$ remain non-negative, the optimal table (F) remains optimal. We express the values of $Z_3 - C_3$, $Z_4 - C_4$ and $Z_5 - C_5$ as a function of C_1

$$Z_3 - C_3 = (C_1, 3)\begin{bmatrix} -1 \\ 2 \end{bmatrix} - 1 \geq 0$$

$$-C_1 + 6 - 1 \geq 0$$

$$-C_1 + 5 \geq 0$$

$$-C_1 \geq -5, C_1 \leq 5$$

$$Thus \quad Z_3 - C_3 \geq 0, \ as\ long\ as\ C_1 \leq 5.$$

and

$$Z_4 - C_4 = (C_1, 3)\begin{bmatrix} 4 \\ -1 \end{bmatrix} - 0 \geq 0$$

$$4C_1 - 3 \geq 0$$

$$4C_1 \geq 3$$

$$C_1 \geq 3/4$$

$$Thus \quad Z_4 - C_4 \geq 0 \ as\ long\ as\ C_1 \geq 3/4.$$

also

$$Z_5 - C_5 = (C_1, 3)\begin{bmatrix} -1 \\ 1 \end{bmatrix} - 0 \geq 0$$

$$-C_1 + 3 \geq 0$$

$$-C_1 \geq -3$$

$$C_1 \leq 3$$

$$Thus \quad Z_5 - C_5 \geq 0 \; as\, long \; as \; C_1 \leq 3.$$

Thus from the above calculations , C_3 lies between [3/4 ,3].The table F remains optimal as long as the changes on C_1 are within 3/4 and 3. Of course, the value of the objective function will change. For example when C_1=1, the optimal solution is given by x_1 = 1, x_2 = 2 and x_3 = 0, but the optimal value is 7. When the value of C_1 goes beyond the limit of [3/4,3], the table F is no more optimal and the Simplex Method has to be applied again.

Case (iii):** **<u>Changing the price of both basic and non-basic variables</u>

A simple case, we may see the changes in both the variables e.g., changing the coefficients of basic variables x_1 and x_2 from 2 and 3 to 1 and 4 and of non basic variable x_3 from 1 to 2

The effect on the optimal product table F remains optimal or not.

$$Z_3 - C_3 = (1,4)\begin{bmatrix} -1 \\ 2 \end{bmatrix} - 2 = 5 \geq 0$$

$$Z_4 - C_4 = (1,4)\begin{bmatrix} 4 \\ -1 \end{bmatrix} - 0 = 0 \geq 0$$

$$Z_5 - C_5 = (1,4)\begin{bmatrix} -1 \\ 1 \end{bmatrix} - 0 = 3 \geq 0$$

Hence the table F remains optimal with x_1 = 2, x_2 = 2 and x_3 = 0 and the optimal value of the objective function is **9**.

However, it has an indication of an alternative optimal solution.

B. Change in right hand side constants b_i :

Let us explore the changes in the optimal product mix if an additional unit of labour is made available. It is clear that this change has no effect on the table F except for changes in the value of the constants. Even after the change, if the right side constants remain non-negative, then the solution given by table F is still a basic feasible solution and the Z-equation of Table F are the same, thus the table F is optimal.

Thus, in order to verify, the effect of changes in the right hand side constants (b_i), it is sufficient to see whether the new vector of constants in the final table stays non-negative. Thus from Table F, we have,

$$B^{-1} = \begin{bmatrix} 4 & -1 \\ -1 & 1 \end{bmatrix}$$

and the values of new right hand vector with increased labour $b_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$$\underline{b} = B^{-1}b1 = \begin{bmatrix} 4 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

which is a positive vector. Hence, Table F is still an optimal table and the new optimal product mix is $x_1 = 5$ and $x_2 = 1$ and the new optimal value of $Z^* = 13$. We note that both the optimal solution and the optimal value have changed due to the changes in the availability of labour but the optimal basis has not changed and the optimal product mix remains unchanged.

Suppose the extra unit of labour is made available by allowing overtime which costs an additional Rs.4/- to the company. The company may evaluate whether this is economical or not by finding $Z_1 = Z^* - Z =$ Rs.13 - Rs.8 = Rs.5 which is more than the cost of overtime (Rs.4/-) and it is therefore profitable to get an additional unit of 1 unit of labour. The increased profit of Rs.5 per unit increase in labour availability is called the shadow price for the labour constraint.

Thus knowing the shadow price of various constraints helps in determining how much one can afford to pay for the increases in the constrained resources. To use the concept of shadow prices, we compute the range on the variation of constraint resources such that the optimal basis remains optimal.

Let $b^* = \begin{bmatrix} b1 \\ 3 \end{bmatrix}$ denotes the new labour made available

$$B^{-1}b = \begin{bmatrix} 4 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} b_1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4b_1 - 3 \\ -b_1 + 3 \end{bmatrix} \geq 0$$

or

Thus

$$4b_1 - 3 \geq 0, or\ b_1 \geq 3/4$$

$$-b1 + 3 \geq 0, or\ b_1 \leq 3$$

The optimal solution is $x_1 = 4b_1 - 3$, $x_2 = -b_1 + 3$ and $x_3 = 0$.

The maximum profit Z = 2(4b13) + 3(-b1+3)

= Rs. 5b1+3

Let us consider the case when the labor availability is increased to 4 units. This means

$$\underline{b} = B^{-1}b1 = \begin{bmatrix} 4 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 13 \\ -1 \end{bmatrix}$$

which is not a non-positive vector as required.

Thus $x_1 = 13$, $x_2 = -1$, $x_3 = 0$, $x_4 = 0$ and $x_5 = 0$ is infeasible .

C : <u>Changes in the Constraint Matrix :</u>

The coefficient matrix A may change due to the changes in the following :

i) Adding a new variable

ii) Changing the resource requirements of the existing activities

iii) Adding new constraints

Case i) Adding a new variable

Suppose, the Cos R & D Department comes out with a proposal to produce another product D which requires 1 unit of labor and 1 unit of material with a unit profit of Rs.3. The Company would like to know whether it is economical to produce product D or not ?.

As we know that inclusion of a new product is equivalent to adding a new variable say x6 and a column with in $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ initial table (Iteration 0) table 1.

Thus table F is optimal as long $Z_6 - C_6$ is non-negative .We use Revised Simplex Method,

$$\pi = C_b B^{-1} = (2,3)\begin{bmatrix} 4 & -1 \\ -1 & 1 \end{bmatrix} = (5,1)$$

$$\pi P_6 - C_6 = (5,1)\begin{bmatrix} 1 \\ 1 \end{bmatrix} - 3 = 3 \geq 0$$

and

$$\overline{C}_6 = \pi P_6 - C_6 = \begin{pmatrix} 5 & 1 \end{pmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} - 3 = 3 \geq 0$$

Since no variable qualifies to enter into the basis, the Table F remains optimal. This means that producing product D will not improve the solution i.e., the profit can not be improved . Similarly, the two other cases in (ii) and (iii) will be dealt with.

Parametric Linear Programming :

We have considered the sensitivity analysis where the changes considered were discrete. We shall now consider the coefficients in the problem with continuous variation as a function of some parameter. The analysis of the effect of this functional dependence hereafter called parametric variations, on the optimal solution of the problem is called parameter linear programming.

Parametric linear programming is essentially based on the same concepts as sensitivity analysis. Assuming that the coefficients, which are varying, are linear functions of a parameter λ, the general strategy adopted is the following. We shall consider the problem where the changes are in the coefficients of non-basic variables. We first compute the optimal solution for $\lambda = 0$. Then using optimality and feasibility conditions, we find the range of values of λ for which the optimal solution remains optimal and feasible. Suppose this range is $(0, \lambda_1)$.

This means that any increase in the value of λ beyond λ_1 will make the present optimal solution non-optimal or infeasible. At $\lambda = \lambda_1$, we determine a new optimal solution and find the range (λ_1, λ_2) of the values of λ for the new optimal solution remains optimal and feasible. The process is repeated at λ_2 and continued till a value of λ is reached beyond which either the optimal solution does not change or does not exist. A similar strategy is adopted for investigating the effect of variations for the negative values of λ.

Example:

$$\textbf{Min } f(\lambda) = (1 + \lambda)x_1 + (-2 - 2\lambda)x_2 + (1 + 5\lambda)x_3$$

Sub to $\quad 2x_1 - x_2 + 2x_3 \le 2$

$x_1 - x_2 \le 3$

$x_1 + 2x_2 - 2x_3 \le 4$

$x_1, x_2, x_3 \ge 0$

We can re-write the objective function as

$$f(\lambda) = f^0 + \lambda f^*$$

where $f^0 = x_1 - 2x_2 + x_3$ and

$f^* = x_1 - 2x_2 + 5x_3$

We introduce the slack variables to convert the inequalities into equations to have the full basis so that we can apply the Regular Simplex Method. The optimal solution is obtained by treating λ=0. However, we will perform the calculations for all the entries of the table.

$$\textbf{Min } f(\lambda) = f(\lambda) = f^0 + \lambda f^*$$

Sub to $\quad 2x_1 - x_2 + 2x_3 + x_4 = 2$

$x_1 - x_2 + x_5 = 3$

$x_1 + 2x_2 - 2x_3 + x_6 = 4$

$x_1, x_2, x_3, x_4, x_5, x_6 \ge 0$ and start

with initial table of the Simplex Method as follows.

Step II: We set up an initial table as :

C_b		$c^*_{j\to}$	1	-2	5	0	0	0	b	Ratios
		$C_{j\to}$	1	-2	1	0	0	0		
f^*	f^0	$X_{b\downarrow}$	y_1	y_2	y_3	y_4	y_5	y_6		
0	0	x_4	2	-1	2	1	0	0	2	
0	0	x_5	1	-1	0	0	1	0	3	
0	0	x_6	1	2	-2	0	0	1	4	4/2=2
		Z_j-$C_j(f^0)$	-1	2	1	0	0	0	Z=0	
		Z_j-$C_j(f^*)$	-1	2	-5	0	0	0	Z=0	

Step II: We set up an initial table as:

C_b		$c^*_{j\to}$	1	-2	5	0	0	0	b	Ratios
		$C_{j\to}$	1	-2	1	0	0	0		
f^*	f^0	$X_{b\downarrow}$	y_1	y_2	y_3	y_4	y_5	y_6		
0	0	x_4	5/2	0	1	1	0	1/2	4	4/1=4
0	0	x_5	3/2	0	-1	0	1	1/2	5	
-2	-2	x_2	1/2	1	-1	0	0	1/2	2	
		Z_j-$C_j(f^0)$	-2	0	1	0	0	-1	Z=-4	
		Z_j-$C_j(f^*)$	-2	0	-3	0	0	-1	Z=-4	

Step II: We set up an initial table as:

C_b		$c^*_{j\to}$	1	-2	5	0	0	0	b	Ratios
		$C_{j\to}$	1	-2	1	0	0	0		
f^*	f^0	$X_{b\downarrow}$	y_1	y_2	y_3	y_4	y_5	y_6		
5	1	x_3	5/2	0	1	1	0	1/2	4	4/1=4
0	0	x_5	4	0	0	1	1	1	9	
-2	-2	x_2	3	1	0	1	0	1	6	
		$Z_j\text{-}C_j(f^0)$	-9/2	0	0	-1	0	-3/2	Z=-8	
		$Z_j\text{-}C_j(f^*)$	11/2	0	3	0	0	1/2	Z=-8	

The table is optimal for $\lambda = 0$ as

$C_1(\lambda) = (-9/2 + (11/2)\lambda)$

$= -9/2 + (11/2)(0) = -9/2 < 0,$

$C4(\lambda) = (-1 + 3\lambda) = (-1 + 3(0)) = -1 < 0$ and

$C6(\lambda) = (-3/2 + (1/2)\lambda) = -3/2 + 0 = -3/2 < 0$

Also, we note that for $\lambda \leq 1/3$

$C_1(\lambda) = -9/2 + (11/2)(1/3) = -9/2 + 11/6 = -16/6 < 0,$

$C_4(\lambda) = (-1 + 3(1/3)) = 0$ and

$C_6(\lambda) = -3/2 + (1/2)(1/3) = (-3/2 + 1/6) = -8/6 < 0$

Thus the basis $x_2 = 6$, $x_3 = 4$ and $x_5 = 9$ is optimal for $0 \leq \lambda \leq 1/3$ and the minimum value of $f(\lambda)$ for this range is given by $-8 + 8\lambda$ as

Min $f(\lambda) = f_0 + \lambda f^*$ = Thus the basis $x_2 = 6$, $x_3 = 4$, $x_5 = 9$ is optimal for

$0 \leq \lambda \leq 1/3$ and the min value of $f(\lambda)$ for this range is given by
$f(\lambda) = (1 + \lambda)x_1 + (-2 - 2\lambda)x_2 + (1 + 5\lambda)x_3$

$= ((1 + \lambda).0 + (-2 - 2\lambda)6 + (1 + 5\lambda)4$

$= -12 - 12\lambda + 4 + 20\lambda.$

$f(\lambda) = -8 + 8\lambda.$

For $\lambda > 1/3$, $C_4(\lambda) = -1 + 3/2 = 12 > 0$. Hence the present solution is no more optimal. Thus, entering x_4 and replacing x_3, we update the table

		$c^*_{j\to}$	1	-2	5	0	0	0		Ratios
C_b		$C_{j\to}$	1	-2	1	0	0	0	b	
f^*	f^0	$X_{b\downarrow}$	y_1	y_2	y_3	y_4	y_5	y_6		
0	0	x_4	5/2	0	1	1	0	1/2	4	4/1=4
0	0	x_5	3/2	1	-1	0	1	1/2	5	
-2	-2	x_2	1/2	0	-1	0	0	-1	2	
		Z_j-$C_j(f^0)$	-2	0	-3	0	0	-1	Z=-4	
		Z_j-$C_j(f^*)$	-2	0	-3	0	0	-1	Z=-4	

Thus for $\lambda > 1/3$, $x_2 = 2$, $x_4 = 4$, $x_5 = 5$ and $x_1 = x_3 = x_6 = 0$ and $f(\lambda)$
Thus for $\lambda > 1/3$, $x_2 = 2$, $x_4 = 4$, $x_5 = 5$, $x_1 = 0$, $x_3 = 0$ and

$$f(\lambda) = (1 + \lambda)x_1 + (-2 - 2\lambda)x_2 + (1 + 5\lambda)x_3$$

$$= ((1 + \lambda).0 + (-2 - 2\lambda)2 + (1 + 5\lambda).0$$

$f(\lambda) =$ **$-4 - 4\lambda$**.

It is also clear from the solution that for, $C_1(\lambda)$, $C_4(\lambda)$ and $C_6(\lambda)$ are non positive for all negative values of λ.
Thus for $\lambda < 0$, the optimal basis is the same for $\lambda = 0$

It is also clear from the solution for $\lambda = 0$, that $C_1(\lambda)$, $C_4(\lambda)$ and $C_6(\lambda)$ are non positive for all negative values of λ.Thus for $\lambda < 0$, the optimal basis is the same for $\lambda = 0$.
The Parametric solution of the present problem may now be summarized as

	x_1	x_2	x_3	$f(\lambda)$
$\lambda \le 1/3$	0	6	4	$-8 + 8\lambda$
$\lambda \ge 1/3$	0	2	0	$-4 - 4\lambda$

$\lambda = 1/3$, the two expression will give the same values

UNIT – III

People travel while commodities are transported.

Transportation Problem :

Farmer's Cooperative has two Central warehouses that supply corn seeds to three regional stores for distribution to farmers. The monthly supply available at the two warehouses is estimated to be 1000 and 2000 sacks of corn seeds. The demand at the three regional stores is estimated to be 1500, 750 and 750 sacks respectively. The cost per sack (in Rs.) for transporting the seed from the warehouses to the stores is given below :

	Stores			Availability
Warehouses	1	2	3	
1	50	100	60	1000
2	30	25	35	2000
Requirement	1500	750	750	

The goal of the cooperative is to satisfy the monthly demand at the regional stores at the least possible transportation cost.

Mathematical Formulation:

We define:

x_{11} be the quantity to be transported from Warehouse 1 to stores 1

x_{12} be the quantity to be transported from Warehouse 1 to stores 2

x_{13} be the quantity to be transported from Warehouse 1 to stores 3

x_{21} be the quantity to be transported from Warehouse 2 to stores 1

x_{22} be the quantity to be transported from Warehouse 2 to stores 2

x_{23} be the quantity to be transported from Warehouse 2 to stores 3

with corresponding costs of transportations $\mathbf{c_{ij}}$, i = 1,2 and j = 1,2,3

	Stores			Availability
Warehouses	1	2	3	
1	x_{11}	x_{12}	x_{13}	1000
2	x_{21}	x_{22}	x_{23}	2000
Requirement	1500	750	750	3000

Minimize **$Z = 50x_{11}+100x_{12} +60x_{13}+30x_{21}+20x_{22}+35x_{23}$**

Sub to

$x_{11} + x_{12} + x_{13} = 1000$

$x_{21} + x_{22} + x_{23} = 2000$

$x_{11} + x_{21} = 1500$

$x_{12} + x_{22} = 750$

$x_{13} + x_{23} = 750$

$x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23} \geq 0$

$a_1 + a_2 = b_1 + b_2 + b_3$

Supply is equal to demand

$$\textbf{Min } Z = CX$$
$$\text{Sub to } AX = b$$
$$X \geq 0$$

Special Case

In general we consider the following:

	Stores				Availability
Warehouses	1	2	…	n	
1	x_{11}	x_{12}	…	x_{1n}	a_1
2	x_{21}	x_{22}	…	x_{2n}	a_2
. . .	. . .	. . .	. . .	. . .	. . .
m	x_{m1}	x_{m2}	…	x_{mn}	a_m
Requirement	b_1	b_2	…	b_n	

$$\textbf{Min } Z=\sum\sum c_{ij}\, x_{ij}=c_{11}x_{11}+c_{12}x_{12}+\ldots+c_{mn}x_{mn}$$

$$\sum_{j=1}^{n} x_{ij} = a_i,\ i=1,2,\ldots,m$$

$$\sum_{i=1}^{m} x_{ij} = b_j,\ j=1,2,\ldots n$$

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j,\quad x_{ij}\geq 0$$

The above model is called the Transportation Model with m-origins and n-destinations. Since $\sum_{i=1}^{m} a_i=\sum_{j=1}^{n} b_j$ i.e. availability is equal to requirement, this is called a balanced transportation problem

Methods of finding feasible solutions:

	Stores			Availability
Warehouses	1	2	3	
1	500 50	500 100	60	1000
2	1000 30	250 25	750 35	2000
Requirement	1500	750	750	

TC = 50X500+100X500+60X0+30X1000+25X250+35X750

= 25000+50000+0+30000+7500+26250

TC = 138750

We may have another solution:

	Stores			Availability
Warehouses	1	2	3	
1	250 50	100	750 60	1000
2	12500 30	750 25	35	2000
Requirement	1500	750	750	

TC = 50X250+100X0+60X750+30X1250+25X750+35X0

TC = 113750

Basic Feasible Solution : The number of allocations must be equal to **m + n - 1** where m is the number of origins and n is the number of destinations.

Example 2.

Factory	**Dealer**				**Supply**
	1	**2**	**3**	**4**	
A	**2**	**2**	2	4	1000
B	**4**	6	4	3	700
C	3	**2**	**1**	**0**	900
Requirement	900	800	500	400	2600

This may have a solution :

Factory	**Dealers**				**Supply**
	1	**2**	**3**	**4**	
A	200 2	800 2	2	4	1000
B	700 4	6	4	3	700
C	3	**2**	500 1	400 0	900
Requirement	900	800	500	400	2600

The solution is not basic feasible solution as the number of allocated cells (5) is less than 6 (m+n-1 = 3+4-1 = 6).
Thus, we will allocate ε a very small quantity with zero cost as per our convenience to have allocated cells equal to 6.

Factory	**Dealers**				**Supply**
	1	**2**	**3**	**4**	
A	200 2	800 2	2	4	1000
B	700 4	ε 6	4	3	700
C	3	**2**	500 1	400 0	900
Requirement	900	800	500	400	2600

Methods of finding basic feasible solutions: There are three methods for finding basic feasible solutions to a T.P.

1. North-West Corner Method
2. Matrix Minima Method:
3. Vogel's Approximation Method (VAM)

North-West Corner Method (NWCM)

Step 1. Select the upper left corner cell of the transportation matrix and allocate as much as you can as per the availability (supply) and requirement (demand).
Step 2. Cross the satisfied row or column of the transportation problem for no more allocation there.
Step 3. In the revised transportation problem, if the row was satisfied, then move down to the next cell to allocate as much as you can else move to the and allocate as much as you can and cross the satisfied row or column of the revised transportation problem.
Step 4. Repeat the steps till all the rows and columns are satisfied.

Example 1.

	Stores				
Warehouses	1	2	3	4	Supply
1	19	30	50	10	7
2	70	30	40	60	9
3	40	8	70	20	18
Demand	5	8	7	14	Total 34

Following the steps, we get the following allocations:

	Stores				
Warehouses	1	2	3	4	Supply
1	5 19	2 30	50	10	7
2	70	6 30	3 40	60	9
3	40	8	4 70	14 20	18
Demand	5	8	7	14	Total 34

Total Cost = 19x5+30x2+30x6+40x3+70x4+20x14

Total Cost = **Rs.1015.**

Matrix Minima Method:

Step 1. In this method, we begin allocations by considering the minimum cost of transportation. We search for the cell with the smallest transportation cost in the T.P. and allocate as much as possible there. Break the ties arbitrarily.

Step 2. We cross the satisfied row or column for no more allocations there and prepare the revised T.P.

Step 3. In the revised T.P., we again search for the cell with minimum cost and make allocations there as much as we can as per the restrictions of availability and requirement and go to step 2 till all allocations are made.

We consider example 1 again.

	Stores				
Warehouses	1	2	3	4	Supply
1	19	30	50	7 10	7
2	2 70	30	7 40	60	9
3	3 40	8 8	70	7 20	18
Demand	5	8	7	14	Total 34

TC = 10x7 + 70x2 + 40x7 + 40x3 + 8x8 + 20x7 = **Rs.814.**

Vogel's Approximation Method (VAM)

This method is preferred over the NWCM and VAM, because the initial basic feasible solution obtained by this method is either optimal solution or very nearer to the optimal solution.

Vogel's Approximation Method (VAM) Steps (Rule)	
Step-1 :	Find the cells having smallest and next to smallest cost in each row and write the difference (called penalty) along the side of the table in row penalty.
Step-2 :	Find the cells having smallest and next to smallest cost in each column and write the difference (called penalty) along the side of the table in each column penalty.
Step-3 :	Select the row or column with the maximum penalty and find cell that has least cost in selected row or column. Allocate as much as possible in this cell. If there is a tie in the values of penalties then select the cell where maximum allocation can be possible
Step-4 :	Adjust the supply & demand and cross out (strike out) the satisfied row or column.
Step-5 :	Repeat this steps until all allocations are made..

Example - 1

1. Find Solution using Vogel's Approximation method

	Destinations				
Stores	D1	D2	D3	D4	Availability
S1	19	30	50	10	7
S2	70	30	40	60	9
S3	40	8	70	20	18
Requirement	5	8	7	14	

Solution:

TOTAL number of supply constraints : 3

TOTAL number of demand constraints : 4

We have

	$D1$	$D2$	$D3$	$D4$	Supply
$S1$	19	30	50	10	7
$S2$	70	30	40	60	9
$S3$	40	8	70	20	18
Demand	5	8	7	14	

We calculate row and column penalties in the following

Table-1

	$D1$	$D2$	$D3$	$D4$	Supply	Row Penalty
$S1$	19	30	50	10	7	9=19-10
$S2$	70	30	40	60	9	10=40-30
$S3$	40	8	70	20	18	12=20-8
Demand	5	8	7	14		
Column Penalty	21=40-19	22=30-8	10=50-40	10=20-10		

The maximum penalty, 22, occurs in column $D2$.
The minimum c_{ij} in this column is $c_{32} = 8$.
The maximum allocation in this cell is min(18,8) = **8**.
It satisfy demand of $D2$ and adjust the supply of $S3$ from 18 to 10 (18 - 8 = 10).

Table-2

	$D1$	$D2$	$D3$	$D4$	Supply	Row Penalty
$S1$	19	30	50	10	7	9=19-10
$S2$	70	30	40	60	9	20=60-40
$S3$	40	8**(8)**	70	20	10	20=40-20
Demand	5	0	7	14		
Column Penalty	21=40-19	--	10=50-40	10=20-10		

The maximum penalty, 21, occurs in column $D1$.
The minimum c_{ij} in this column is $c_{11} = 19$.
The maximum allocation in this cell is min(7,5) = **5**.
It satisfy demand of $D1$ and adjust the supply of $S1$ from 7 to 2 (7 - 5 = 2).

Table-3

	$D1$	$D2$	$D3$	$D4$	Supply	Row Penalty
$S1$	19**(5)**	30	50	10	2	40=50-10
$S2$	70	30	40	60	9	20=60-40
$S3$	40	8**(8)**	70	20	10	50=70-20
Demand	0	0	7	14		
Column Penalty	--	--	10=50-40	10=20-10		

The maximum penalty, 50, occurs in row $S3$.
The minimum c_{ij} in this row is c_{34} = 20.
The maximum allocation in this cell is min(10,14) = **10**.
It satisfy supply of $S3$ and adjust the demand of $D4$ from 14 to 4 (14 - 10 = 4).

Table-4

	$D1$	$D2$	$D3$	$D4$	Supply	Row Penalty
$S1$	19**(5)**	30	50	10	2	40=50-10
$S2$	70	30	40	60	9	20=60-40
$S3$	40	8**(8)**	70	20**(10)**	0	--
Demand	0	0	7	4		
Column Penalty	--	--	10=50-40	50=60-10		

The maximum penalty, 50, occurs in column $D4$.
The minimum c_{ij} in this column is c_{14} = 10.
The maximum allocation in this cell is min(2,4) = **2**.
It satisfy supply of $S1$ and adjust the demand of $D4$ from 4 to 2 (4 - 2 = 2).

Table-5

	$D1$	$D2$	$D3$	$D4$	Supply	Row Penalty
$S1$	19**(5)**	30	50	10**(2)**	0	--
$S2$	70	30	40	60	9	20=60-40
$S3$	40	8**(8)**	70	20**(10)**	0	--
Demand	0	0	7	2		
Column Penalty	--	--	40	60		

The maximum penalty, 60, occurs in column $D4$.
The minimum cij in this column is $c24 = 60$.
The maximum allocation in this cell is min(9,2) = **2**.
It satisfy demand of $D4$ and adjust the supply of $S2$ from 9 to 7 (9 - 2 = 7).

Table-6

	$D1$	$D2$	$D3$	$D4$	Supply	Row Penalty
$S1$	19**(5)**	30	50	10**(2)**	0	--
$S2$	70	30	40	60**(2)**	7	40
$S3$	40	8**(8)**	70	20**(10)**	0	--
Demand	0	0	7	0		
Column Penalty	--	--	40	--		

The maximum penalty, 40, occurs in row $S2$.
The minimum c_{ij} in this row is $c_{23} = 40$.
The maximum allocation in this cell is min(7,7) = **7**.
It satisfy supply of $S2$ and demand of $D3$.
Initial feasible solution is

	D_1	D_2	D_3	D_4	Supply	Row Penalty
S_1	19**(5)**	30	50	10**(2)**	7	9 | 9 | 40 | 40 | -- | -- |
S_2	70	30	40**(7)**	60**(2)**	9	10 | 20 | 20 | 20 | 20 | 40 |
S_3	40	8**(8)**	70	20**(10)**	18	12 | 20 | 50 | -- | -- | -- |
Demand	5	8	7	14		
Column Penalty	21 21 -- -- -- --	22 -- -- -- -- --	10 10 10 10 40 40	10 10 10 50 60 --		

TC = 19×5 + 10×2 + 40×7 + 60×2 + 8×8 + 20×10 = **Rs.779**

MODI Method for Testing Optimality :

	Stores					
Warehouses	1	2	3	4	Supply	
1	5 19	d_{12} 30	d_{13} 50	2 10	7	U_1=-10
2	d_{21} 70	d_{22} 30	7 40	2 60	9	U_2=40
3	d_{31} 40	8 8	d_{33} 70	10 20	18	U_3=0
Demand	5	8	7	14		
	V_1=29	V_2=8	V_3=0	V_4=20		

First we find u_i and v_j for allocated cells.

We set u_i or v_j for having maximum number allocated cells (basic cells).

We set $\mathbf{u_3 = 0}$.

Then using $\mathbf{c_{ij} = u_i + v_j}$ (for basic cells)

$c_{34} = u_3 + v_4$

$20 = 0 + v_4 \Rightarrow \mathbf{v_4 = 20}$

$c_{32} = u_3 + v_2$

$8 = 0 + v_2 \Rightarrow \mathbf{v_2 = 8}$

$c_{24} = u_2 + v_4$

$60 = u_2 + 20 \Rightarrow \mathbf{u_2 = 40}$

$c_{23} = u_2 + v3$

$40 = 40 + v_3 \Rightarrow \mathbf{v_3 = 0}$

$c_{14} = u_1 + v_4$

$10 = u_1 + 20 \Rightarrow \mathbf{u_1 = -10}$

$c_{11} = u_1 + v_1$

$19 = -10 + v_1 \Rightarrow \mathbf{v_1 = 29}$

We find d_{ij}'s for non allocated cells (non basic cells) using

$$\mathbf{d_{ij} = u_i + v_j - c_{ij} \leq 0}$$

$d_{12} = u_1 + v_2 - c_{12}$

$= -10 + 8 - 30 = -32$

$d_{13} = u_1 + v_3 - c_{13}$

$= -10 + 0 – 50 = -60$

$d_{21} = u_2 + v_1 - c_{21}$

$= 40 + 29 – 70 = -1$

$d_{22} = u_2 + v_2 - c_{22} = 40 + 8 – 30 = 18$

$d_{31} = u_3 + v_1 - c_{31}$
$= 0 + 29 – 40 = -11$
$d_{33} = u_3 + v_3 - c_{33}$
$= 0 + 0 – 70 = -70$

	Stores					
Warehouses	1	2	3		Supply	
1	5 19	30	50	2 10	7	U1=-10
2	70	30	7 40	2 60	9	U2=40
3	40	8 8	70	10 20	18	U3=0
Demand	5	8	7	14		
	V1=29	V2=8	V3=0	V4=20		

Warehouses	1	2	3		Supply
1	5 19	30	50	2 10	7
2	70	$+\theta$ 30	7 40	$2-\theta$ 60	9
3	40	$8-\theta$ 8	70	$10+\theta$	18
Demand	5	8	7	14	

Min $[8-\theta, 2-\theta] = 0$ or **$2-\theta = 0$ or $\theta = 2$**

Warehouses	1	2	3		Supply
1	5 19	30	50	2 10	7
2	70	2 30	7 40	60	9
3	40	6 8	70	12 20	18
Demand	5	8	7	14	

TC = 19x5 + 10x2 + 30x2 + 40x7 + 8x6 + 20x12 = **Rs.743**

We will again find $\mathbf{u_i}$ and $\mathbf{v_j}$ and calculate $\mathbf{d_{ij}}$**'s** for non basic cells.

The final optimal table is given blow:

Warehouses	1	2	3	4	Supply	
1	$\boxed{5}$ 19	30	50	$\boxed{2}$ 10	7	U1=-10
2	70	$\boxed{2}$ 30	$\boxed{7}$ 40	60	9	U2=22
3	40	$\boxed{6}$ 8	70	$\boxed{12}$ 20	18	U3=0
Demand	5	8	7	14		
	V1=29	V2=8	V3=18	V4=20		

$c_{34}=u_3+v_4$; $20=0+v_4$ or $\mathbf{v_4=20}$

$8=0+v_2$ or $\mathbf{v_2=8}$

$c_{14}=u_1+v_4$; $10= -10+v_1$ or $\mathbf{v_1=20}$

$c_{11}=u_1+v_1$; $19=-10+v_1$ or $\mathbf{v_1= 29}$

$c_{11}=u_1+v_1$; $c_{22}=u_2+v_2$; $30=u_2+8$ or $\mathbf{u_2=22}$

$c_{23}=u_2+v_3$; $40=22+v_3$ or $\mathbf{v_3=18}$

$\mathbf{d_{12}=u_1+v_2-c_{12}}$ = -10+8-30 =-32

$d_{13}=u_1+v_3-c_{13}$ =-10+18-50 =-42

$d_{21}=u_2+v_1-c_{21}$ = 22+29-70 =-29

$d_{24}=u_2+v_4-c_{24}$ = 22+20-60 =-18

$d_{31}=u_2+v_1-c_{31}$ = 0+29-40 =-11

$d_{33}=u_3+v_3-c_{33}$ = 0+18-70 = - 52

Warehouses	1	2	3	4	Supply
1	$\boxed{5}$ 19	d_{12}=-32 30	d_{13}=-42 50	$\boxed{2}$ 10	7
2	d_{21}=-29 70	$\boxed{2}$ 30	$\boxed{7}$ 40	d_{24}=-18 60	9
3	d_{31}=-11 40	$\boxed{6}$ 8	d_{33}=-52 70	$\boxed{12}$ 20	18
Demand	5	8	7	14	

Since all d_{ij}'s are less or equal to zero, then the given transportation table is optimal.
Thus, the optimal transportation system is to transport:

5 units from store 1 to destination 1
2 units from store 1 to destination 4
2 units from store 2 to destination 2
7 units from store 2 to destination 3
6 units from store 3 to destination 2
12 units from store 3 to destination 4
With total cost of transportation of ***Rs.743/-***

TC = 19x5 + 10x2 + 30x2 + 40x7 + 8x6 + 20x12 = **Rs.743**.

....**Theorem 1: A necessary and sufficient condition for the existence of a feasible solution of a transportation problem is**
$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$, i=1,2,…,m and j=1,2,…,n

Proof : The condition is necessary:

Let there exists a feasible solution to the T.P., then
$\sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} = \sum_{i=1}^{m} a_i, \sum_{j=1}^{n} \sum_{i=1}^{m} x_{ij} = \sum_{j=1}^{n} b_j \Leftrightarrow \sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$

The condition is sufficient:

Let $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j = k$

If there exists a real number $\lambda \neq 0$ such that
$x_{ij} = \lambda_i b_j$, ________________________ (1)
then the value of λ_i is given by

$$\sum_{j=1}^{n} x_{ij} = \sum_{j=1}^{n} \lambda_i b_j$$
$$= \lambda_i \sum_{j=1}^{n} b_j$$
$$= \lambda_i \; k \; (\text{as} \sum_{j=1}^{n} b_j = k, \text{a constant})$$
$$\Rightarrow \lambda_i = \frac{\sum_{j=1}^{n} x_{ij}}{k}$$
$$= \frac{a_i}{K}$$

Thus from (1), $x_{ij} = \lambda_i b_j = \frac{a_i b_j}{k} \geq 0$, since $a_i, b_j > 0$ and $k > 0$,
therefore $x_{ij} \geq 0$. Hence feasible solution exists.

Theorem 2 : The number of basic variables in a T.P are at most m+n-1.

Proof : To prove the theorem, let us consider first m+n-1 constraints of the T.P. as

$\sum_{i=1}^{m} x_{ij}=b_j$, j=1,2,…,n-1 -------------------- (1)

$\sum_{j=1}^{n} x_{ij}=a_i$, i=1,2,…,m --------------------- (2)

Now adding (n-1) destination constraints given in (1)

$\sum_{j=1}^{n-1}\sum_{i=1}^{m} x_{ij}=\sum_{j=1}^{n-1} b_j$---------------------------- (3)

Also, adding m-origin constraints given in (2), we get

$\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij}=\sum_{i=1}^{m} a_i$--------------------------------- (4)

Subtracting (3) from (4)

$$\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij}-\sum_{j=1}^{n-1}\sum_{i=1}^{m} x_{ij}=\sum_{i=1}^{m} a_i-\sum_{j=1}^{n-1} b_j$$

or

$$\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij}-\left(\sum_{j=1}^{n}\sum_{i=1}^{m} x_{ij}-\sum_{i=1}^{m} x_{in}\right)=\sum_{i=1}^{m} a_i-\left(\sum_{j=1}^{n} b_j-b_n\right)$$

or

$$\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij}-\sum_{j=1}^{n}\sum_{i=1}^{m} x_{ij}+\sum_{i=1}^{m} x_{in}=\sum_{i=1}^{m} a_i-\sum_{j=1}^{n} b_j+b_n)$$

or

$$\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij}-\sum_{j=1}^{n}\sum_{i=1}^{m} x_{ij}+\sum_{i=1}^{m} x_{in}=\sum_{i=1}^{m} a_i-\sum_{i=1}^{m} a_i+b_n)$$

$$(\text{as} \sum_{i=1}^{m} a_i=\sum_{j=1}^{n} b_j)$$

or

$\sum_{i=1}^{m} x_{in}=b_n$ which is exactly the last nth destination constraint that stands automatically satisfied.

……………………………………………………………………

Assignment problem Hungarian Method

Step 1 *:* In a given problem, if the number of rows is not equal to the number of columns and vice versa, then add a dummy row or a dummy column. The assignment costs for dummy cells are always assigned as zero.
Step 2 *:* Reduce the matrix by selecting the smallest element in each row and subtract with other elements in that row.
Step 3 : Reduce the new matrix column-wise using the same method as given in step 2.
Step 4 : Draw minimum number of lines to cover all zeros.
Step 5 : If Number of lines drawn = order of matrix, then optimally is reached, so proceed to step 7. If optimality is not reached, then go to step 6.
Step 6 *:* Select the smallest element of the whole matrix, which is **NOT COVERED** by lines. Subtract this smallest element with all other remaining elements that are **NOT COVERED** by lines and add the element at the intersection of lines. Leave the elements covered by single line as it is. Now go to step 4.
Step 7 *:* Take any row or column which has a single zero and assign by squaring it. Strike off the remaining zeros, if any, in that row and column (X). Repeat the process until all the assignments have been made.
Step 8 *:* Write down the assignment results and find the minimum cost/time.
Note : While assigning, if there is no single zero exists in the row or column, choose any one zero and assign it. Strike off the remaining zeros in that column or row, and repeat the same for other assignments also. If there is no single zero allocation, it means multiple numbers of solutions exist. But the cost will remain the same for different sets of allocations.

Example : Assign the four tasks to four operators. The assigning costs are given in the Table given below.
Assignment Problem

$$
\begin{array}{ccc}
 & & \text{Operators} \\
 & & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\
\text{Tasks} & \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{pmatrix} 20 & 28 & 19 & 13 \\ 15 & 30 & 31 & 28 \\ 40 & 21 & 20 & 17 \\ 21 & 28 & 26 & 12 \end{pmatrix}
\end{array}
$$

Solution:
Step 1: The given matrix is a square matrix and it is not necessary to add a dummy row/column
Step 2: Reduce the matrix by selecting the smallest value in each row and subtracting from other values in that corresponding row. In row A, the smallest value is 13, row B is 15, row C is 17 and row D is 12. The row wise reduced matrix is shown in table below.
Row-wise Reduction

$$
\begin{array}{ccc}
 & & \text{Operators} \\
 & & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\
\text{Tasks} & \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{pmatrix} 7 & 15 & 6 & 0 \\ 0 & 15 & 16 & 13 \\ 23 & 4 & 3 & 0 \\ 9 & 16 & 14 & 0 \end{pmatrix}
\end{array}
$$

Step 3: Reduce the new matrix given in the following table by selecting the smallest value in
each column and subtract from other values in that corresponding column. In column 1, the smallest value is 0, column 2 is 4, column 3 is 3 and column 4 is 0. The column-wise reduction matrix is shown in the following table.

Column-wise Reduction Matrix

$$
\begin{array}{ccc}
 & & \text{Operators} \\
 & & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\
\text{Tasks} & \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{pmatrix} 7 & 11 & 3 & 6 \\ 0 & 11 & 13 & 13 \\ 23 & 0 & 0 & 0 \\ 9 & 12 & 11 & 0 \end{pmatrix}
\end{array}
$$

Step 4: Draw minimum number of lines possible to cover all the zeros in the matrix given in Table

Matrix with all Zeros Covered

$$
\begin{array}{ccc}
 & & \text{Operators} \\
 & & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\
\text{Tasks} & \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{pmatrix} 7 & 11 & 3 & 0 \\ 0 & 11 & 13 & 13 \\ 23 & 0 & 0 & 0 \\ 9 & 12 & 11 & 0 \end{pmatrix}
\end{array}
$$

No. of lines drawn $\neq$ order of matrix

The first line is drawn crossing row C covering three zeros, second line is drawn crossing column 4 covering two zeros and third line is drawn crossing column 1 (or row B) covering a single zero.

Step 5*:* Check whether number of lines drawn is equal to the order of the matrix, i.e., $3 \neq 4$. Therefore optimality is not reached. Go to step 6.

Step 6: Take the smallest element of the matrix that is not covered by single line, which is 3. Subtract 3 from all other values that are not covered and add 3 at the intersection of lines. Leave the values which are covered by single line. The following table shows the details.

Subtracted or Added to Uncovered Values and Intersection Lines Respectively

Operators

$$
\text{Tasks}\quad
\begin{array}{c|cccc}
 & 1 & 2 & 3 & 4 \\
\hline
A & 7 & 9 & 0 & 0 \\
B & 0 & 9 & 10 & 13 \\
C & 26 & 0 & 0 & 3 \\
D & 9 & 9 & 8 & 0
\end{array}
$$

Step 7*:* Now, draw minimum number of lines to cover all the zeros and check for optimality. Here in table minimum number of lines drawn is 4 which are equal to the order of matrix. Hence optimality is reached.

Optimality Matri

Operators

$$
\text{Tasks}\quad
\begin{array}{c|cccc}
 & 1 & 2 & 3 & 4 \\
\hline
A & 7 & 9 & 0 & 0 \\
B & 0 & 9 & 10 & 13 \\
C & 26 & 0 & 0 & 3 \\
D & 9 & 9 & 8 & 0
\end{array}
$$

No. of lines drawn = order of matrix

Step 8: Assign the tasks to the operators. Select a row that has a single zero and assign by squaring it. Strike off remaining zeros if any in that row or column. Repeat the assignment for other tasks. The final assignment is shown in table below.

Final Assignment

$$
\begin{array}{cc}
 & \textbf{Operators} \\
 & \begin{array}{cccc} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} \end{array} \\
\textbf{Tasks} \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{pmatrix} 7 & 9 & \boxed{0} & \xcancel{0} \\ \boxed{0} & 9 & 10 & 13 \\ 26 & \boxed{0} & \xcancel{0} & 3 \\ 9 & 9 & 8 & \boxed{0} \end{pmatrix}
\end{array}
$$

Therefore, optimal assignment is:

Task	Operator	Cost
A	3	19
B	1	15
C	2	21
D	4	12
	Total Cost = Rs. 67.00	

Example : Solve the following assignment problem shown in Table using Hungarian method. The matrix entries are processing time of each man in hours.

Assignment Problem

$$
\begin{array}{ccc}
 & & \textbf{Men} \\
 & & \begin{array}{ccccc} \textbf{1} & \textbf{2} & \textbf{3} & \textbf{4} & \textbf{5} \end{array} \\
\textbf{Job} & \begin{array}{c} \textbf{I} \\ \textbf{II} \\ \textbf{III} \\ \textbf{IV} \\ \textbf{V} \end{array} & \begin{pmatrix} 20 & 15 & 18 & 20 & 25 \\ 18 & 20 & 12 & 14 & 15 \\ 21 & 23 & 25 & 27 & 25 \\ 17 & 18 & 21 & 23 & 20 \\ 18 & 18 & 16 & 19 & 20 \end{pmatrix}
\end{array}
$$

Solution: The row-wise reductions are shown in Table

Row-wise Reduction Matrix

$$
\begin{array}{ccc}
 & & \textbf{Men} \\
 & & \begin{array}{ccccc} \textbf{1} & \textbf{2} & \textbf{3} & \textbf{4} & \textbf{5} \end{array} \\
\textbf{Job} & \begin{array}{c} \textbf{I} \\ \textbf{II} \\ \textbf{III} \\ \textbf{IV} \\ \textbf{V} \end{array} & \begin{pmatrix} 5 & 0 & 3 & 5 & 10 \\ 6 & 8 & 0 & 2 & 3 \\ 0 & 2 & 4 & 6 & 4 \\ 0 & 1 & 4 & 6 & 3 \\ 2 & 2 & 0 & 3 & 4 \end{pmatrix}
\end{array}
$$

The column wise reductions are shown in Table.

Column-wise Reduction Matrix

$$
\begin{array}{ccc}
 & & \textbf{Men} \\
 & & \begin{array}{ccccc} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{5} \end{array} \\
\textbf{Job} & \begin{array}{c} \mathbf{I} \\ \mathbf{II} \\ \mathbf{III} \\ \mathbf{IV} \\ \mathbf{V} \end{array} & \begin{pmatrix} 5 & 0 & 3 & 3 & 7 \\ 6 & 8 & 0 & 0 & 0 \\ 0 & 2 & 4 & 4 & 1 \\ 0 & 1 & 4 & 4 & 0 \\ 2 & 2 & 0 & 1 & 1 \end{pmatrix}
\end{array}
$$

Matrix with minimum number of lines drawn to cover all zeros is shown in Table.

Matrix will all Zeros Covered

$$
\begin{array}{ccc}
 & & \textbf{Men} \\
 & & \begin{array}{ccccc} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{5} \end{array} \\
\textbf{Job} & \begin{array}{c} \mathbf{I} \\ \mathbf{II} \\ \mathbf{III} \\ \mathbf{IV} \\ \mathbf{V} \end{array} & \begin{pmatrix} 5 & 0 & 3 & 3 & 7 \\ 6 & 8 & 0 & 0 & 0 \\ 0 & 2 & 4 & 4 & 1 \\ 0 & 1 & 4 & 4 & 0 \\ 2 & 2 & 0 & 1 & 1 \end{pmatrix}
\end{array}
$$

The number of lines drawn is 5, which is equal to the order of matrix. Hence optimality is reached. The optimal assignments are shown in Table.

Optimal Assignment

$$
\begin{array}{ccc}
 & & \textbf{Men} \\
 & & \begin{array}{ccccc} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{5} \end{array} \\
\textbf{Job} & \begin{array}{c} \mathbf{I} \\ \mathbf{II} \\ \mathbf{III} \\ \mathbf{IV} \\ \mathbf{V} \end{array} & \begin{pmatrix} 5 & \boxed{0} & 3 & 3 & 7 \\ 6 & 8 & \cancel{0} & \boxed{0} & \cancel{0} \\ \boxed{0} & 2 & 4 & 4 & 1 \\ \cancel{0} & 1 & 4 & 4 & \boxed{0} \\ 2 & 2 & \boxed{0} & 1 & 1 \end{pmatrix}
\end{array}
$$

Therefore, the optimal solution is:

Job	Men	Time
I	2	15
II	4	14
III	1	21
IV	5	20
V	3	16
	Total time =	86 hours

Theorem : In an assignment problem, if we add (or subtract) a constant from/to every element of a row or a column of the cost matrix [cij], then an assignment plan that minimizes the total cost for the new cost matrix also minimizes the total cost of the original cost matrix.

Proof: Let $x_{ij} = \hat{x}_{ij}$ minimizes $Z=\sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}x_{ij}$, $x_{ij} \geq 0$ such that $\sum_{i=1}^{n} x_{ij} = 1$ and $\sum_{j=1}^{n} x_{ij} = 1$, then $x_{ij} = \hat{x}_{ij}$ also minimizes $Z^{*}=\sum_{i=1}^{n}\sum_{j=1}^{n} c^{*}_{ij} x_{ij}$ where $c^{*}_{ij} = (c_{ij} - u_i - v_j)$ for all i,j=1,2,…,n and u_i and v_j are real numbers. We have,

$$Z^{*}=\sum_{i=1}^{n} \sum_{j=1}^{n} (c_{ij} - u_i - v_j)\, x_{ij}$$

$$=\sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} - \sum_{i=1}^{n} \sum_{j=1}^{n} u_i x_{ij} - \sum_{i=1}^{n} \sum_{j=1}^{n} v_j x_{ij}$$

$$=Z-\sum_{i=1}^{n} u_i - \sum_{i=1}^{n} v_j \text{ as } \sum_{i=1}^{n} x_{ij} = 1 \text{ and } \sum_{j=1}^{n} x_{ij} = 1$$

Since the terms that are subtracted from Z to give z^* minimum are independent of $x_{ij}s$, it follows that z^* is minimized.

Project Networks

Project Networks: Network models can be used as an aid in scheduling large complex projects that consist of many activities. If the duration of each activity is known with certainty, then we use the critical path method (CPM) to determine the length of time required to complete a project. CPM is also used to determine how long an each activity in the project can be delayed without delaying the completion of the project. The method was developed in the late 1950. If the duration of the completion of an activity is not known with certainty, then we make use of Program and Evaluation and Review Technique (PERT). CPM and PERT have been used in many applicationsincluding:

1. Scheduling construction projects such as

(i) Residential complexes

(ii) Highways

(iii) Railway Bridges

(iv) Power Plants

(v) Nuclear Reactors

(vi) Dams

Even though the method was developed for construction projects but it has found applications in many projectswhere there are interdependent activities namelyconduction of big examination namely for civil services etc.

Network : The network of the project is the graphical representation of project operations (activities)

Activity : In each network, an activity is shown by an arrow.The arrow head indicates the direction of the progress of the activity. The number on the arrow indicates the duration of the activity.

Node : The activity starting points are called nodes which are shown by circles. Each network should satisfy thefollowing conditions.

(i) Each activity must be represented by one and only one arrow.

(ii) Two activities cannot have initial and same terminal nodes i.e., each activity must be identified by two distinct nodes.

Example : Construct the network where the activities satisfy the requirements

i) activities A and B are the first activities of theproject which an start simultaneously.

ii) A and B precede C

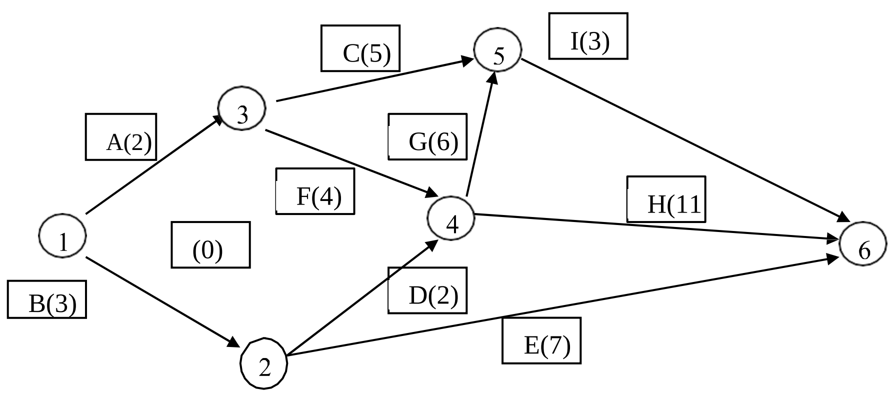
iii) B precedes D and E

iv) F and D precede G and H

v) C and G precede I

vi) E, H and I are the terminal activities

The duration of the activities A,B,…H and I are2,3,5,2,7,4,6,11 and 3



Network Diagram

Critical Path Method : First, we consider CPM. In the critical path method, the critical activities of a program or a project are identified. These are the activities that have a direct impact on the completion date of the project.

The process of using critical path method in project planning phase has the following main steps.

Step 1: Activity sequence establishment
In this step, the correct activity sequence is established.We should know

- Which activity should take place before this activity happens?
- Which activity should be completed at the same timeas this activity?
- Which activity should happen immediately after this activity?

Step 2 : Network Diagram :
Once the activity sequence iscorrectly identified, the network diagram can be drawn using activities with nodes.

Step 3: Estimates for each activity
These are assumed to be known either from the past practice or can be estimated
Step 4: Identification of the critical path
For this, we need to determine four parameters of each activity of the network.

- **Earliest start time (ES) :** The earliest time an activity can start once the previous dependent activities are over.
- **Earliest finish time (EF)** = ES + activity duration.
- **Latest finish time (LF) :** The latest time an activity canfinish without delaying the project.
- **Latest start time (LS)** = LF - activity duration.
- **Float or Slack values :** There are generally three types offloats namely total float, free float and independent float.

1. **Total Float :** Total float is the extra surplus time which can be allocated to an activity or this is the period of time up to which an activity can be delayed beyond its earliest finish time without extending the overall project time. It can be calculated as

$$\mathbf{TF_{ij} = LF_j - ES_i - D_{ij}}$$

2. **Free Float :** It is defined by assuming that all theactivities start as early as possible. In this case, $\mathbf{FF_{ij} = ES_j - ES_i - D_{ij}}$, where Dij s the normal duration ofcompletion of an activity.

3. **Independent Float :** This is the time by which an activity can be rescheduled without affecting the preceding orsucceeding activities. It is calculated as

IF = Free Float –Tail event slack = Free float - (LF_i - ES_i)

The critical path calculations involve two passes-Forward Pass and Backward Pass:

(A) Forward Pass :

(i) We set **□1=0 (ES1=0)** i.e., Project starts at time 0 and

(iii) Given that nodes p,q,… and v are linked to node j by incoming activities (p,j), (q,j),…, and (v,j) and that the earliest occurrence times of events (nodes)p,q,…,and v have already been calculated, then theearliest occurrence time of even j is computed as

$\mathbf{\square_j = Max\ \{\square_p + D_{pj},\ \square_q + D_{qj}, \ldots, \square_v + D_{vj}\}}$ The forward pass is complete when □n at node n iscalculated. By definition, □j represents thelongest path (duration) to node j.

(B) Backward Pass :

(Latest Occurrence Time), Following the completion of the forward pass, the backward pass computation starts at node n and regress recursively by node 1.

(i) We set $\mathbf{\Delta n = \square_n}$ to indicate that the earliest and latest occurrence of the last of the project are the same.

(ii) Given that nodes p,q,… and v are linked to node j by incoming activities (j,p), (j,q),…, and (j,v) and that thelatest occurrence times of events (nodes) p,q,…,and vhave already been calculated, then the latest occurrence time of even j is computed as

Δ_j **=Min {Δ_p-D_{jp}, Δ_q-D_{jq},…, Δ_v-D_{jv}}** The backwardis complete when Δ1 at node 1 is calculated. By definition, □j represents the longest path (duration) to node j.

Based on the above calculations, an activity (I,j) is said becritical if it satisfies the following three conditions

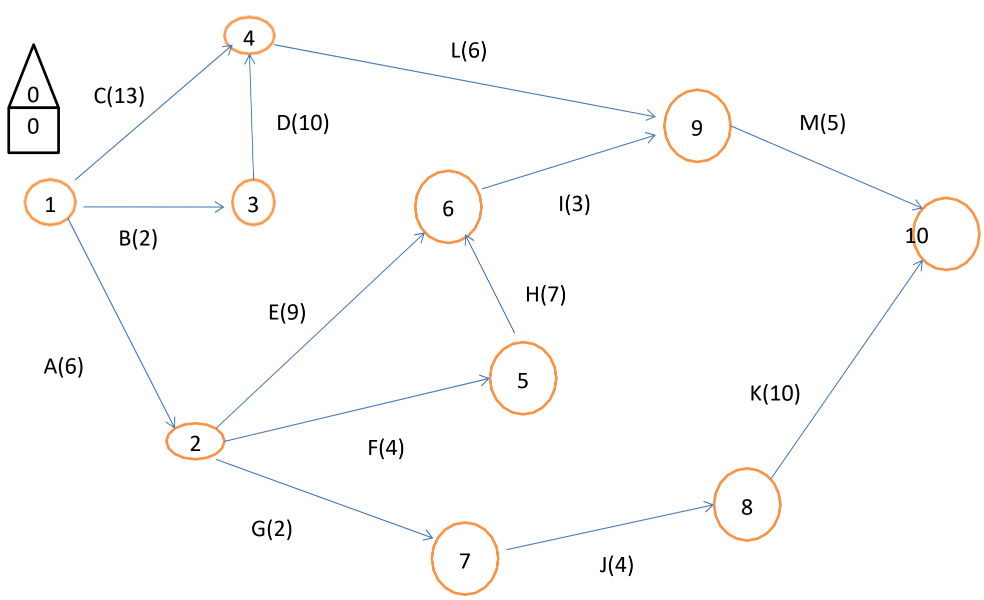
a. $\square_i = \Delta_i$

b. $\square_j = \Delta_j$

c. $\square_j - \square_j = \Delta_j - \Delta_i = D_{ij}$

An activity that does not satisfy the above threeconditions is called non critical.

Example : **NETWORK DIAGRAM**



Find the critical path and the duration of completion of the the above network of a Project

Solution : We first find the critical path : Forward Pass:

We set E1=0

E2=E1+D12=0+6=6

E3=E1+D13=0+2=2

E4=Max{E1+D14, E3+D34}=Max{0+13,2+10}=13

E5=E2+D25=6+4=10 E6=Max{E2+D26,E5+D56}=Max{6+9,10+7}=17

E7=E2+D27=6+2=8

E8=E7+D78=8+4=12

E9=Max{E6+D69, E4+D49}=Max{17+3,13+6}=20

E10=Max{E9+D9,10, E8+D8,10}

=Max{20+5,12+10}=25

Backward Pass :

We set L10=E10=25=0

L9=L10-D9,10=25-5=20

L8=L10-D8,10=25-10=15

L7=L8-D78=15-4=11 L6=L9-

D69=20-3=17

L5=L6-D56=17-7=10L4=L9-

D49=20-6=14 L3=L4-

D34=14-10=4

L2=Min{L5-D25,L6-D26,L7-D27}

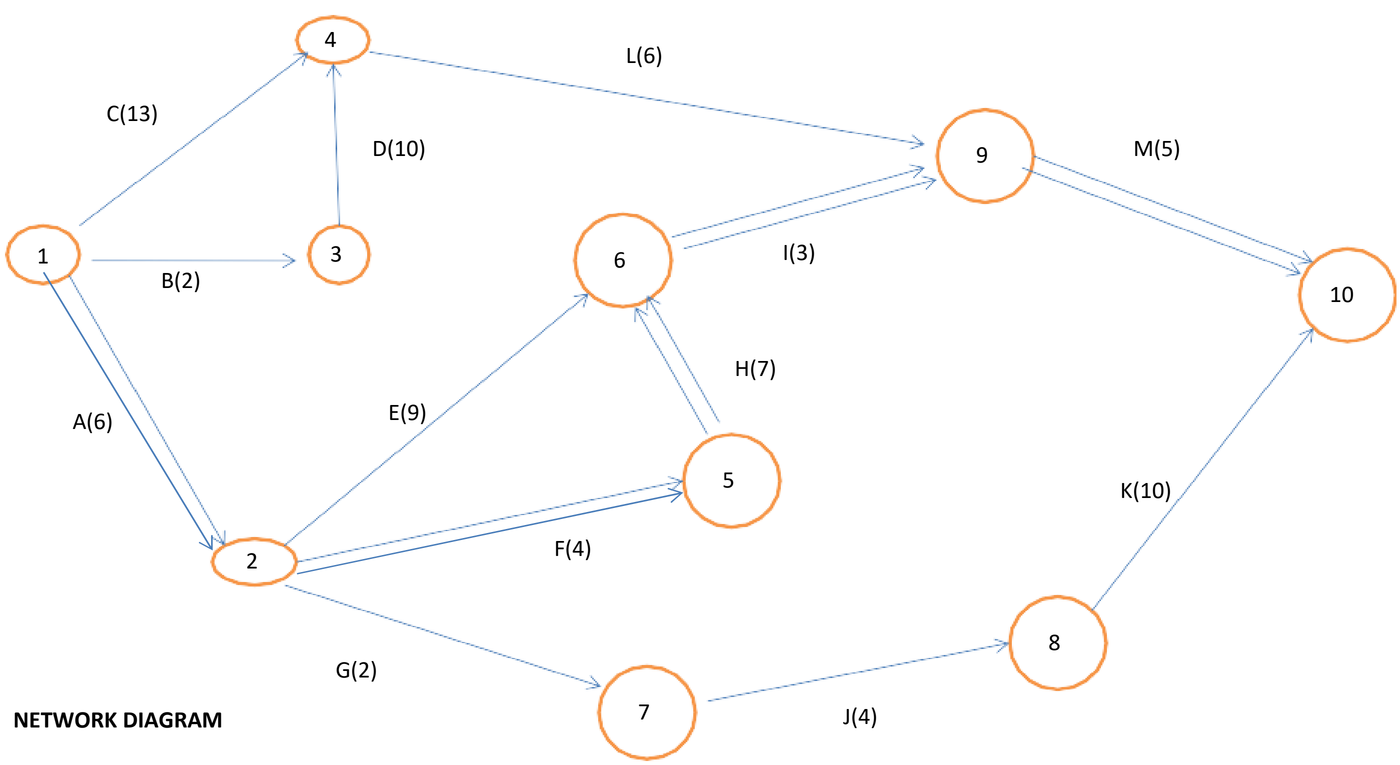
=Min{10-4,17-9,11-2}=6 L1=Min{L2-D12,L3-

D13,L4-D14}

=Min{6-6,4-2,14-13}=0

Activity	Dij	Earliest Time Start ESi	Finish Esi+ Dij	Latest Time Start LFj-Dij	Finish = LFj	TFij=LFj-(ESi+Dij)	FFij=ESj-(ESi+Dij)
(1,2)	6	0	6	0	6	0	0
(1,3)	2	0	2	2	4	2	0
(1,4)	13	0	13	1	14	1	0
(2,5)	4	6	10	6	10	0	0
(2,6)	9	6	15	8	17	2	2
(2,7)	2	6	8	9	11	3	0
(3,4)	10	2	12	4	14	2	1
(4,9)	6	13	19	14	20	1	1
(5,6)	7	10	17	10	17	0	0
(6,9)	3	17	20	17	20	0	0
(7,8)	4	8	12	11	15	3	0
(8,10)	10	12	22	15	25	3	3
(9,10)	5	20	25	20	25	0	0

We have the Critical Path with Double Arrows i.e., **1→2→5→6→9→10**
And the total duration of completion of the project is **25 days.**



NETWORK DIAGRAM

PERT : If the duration of the completion of an activity is not known with certainty, then we make use of Program and Evaluation and Review Technique (PERT). The method was developed in the late 1950 also. The two methods CPM and PERT were developed in the same year on the same analogy but both the methods used different time estimates and the two developers did not know each other. PERT uses probability approach in estimates and thus enables us in finding the probability of expected duration of completion of the project. The steps of drawing network and finding of Forward and Backward passes are the same except that we will be dealing with the expected time of duration instead of certain time.

Under the conditions of uncertainty, the estimated time for each activity for PERT network is represented by a probability distribution. This probability distribution of activity time is based upon three different time estimates made for each activity. These are as follows:

(i) **The Optimistic Time (t_o):** This is the shortest possible time to complete the activity if everything goes normal.

(ii) **The Pessimistic Time (t_p):** This is the longest time an activity would take

(iii) **Most Likely time (t_m):** This is the estimate of the normal time an activity

Keeping in view the above, it is justified to assume that the duration of each activity may follow Beta distribution with its uni-model point occurring at t_m and its end points t_o and t_p.

The expected value of the activity duration can be approximated as the arithmetic mean of $(t_o + t_p)/2$ and $2t_m$. Thus we have,

$$\mathbf{t_e} = \tfrac{1}{3}[2t_m+(t_o + t_p)/2] = \mathbf{[t_o + 4t_m + t_p)/6]} \text{ and}$$

the variance $\{(t_p - t_o)/6\}^2$

Probability of Meeting the schedule time : With PERT, it is possible to determine the probability of completing a contract on schedule..The scheduled dates are expressed as a number of time units from the present time, initially they may be the latest times for each event, but after a project is started we shall know how far, t has progressed at any given date and the scheduled time will be the latest time if the project is to be completed on its original schedule.

The probability distribution of times for completing an event can be approximated by the distribution due to Central Limit Theorem (CLT). Thus the probability of completing the project schedule time (Ts) is given by

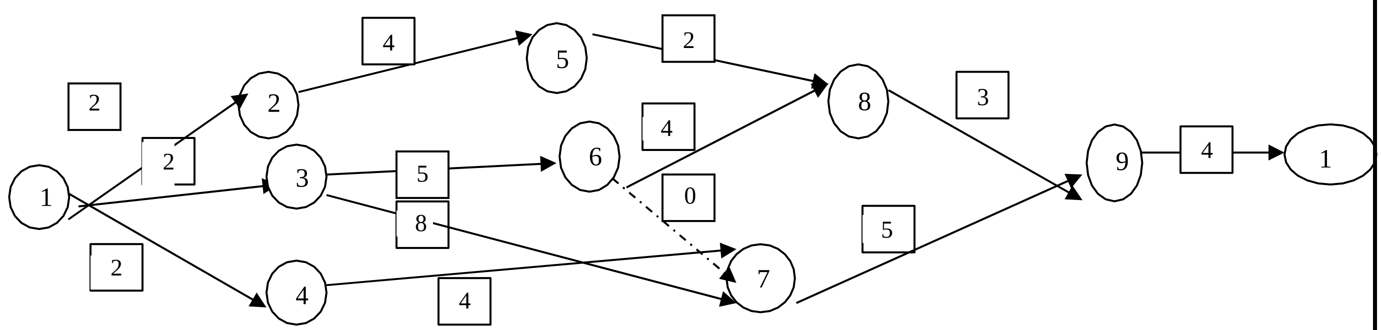
Prob (Z< $\frac{Ts-Te}{\sigma}$) where Z is N(0,1) and Te is the expected completion time of the project

Example : A project consists of thirteen activities with the following relevant information.

Activity	t_o	t_m	t_p
(1,2)	1.0	2.0	3.0
(1,3)	1.5	2.0	2.5
(1,4)	1.5	1.75	3.5
(2,5)	3.0	3.75	7.0
(3,6)	4.0	4.5	8.0
(3,7)	6.0	8.25	9.0
(4,7)	3.0	3.5	7.0
(5,8)	2	2	2
(6,7)	0	0	0
(6,8)	2.0	4.0	6.0
(7,9)	2	4.5	8.0
(8,9)	2.0	3.0	4.0
(9,10)	2.5	4.5	4.5

(i) Draw the network diagram of the project.
(ii) Find the critical path of the completion of the Project.
(iii) Calculate expected duration of the completion of the Project.
(iv) Calculate the probability of completing the project in 20 days.

Solution: The network Diagram of the above project is produced below:



Now, we calculate the expected duration t_e and variances for each activity.

Activity	t_o	t_m	t_p	$t_e = (t_o + 4t_m + t_p)/6$	$\sigma^2 = \{(t_p - t_o)/6\}^2$
(1,2)	1.0	2.0	3.0	2	4/36
(1,3)	1.5	2.0	2.5	2	1/36
(1,4)	1.5	1.75	3.5	2	4/36
(2,5)	3.0	3.75	7.0	4	16/36
(3,6)	4.0	4.5	8.0	5	16/36
(3,7)	6.0	8.25	9.0	8	9/36
(4,7)	3.0	3.5	7.0	4	16/36
(5,8)	2	2	2	2	0
(6,7)	0	0	0	0	0
(6,8)	2.0	4.0	6.0	4	16/36
(7,9)	2	4.5	8.0	5	36/36
(8,9)	2.0	3.0	4.0	3	4/36
(9,10)	2.5	4.5	4.5	4	4/36

Rules or finding the variance:

(i) Set $v_1=0$

(ii) v_j the variance of the succeeding vent is obtained by adding activities variance to the variance of the predecessor event except at merge point $v_j = v_j + \sigma^2_{ij}$

(iii) At merge points, the variance is calculated along the longest path. In case of the two paths having the same length, the larger of the two variances is chosen as the variance for that event.

We have

$v1=0$, $v2=4/36$ $v3=1/36$ $v4=4/36$

$v5=v_2+\sigma^2_{2,5}=4/36+16/36=20/36$

$v6=v_3+\sigma^2_{3,6}=1/36+16/36=17/36$

$v7=v_3+\sigma^2_{3,7}=1/36+16/36=10/36^{*}$,

$v8=v_6+\sigma^2_{6,8}=33/36^{*}$

$v9=v_7+\sigma^2_{7,9}=10/36+36/36=46/36$

$v10=v_9+\sigma^2_{9,10}=46/36+4/36=50/36$

*along the longest path

Now we perform **Forward Pass**

$E1=0$ $E2=E1+t_{12}=0+2=2$, $E3=E1+t_{13}=0+2=2$

$E4=E1+t_{14}=0+2=2$ $E5=E2+t_{25}=2+4=6$

$E6=E3+t_{36}=2+5=7$,

$E7=Max\{E3+t_{37}, E4+t_{47}\}=Max\{2+8,2+4\}=10$

$E8=Max\{E5+t_{58}, E6+t_{68}\}= Max\{6+2,7+4\}=11$

$E9=Max\{E7+t_{79},E8+t_{89}\}= Max\{10+5,11+3\}=15$

$E10=E9+t_{9,10}=15+4=19$

Similarly, we perform **Backward Pass:**

Let $L10=E10=19$

$L9=L10-t_{9,10}=19-4=15$, $L8=L9-t_{9,10}=15-3=12$

$L7=L9-t_{79}=15-5=10$

$L6=Min\{L8-t_{68}, L7-t_{67}\} = Min\{12-4,10-0\}=8$

$L5=L8-t_{58}=12-2=10$

$L4=L7-t_{47}=10-4=6$

$L3=Min\{ L6-t_{36},L7-t_{37}\}= Min\{ 8-5,10-8\}=2$

$L2=L5-t_{25}=10-4=6$

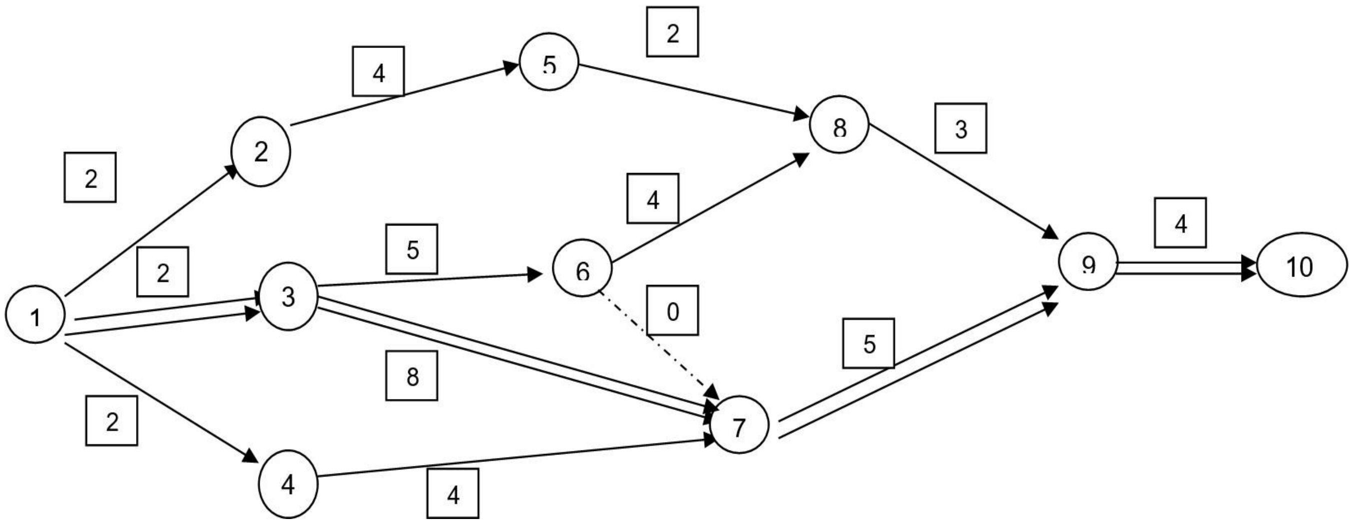
$L1= Min\{ L2-t_{12},L3-t_{13}, L4-t_{14}\} = Min\{ 6-2,2-2,6-4\}=0$

Activity	t_{ij}	Earliest Time E_i	Earliest Finish Time E_i+t_{ij}	Latest Time Start L_j-t_{ij}	Latest Start Finish L_j	Total Float Time $TF_{ij}=L_j-E_i-t_{ij}$	Free Flaot Time $FF_{ij}=E_j-E_i-t_{ij}$
(1,2)	2	0	2	4	6	4	0
(1,3)	2	0	2	0	2	0	0
(1,4)	2	0	2	4	6	4	0
(2,5)	4	2	6	6	10	4	0
(3,6)	5	2	7	3	8	1	0
(3,7)	8	2	10	2	10	0	0
(4,7)	4	2	6	6	10	4	4
(5,8)	2	6	8	10	12	4	3
(6,7)	0	0	0	0	0	0	0
(6,8)	4	7	11	8	12	1	0
(7,9)	5	10	15	10	15	0	0
(8,9)	3	11	14	12	15	1	1
(9,10)	4	15	19	15	19	0	0

***Note i) For an activity (i,j), we have corresponding Ei,Ej,Li and Lj for example say for an activity (3,4) we have Ei=E3, Ej=E4 and Li=L3 and Lj=L4.**

FF is always less or equal to TF and critical activities will have total float equal to zero.

ii) The critical path is 1→3→7 →9→10



iii) **Expected duration of completing the project is 19 days and the variance of the path is** 50/36.

iv) **We have** $P\left[\frac{Ts-\mu}{\sigma} \le \frac{20-\mu}{\sigma}\right]$ **where** μ **and** σ **are the mean and standard deviation of the distribution.**

Substituting the values, we have

$$P\left[Z \le \frac{20-19}{\sqrt{\frac{50}{36}}}\right]$$

$$= P\left[Z \le \frac{1}{\frac{7.07}{6}}\right]$$

$= P[Z \le .85] = 0.80$ **(from tables of N(0,1)**

We may also like to know how many chances are there of completing a project with a specified confidence level. Let Ts be the scheduled duration, then

$P[T \le t] = 0.95$ **(We wish to be 95% confident)**

$$= P\left(\frac{t-\mu}{\sigma} \le \frac{Ts-\mu}{\sigma}\right] = .95$$

$$= P\left(Z \le \frac{Ts-\mu}{\sigma}\right] = .95$$

or $P(Z \le 1.64] = .95$ **using Table of SNV (Z)**

or $\frac{Ts-\mu}{\sigma} = 1.64$ **or** $Ts = \mu + 1.64\,\sigma$

or $Ts = 19 + 1.64 \times \sqrt{\frac{50}{36}} = 20.90 = 21$ **days**

Game Theory

We have considered many problems in LPP for finding optimal solutions. In all these problems, we had only one decision maker and he was free to decide what to produce and what not to produce. However, this is a very idealistic situation and in may real life problems, there are more than one decision makers whose decisions are conflicting with each other. In such a situation, one competitor called a Player has many options called strategies and another competitor called player B also has many counter options to the strategies of player B. This is called a game problem of two players.

Pay off Matrix :- Each combination of alternatives of Players A and B is associated with an outcome $\mathbf{a_{ij}}$. If $\mathbf{a_{ij}}$ is positive, then it represents a gain to player A and loss to player B. If it is negative, it represents a loss to A and gain to B. A sample pay off matrix of player is given below:

		Player B			
		I	II	…	n
	I	a_{11}	a_{12}	…	a_{1n}
	II	a_{21}	a_{22}	…	a_{2n}
Player A	. . .	. . .	. . .	. . .	. . .
	m	a_{m1}	a_{m2}	…	a_{mn}

Max Min Principle :- The principle maximizes the minimum guaranteed gains of Player A. The minimum gains with respect to different alternatives of A, irrespective of B's alternatives are obtained first. The maximum of these minimum gains is known as the maximum value and the corresponding alternative is called as maxmin strategy.

MiniMax Principle :- The principle minimizes the maximum losses of Player B. The maximum losses with respect to different alternatives of Player B, irrespective of Player A's alternatives are obtained first. The minimum of these maximum losses is known as the minimax value and the corresponding alternative is called as minimax strategy.

Saddle Point :- If in a game, Maxmin value is equal to minimax value, then the game is said to have a saddle point and the game is said to be stable. In this case each player will choose his pure strategy. i.e. **MaxMin = MiniMax ,** game is stable.

Value of the game :- If the game has a saddle point, then the value of the cell at the saddle point in the payoff matrix is called the value of the game.`

Two-person Zero-sum game :- In a game with two players, if the gain of bone is equal to loss of the other, then that game is called two-person zero sum game. i.e. **Gain = Loss**

Example - 1

		Player B			Min of each row	max
		I	II	III		
Player A	I	1	3	6	1	
	II	2	1	3	1	1
	III	6	2	1	1	
	Max of each column	6	3	6		
	min		3			

Saddle Point :- In our case Minimax is not equal to MaxMin. Hence, no Saddle Point exist.

Example - 2

		Player B			Min of each row	max
		I	II	III		
Player A	I	20	15	22	15	
	II	35	45	40	35	35
	III	18	20	25	18	
	Max of each column	35	45	40		
	min	35				

Saddle Point :- In this case, Minmax is equal to MaxMin. Hence, Saddle Point exist. Game is stable.
The common cell (A2,B1) in the game matrix has the maxmin = minmax. Therefore the optimal pure strategies of both the players are II and I and the value of the is 35.The player A has a gain of Rs.35 and player B has a loss of Rs.35.The strategies being used by both the players are pure strategies.

Example - 3

		Player B				Min of each row	max
		I	II	III	IV		
Player A	I	1	7	3	4	1	
	II	5	6	4	5	4	4
	III	7	2	0	3	0	
	Max of each column	7	7	4	5		
	min				4		

Saddle point at $\mathbf{A_2, B_3}$ and optimal value of the game is **4.** The player A has a gain of Rs.4 and Player B has a loss of **Rs. 4**.

Graphical method

The graphical method is used to solve the games whose payoff matrix has two rows and n columns (2 x n) or m rows and two columns (m x 2)

Algorithm for solving 2 x n matrix games

Draw two vertical axes 1 unit apart. The two lines are $\mathbf{x_1 = 0, x_1 = 1}$
Take the points of the first row in the payoff matrix on the vertical line $x_1 = 1$ and the points of the second row in the payoff matrix on the vertical line $x_1 = 0$.
The point a_{1j} on axis $x_1 = 1$ is then joined to the point a_{2j} on the axis $x_1 = 0$ to give a straight line. Draw 'n' straight lines for j=1, 2… n and determine the highest point of the lower envelope obtained. This will be the **maximin point**.
The two or more lines passing through the maximin point determines the required 2 x 2 payoff matrix. This in turn gives the optimum solution by making use of analytical method.

Example - 4

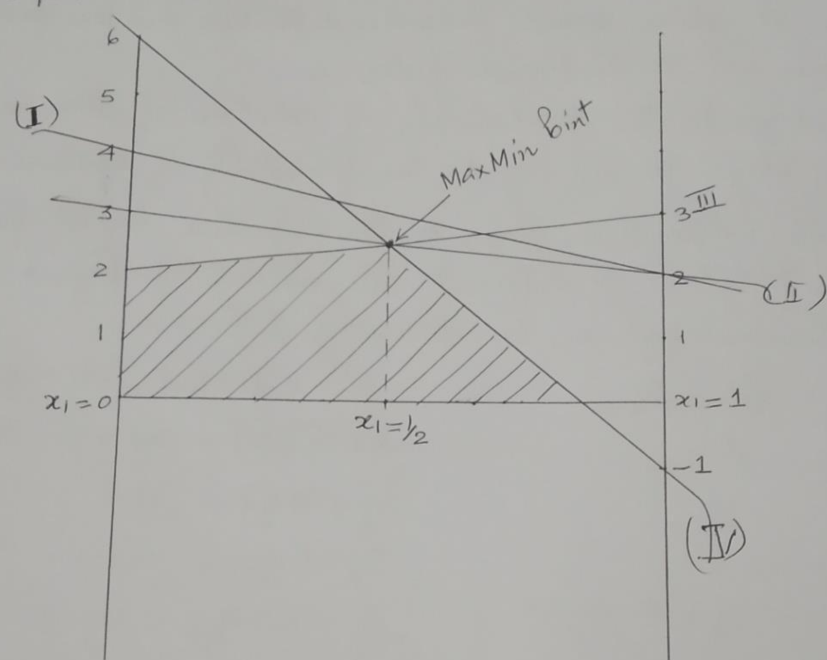
		Player B				Min of each row	max
		I	II	III	IV		
Player A	I (x1)	2	2	3	-1	-1	2
	II (x2)	4	3	2	6	2	
	Max of each column	4	3	3	6		
	min		3				

Thus MaxMin =2 and MinMax=3. Thus the game does not have saddle point. Let the player A choose his strategies I and II with probabilities x_1 and x_2 where $\mathbf{x_2 = 1 - x_1}$ as $\mathbf{x_1 + x_2 = 1}$.
We can solve a game problem if we have a game of the dimension 2 x n or m x 2.
A's expected payoffs corresponding to B's strategies are given as follows.

B's pure strategies	Expected payoff	$x_1 = 0$	$x_1 = 1$
1	$2x_1 + 4x_2$ or $2x_1 + 4(1-x_1) = -2x_1 + 4$	4	2
2	$2x_1 + 3x_2$ or $2x_1 + 3(1-x_1) = -x_1 + 3$	3	2
3	$3x_1 + 2x_2 = x_1 + 2$	2	3
4	$-x_1 + 6x_2 = -7x_1 + 6$	6	-1

	for $x_1=0$	$x_1=1$
$-2x_1+4$ gives	4	2
$-x_1+3$ "	3	2
x_1+2 "	2	3
$-7x_1+6$ "	6	-1

We plot the above four linear equations



The maxmin occurs at $x_1=1/2$ i.e. the upper most (highest point) point of the ~~upper~~ lower envelop (shaded region) and A's optimal strategies are $x_1^*=1/2$ and $x_2=1-x_1=1/2$.
The value of the game is obtained for x_1 in the equation of any of the lines passing through the maximum point. Thus

$$V^* = \begin{cases} -\frac{1}{2}+3 = 5/2 & \text{line II} \\ \frac{1}{2}+2 = 5/2 & \text{line III} \\ -7\cdot\frac{1}{2}+6 = 5/2 & \text{line IV} \end{cases}$$

To determine B's optimal strategies, it should be noticed that three lines pass through the Maxmin point. This is an indication that B can mix all three strategies. Any two lines having opposite signs for their slopes, define an alternative optimum. Thus, the three combinations (II,III), (II,IV) and (III,IV) are optimal whereas (I,IV) is non optimal.

	Player B		
		II	III
Player A	I	2	3
	II	3	2

or

	Player B		
		II	IV
Player A	I	2	3
	II	3	2

or

	Player B		
		III	IV
Player A	I	3	-1
	II	2	6

We will use analytical method, an alternative approach to find the probabilities associated with the strategies using any of the above combinations.

Using the first option, we have

	Player B		
		II (y_2)	III (y_3)
Player A	I (x_1)	2	3
	II (x_2)	3	2

We write as

$2x_1 + 3x_2 = \nu$
$3x_1 + 2x_2 = \nu$

Or $2x_1 + 3(1-x_1) = \nu$
$3x_1 + 2(1-x_1) = \nu$

Or $-x_1 + 3 = \nu$
$x_1 + 2 = \nu$
or $-2x_1 + 1 = 0$ or $x_1 = 1/2$ and $x_2 = 1-x_1 = 1-1/2 = 1/2$

For finding probabilities associated with Player B' strategies,

We write as
$2y_2 + 3y_3 = \nu$
$3y_2 + 2y_3 = \nu$
Or $2y_2 + 3(1-y_2) = \nu$
$3y_2 + 2(1-y_2) = \nu$

Or $-y_2 + 3 = \nu$
$y_2 + 2 = \nu$

or $-2y_2 + 1 = 0$

or $y_2 = 1/2$ and $y_3 = 1-y_2 = 1-1/2 = 1/2$
Thus player A will play his strategies I and II with probabilities 1/2 each and player B plays his strategies II and III with probabilities 1/2 each.

Thus, We writes as (A1, A2) with $(x_1^*, x_2^*) = (1/2, 1/2)$
and (B2,B3) = $(y_2^*, y_3^*) = (1/2, 1/2)$
We also calculate the value of the game as
$V^* = a_{12}.\ x_1^*.\ y_2^* + a_{13}.\ x_1^*.y_3^* + a_{22}.x_2^*.y_2^* + a_{23}.x_2^*.y_3^*$
$= 2.1/2.1/2 + 3.1/2.1/2 + 3.1/2.1/2 + 2.1/2.1/2$
$= 2/4 + 3/4 + 3/4 + 2/4 = 10/4$
V^* = 5/2 (The same result as obtained earlier)

Algorithm for solving m x 2 matrix games

Draw two vertical axes 1 unit apart. The two lines are $x_1 = 0$, $x_1 = 1$

Take the points of the first row in the payoff matrix on the vertical line $x_1 = 1$ and the points of the second row in the payoff matrix on the vertical line $x_1 = 0$.
The point a_{1j} on axis $x_1 = 1$ is then joined to the point a_{2j} on the axis $x_1=0$ to give a straight line. Draw 'n' straight lines for j=1, 2… n and determine the lowest point of the upper envelope obtained. This will be the **minimax point**.
The two or more lines passing through the minimax point determines the required 2 x 2 payoff matrix. This in turn gives the optimum solution by making use of analytical method.

Example 4

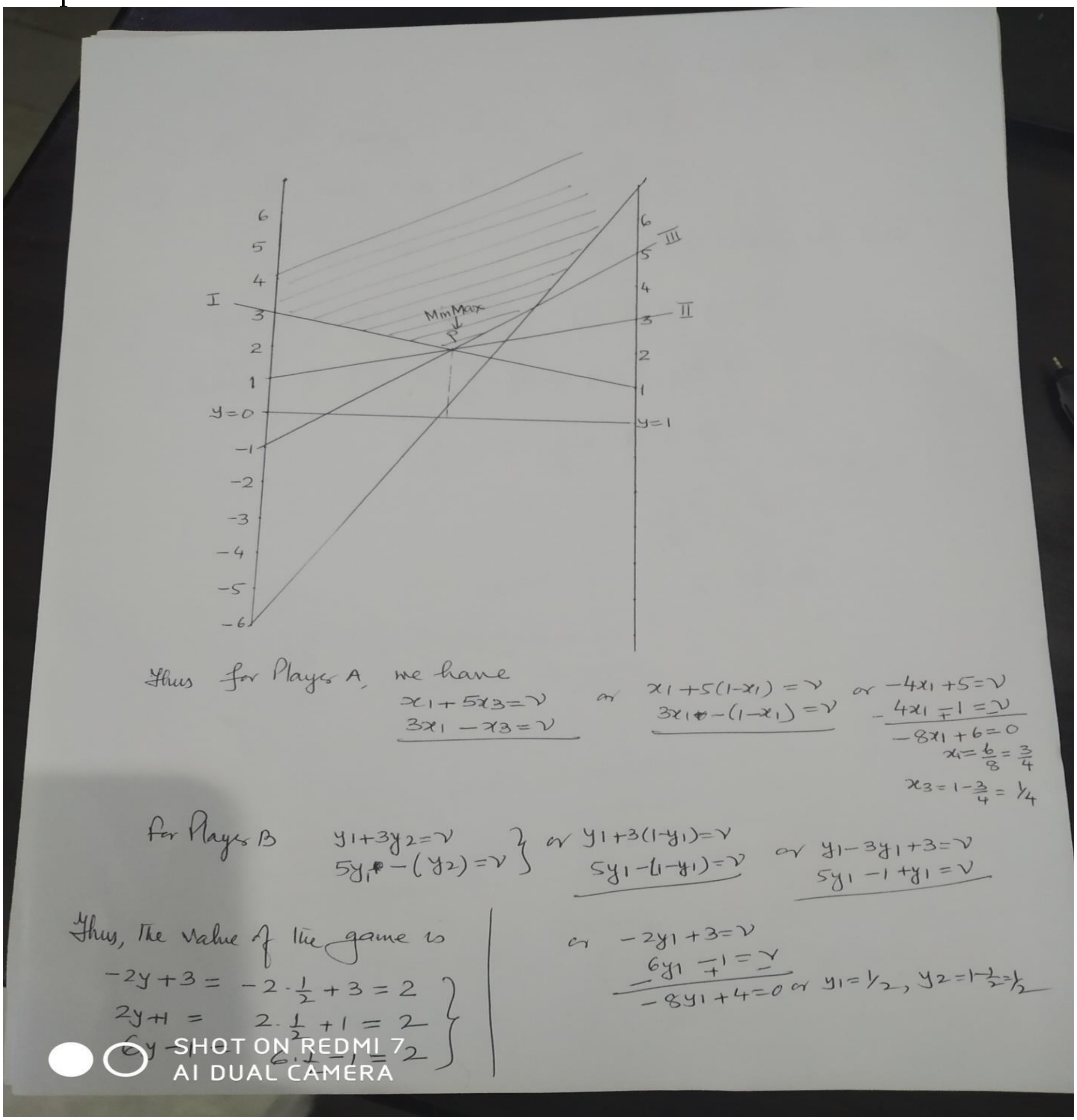
		Player B			
		I (y)	II (1-y)	Min of each row	Max
	I	1	3	1	
Player A	II	3	1	1	
	III	5	-1	-1	1
	IV	6	-6	-6	
	Max of each column	6	3	1	
	min	3			

Here the saddle point does not exist. Since we have only two strategies for player B, We can apply Graphical Method.

Expected Payoff B

A's Alternative	B's expected Payoff	y = 0	y = 1
I	y + 3(1 - y) = -2y + 3	3	1
II	3y + (1 - y) = 2y + 1	1	3
III	5y - (1 - y) = 6y - 1	-1	5
IV	6y – 6(1 - y) = 12y - 6	-6	6

We plot the above four lines as:



Thus for Player A, we have

$$\begin{aligned} x_1 + 5x_3 &= v \\ 3x_1 - x_3 &= v \end{aligned} \quad \text{or} \quad \begin{aligned} x_1 + 5(1-x_1) &= v \\ 3x_1 - (1-x_1) &= v \end{aligned} \quad \text{or} \quad \begin{aligned} -4x_1 + 5 &= v \\ 4x_1 - 1 &= v \\ \hline -8x_1 + 6 &= 0 \end{aligned}$$

$$x_1 = \frac{6}{8} = \frac{3}{4}$$

$$x_3 = 1 - \frac{3}{4} = \frac{1}{4}$$

for Player B

$$\left.\begin{aligned} y_1 + 3y_2 &= v \\ 5y_1 - (y_2) &= v \end{aligned}\right\} \quad \text{or} \quad \begin{aligned} y_1 + 3(1-y_1) &= v \\ 5y_1 - (1-y_1) &= v \end{aligned} \quad \text{or} \quad \begin{aligned} y_1 - 3y_1 + 3 &= v \\ 5y_1 - 1 + y_1 &= v \end{aligned}$$

$$\text{or} \quad \begin{aligned} -2y_1 + 3 &= v \\ 6y_1 - 1 &= v \\ \hline -8y_1 + 4 &= 0 \end{aligned} \quad \text{or} \quad y_1 = 1/2, \; y_2 = 1 - \frac{1}{2} = 1/2$$

Thus, The value of the game is

$$\left.\begin{aligned} -2y + 3 &= -2 \cdot \tfrac{1}{2} + 3 = 2 \\ 2y + 1 &= 2 \cdot \tfrac{1}{2} + 1 = 2 \\ 6y - 1 &= 6 \cdot \tfrac{1}{2} - 1 = 2 \end{aligned}\right\}$$

Thus we select any two lines with opposite slopes passing through the minmax point P namely lines I and III with following payoff matrix

		Player B	
		I(y1)	II(y2)
Player A	I (x1)	1	3
	III (x3)	5	-1

For optimal strategies of player A, we have

$x_1 + 5x_3 = \nu$

$3x1\text{-}x3 = \nu$

Or $x_1 + 5(1 - x_1) = \nu$

$3x_1 - (1 - x_1) = \nu$

Or $-4x_1 + 5 = \nu$

$4x_1 – 1 = \nu$

or $-8x_1 + 6 = 0$ or $x_1 = 3/4$ and $x_3 = 1 - x_1 = 1 - ¾ = 1/4$

We write as

$y_1 + 3y_2 = \nu$

$5y_1 - y_2 = \nu$

Or $y_1 + 3(1 - y_1) = \nu$

$5y_1 - (1 - y_1) = \nu$

Or $-2y_1 + 3 = \nu$

$6y_1 - 1 = \nu$

or $–8y_1 + 4 = 0$

or $y_1 = 1/2$ and $y_2 = 1 - y_1 = 1 - ½ = 1/2$

Thus player A will play his strategies I and III with probabilities 3/4 and 1/4 respectively while player B plays his strategies I and II with probabilities 1/2 each.

Thus, We write optimal strategies as (A1, A3) with $(x_1^*, x_3^*) = (3/4, 1/4)$
and $(B2,B3) = (y_1^*, y_2^*) = (1/2, 1/2)$

We also calculate the value of the game as

$\mathbf{V^* = a_{11}.\, x_1^*.\, y_1^* + a_{12}.x_1^* .y_2^* + a_{31} .x_3^* .y_1^* + a_{32}.x_3^* .y_2^*}$

$=1.3/4.1/2 + 3.3/4.1/2 + 5.1/4.1/2 + (-1)1/4.1/2$

$= 3/8 + 9/8 + 5/4 - 1/8$

$\mathbf{V^* = 2}$ (The same result as obtained earlier)

The Principle of Dominance

In solving a game without a saddle point, one comes across the phenomenon of the dominance of a row over another row or a column over another column in the pay-off matrix of the game.

Such a situation is discussed in the sequel.

In a given pay-off matrix A, we say that the ith row dominates the kth row if

$$a_{ij} \geq a_{kj} \text{ for all } j = 1,2,\ldots,n$$

and

$$a_{ij} > a_{kj} \text{ for at least one } j.$$

In such a situation player A will never use the strategy corresponding to kth row, because he will gain less for choosing such a strategy.

Similarly, we say the pth column in the matrix dominates the qth column if

$$a_{ip} \leq a_{iq} \text{ for all } i = 1,2,\ldots,m$$

and

$$a_{ip} < a_{iq} \text{ for at least one } i.$$

In this case, the player B will loose more by choosing the strategy for the qth column than by choosing the strategy for the pth column. So he will never use the strategy corresponding to the qth column. When dominance of a row (or a column) in the pay-off matrix occurs, we can delete a row (or a column) from that matrix and arrive at a reduced matrix. This principle of dominance can be used in the determination of the solution for a given game.

Let us consider an illustrative example involving the phenomenon of dominance in a game.

Example 1

Solve the game with the following pay-off matrix

$$\text{Player A} \quad \overset{\text{Player B}}{\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{cccc} I & II & III & IV \end{array} \\ \begin{bmatrix} 4 & 2 & 3 & 6 \\ 3 & 4 & 7 & 5 \\ 6 & 3 & 5 & 4 \end{bmatrix} \end{array}}$$

Solution

We see that there is no saddle point for the game.

Compare columns II and III

Column II	Column III
2	3
4	7
3	5

We see that each element in column III is greater than the corresponding element in column II. The choice is for player B. Since column II dominates column III, player B will discard his strategy 3.

Now we have the reduced game

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} I & II & IV \end{array} \\ \begin{bmatrix} 4 & 2 & 6 \\ 3 & 4 & 5 \\ 6 & 3 & 4 \end{bmatrix} \end{array}$$

For this matrix again, there is no saddle point. Column II dominates column IV. The choice is for player B. So player B will give up his strategy 4
The game reduces to the following:

$$\begin{array}{c c} & \begin{array}{cc} I & II \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 4 & 2 \\ 3 & 4 \\ 6 & 3 \end{bmatrix} \end{array}$$

This matrix has no saddle point.

The third row dominates the first row. The choice is for player A. He will give up his strategy 1 and retain strategy 3. The game reduces to the following:

	Player B		
		I(y1)	II(y2)
Player A	II (x2)	3	4
	III (x3)	6	3

Thus, the optimal strategies for Player A and B are A2, A3 and B1, B2 with
$(x_2^*, x_3^*) = (3/4, 1/4)$ and $(B2, B3) = (y_2^*, y_3^*) = (1/4, 3/4)$

The value of the game is **15/4**

Equivalence of a Game Problem with LPP :-

If the size of the game can not be reduced to 2 x 2 or 2 x n or m x 2, then the problem is to be solved using LPP.

Example:

		Player B			Min of each row	max
		I	II	III		
Player A	I	1	-1	-1	-1	
	II	-1	-1	3	-1	-1
	III	-1	2	-1	-1	
	Max of each column	1	2	3		
	min		3			

Thus the game does not a saddle point. Moreover, it cannot be reduced further by using the rule of dominance.

Model of LPP w.r.t. Player A :- The expected gain to player A with respect to the selection of each of the alternatives of B is presented in the following table.

Player B's Alternatives **Expected gain functions of player A**

1 $a_{11}x_1 + a_{21}x_2 + \ldots + a_{i1}x_i + \ldots + a_{m1}x_m = \sum_{i=1}^{m} a_{i1}x_i$

2 $a_{12}x_1 + a_{22}x_2 + \ldots + a_{i2}x_i + \ldots + a_{m2}x_m = \sum_{i=1}^{m} a_{i2}x_i$

.

.

.

j $a_{1j}x_1 + a_{2j}x_2 + \ldots + a_{ij}x_i + \ldots + a_{mj}x_m = \sum_{i=1}^{m} a_{ij}x_i$

.

.

n $a_{1n}x_1 + a_{2n}x_2 + \ldots + a_{in}x_i + \ldots + a_{mn}x_m = \sum_{i=1}^{m} a_{in}x_n$

Since player A is a maximizing player, he tries to maximize the value of the game. Thus, we have

$$\text{Max}\ \{\min(\textstyle\sum_{i=1}^{m} a_{i1}x_i,\ \sum_{i=1}^{m} a_{i2}x_i, \ldots, \sum_{i=1}^{m} a_{ij}x_i, \ldots, \sum_{i=1}^{m} a_{in}x_n)\}$$

Sub to

$$x_1 + x_2 + \ldots, x_i + \ldots.\ x_m = 1$$
$$x_1, x_2, \ldots, x_i, \ldots, x_m \geq 0$$

The above model is known as descriptive model as the objective function is not in linear form. Since the above model is not a workable model, the same is converted into a linear model by using the following transformation. Also, let

$$V = \min(\textstyle\sum_{i=1}^{m} a_{i1}x_i, \sum_{i=1}^{m} a_{i2}x_i \ldots, \sum_{i=1}^{m} a_{ij}x_i, \ldots \sum_{i=1}^{m} a_{in}x_n)$$

Thus we can write

Max Z=V

Sub to

$$\sum_{i=1}^{m} a_{i1}x_i \geq V,$$
$$\sum_{i=1}^{m} a_{i2}x_i \geq V,$$

.

.

.

$$\sum_{i=1}^{m} a_{ij}x_i \geq V,$$

.

.

.

$$\sum_{i=1}^{m} a_{in}x_n \geq V,$$
$$x_1 + x_2 + \ldots, x_i + \ldots. + x_m = 1$$
$$x_1, x_2, \ldots, x_i, \ldots, x_m \geq 0$$

The system of constraints of the above model can be simplified by dividing it by V as

$\sum_{i=1}^{m} a_{i1}x_i/V \geq 1,$
$\sum_{i=1}^{m} a_{i2}x_i/V \geq 1,$

.
.
.
$\sum_{i=1}^{m} a_{ij}x_i/V \geq 1,$

.
.
.$\sum_{i=1}^{m} a_{in}x_n/V \geq 1,$

$\frac{x_1}{V}+\frac{x_2}{V}+\ldots+\frac{x_i}{V}+\ldots\frac{x_m}{V}=1/V$
$x_1,x_2,\ldots,x_i,\ldots,x_m \geq 0.$

In this modified model

i) if the value of the game is less than zero, the type of each constraint will get changed

ii) if the value of the game is zero, the terms of the constraints will become infinite.

Therefore to avoid these problems, a constant K which is equal to the absolute value of the maximum of the negative values of the payoff matrix plus 1 is added to each of the entries in the payoff matrix. After solving the game, the true value of the game is obtained by subtracting from the value of the game.

Let $\frac{x_i}{V} = X_i$, i=1,2,…m.

Therefore

Max V = min$\frac{1}{V}$

$= \min\left(\frac{x_1}{V}+\frac{x_2}{V}+\ldots+\frac{x_i}{V}+\ldots.+\frac{x_m}{V}\right)$

Max V = min($X_1 + X_2 + \ldots + X_i + \ldots. + X_m$)

Then by substituting the above function in the model, we get the revised model as

Minimize $Z_1 = X_1 + X_2 + \dots + X_i + \dots. + X_m$

Sub to

$\sum_{i=1}^{m} a_{i1}X_i \geq 1,$

$\sum_{i=1}^{m} a_{i2}X_i \geq 1,$

.

.

.

$\sum_{i=1}^{m} a_{ij}X_i \geq 1,$

.

.

.$\sum_{i=1}^{m} a_{in}X_n \geq 1$

$x_1, x_2, \dots, x_i, \dots, x_m \geq 0.$

The values of x_i, I =1,2,…,m and V are obtained using the following formulae

$V = \frac{1}{Z_1}$ and $x_i = V$

Similarly, the model for Player B can be developed as given below:

Model of LPP w.r.t. Player B:- The expected loss to Player B with respect to the selection of each of the alternatives of player A is presented in the following table.

A's Alternatives	Expected gan functions of player B
1	$a_{11}y_1 + a_{12}y_2 + \ldots + a_{1j}y_j + \ldots + a_{1n}y_n = \sum_{j=1}^{n} a_{1j}\, y_j$
2	$a_{21}y_1 + a_{22}y_2 + \ldots + a_{2j}y_j + \ldots + a_{2n}y_n = \sum_{j=1}^{n} a_{2j}\, y_j$
.	
.	
.	
i	$a_{i1}y_1 + a_{i2}y_2 + \ldots + a_{ij}y_j + \ldots + a_{in}y_n = \sum_{j=1}^{n} a_{ij}\, y_j$
.	
.	
m	$a_{m1}y_1 + a_{m2}y_2 + \ldots + a_{mj}y_j + \ldots + a_{mn}y_n = \sum_{j=1}^{n} a_{mj}\, y_j$

Since player B is a minimizing player, he tries to minimize the value of the game. Thus, we have

$$\textbf{Min } \{\textbf{max}(\textstyle\sum_{j=1}^{n} a_{1j}y_j, \sum_{j=1} a_{2j}y_j \ldots, \sum_{j=1}^{n} a_{ij}y_j, \ldots \sum_{j=1}^{n} a_{mj}y_j)\}$$

Sub to

$$y_1 + y_2 + \ldots, + y_j + \ldots. + y_m = 1$$
$$y_1, y_2, \ldots, y_j, \ldots, y_m \geq 0$$

The above model is known as descriptive model as the objective function is not in linear form. Since the above model is not a workable model, the same is converted into a linear model by using the following transformation. Also, let

$$\mathbf{V = max}(\textstyle\sum_{j=1}^{n} a_{1j}y_j, \sum_{j=1} a_{2j}y_j \ldots, \sum_{j=1}^{n} a_{ij}y_j, \ldots \sum_{j=1}^{n} a_{mj}y_j)$$

Thus we can write

Min Z=V

Sub to

$\sum_{j=1}^{n} a_{1j}y_j \leq V,$

$\sum_{j=1}^{n} a_{2j}y_j \leq V,$

.

.

.

$\sum_{j=1}^{n} a_{mj}y_j \leq V,$

.

.

.

$\sum_{j=1}^{n} a_{ij}y_j \leq V,$

$y_1+y_2+\ldots,y_j+\ldots.y_n=1$

$y_1,y_2,\ldots,y_i,\ldots,y_n \geq 0$

The system of constraints of the above model can be simplified by dividing it by V as

$\sum_{j=1}^{n} a_{1j}y_j /V \leq 1,$

$\sum_{j=1}^{n} a_{2j}y_j /V \leq 1,$

.

.

.

$\sum_{i=1}^{m} a_{ij}x_i/V \leq 1,$

.

.

.$\sum_{i=1}^{m} a_{mj}x_n/V \leq 1$

$\frac{y_1}{V}+\frac{y_2}{V}+\ldots+\frac{y_j}{V}{}_i+\ldots.\frac{y_n}{V}=1/V$

$y_1,y_2,\ldots,y_j,\ldots,y_m \geq 0.$

In this modified model

iii) if the value of the game is less than zero, the type of each constraint will get changed

iv) if the value of the game is zero, the terms of the the constraints will become infinite.

Therefore to avoid these problems, a constant K which is equal to the absolute value of the maximum of the negative values of the payoff matrix plus 1 is added to each of the entries in the payoff matrix. After solving the game, the true value of the game is obtained by subtracting from the value of the game.

Let $\frac{y_i}{V} = Y_j$, i=1,2,…n.

Therefore

Max V=min$\frac{1}{V}$

$$=\min \left(\frac{y_1}{V}+\frac{y_2}{V}+\ldots+\frac{y_j}{V}+\ldots.\frac{y_n}{V}\right)$$

Max V= min($Y_1+Y_2+\ldots+Y_j+\ldots.+Y_n$)

Then by substituting the above function in the model, we get the revised model as

Minimize $Z_2 = Y_1 + Y_2 + \ldots + Y_j + \ldots. + Y_n$

Sub to

$\sum_{i=1}^{m} a_{i1}Y_i \le 1,$

$\sum_{i=1}^{m} a_{i2}Y_i \le 1,$

.

.

.

$\sum_{i=1}^{m} a_{ij}Y_i \le 1,$

.

.$\sum_{i=1}^{m} a_{mj}Y_j \le 1$

$Y_1, Y_2, \ldots, Y_i, \ldots, Y_n \geq 0$.

The values of y_i, $i=1,2,\ldots,n$ and V are obtained using the following formulae

$V=\frac{1}{Z_1}$ and $y_i=VY_i$

Example:

		Player B			Min of each row	max
		I	II	III		
Player A	¤ I	1	-1	-1	-1	
	II	-1	-1	3	-1	-1
	III	-1	2	-1	-1	
	Max of each column	1	2	-1		
	min		2			

We will use LPP technique here:

Since, the matrix has negative values, the absolute value of the most negative value plus 1 i.e. K=1+1 =2 is added to every element of the matrix to obtain the revised matrix given below:

		Player B		
		I	II	III
Player A	I	3	1	1
	II	1	1	5
	III	1	4	1

Since the LPP with respect to Player B will have only ≤ constraints, it is advisable to develop a model for Player B and solve it. We will find the optimal strategies by using the concept of duality. The true value of the game will be obtained by subtracting K from the value of the modified game.

We write the LPP w.r.t. to Player B as

$$3y_1 + y_2 + y_3 \leq V$$
$$y_1 + y_2 + 5y_3 \leq V$$
$$y_1 + 4y_2 + y_3 \leq V$$

and $y_1 + y_2 + y_3 = 1$

Dividing the above constraints by V , we get

$$3y_1/V + y_2/V + y_3/V \leq V/V$$

$$y_1/V + y_2/V + 5y_3/V \leq V/V$$

$$y_1/V + 4y_2/V + y_3/V \leq V/V$$

and $y_1/V + y_2/V + y_3/V = 1/V$

Substituting $y_j/V = Y_j$, $j = 1,2,\ldots,n$ in the above system, we have

$$3Y_1+Y_2 +Y_3 \leq 1$$

$$Y_1+Y_2 +5Y_3 \leq 1$$

$$Y_1+4Y_2 +Y_3 \leq 1$$

and $Y_1+Y_2 +Y_3 =1$

From the above system, we construct the LPP as given below:

Max $Z_2 = Y_1 + Y_2 + Y_3 + S_1 + S_2 + S_3$

Sub to

$$3Y_1+Y_2+Y_3+S_1=1$$

$$Y_1+Y_2+5Y_3+S_2=1$$

$$Y_1+4Y_2+Y_3+S_3=1$$

$$Y_1, Y_2, Y_3, S_1+S_2+S_3 \geq 0$$

Solution using Simplex Method:-

Max $Z_2 = Y_1 + Y_2 + Y_3 + 0.Y_4 + 0.Y_5 + 0.Y_6$

Sub to

$$3Y_1+Y_2+Y_3+Y_4=1$$

$$Y_1+Y_2+5Y_3+Y_5=1$$

$$Y_1+4Y_2+Y_3+Y_6=1$$

$$Y_1, Y_2, Y_3, Y_4, Y_5, Y_6 \geq 0$$

Iteration 0

C_b	**$C_j \rightarrow$ $X_b \downarrow$**	**1 Y_1**	**1 Y_2**	**1 Y_3**	**0 Y_4**	**0 Y_5**	**0 Y_6**	**b**	**Ratio**
0	Y_4	**3**	1	1	1	0	0	1	1/3 →
0	Y_5	1	1	5	0	1	0	1	1/1=1
0	Y_6	1	4	1	0	0	1	1	1/1=1
Zj-Cj		-1↑	-1	-1	0	0	0		**Z=0**

Iteration 1.

C_b	$C_j \rightarrow$ / X_b	1 / y_1	1 / y_2	1 / y_3	0 / y_4	0 / y_5	0 / y_6	b	Ratio
1	y_1	1	1/3	1/3	1/3	0	0	1/3	1
0	S_2	0	2/3	14/3	-1/3	1	0	2/3	1
0	S_3	0	11/3	-1/3	-1/3	0	1	2/3	2/11 →
$Z_j - C_j$		0	-2/3 ↑	-2/3	1/3	0	0	Z = 1/3	

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Iteration-2.

C_b	$C_j \rightarrow$ / $X_b \downarrow$	1 / y_1	1 / y_2	1 / y_3	0 / y_4	0 / y_5	0 / y_6	b	Ratio
1	y_1	1	0	3/11	4/11	0	-1/11	3/11	1
0	S_2	0	0	50/11	-3/11	1	-2/11	6/11	3/35 →
1	y_2	0	1	2/11	-1/11	0	3/11	2/11	1
$Z_j - C_j$		0	0	-6/11 ↑	3/11	0	2/11	Z = 5/11	

Iteration 3

C_b	$C_j \rightarrow$ / $X_b \downarrow$	1 / y_1	1 / y_2	1 / y_3	0 / y_4	0 / y_5	0 / y_6	b
1	y_1	1	0	0	19/50	-3/50	-1/11	6/25
1	y_3	0	0	1	-3/50	11/50	-2/11	3/25
1	y_2	0	1	0	-2/25	-1/25	3/11	4/25
$Z_j - C_j$		0	0	0	6/25	3/25	4/25	Z_2 = 13/25

In Table 3, since all **Zj-Cj ≥ 0**, the table is optimal and the optimal solution is

Y1 = 6/25 , Y2 = 4/25 , Y3 = 3/25 , Z2 = 13/25.

We compute the value of V and y_1, y_2, y_3 using the following formulae $V = 1/Z_2$ and $y_j = VY_j$, j = 1,2,3

V = 1/(13/25) = **25/13**

Where the value of the original game =(25/13)-K = (25/13)-2 = **-1/13**

and **y1** = $V.Y_1$ = (25/13).(6/25) = **6/13**

y2 = V.Y2 = (25/13).(4/25) = **4/13**

y3 = V.Y3 = (25/13).(3/25) = **3/13**

Similarly, we can solve the problem of Player A given below:

Min $Z_1 = X_1 + X_2 + X_3$

Sub to

$3X_1 + X_2 + X_3 \geq$

$1\ X_1 + X_2 + 4X_3$

$\geq 1\ X_1 + 5X_2 +$

$X_3 \geq 1\ X_1, X_2, X3$

≥ 0

Using Big M Method or two Phase Method or using duality concept, we obtain optimal solution to the above problem as

$X_1 = 6/25$

$X_2 = 3/25$

$X_3 = 4/25$

and **$Z_1 = 13/25$**

We compute the value of V and x1,x2,x3using the following formulae

$V = 1/Z_1$

$V = 1/(13/25) = 25/13$

Where the value of the original game = (25/13) - K = (25/13) - 2 =

-1/13 and $x_i = VX_i$, i = 1,2,3 and

$\mathbf{x_1} = V.X_1 = (25/13).(6/25)$

= **6/13** $\mathbf{x_2}$ = V.X2 =

(25/13).(3/25) = **3/13** $\mathbf{x_3}$ =

V.X3 = (25/13).(4/25) =

4/13

The Optimal strategies of Players A and B are summarized as

A(I,II,III) with probabilities (6/13,3/13,4/13) &

B(I,II,III) with probabilities (6/13,4/13,3/13) and the value of the game is **-1/13**

UNIT - IV

INTEGER LINEAR PROGRAMMING

In linear programming problems, decision variables are non-negative values which are restricted to zero or more than zero. This demonstrates one of the basic properties of linear programming, namely, continuity, which means that fractional values of the decision variables are possible in the solution of a linear programming model. For problems like, product mix problem, e.g., the production volume of different types of fertilizer in tonnage may satisfy the continuity assumption. But if the products are high valued, which means, the profit contribution per unit of them will be very high (e.g., ship, bucket wheel excavator, crane etc), the assumption of the continuity may lead to some practical difficulties.

Some items cannot be produced in fractions, like 101.5 cranes, 3.2 bucket wheel excavator, etc, if we round off the production integer value, the corresponding solution would be different from the optimal solution based on the assumption of continuity. The difference would be significant of the profit per unit of each of the products in the product mix problem is very high. Hence, there is a need for integer programming method to overcome this difficulty.

Example:- Consider the following production planning situation of a company manufacturing mixes. The company plans to manufactures two types of mixes. The selling prices for these mixes are Model A costs Rs 1,750 per unit and Model B costs Rs 2,000 per unit. Daily production volume of each type of these mixes is constrained by available man hours and available machine hours. The production specification for the given problem situation are presented in table.

Resource requirement/unit			
Resource	Model A	Model B	Availability (in hours)
Man hours	4	8	32
Machine hours	6	4	36

Find the optimal production plan for the above problem.

Solution: - Let x_1 and x_2 be the respective number of Model A and Model B to be manufactured. An integer linear programming model for above problem is represented as given below :

Maximize $Z = 1750x_1 + 2000x_2$

Sub to $4x_1 + 8x_2 \leq 32$

$6x_1 + 4x_2 \leq 36$

x_1 **and** $x_2 \geq 0$ **and integer**

The optimal linear programming solution of the above problem is given below :

$x_1 = 5.0$, $x_2 = 1.5$, Z(optimum) = **Rs 11,750.**

In this solution, the value of x_1 is an integer and that of x_2 is non-integer. But the solution of this problem will be meaningful only when the values of al the decision variables are integer. A simple approach may be to round off the value of x_2 to the previous integer value of 1 to maintain feasibility. After rounding off the value of x_2, the values of the decision variables, x_1 and x_2 becomes 5 and 1, respectively. The corresponding total profit is **Rs. 10750** which is less than that of the optimum value of the linear programming solution.

However, the optimum integer solution for the given problem is as follows which is better than the rounded off solution of the linear programming problem.

$x_1 = 4$, $x_2 = 2$ and Z (optimum) = **Rs 11,000**.

We will discuss the following two Methods

1. **The Branch and Bound** – Method developed by A. H. Land and A. G. Doing in 1960.
2. **The Cutting Plane Algorithm -** developed by Ralph-E Gomory in 1958.

Branch–and–Bound Method (B&B Method) :

If the number of decision variables in an integer programming problem is only two, a branch-and-bound technique can be used to find its solution graphically. Various terminologies of B&B Method are explained as under.

Branching :-

If the solution to the linear programming problem contains non-integer values for some or all decision variables, then the solution space is reduced by introducing constraints with respect to any one of those decision variables. If the value of a decision variable x_1 is 2.5 (say), then two more problems are created by using each of the following constraints:

$x_1 \leq 2$ and $x_1 \geq 3$

Lower bound : -

This is a limit to define a lower value for the objective function at each and every node. The lower bound at a node is the value of the objective function corresponding to the truncated value (integer parts) of the decision variables of the problem in that node.

Upper bound : -

This is a limit to define an upper value for the objective function at each and every node. The upper bound at a node is the value of the objective function corresponding to the linear programming solution in the node.

Fathomed Sub-problem / node : -

A problem is said to be fathomed if any one of the following three conditions is true.

1. The values of the decision variables of the problem are integer.
2. The upper bound of the problem which has non-integer values for its decision variables is not greater than the current best lower bound.
3. The problem has infeasible solution.

This means that further branching from this type of fathomed node is not necessary.

Current Best Lower Bound: -

This is the best lower bound (highest in the case of maximization problem and lowest in the case of minimization problem) among the lower bounds of all the fathomed nodes. Initially, it is assumed as infinity for the root node.

Branch and Bound Method :

Step 1 :- Solve the given linear programming problem graphically. Set, the current best lower bound, ZB as ∞.

Step 2 :- Check, whether the problem, has integer solution. If yes, print the current solution as the optimal solution and stop; otherwise go to step 3.

Step 3 :- Identify the variable x_k which has the max fractional part as the branching variable. (In case of tie, select the variable which has the highest objective function coefficients).

Step 4 :- Create two more problems by including each of the following constraints to the current problem and solve them

$x_k \leq$ integer part of x_k.

$x_k \geq$ Next integer of x_k

Step 5: - If any one of the new sub-problems has infeasible solution or fully integer values for the decision variables, the corresponding node is fathomed. If a new node has integer values for the decision variables, update the current best lower bound as the lower bound of that node of if its lower bound is greater than the previous current best lower bound.

Step 6 :- Are all terminal nodes fathomed? If the answer is yes, go to step 7: otherwise, identify the node with the highest lower bound and go to step 3.

Step 7 :- Select the solution of the problem with respect to the fathomed node whose lower bound is equal to the current best lower bound as the optimal solution.

Example 1:

Maximize $Z = 1750x_1 + 2000x_2$

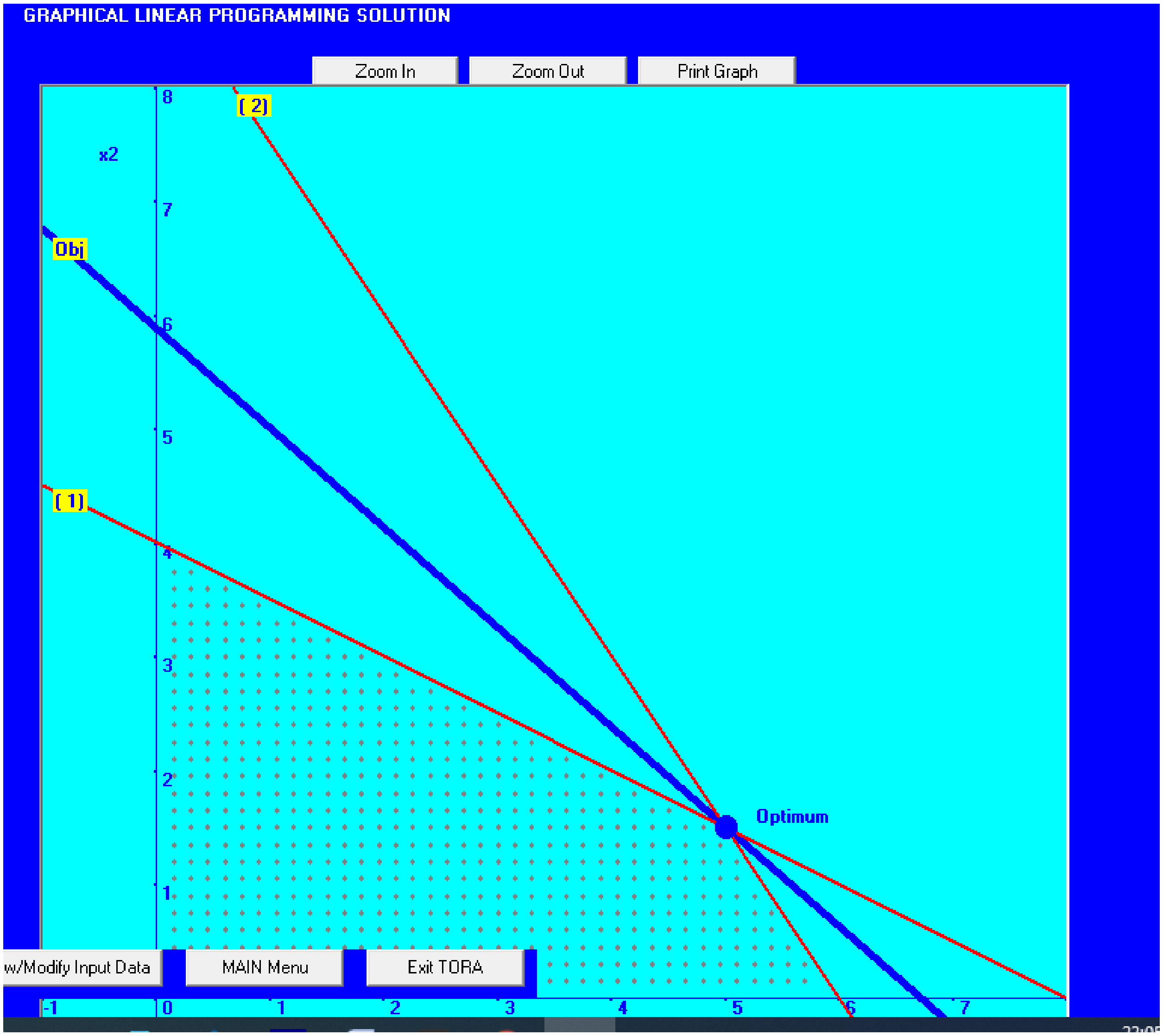
Sub to $\quad 4x_1 + 8x_2 \leq 32$

$6x_1 + 4x_2 \leq 36$

x_1 and $x_2 \geq 0$ and integer

Solution : We use B & B Method as the ILP is in two variables.

We solve it graphically; the feasible region is produced below:



The optimal solution to the above linear programming is given below:

x_1 = 5.0, x_2 = 1.5, and optimal value of **Z = Rs 11,750**.

The notations for different types of lower bound are defines as follows.

Zu = upper bound = optimum value of LPP with $Z_u = 11750$

Z_L=Lower bound w.r.t. the truncated values of the decision variables with

$Z_L = 10750$ with $x_1 = 5$, $x_2 = 1$

Z_B = current best lower bound with $Z_B = \infty$

Since both the values of x_1 and x_2 are not integers, the solution is not optimum from the view point of the given problem. So, the problem is to be branched out into two problems by including integer constraints one by one. The lower bound of the solution of P_1 is 10750. This is nothing but the value of objective function for the truncated values of the decision variables.

We split the LPP: P1 as

P_1

$x_2 \le 1$ $\qquad$ $x_2 \ge 2$

P_2 $\qquad\qquad\qquad$ P_3

P_2:

Max. $Z = 1750x_1 + 20006x_2$

sub to $4x_1 + 8x_2 \le 32$

$6x_1 + 4x_2 \le 36$

$x_2 \le 1$

$x_1, x_2 \ge 0$ and integer

P_3:

Max. $Z = 5x_1 + 6x_2$

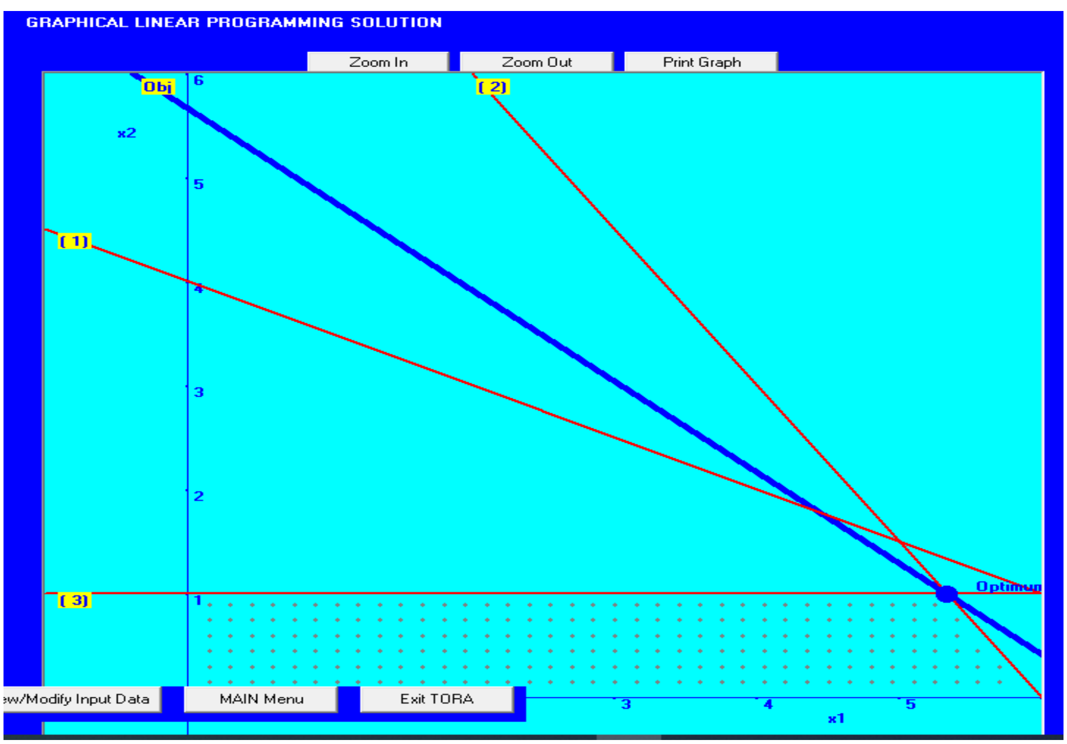
$4x1+8x2 \le 32$

$6x1+4x2 \le 36$

$x_2 \ge 2$

$x_1, x_2 \ge 0$ and integer

The feasible region to P2 is

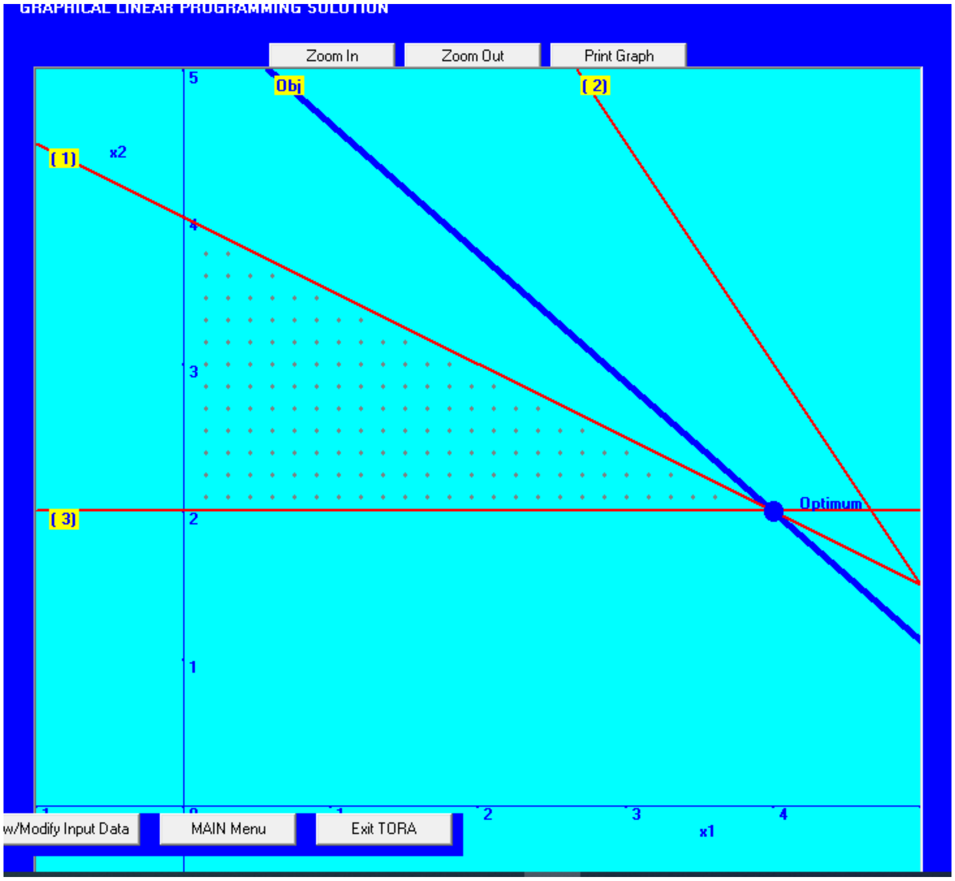


and the optimal solution is $\mathbf{x_1 = 5.3333, x_2 = 1}$ with optimal value $\mathbf{Z = 11333.33 = Zu}$

and **Zlb = 10750** with $\mathbf{x_1 = 5}$ **and** $\mathbf{x_2 = 1}$

Z_B = current best lower bound with $\mathbf{Z_B = 10750}$

The feasible region to P3 is



and the optimal solution is $\mathbf{x_1 = 4, x_2 = 2}$ and the optimal value is $\mathbf{Z = 11000 = Zu}$

Since, the solution is integer, the problem is fathomed. Z_B = current best lower bound with $\mathbf{Z_B = 11000}$.

We split the LPP : P4 as

P$_4$

$x_1 \le 5$ $x_1 \ge 6$

P$_5$ P$_6$

Max. $Z = 1750x_1 + 20006x_2$

sub to $4x_1 + 8x_2 \le 32$

$6x_1 + 4x_2 \le 36$

$x_2 \le 1$

$x_1 \le 5$

$x_1, x_2 \ge 0$ and integer

Max. $Z = 5x_1 + 6x_2$

$4x1+8x2 \le 32$

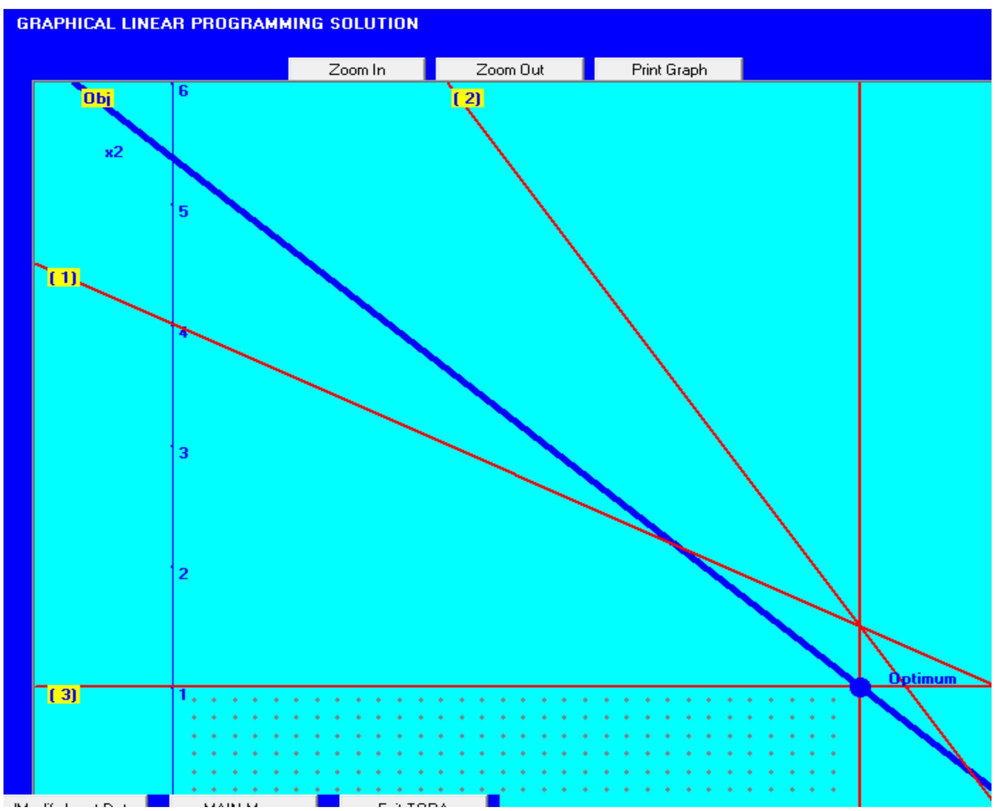
$6x1+4x2 \le 36$

$x_2 \ge 2$

$x_1 \ge 6$

$x_1, x_2 \ge 0$ and integer

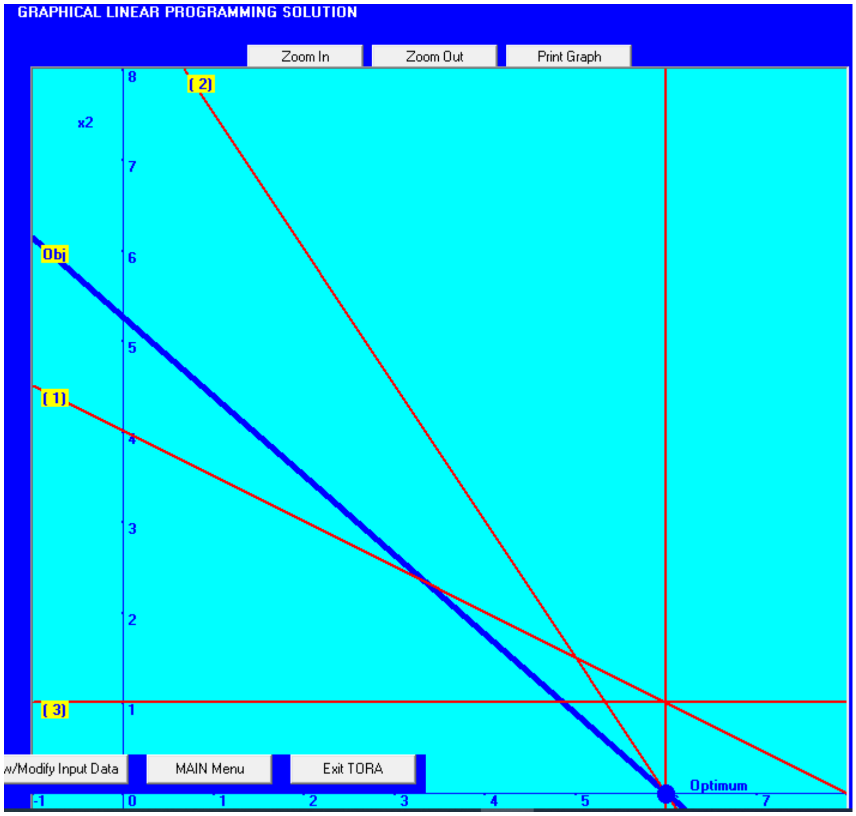
The feasible region to P4



The optimal solution to P4 is **$x_1 = 5, x_2 = 1$** and the optimal value is **$Z = 10750 = Zu$**

The Problem is fathomed.

The feasible region to P5



and the optimal solution is **$x_1 = 6, x_2 = 0$** with **$Z = 10500 = Zu$**

The Problem is fathomed.

Example 2.

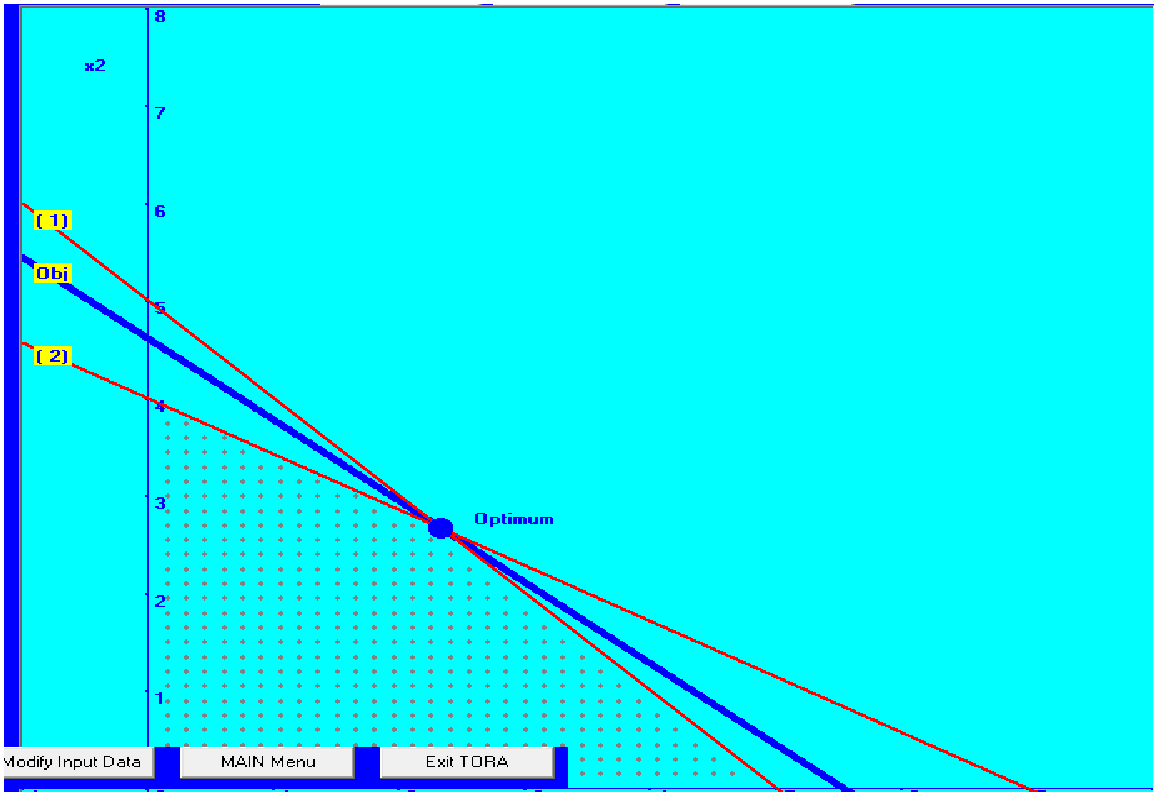
P1 : **Max. $Z = 5x_1 + 6x_2$**

Sub to **$x_1 + x_2 \leq 5$ ------ --I**

$4x_1 + 7x_2 \leq 28$-----------II

$x_1, x_2 \geq 0$ and integers

Solution: We plot the constraint I & II of P1 as shown in figure A.



The dotted region is the feasible region.

Then optimal solution is $\mathbf{x_1 = 7/3 = 2.33}$ and $\mathbf{x_2 = 8/3 = 2.67}$ is the optimal solution with $\mathbf{Z = 27.67}$

Zu = upper bound = optimum value of LPP with $\mathbf{Z_u = 27.67}$

Z_L = Lower bound w.r.t. the truncated values of the decision variables with $\mathbf{Z_L = 22}$ with $\mathbf{x_1 = 2, x_2 = 2}$

Z_B = current best lower bound with $\mathbf{Z_B = \infty}$

Since both the values of x_1 and x_2 are not integers, the solution is not optimum from the view point of the given problem. So, the problem is to be branched out into two problems by including integer constraints one by one. The lower bound of the solution of **P_1 is 22**. This is nothing but the value of objective function for the truncated values of the decision variables.

In the continuous solution of the given linear programming problem P_1, the variable x_2 has the highest fractional part (.67). Hence the variable is selected for further branching as shown is below as per the rule:

$x_k \leq$ integer part of x_k and $x_k \geq$ Next integer of x_k i.e., We take $x_2 \leq 2$ and $x_2 \geq 3$. We split the LPP: P1 as

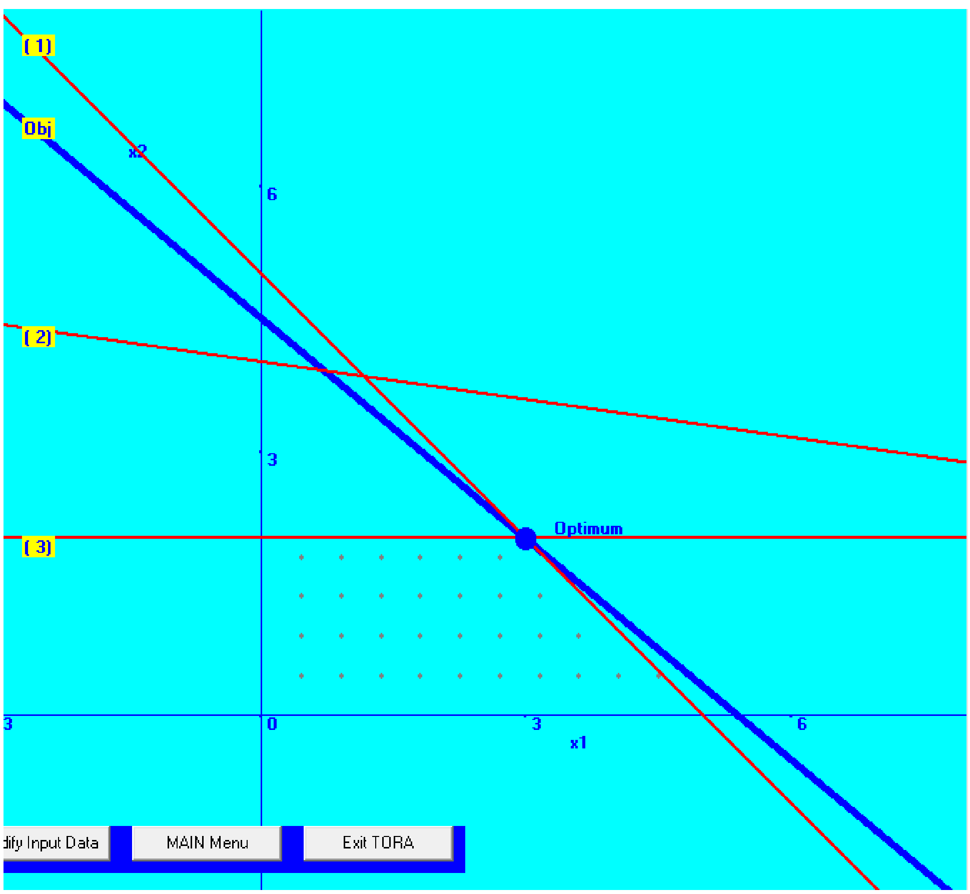
P_1

$x_2 \leq 2$ $\qquad$ $x_2 \geq 3$

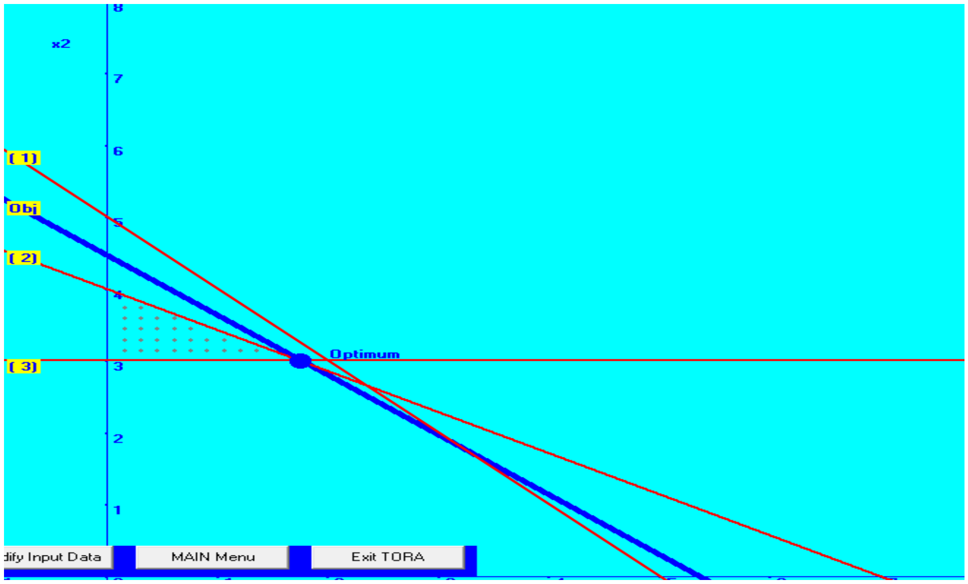
P_2 $\qquad$ P_3

P_2	P_3
Max. $Z = 5x_1 + 6x_2$	**Max.** $Z = 5x_1 + 6x_2$
sub to $x_1 + x_2 \leq 5$	$x_1 + x_2 \leq 5$
$4x_1 + 7x_2 \leq 28$	$4x_1 + 7x_2 \leq 28$
$x_2 \leq 2$	$x_2 \geq 3$
$x_1, x_2 \geq 0$ and integer	$x_1, x_2 \geq 0$ and integer

Optimal solution to P2 is $x_1 = 3$ and $x_2 = 2$ and $Z_L=27$ (LB)



The optimal solution to P3 is **$x_1 = 1.75$** and **$x_2 = 3$** and **$Z = 26.75$**



P_3

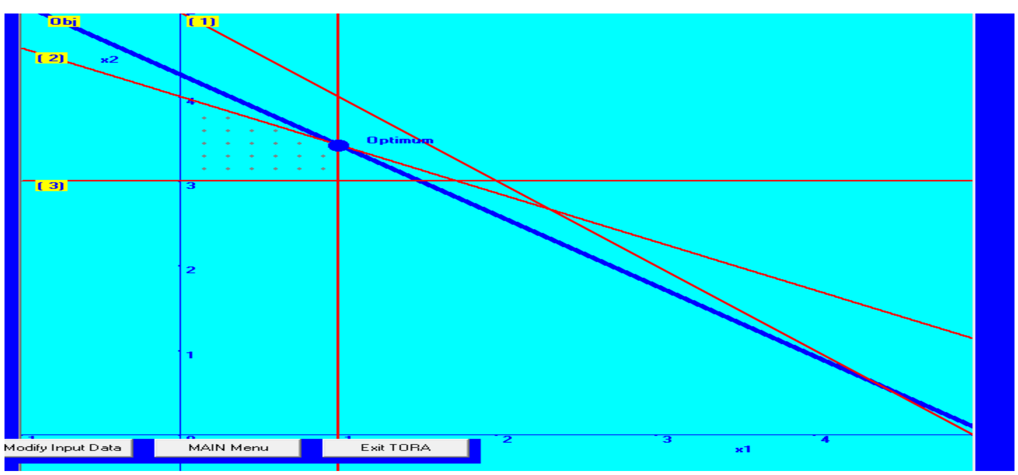
$x_1 \leq 1$ $\quad$ $x_1 \geq 2$

P_4 $\quad$ P_5

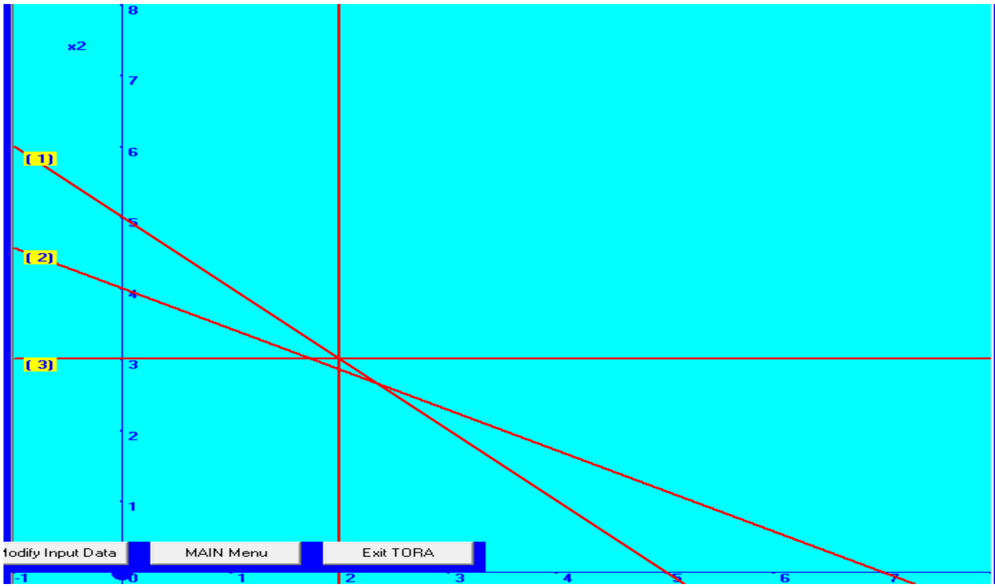
P_4	P_5
Max. Z = $5x_1 + 6x_2$	**Max. Z = $5x_1 + 6x_2$**
sub to $x_1 + x_2 \leq 5$	**$x1+x2\leq 5$**
$4x_1 + 7x_2 \leq 28$	**$4x1+7x2\leq 28$**
$x_2 \geq 3$	**$x_2 \geq 3$**
$x_1 \leq 1$	**$x_1 \geq 2$**

$x_1, x_2 \geq 0$ and integer $\qquad$ $x_1, x_2 \geq 0$ and integer

Optimal solution to P4 is $x_1 = 1$ and $x2 = 3.42$ and Z_L=25.57



and P5 infeasible



Since in the optimal solution to P4, $\mathbf{x_2 = 3.42}$ is not an integer, we will further branch it out with two more sub problems by adding two constraints namely $\mathbf{x_2 \leq 3}$ and $\mathbf{x_2 \geq 4}$ to get the following two sub-problems P6 and P7

P_4

$X_2 \leq 3$ $x_2 \geq 4$

P_6

Max. Z = $5x_1 + 6x_2$

sub to $x_1 + x_2 \leq 5$

$4x_1 + 7x_2 \leq 28$

$x_2 \geq 3$

$x_1 \leq 1$

$x_2 \leq 3$

$x_1, x_2 \geq 0$ and integer

P_7

Max. Z = $5x_1 + 6x_2$

$x1+x2\leq 5$

$4x1+7x2\leq 28$

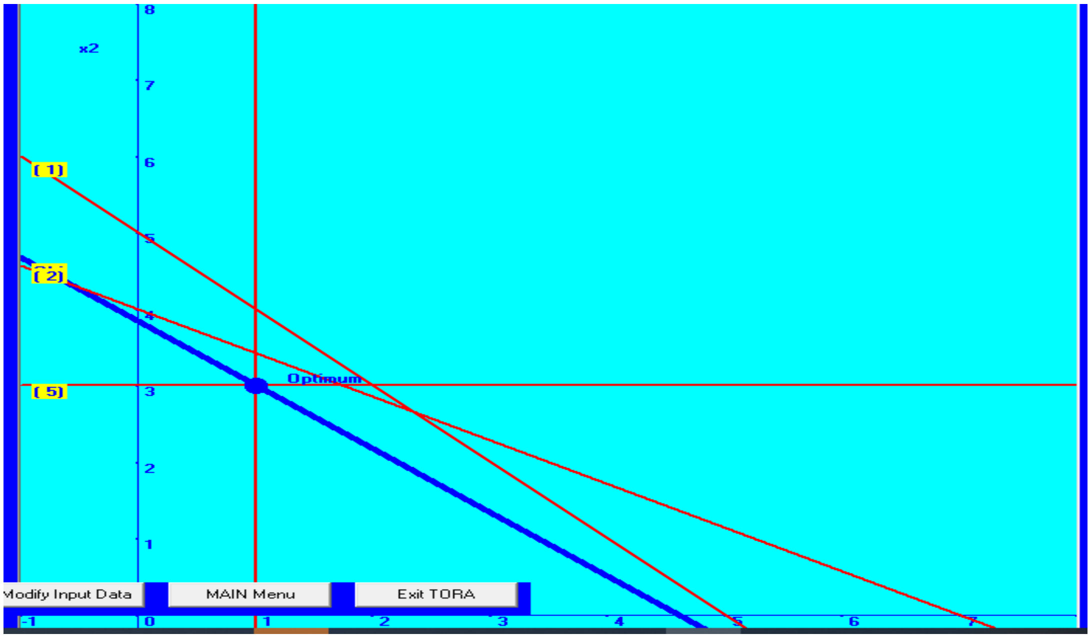
$x_2 \geq 3$

$x_1 \geq 2$

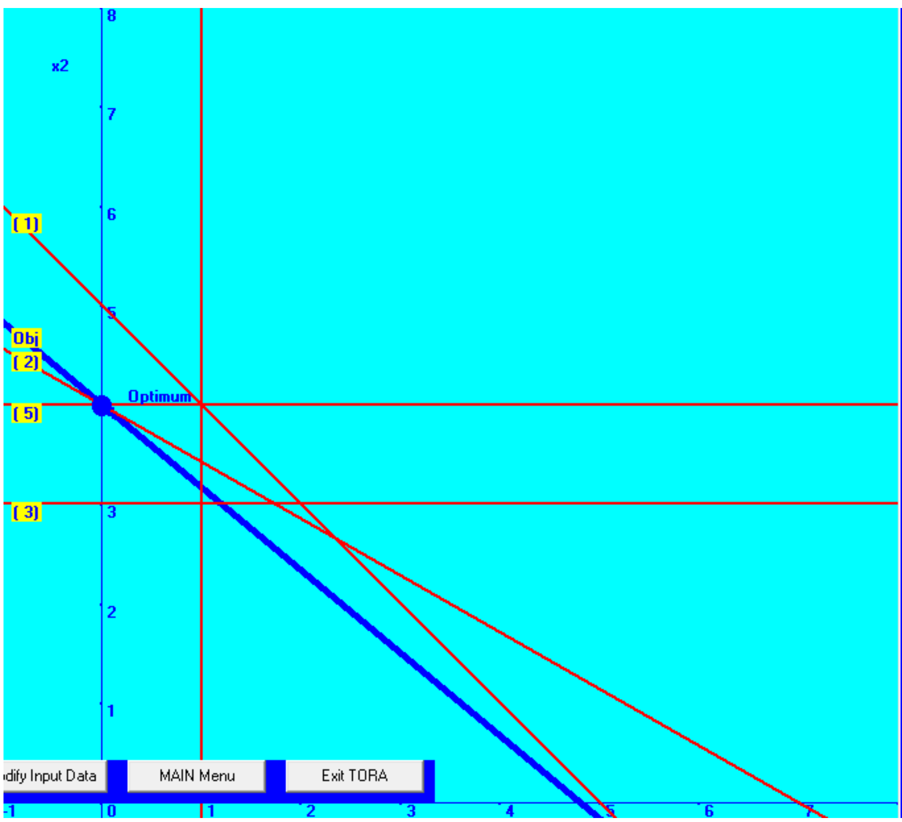
$x_2 \geq 4$

$x_1, x_2 \geq 0$ and integer

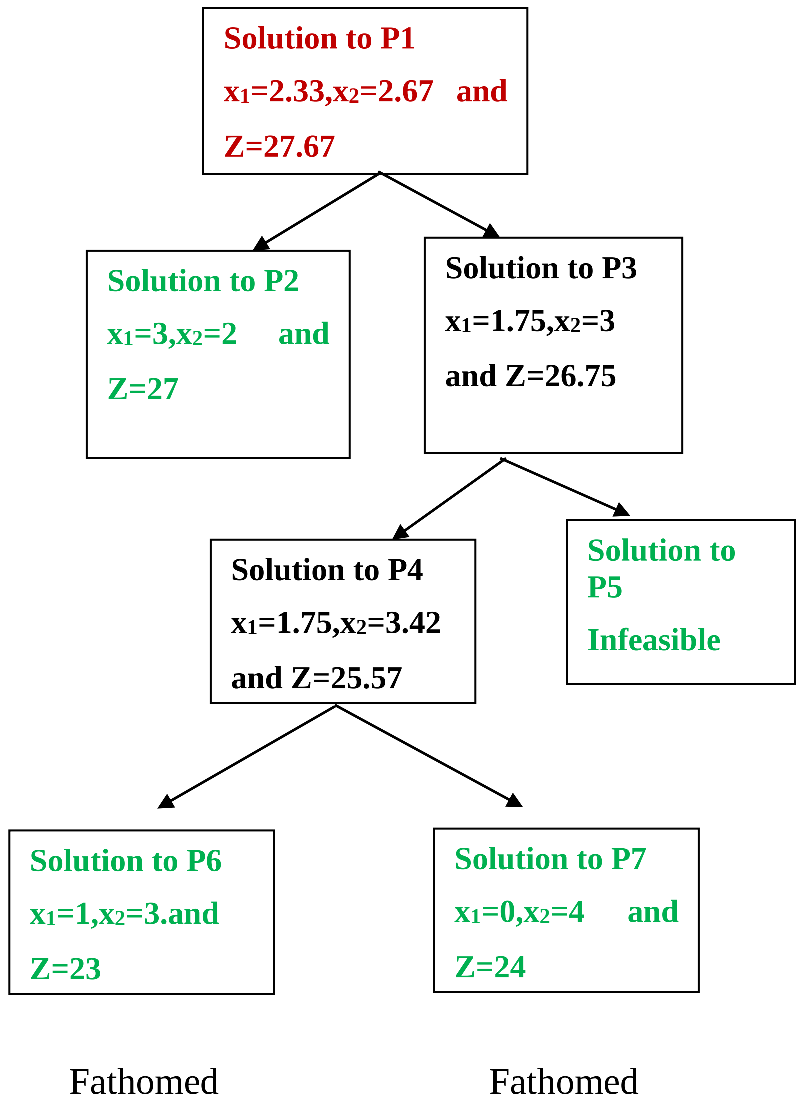
The solution to Sub problem P6 is $\mathbf{x_1 = 1}$, $\mathbf{x_2 = 3}$ and $\mathbf{Z = 23}$



The solution Sub problem P7 is $x_1 = 0$, $x_2 = 4$ and $Z = 24$



Summary of the Method



Thus we choose the solution of P2 as the best integer solution $\mathbf{x_1 = 3,\ \ x_2 = 2}$ as it gives the integer solution and the value of the objective function is maximum **(Z = 27)** and **Z = 27 .**

Gomory's Cutting Plane Method

Step I: Ignoring integer value requirement, solve the given ILP using Simplex Method

Step II : If all the basic variables have integer values, then we stop, the given optimal solution to LPP is also optimal for the ILP

Step III : If any of the basic variables is having fractional value, then generate the cutting plane with the variable having the largest fractional value.

Step IV : Add the cutting plane to the bottom of the optimal Simplex table and find the new optimal solution using the Dual Simplex Method. Steps from II to IV are repeated till all the basic variables are integer valued variables

Example : A Readymade Garment store sells two types of shirts-Zee-shirts and Button down shirts. He makes a profit of Rs.3 and Rs.12 per shirt on Zee-shirts and Button-down shirts respectively. He has two tailors A and B at his disposal for stitching the shirts. Tailors A and B can devote at the most 7 hours and 15 hours per day, respectively. Both of these two shirts are to be stitched by both the tailors. Tailors A and B spend 2 hours and 5 hours respectively in stitching one Zee-shirt and 4 hours and 3 hours respectively on stitching a Button-down shirt. How many shirts of both types should be stitched in order to maximize the daily profit.
Let x_1 and x_2 denote the numbers Zee-shirts and Button-down shirts to be stitched daily respectively.

Maximize $Z = 3x_1 + 12x_2$

Sub to

$$2x_1 + 4x_2 \leq 7$$
$$5x_1 + 3x_2 \leq 15$$
$$x_1, x_2 \geq 0 \text{ and integers.}$$

Adding slack variables x_3, x_4, then the problem in the standard form is given below:

Maximize $Z = 3x_1 + 12x_1 + 0 \cdot x_3 + 0 \cdot x_4$

Sub to

$$2x_1 + 4x_2 + x_3 = 7$$
$$5x_1 + 3x_2 + x_4 = 15$$
$$x_1, x_2, x_3, x_4 \geq 0 \text{ and integers.}$$

The optimal table by using the Simplex Method is given below with optimal solution **$x_1 = 0$, $x_2 = 7/4$ Max $Z = 21$**.

C_b	X_b	3	12	0	0	b
		y_1	y_2	y_3	y_4	
12	x_2	1/2	1	1/4	0	7/4
0	x_4	7/2	0	-3/4	1	39/4
	Z_j-C_j	**3**	**0**	**30**	**0**	**Z=21**

Since the solution is a non integer one, we construct a Gomory's fractional cut with the help of the source row. We choose the source row with the largest fractional value of the basic variable. If there is tie, we choose arbitrarily. In our case, we have , two basic variables x_2 and x_4 in the optimal table of the Simplex, with their integer and fractional values as
$Xb_1 = I_{B1} + f_{B1} = 7/4 =$ **(1+3/4)**, (for x_2) and
$Xb_2 = I_{B2} + f_{B2} = 39/4 =$ **(9+3/4)** (for x_4)

Thus, **max(f_{B1}, f_{B2})=(3/4, 3/4)** Since there is a tie, we can select any of x_2 and x_4 and x_2 being the decision variable, we choose x_2 as source row.
Using the source row x_2, we have
$\mathbf{7/4 = 1/2x_1 + x_2 + 1/4x_3}$

Writing in terms of integers and fractional values (must be positive)
$(1 + 3/4) = (0 + 1/2)x_1 + (1 + 0)x_2 + (0 + 1/4) x_3$
$3/4 + (1 - x_2) = 1/2x_1 + 1/4 x_3$ (collecting integers on left hand side)
or
$\mathbf{3/4 \leq 1/2x_1 + 1/4 x_3}$

(as we have deleted something from equation)
Now, on adding Gomory's Slack variable (g_1) to make it an equality),say g_1, it becomes
$\mathbf{3/4 + g_1 = 1/2x_1 + 1/4 x_3}$
$\mathbf{g_1 - 1/2x_1 - 1/4 x_3 = -3/4}$

Adding this additional constraint to the bottom of the optimal Simplex Table , we get a new table

C_b	X_b	3	12	0	0	0	b
		y_1	y_2	y_3	y_4	y_5	
12	x_2	1/2	1	1/4	0	0	7/4
0	x_3	7/2	0	-3/4	1	0	39/4
0	g_1	-1/2	0	-1/4	0	1	-3/4
	Z_j-C_j	**3**	**0**	**3**	**0**	**0**	**Z=21**
	Ratios $(Z_j-C_j)/(y_{3j})$ $y_{3j<0}$	**-6**	**-**	**-12**	**-**	**-**	

Applying the Dual Simplex Method, we get ,

C_b	X_b	3	12	0	0		b
		y_1	y_2	y_3	y_4	y_5	
12	x_2	0	1	0	0	1	1
0	x_3	0	0	-5/2	1	7	9/2
3	x_1	1	0	1/2	0	-2	3/2
	Z_j-C_j	**0**	**0**	**3/2**	**0**	**0**	**Z=33**

Since the solution is still non integer one, we construct a one more Gomory's fractional cut with the help of x_1 as source row with two basic variable x_1 and x_3 in the optimal table of Simplex, with their integer and fractional values as

$Xb_3 = I_{B3} + f_{B3}$

$9/2 = (4 + 1/2)$ (for x_3) and

$Xb_4 = I_{B4} + f_{B4}$

$3/2 = (3 + 1/2)$ (for x_1)

Thus, **max(f_{B2}, f_{B3}) = (1/2, 1/2)** Since there is a tie, we can select any of x_1 and x_3 and x_1 being the decision variable, we choose x_1 as source row and proceed as under

$3/2 = x_1 + 1/2x_3 - 2g_1$ (x_1 source row)

$(1 + 1/2) = (1 + 0)x_1 + (0 + 1/2)x_3 + (-2 + 0)g_1$

$1/2 + (1 - x_1 + 2g_1) = 1/2x_3$ (collecting integers terms o LHS

or

$1/2 \leq 1/2x_3$

on adding Gomory's Slack variable g_2, it becomes

$1/2 + g_2 = 1/2x_3$,

$g_2 - 1/2x_3 = -1/2$

Applying this cut in to the Simplex Table and applying again Dual Simplex Method, we get ,

C_b	**X_b**	**3**	**12**	**0**	**0**	**0**	**0**	**b**
		y_1	y_2	y_3	y_4	y_5	y_6	
12	x_2	0	0	0	0	1	0	1
0	x_4	0	1	-5/2	1	7	0	9/2
3	x_1	1	0	1/2	0	-2	0	3/2
0	g_2	0	0	**-1/2**	0	0	1	-1/2
	Z_j-C_j	**0**	**0**	**3/2**	**0**	**0**	**6**	**Z=33**
	Ratios(Z_j-C_j)/(y_{3j}), $y_{3j<0}$	**0**	**0**	**3**	**0**	**6**		

Updating the Table, we get ,

C_b	X_b	**3**	**12**	**0**	**0**	**0**	**0**	**b**
		y_1	y_2	y_3	y_4	y_5	y_6	
12	x_2	0	0	0	0	0	0	1
0	x_4	0	1	0	1	7	-5	7
3	x_1	1	0	0	0	2	1	1
0	x_3	0	0	1	0	0	2	1
	Z_j-C_j	**0**	**0**	**0**	**0**	**6**	**3**	**Z=15**

The table is optimal and has integer solution. Thus the,- the store must stitch 1 Zee- shirt and 1 Button-shirt in order to get the profit of **Rs.15**

C_b	X_b	**3**	**12**	**0**	**0**	**0**	**0**	**b**
		y_1	y_2	y_3	y_4	y_5	y_6	
12	$\mathbf{x_2}$	0	0	0	0	0	0	1
0	$\mathbf{x_4}$	0	1	0	1	7	-5	7
3	$\mathbf{x_1}$	1	0	0	0	2	1	1
0	$\mathbf{x_3}$	0	0	1	0	0	-2	1
	Z_j-C_j	**0**	**0**	**0**	**0**	**6**	**3**	**Z=15**