NC Formula Sheet

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Error Analysis

Absolute True Error

$$Absolute\ True\ Error = |Actual\ Value - Approximate\ Value|$$

Relative True Error

$$Relative\ True\ Error = \frac{|Actual\ Value - Approximate\ Value|}{|Actual\ Value|}$$

Percentage Relative True Error

$$Percentage\ Relative\ True\ Error = \frac{|Actual\ Value - Approximate\ Value|}{|Actual\ Value|} \times 100$$

Absolute Approximate Error

$$Absolute\ Approximate\ Error = \ |Current\ Approximate - Previous\ Approximate|$$

Relative Approximate Error

$$Relative\ Approximate\ Error = \frac{|Current\ Approximate - Previous\ Approximate|}{|Current\ Approximate|}$$

Percentage Relative Approximate Error

$$Percenatge\ Relative\ Approximate\ Error = \frac{|Current\ Approximate - Previous\ Approximate|}{|Current\ Approximate|} \times 100$$

Taylor Series

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)(x - x_0)}{2!} + \cdots$$

$$f(x) = \sum_{i=0}^{n} \frac{f^{(i)}(x_0)(x - x_0)^i}{i!}$$

Taylor Polynomial of n^{th}degree

$$f(x) = \ P_n(x) + E_n(x)$$
$$where\ P_n\ is\ the\ Taylor\ Polynomial,$$
$$and\ E_n(x) is\ the\ Error\ Function$$

Floating Point Representation

$$(-1)^s 2^{c-1023}(1 + f)$$
$$where\ s\ is\ the\ sign\ indicator,$$
$$c\ is\ the\ characteristic,$$
$$and\ f\ is\ the\ mantissa$$

$$Example{:}\quad \underbrace{0}_{s(1\ bit)}\ \underbrace{10011011010}_{c\ (11\ bits)}\ \underbrace{00000100 \ldots 01}_{f\ (52\ bits)}$$
$$c = 1242$$
$$f = 2^{-6} + 2^{-52}$$
$$\Rightarrow (-1)^0 2^{1242-1023}(1 + 2^{-6} + 2^{-52})$$

Interpolation and Polynomial approximation

Lagrange Interpolation

$$P(x) = L_0 f(x_0) + L_1 f(x_1) + \cdots + L_n f(x_n)$$

$$P(x) = \sum_{k=0}^{n} f(x_k) L_k$$

$$L_k = \prod_{\substack{i=0 \\ i\neq k}}^{n} \frac{(x - x_i)}{(x_k - x_i)}$$

Newton Divided Difference

$$P(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots a_n(x - x_0)(x - x_1) \dots (x - x_{n-1})$$

$$where\ a_0 = f[x_0]$$

$$a_1 = f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0}$$

$$a_2 = f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

$$a_n = f[x_0, x_1, x_2, \dots, x_n] = \frac{f[x_1, x_2, \dots, x_n] - f[x_0, x_1, \dots, x_{n-1}]}{x_n - x_0}$$

Newton Forward Difference

$$P(x) = f(x_0) + s\Delta f(x_0) + \frac{s(s-1)}{2!}\Delta^2 f(x_0) + \cdots + \frac{s(s-1)(s-(n-1))}{n!}\Delta^n f(x_0)$$

$$s = \frac{x - x_0}{h}$$

$$h = step\ size$$

Newton Backward Difference

$$P(x) = f(x_n) + s\nabla f(x_n) + \frac{s(s+1)}{2!}\nabla^2 f(x_n) + \cdots + \frac{s(s+1)(s+(n-1))}{n!}\nabla^n f(x_n)$$

$$s = \frac{x - x_n}{h}$$

Stirling's Interpolation

$$P(x) = f[x_0] + \frac{sh}{2}(f[x_{-1}, x_0] + f[x_0, x_1]) + s^2h^2 f[x_{-1}, x_0, x_1] +$$
$$\frac{s(s^2-1)h^3}{2}(f[x_{-2}, x_{-1}, x_0, x_1] + f[x_{-1}, x_0, x_1, x_2]) + \cdots +$$
$$s^2(s^2-1)(s^2-4)\dots(s^2-(m-1)^2)h^{2m} f[x_{-m}, \dots, x_m] +$$
$$\frac{s(s^2-1)\dots(s^2-m^2)h^{2m+1}}{2}(f[x_{-m-1}, \dots, x_m] + f[x_{-m}, \dots, x_{m+1}])$$

$$The\ above\ formula\ uses\ Divided\ Difference$$

$$P(x) = f(x_0) + \frac{s}{2}(\Delta f_{-1} + \Delta f_0) + \frac{s^2}{s!}\Delta^2 f_{-1} + \frac{s(s^2-1)}{3! \times 2}(\Delta^3 f_{-2} + \Delta^3 f_{-1}) + \frac{s^2(s^2-1)}{4!}\Delta^4 f_{-2} + \cdots$$

$$The\ above\ formula\ uses\ Forward\ Difference$$

Numerical Differentiation

Forward Difference Method

$$f'(x_0) = \frac{f(x_0 + h) - f(x_0)}{h} - \frac{h}{2} f''(\zeta)$$

Backward Difference Method

$$f'(x_0) = \frac{f(x_0) - f(x_0 - h)}{h} + \frac{h}{2} f''(\zeta)$$

Three Point Endpoint Method

$$f(x_0) = \frac{1}{2h}\big(-3f(x_0) + 4f(x_0 + h) - f(x_0 + 2h)\big) + \frac{h^2}{3} f'''(\zeta)$$

Three Point Midpoint Method

$$f'(x_0) = \frac{1}{2h}\big(f(x_0 + h) - f(x_0 - h)\big) + \frac{h^2}{6} f'''(\zeta)$$

Five Point Endpoint Method

$$f'(x_0) = \frac{1}{12h}\big(-25f(x_0) + 48f(x_0 + h) - 36f(x_0 + 2h) + 16f(x_0 + 3h) - 3f(x_0 + 4h)\big) + \frac{h^4}{5} f^{(5)}(\zeta)$$

Five Point Midpoint Method

$$f'(x) = \frac{1}{12h}\big(f(x_0 - 2h) - 8f(x_0 - h) + 8f(x_0 + h) - f(x_0 + 2h)\big) + \frac{h^4}{30} f^{(5)}(\zeta)$$

Second Derivative Midpoint Formula

$$f''(x_0) = \frac{1}{h^2}\big(f(x_0 - h) - 2f(x_0) + f(x_0 + h)\big) - \frac{h^2}{12} f^{(4)}(\zeta)$$

$$If\ points\ are\ not\ ordered\ in\ order\ of\ \mathrm{h}, then\ break\ data.$$

Numerical Integration

Closed Newton-Cotes

$$Trapezoidal\ Rule\ (n = 1)$$

$$\int_{x_0}^{x_1} f(x)dx = \frac{h}{2}\big(f(x_0) + f(x_1)\big) - \frac{h^3}{12} f''(\xi)$$

$$Simpson's\ {}^1/_3\ Rule\ (n = 2)$$

$$\int_{x_0}^{x_2} f(x)dx = \frac{h}{3}\big(f(x_0) + 4f(x_1) + f(x_2)\big) - \frac{h^5}{90} f^{(4)}(\xi)$$

$$Simpson's\ {}^3/_8\ Rule\ (n = 3)$$

$$\int_{x_0}^{x_3} f(x)dx = \frac{3h}{8}\big(f(x_0) + 3f(x_1) + 3f(x_2) = f(x_3)\big) - \frac{3h^5}{80} f^{(4)}(\xi)$$

$$where\ h = \frac{b-a}{n}$$

Open Newton-Cotes

$$n = 0, \qquad \int_{x_{-1}}^{x_1} f(x)dx = 2hf(x_0) + \frac{h^3}{3} f''(\xi)$$

$$n = 1, \qquad \int_{x_{-1}}^{x_2} f(x)dx = \frac{3h}{3}\big(f(x_0) + f(x_1)\big) + \frac{3h^3}{4} f'''(\xi)$$

$$n = 2, \qquad \int_{x_{-1}}^{x_3} f(x)dx = \frac{4h}{3}\big(2f(x_0) - f(x_1) + 2f(x_2)\big) + \frac{14h^5}{45} f^{(4)}(\xi)$$

$$where\ h = \frac{b-a}{n+2}$$

Composite Numerical Integration

$$Composite\ Trapezoidal\ Rule$$

$$\int_a^b f(x)dx = \frac{h}{2}\left[f(a) + 2\sum_{j=1}^{n-1} f(x_j) + f(b)\right] - \frac{b-a}{12} h^2 f''(\mu)$$

$$Composite\ Simpson's\ {}^1/_3\ Rule$$

$$\int_a^b f(x)dx = \frac{h}{3}\left[f(a) + 2\sum_{j=1}^{\frac{n}{2}-1} f(x_{2j}) + 4\sum_{j=1}^{\frac{n}{2}} f(x_{2j-1}) + f(b)\right] - \frac{b-a}{180} h^4 f^{(4)}(\mu)$$

Initial-Value Problem for ODEs

Euler's Method

$$y_{i+1} = y_i + hf(t_i, y_i)$$

Heun's Method (special case of RK-2 method with $a_1 = a_2 = \frac{1}{2}, p_1 = q_{11} = 1$)

$$y_{i+1} = y_i + \frac{h}{2}(k_1 + k_2)$$

$$where\ k_1 = f(t_i, y_i)$$

$$k_2 = f(t_i + h, y_i + k_1 h)$$

Midpoint Method (special case of RK-2 with $a_1 = 0, a_2 = 1, p_1 = q_{11} = \frac{1}{2}$)

$$y_{i+1} = y_i + k_2 h$$

$$where\ k_1 = f(t_i, y_i)$$

$$k_2 = f(t_i + \frac{h}{2}, y_i + \frac{k_1 h}{2})$$

Rk-4 Method ($a_1 = a_4 = \frac{1}{6}, a_2 = a_3 = \frac{2}{6}, p_1 = p_2 = \frac{1}{2}, p_3 = 1, q_{11} = q_{22} = \frac{1}{2}, q_{21} = q_{31} = q_{32} = 0, q_{33} = 1$)

$$y_{i+1} = y_i + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$where\ k_1 = f(t_i, y_i)$$

$$k_2 = f\left(t_i + \frac{h}{2}, y_i + \frac{k_1 h}{2}\right)$$

$$k_3 = f\left(t_i + \frac{h}{2}, y_i + \frac{k_2 h}{2}\right)$$

$$k_4 = f(t_i + h, y_i + k_3 h)$$

Direct Method for solving linear system

LU Decomposition (Doo Little

$$A = LU$$

$$U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}, \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$$

Crout's Approach

$$A = LU$$

$$U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}, \qquad L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$$

LDL^t Factorization

$$A = LDL^t$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix}$$

Cholesky Method

$$A = LL^t$$

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$$

Iterative Method for solving linear system

Gauss Seidel & Jacobi

$$No\ such\ formula, only\ recursive\ technique$$

Vector Norms

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \end{bmatrix}, \qquad \|X\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}$$

$$\|X\|_\infty = \underset{1 \le i \le n}{Max(|x_i|)}$$

Approximating Eigen values and Eigen vectors

Power Method

$$\lambda^{(k+1)} X_{(k+1)} = A X_k$$

$$No\ other\ formula, only\ recursive\ technique$$

Numerical Optimization

Gradient Descent

$$h_{\theta_0,\theta_1}(x) = \theta_0 + \theta_1 x_1$$

$$J(\theta_0, \theta_1) = \frac{1}{2m} \sum_{i=1}^{m} \left(h_{\theta_0,\theta_1}(x_i) - y_i \right)^2$$

$$\frac{\partial J}{\partial \theta_0} = \frac{1}{m} \sum_{i=1}^{m} \left(h_{\theta_0,\theta_1}(x_i) - y_i \right)$$

$$\frac{\partial J}{\partial \theta_1} = \frac{1}{m} \sum_{i=1}^{m} x_i \left(h_{\theta_0,\theta_1}(x_i) - y_i \right)$$

$$\theta_0^{k+1} = \theta_0^k - \alpha \frac{\partial J}{\partial \theta_0}$$

$$\theta_1^{k+1} = \theta_1^k - \alpha \frac{\partial J}{\partial \theta_1}$$