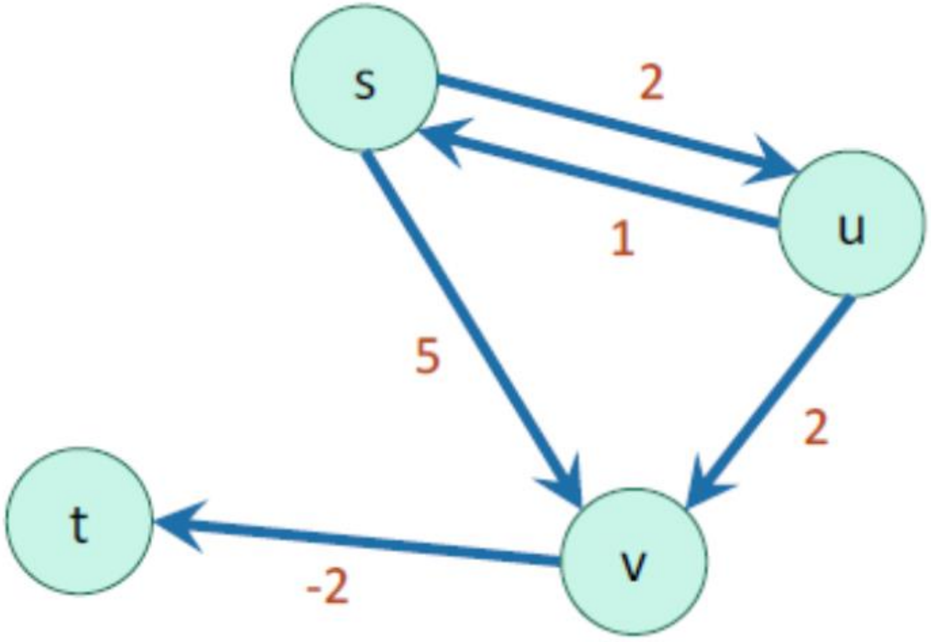
Floyd-Warshall Algorithm (All Pairs Shortest Path)

Floyd-Warshall

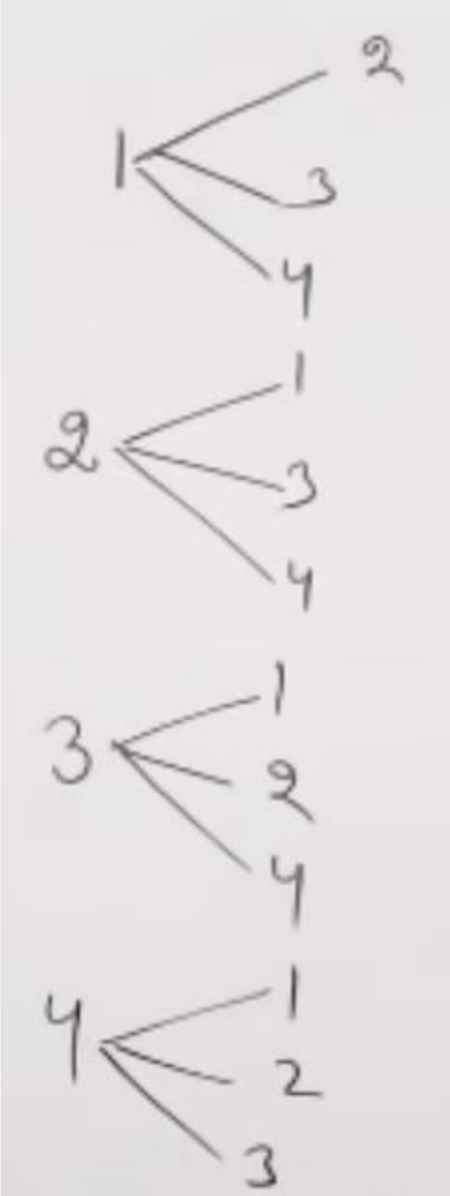
- **Floyd-Warshall Algorithm**
- This is an algorithm for All-Pairs Shortest Paths (APSP)
  - That is, I want to know the shortest path from u to v for **ALL pairs** u,v of vertices in the graph.
  - Not just from a special single source s.

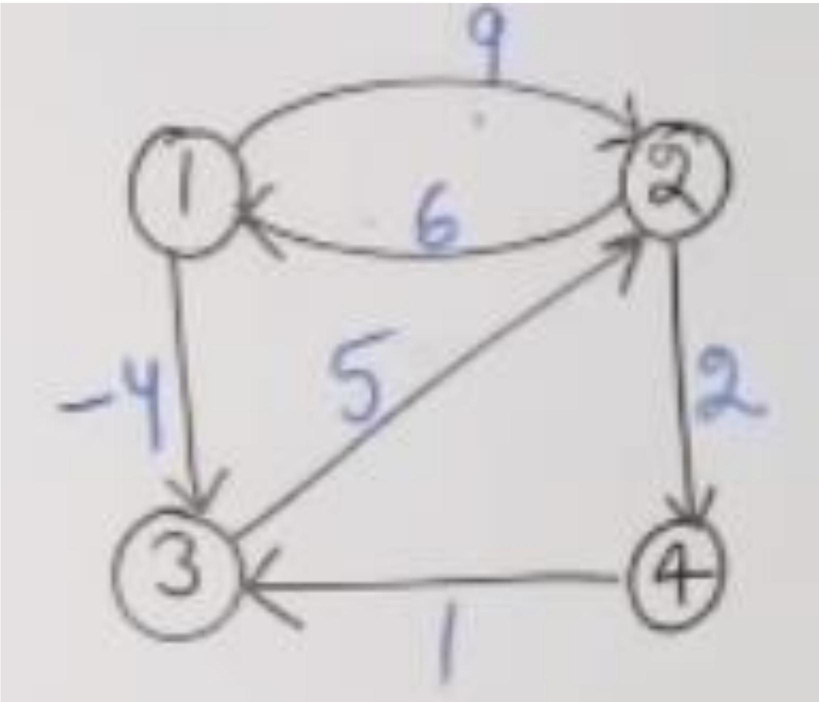
Destination

Source	s	u	v	t
s	0	2	4	2
u	1	0	2	0
v	∞	∞	0	-2
t	∞	∞	∞	0



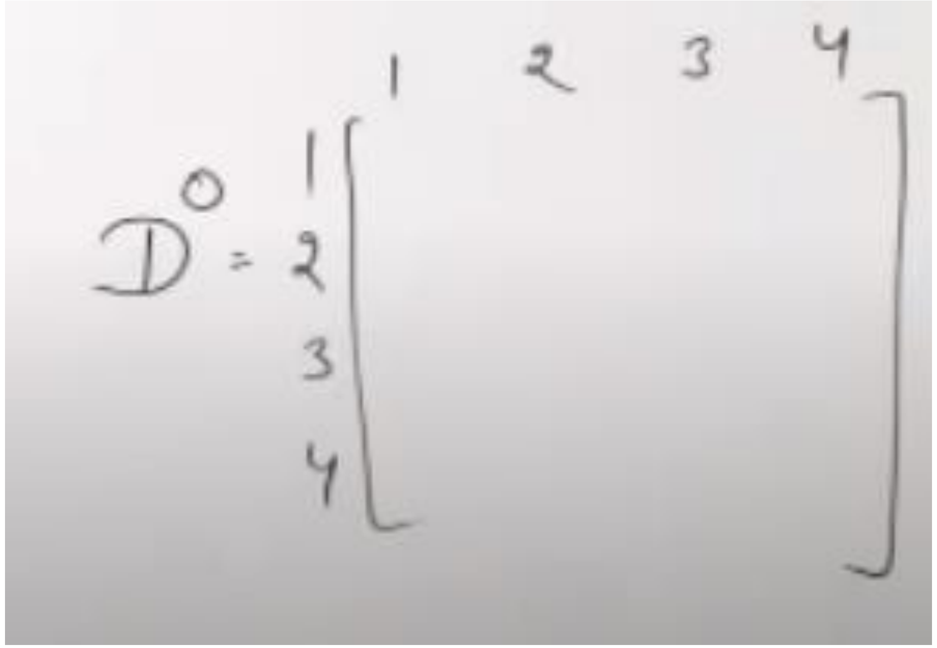
Floyd-Warshall





Floyd-Warshall

Adjacency Matrix/Distance Matrix



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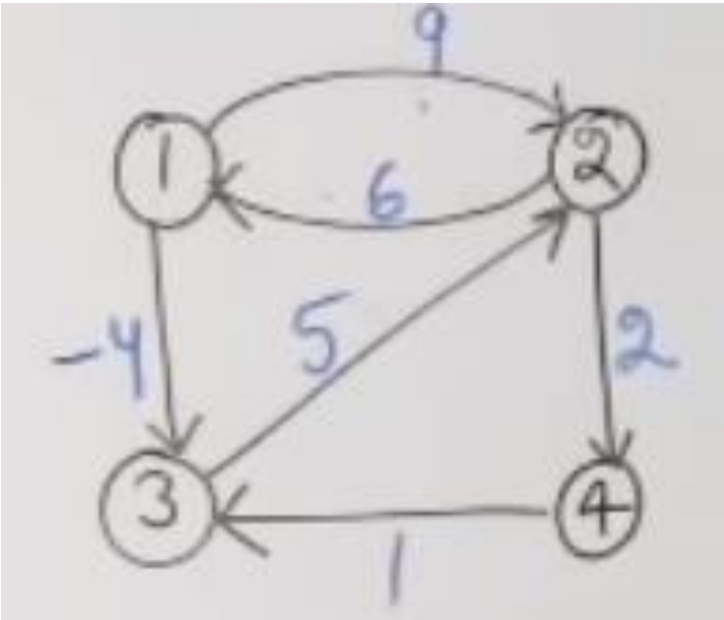


$$D^0 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & & & & \\ 2 & & & & \\ 3 & & & & \\ 4 & & & & \end{array}$$

$$D^0 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 9 & -4 & \infty \\ 6 & 0 & \infty & 2 \\ \infty & 5 & 0 & \infty \\ \infty & \infty & 1 & 0 \end{bmatrix} \end{array}$$

Floyd-Warshall

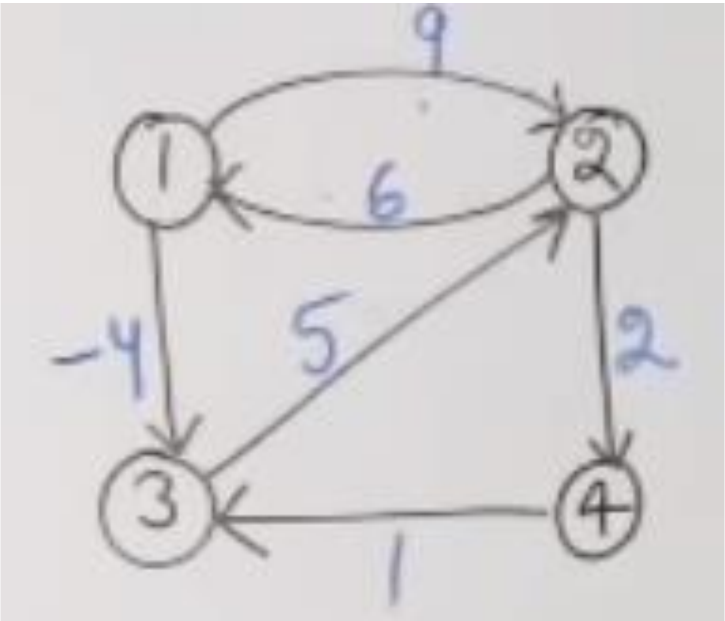
How can we find $D^{(k)}[u,v]$ using $D^{(k-1)}$?

- $D^{(k)}[u,v] = \min\{ D^{(k-1)}[u,v], D^{(k-1)}[u,k] + D^{(k-1)}[k,v] \}$

Case 1: Cost of shortest path through {1,…,k-1}

Case 2: Cost of shortest path from u to k and then from k to v through {1,…,k-1}

Floyd-Warshall



- $D^{(k)}[u,v] = \min\{\, D^{(k-1)}[u,v],\ D^{(k-1)}[u,k] + D^{(k-1)}[k,v] \,\}$

$$D^{0} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix} \end{array}$$

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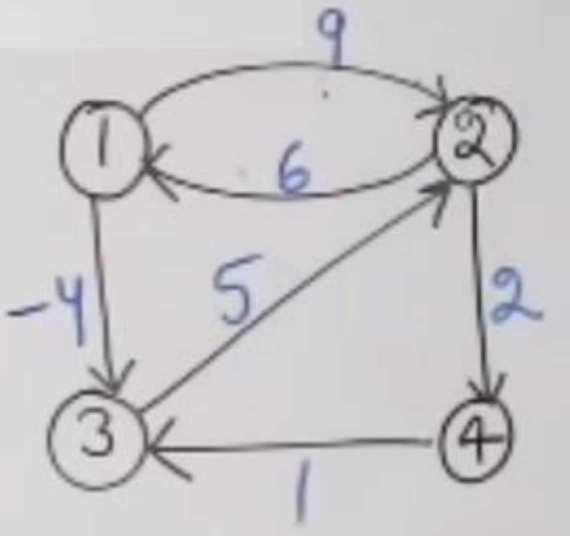
$$D^1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 9 & -4 & \infty \\ 6 & 0 & & \\ \infty & & 0 & \\ \infty & & & 0 \end{bmatrix} \end{array}$$

$$D^1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 9 & -4 & \infty \\ 6 & 0 & 2 & 2 \\ \infty & 5 & 0 & \infty \\ \infty & \infty & 1 & 0 \end{bmatrix} \end{array}$$



$D^0[2,3] \quad D^0[2,1] + D^0[1,3]$

$\infty \quad > \quad 6 + (-4) \quad 2$

$D^0[2,4] = D^0[2,1] + D^0[1,4]$

$2 \quad < \quad 6 + \infty = \infty$

$D^0[3,2] = D^0[3,1] + D^0[1,2]$

$5 \quad < \quad \infty + 9 = \infty$

$$D^0 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 9 & -4 & \infty \\ 6 & 0 & \infty & 2 \\ \infty & 5 & 0 & \infty \\ \infty & \infty & 1 & 0 \end{bmatrix} \end{array}$$

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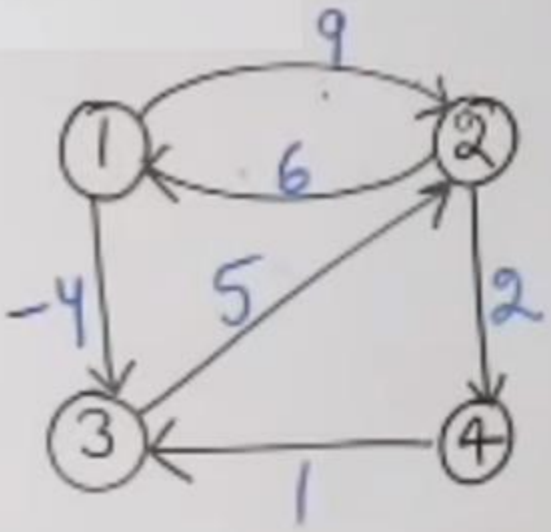
$$D^1 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 9 & -4 & \infty \\ 2 & 6 & 0 & 2 & 2 \\ 3 & \infty & 5 & 0 & \infty \\ 4 & \infty & \infty & 1 & 0 \end{array}$$

$$D^2 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 9 & -4 & 11 \\ 2 & 6 & 0 & 2 & 2 \\ 3 & 11 & 5 & 0 & 7 \\ 4 & \infty & \infty & 1 & 0 \end{array}$$



$$D^0 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 9 & -4 & \infty \\ 2 & 6 & 0 & \infty & 2 \\ 3 & \infty & 5 & 0 & \infty \\ 4 & \infty & \infty & 1 & 0 \end{array}$$

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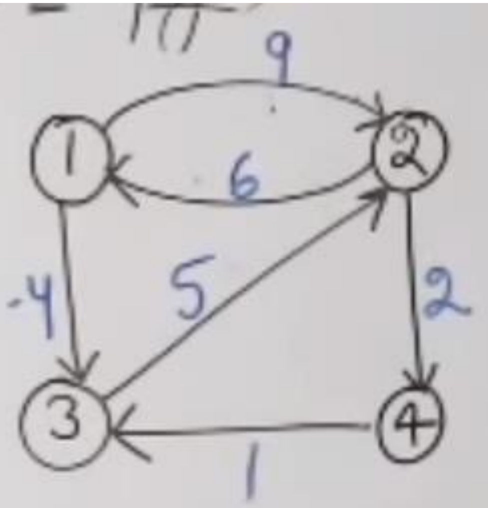
$$D^1 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 9 & -4 & \infty \\ 2 & 6 & 0 & 2 & 2 \\ 3 & \infty & 5 & 0 & \infty \\ 4 & \infty & \infty & 1 & 0 \end{array}$$

$$D^4 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1 & -4 & 3 \\ 2 & 6 & 0 & 2 & 2 \\ 3 & 11 & 5 & 0 & 7 \\ 4 & 12 & 6 & 1 & 0 \end{array}$$



$$D^2 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 9 & -4 & 11 \\ 2 & 6 & 0 & 2 & 2 \\ 3 & 11 & 5 & 0 & 7 \\ 4 & \infty & \infty & 1 & 0 \end{array}$$

$$D^3[1,2] = D^3[1,4] + D^3[4,2]$$

$$1 < 3 + 6$$

$$D^3[1,3] = D^3[1,4] + D^3[4,3]$$

$$-4 \qquad 3 + 1$$

$$D^0 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 9 & -4 & \infty \\ 2 & 6 & 0 & \infty & 2 \\ 3 & \infty & 5 & 0 & \infty \\ 4 & \infty & \infty & 1 & 0 \end{array}$$

$$D^3 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1 & -4 & 3 \\ 2 & 6 & 0 & 2 & 2 \\ 3 & 11 & 5 & 0 & 7 \\ 4 & 12 & 6 & 1 & 0 \end{array}$$

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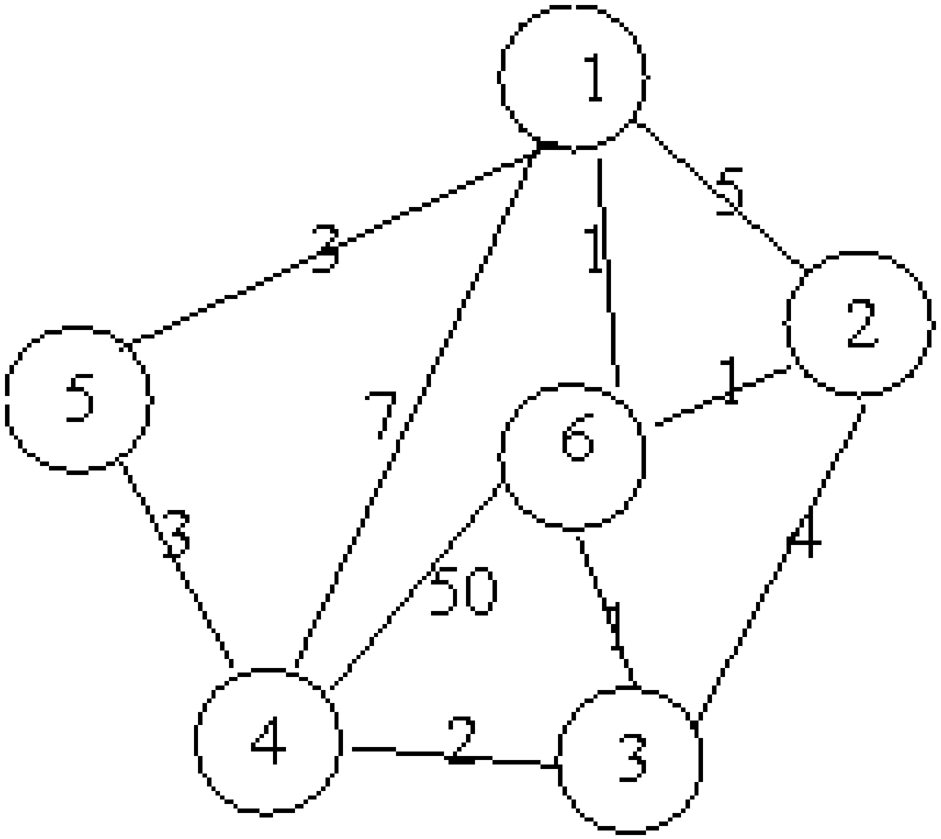

Floyd-Warshall

FLOYD-WARSHALL(W)

$n \leftarrow rows[W]$

$D^{(0)} \leftarrow W$

for $k \leftarrow 1$ **to** n

    do for $i \leftarrow 1$ **to** n

        do for $j \leftarrow 1$ **to** n

            $d_{ij}^{(k)} \leftarrow min(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)})$

return $D^{(n)}$

- Running time: $O(n^3)$
  - Better than running BF n times!
  - Not really better than running Dijkstra n times.
    - But it's simpler to implement and handles negative weights.