Question 1

a) Use any method to determine if the series converges or diverges

$$\sum_{n=1}^{\infty} \frac{n!\, 2^n}{n^n}$$

Solution Here $a_n = \dfrac{n!\, 2^n}{n^n}$

$$a_{n+1} = \frac{(n+1)!\, 2^{n+1}}{(n+1)^{n+1}}$$

Now

$$\left|\frac{a_{n+1}}{a_n}\right| = \left| \frac{(n+1)!\, 2^{n+1}}{(n+1)^{n+1}} \times \frac{n^n}{n!\, 2^n} \right|$$

$$= \frac{(n+1)\, n!\, 2^n \times 2}{(n+1)^n (n+1)} \times \frac{n^n}{n!\, 2^n}$$

$$= \frac{2n^n}{(n+1)^n} = \frac{2n^n}{n^n(1+1/n)^n} = \frac{2}{(1+1/n)^n}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{2}{(1+1/n)^n} = \frac{2}{e} < 1$$

So by Ratio Test, the series Converges absolutey

$\Rightarrow \sum_{n=1}^{\infty} \frac{n!\,2^n}{n^n}$ Converges

b) Find Fourier series of f on the given interval

$$f(x) = \begin{cases} 0 & -\pi < x < 0 \\ 1 & 0 \le x < \pi \end{cases}$$

Solution: The Fourier series of a function f defined on interval $(-p, p)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{p}x + b_n \sin\frac{n\pi}{p}x\right),$$

where

$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)\,dx \qquad b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin\frac{n\pi}{p}x\,dx$$

$$a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos\frac{n\pi}{p}x\,dx$$

Here $p = \pi$

So

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_{0}^{\pi} 1\,dx\right]$$

$$= \frac{1}{\pi}\left\{ 0 + x\Big|_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left\{(\pi - 0)\right\} = \frac{\pi}{\pi} = 1$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos\frac{n\pi}{\pi}x\,dx$$

$$= \frac{1}{\pi}\left\{\int_{-\pi}^{\pi} f(x)\cos nx\,dx\right\}$$

$$= \frac{1}{\pi}\left\{\int_{-\pi}^{0} 0\,dx + \int_{0}^{\pi}\cos nx\,dx\right\}$$

$$= \frac{1}{\pi}\left\{\int_{0}^{\pi}\cos nx\,dx\right\}$$

$$= \frac{1}{\pi}\left(\frac{\sin nx}{n}\Big|_0^{\pi}\right)$$

$$= \frac{1}{\pi}\left(\frac{\sin n\pi}{n} - \frac{\sin 0}{n}\right)$$

$$= \frac{1}{\pi}(0-0) = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin \frac{n\pi}{\pi} x \, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

$$= \frac{1}{\pi}\left\{\int_{-\pi}^{0} 0 \, dx + \int_{0}^{\pi} \sin nx \, dx\right\}$$

$$= \frac{1}{\pi}\left\{\int_{0}^{\pi} \sin nx \, dx\right\}$$

$$= \frac{1}{\pi}\left(-\frac{\cos nx}{n}\Big|_0^{\pi}\right)$$

$$= \frac{1}{\pi}\left(-\frac{\cos n\pi}{n} + \frac{\cos 0}{n}\right)$$

$$= \frac{1}{\pi}\left(-\frac{(-1)^n}{n} + \frac{1}{n}\right)$$

$$= \frac{1}{n\pi}\left(1-(-1)^n\right)$$

So Fourier Series is

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \left(0 + \frac{1}{n\pi} \left(1-(-1)^n \right) \sin \frac{n\pi}{\pi} x \right)$$

$$= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n\pi} \sin nx$$

① $(y^2+yx)dx + x^2 dy = 0$ —①

$M(x,y) = y^2+yx$, $N(x,y) = x^2$

$M(tx,ty) = t^2y^2 + ty\cdot tx$ $N(tx,ty) = t^2 N(x,y)$

$= t^2(y^2+yx) = t^2 M(x,y)$

So both M and N are homogeneous functions of degree 2.

So suppose. $\boxed{y = ux}$

$$\frac{dy}{dx} = u + x\frac{du}{dx}$$

$$\Rightarrow \boxed{dy = u\,dx + x\,du}$$

So ① becomes,

$$(u^2x^2 + ux^2)dx + x^2(u\,dx + x\,du) = 0$$

$$x^2(u^2+u)\,dx + x^2u\,dx + x^3\,du = 0$$

$$(u^2+2u)\,dx + x\,du = 0$$

$$-\int\frac{dx}{x} = +\int\frac{du}{u(u+2)}$$

$$\frac{1}{u(u+2)} = \frac{A}{u+2} + \frac{B}{u}$$

$$1 = A(u) + B(u+2)$$

$u=0$, $u=-2$

$B = 1/2$. $A = -1/2$

$$-\int\frac{dx}{x} = \frac{1}{2}\int\left(\frac{1}{u} - \frac{1}{u+2}\right)du$$

$$-\ln x = \frac{1}{2}\left(\ln u - \ln(u+2)\right)$$

$$\ln c = \frac{1}{2}\ln\left(\frac{u}{u+2}\right) + \ln x \Rightarrow \ln c = \ln\left(\frac{ux^2}{u+2}\right)$$

$$c = \frac{ux^2}{u+2}$$

$$c = \frac{yx^2}{y+2x}$$

(2) $y\,dx = 4(x+y^6)\,dy \quad , \quad y(1) = 1$

$$\frac{dx}{dy} = 4\left(\frac{x+y^6}{y}\right)$$

$$\frac{dx}{dy} - \frac{4x}{y} = 4y^5$$

$$I.F = e^{\int -\frac{4}{y}\,dy} = e^{-4\ln y} = y^{-4}$$

$$\Rightarrow \quad y^{-4}\frac{dx}{dy} - 4xy^{-5} = 4y$$

$$\frac{d}{dy}(xy^{-4}) = 4y$$

$$\Rightarrow \quad xy^{-4} = 2y^2 + c$$

$$y(1) = (1) \Rightarrow 1 = 2 + C \Rightarrow C = -1$$

$$\Rightarrow \quad xy^{-4} = 2y^2 - 1$$

$$\boxed{x = 2y^6 - y^4}$$

Solution

(a) Let the size of bacteria culture be P

As given in question

$$\frac{dP}{dt} \propto P$$

$$\frac{dP}{dt} = kP$$

$$\frac{1}{P}\, dP = k\, dt$$

Integrating

$$\int \frac{1}{P}\, dP = \int k\, dt$$

$$\ln P = kt + c_1 \qquad c = e^{1}$$

$$\boxed{P = c\, e^{kt}} \text{ --- } (1)$$

The above equation shows size of bacteria at any time t.

Conditions

At $t = 0$ $\quad P = P_0$

$t = 4$ $\quad P = 2P_0$

Using 1st condition in eq (1)

$$P_0 = c\, e^{k(0)} \Rightarrow c = P_0$$

$$(1) \Rightarrow \quad P = P_0\, e^{kt}$$

Using 2nd condition

$$2P_0 = P_0 e^{4k}$$

$$e^{4k} = 2$$

$$4k = \ln 2$$

$$k = \ln 2/4 \quad = 0.1732.$$

Now after two conditions

$$P = P_0 e^{0.1732t}$$

$$10P_0 = P_0 e^{0.1732t}.$$

$$\Rightarrow \quad 10 = e^{0.1732t}.$$

$$t = \frac{\ln 10}{0.1732} = 13.29$$

It takes $\boxed{\text{13 days}}$ for the bacteria culture to grow 10 times to its initial size.

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Q3

b Solve the following Cauchy Euler Equation

$$x^3 \frac{d^3y}{dx^3} + 2x^2 \frac{d^2y}{dx^2} + 2y = 0 \quad \text{--- (1)}$$

Using substitution

$$y = x^m$$

$$\frac{dy}{dx} = m x^{m-1}$$

$$\frac{d^2y}{dx^2} = m(m-1)x^{m-2}$$

$$\frac{d^3y}{dx^3} = m(m-1)(m-2)x^{m-3}$$

in above diff eq (1)

$$m(m-1)(m-2)x^m + 2m(m-1)x^m + 2x^m = 0$$

$$x^m \left[(m^2-m)(m-2) + 2m^2 - 2m + 2\right] = 0$$

$$x^m \left[m^3 - \cancel{2m^2} - m^2 + \cancel{2m} + \cancel{2m^2} - \cancel{2m} + 2\right] = 0$$

In this case we say that $y = x^m$ will be a solution of diff eq for

$$m^3 - m^2 + 2 = 0$$

i-e

$m = -1$, $m = 1 \pm i$

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$$y = c_1 x^{-1} + x\left[c_2 \cos \ln x + c_3 \sin \ln x\right]$$

Method of Undetermined Coefficient

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(Review Exercise Chapter 4)

Q31 $y'' - y = x + \sin x \; ; \; y(0) = 2, \; y'(0) = 3$

Sol.

y_c Complementary solution :

~~the Associated~~

$\cancel{y'' - y = 0}$

the Associated Homogeneous form.

$$y'' - y = 0$$

$$m^2 - 1 = 0$$

$$(m-1)(m+1) = 0$$

$$m_1 = 1, \quad m_2 = -1$$

Real And Distinct Roots

$$y_c = c_1 e^{x} + c_2 e^{-x}$$

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For particular Solution y_p:

$$g(x) = x + \sin x$$

So,

$$y_p = (Ax + B) + (C\cos x + D\sin x)$$

$$y_p' = A + (-C\sin x + D\cos x)$$

$$y_p'' = -C\cos x - D\sin x$$

$$y_p'' - y_p = x + \sin x$$

$$(-C\cos x - D\sin x) - [(Ax+B) + (C\cos x + D\sin x)] = x + \sin x$$

$$-C\cos x - D\sin x - Ax - B - C\cos x - D\sin x = x + \sin x$$

$$-2C\cos x - 2D\sin x - Ax - B = x + \sin x$$

Comparing Coefficients of "x" and "$\sin x$"

$$-2D = 1$$

$$\boxed{D = -1/2}$$

$$-A = 1$$

$$\boxed{A = -1}$$

$$2C = 0$$

$$\boxed{C = 0}$$

$$\boxed{B = 0}$$

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$$y_p = (-x+0) + (0\cos x + (-\tfrac{1}{2}\sin x))$$

$$\boxed{y_p = -x - \frac{1}{2}\sin x}$$

The general solution is :

$$y = y_c + y_p$$

$$\boxed{y = c_1 e^{x} + c_2 e^{-x} - x - \frac{1}{2}\sin x}$$

Given conditions are :

$$y(0) = 2 \quad \& \quad y'(0) = 3,$$

we obtain,

$$y(0) = c_1 e^{0} + c_2 e^{-0} - 0 - \frac{1}{2}\sin(0)$$

$$2 = c_1 + c_2$$

$$c_1 + c_2 = 2$$

$$\cancel{y(0) = c_1 e^{0} + c_2 e^{0} - 0 - \frac{1}{2}\sin(0)}$$

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$$y' = c_1 e^{x} - c_2 e^{-x} - 1 - \frac{1}{2}\cos x$$

$$y'(0) = c_1 e^{0} - c_2 e^{-0} - 1 - \frac{1}{2}\cos(0)$$

$$3 = c_1 - c_2 - 1 - \frac{1}{2}(1)$$

$$3 = c_1 - c_2 - 1 - \frac{1}{2}$$

$$c_1 - c_2 = 3 + 1 + \frac{1}{2}$$

$$c_1 - c_2 = \frac{6+2+1}{2}$$

$$c_1 - c_2 = \frac{9}{2}$$

Now solving simultaneously, we get

$$c_1 + c_2 = 2$$

$$c_1 - c_2 = 9/2$$

$$2c_1 = 2 + 9/2$$

$$2c_1 = \frac{4+9}{2}$$

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Golden Pearl Cosmetics

$$2C_1 = \frac{13}{2}$$

$$\boxed{C_1 = \frac{13}{4}}$$

$$C_1 + C_2 = 2$$

$$\frac{13}{4} + C_2 = 2$$

$$C_2 = 2 - \frac{13}{4}$$

$$C_2 = \frac{8 - 13}{4}$$

$$\boxed{C_2 = \frac{-5}{4}}$$

The general solution becomes

$$\boxed{y = \frac{13}{4}e^{x} - \frac{5}{4}e^{-x} - x - \frac{1}{2}\sin x}$$

Solution

Using Method of Separation of Variables Solution of p.d.e ① is of the form

$$u(x,t) = X(x)\,T(t)$$

$$\frac{\partial^2 u}{\partial x^2} = X''T \quad , \quad \frac{\partial u}{\partial t} = XT'$$

Using above assumption in heat eq.

$$X''T = \frac{1}{K} XT'$$

$$\frac{X''}{X} = \frac{T'}{KT}$$

A function of x can be equal to a function of y if only ~~...~~ we equate both of them equal to same constant say $-\lambda$, where $-\lambda$ is separation constant and arbitrary.

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$$\frac{X''}{X} = \frac{T'}{kT} = -\lambda$$

$$X'' + \lambda X = 0 \quad \text{--- (2)} \qquad T' + \lambda k T = 0 \quad \text{--- (3)}$$

O.d.e (2) has solutions

for $\boxed{\lambda = 0}$

$$X = C_1 + C_2 x$$

for $\boxed{\lambda = -\alpha^2 < 0}$

$$X = C_3 \cosh \alpha x + C_4 \sinh \alpha x$$

for $\boxed{\lambda = \alpha^2 > 0}$

$$X = C_5 \cos \alpha x + C_6 \sin \alpha x$$

The boundary conditions will also be transformed using method of product solutio

$u(0,t) = 0$, $u(L,t) = 0$

$X(0)\,T(t) = 0$

$T(t) \neq 0$ for non trivial sol.

$\Rightarrow \boxed{X(0) = 0}$ Similarly $\boxed{X(L) = 0}$

Using the derived boundary conditions in the solutions of O.d.e 2, to find particular solutions.

for $\lambda = 0$.

we get $c_1 = c_2 = 0$; i-e trivial solution.

for $\lambda = -\alpha^2 < 0$

we get $c_3 = c_4 = 0$, i-e trivial solution

for $\lambda = \alpha^2 > 0$

$$X(x) = c_5 \cos \alpha x + c_6 \sin \alpha x.$$

$X(0) = 0$

$$0 = c_5 + 0 \Rightarrow c_5 = 0$$

$$X(x) = c_6 \sin \alpha x$$

$X(L) = 0$

$$X(L) = c_6 \sin \alpha L = 0$$

$$c_6 \sin \alpha L = 0$$

$$c_6 \sin \sqrt{\lambda} L = 0$$

$c_6 \neq 0$ for non trivial solution.

$$\sin \sqrt{\lambda} L = 0$$

$$\sin \sqrt{\lambda} L = \sin n\pi$$

$$\sqrt{\lambda} = \frac{n\pi}{L} \qquad \lambda_n = \left(\frac{n\pi}{L}\right)^2 \quad n = 1, 2, 3, \dots$$

It means the only non trivial solution for o.d.e ② with satisfied boundary condition is

$$\boxed{X_n(x) = C_1 \sin \frac{n\pi}{L} x} \quad , n = 1, 2, 3, \ldots$$

For solution of o.d.e ③

$$T' + \lambda k T = 0$$

$$\frac{dT}{dt} = -\lambda k T$$

$$\int \frac{1}{T} \, dT = \int -\lambda k \, dt \qquad \text{(Seperable)}$$

$$\ln T = -\lambda k t + a_1$$

$$T = e^{-\lambda k t} \cdot e^{a_1}$$

$$T = a e^{-\lambda k t}$$

$$\boxed{T_n(t) = a e^{-\left(\frac{n\pi}{L}\right)^2 k t}} \qquad n = 1, 2, 3, \ldots$$

It follows that

$$U_n(x, t) = X_n(x) \, T_n(t)$$

$$U_n(x,t) = A_n e^{-\left(\frac{n\pi}{L}\right)^2 k t} \sin \frac{n\pi}{L} x \quad , n = 1, 2, 3, 4, \ldots$$

④

$\cancel{u(x,0) = x(L-x)}$

$\cancel{u_t(x,0) = f(x)}$

$u(x,0) = f(x)$

$$\Rightarrow f(x) = A_n \sin \frac{n\pi}{L} x$$

$$\int_0^L \sin \frac{m\pi}{L} x \; f(x)\,dx = \int_0^L A_n \sin \frac{n\pi}{L} x \sin \frac{m\pi}{L} x \, dx$$

$$\text{As } \int_0^L \sin \frac{n\pi}{L} x \sin \frac{m\pi}{L} x \, dx = \begin{cases} 0 & n \neq m \\ L/2 & n = m \end{cases}$$

$$\Rightarrow \boxed{A_n = \frac{2}{L} \int_0^L \sin \frac{n\pi}{L} x \; f(x)\,dx}$$

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Work

As $f(x) = x(L-x)$

$$A_n = \frac{2}{L}\int_0^L \sin\frac{n\pi}{L}x\,(xL - x^2)\,dx$$

$$= \frac{2}{L}\cdot L\int_0^L x\sin\frac{n\pi}{L}x\,dx - \frac{2}{L}\int_0^L x^2\sin\frac{n\pi x}{L}\,dx$$

$$= 2\left[-x\cos\frac{n\pi}{L}x\,\frac{L}{n\pi} + \frac{L}{n\pi}\int\cos\frac{n\pi}{L}x\,dx\right]_0^L$$

$$-\frac{2}{L}\left[-x^2\cos\frac{n\pi}{L}x\,\frac{L}{n\pi} + \frac{2L}{n\pi}\int\cos\frac{n\pi}{L}x\;x\,dx\right]_0^L$$

$$= 2\left[-\frac{xL}{n\pi}\cos\frac{n\pi}{L}x + \frac{L^2}{n^2\pi^2}\sin\frac{n\pi}{L}x\right]_0^L$$

$$-\frac{2}{L}\left[-\frac{x^2L}{n\pi}\cos\frac{n\pi}{L}x + \frac{2L}{n\pi}\left\{\frac{xL}{n\pi}\sin\frac{n\pi}{L}x - \frac{L}{n\pi}\int\sin\frac{n\pi}{L}x\,dx\right\}\right]_0^L$$

$$= 2\left[-\frac{L^2}{n\pi}\cos n\pi + \frac{L^2}{n^2\pi^2}\sin n\pi\right] - \frac{2}{L}\left[-\frac{x^2L}{n\pi}\cos\frac{n\pi}{L}x + \frac{2L^2x}{n^2\pi^2}\sin\frac{n\pi}{L}x\right.$$

$$\left. + \frac{2L^3}{n^3\pi^3}\cos\frac{n\pi}{L}x\right]_0^L$$

$$= 2\left[\frac{-L^2}{n\pi}(-1)^n\right] - \frac{2}{L}\left[-\frac{L^3}{n\pi}\cos n\pi + \frac{2L^3}{n^2\pi^2}\sin n\pi + \frac{2L^3}{n^3\pi^3}\cos n\pi\right.$$

$$\left. - \frac{2L^3}{n^3\pi^3}\right]$$

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$$= \frac{2(-1)^{n+1}}{n\pi} L^2 + \frac{2L^2}{n\pi}(-1)^n + \frac{4L^2}{n^3\pi^3}(-1)^{n+1}$$

$$+ \frac{4L^2}{n^3\pi^3}$$

$$= \frac{4L^2}{n^3\pi^3}\left[1 + (-1)^{n+1}\right]$$

$$U(x,t) = \sum_{n=1}^{\infty} U_n(x,t)$$

$$= \sum_{n=1}^{\infty} A_n e^{-(n\pi/L)^2 kt} \sin \frac{n\pi}{L} x$$

$$U(x,t) = \sum_{n=1}^{\infty} \frac{4L^2}{n^3\pi^3} e^{-(n\pi/L)^2 kt} \sin \frac{n\pi}{L} x$$

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