Aalto University
School of Electrical Engineering

Exercise Round 7

The deadline of this exercise round is **Wednesday March 11, 2020**. The solutions will be gone through during the exercise session in room T2 in Konemiehentie 2 (CS) on that day starting at 14:15.

The problems should be *solved before the exercise session*, and during the session those who have completed the exercises may be asked to present their solutions on the board/screen.

Exercise 1. (Rao–Blackwellization)

Let $\mathbf{x}_k,\ k = 1, \ldots, K$ be independent draws from $\mathrm{N}(\mathbf{m}, \mathbf{P})$ with unknown $\mathbf{m}$ and known $\mathbf{P}$. A crude estimator for $\mathbf{m}$ is given by

$$\hat{\mathbf{m}}(\mathbf{x}_{1:K}) = \mathbf{x}_1, \tag{1}$$

and a sufficient statistic for $\mathbf{m}$ is given by

$$\mathbf{s}(\mathbf{x}_{1:K}) = \sum_{k=1}^{K} \mathbf{x}_k. \tag{2}$$

Do the following steps:

(a) The vector

$$\begin{bmatrix} \hat{\mathbf{m}}(\mathbf{x}_{1:K}) \\ \mathbf{s}(\mathbf{x}_{1:K}) \end{bmatrix} \tag{3}$$

has a Gaussian distribution. Compute the mean vector and the covariance matrix.

(b) Compute the conditional mean

$$\mathrm{E}[\hat{\mathbf{m}}(\mathbf{x}_{1:K}) \mid \mathbf{s}(\mathbf{x}_{1:K})]. \tag{4}$$

(c) Compare the variance of $\hat{\mathbf{m}}(\mathbf{x}_{1:K})$ with the variance of $\mathrm{E}[\hat{\mathbf{m}}(\mathbf{x}_{1:K}) \mid \mathbf{s}(\mathbf{x}_{1:K})]$, which estimator is better?

Exercise 2. (Rao–Blackwellized Particle Filter I)

Consider the following clutter model:

$$\begin{aligned} x_k \mid x_{k-1} &\sim \mathrm{N}(x_{k-1}, 1), && (5a) \\ u_k &\sim \mathrm{Bernoulli}(0.1), && (5b) \\ y_k \mid x_k, u_k &\sim (1 - u_k)\mathrm{N}(x_k \mid 1) + u_k \mathrm{N}(0, 10), && (5c) \end{aligned}$$

where $u_k \sim \mathrm{Bernoulli}(p)$ means that u_k equals 1 with probability p and 0 with probability $1 - p$, respectively.

(a) Write down the Rao–Blackwellized particle filter equations for this model.

(b) Implement the bootstrap version of the Rao–Blackwellized particle filter. That is, select the importance distribution:

$$\pi(u_k \mid u_{1:k-1}, y_{1:k}) = \mathrm{Bernoulli}(u_k \mid 0.1). \tag{6}$$

Test your algorithm on simulated data.

Exercise 3. (Rao–Blackwellized Particle Filter II)

Consider the model in Eq. (5) again. Recall that the optimal proposal is given by

$$\pi(u_k \mid u_{1:k-1}, y_{1:k}) \propto p(y_k \mid u_{1:k}, y_{1:k-1}) p(u_k \mid u_{k-1}). \tag{7}$$

Do the following:

(a) Derive the optimal importance proposal for u_k

(b) (Bonus) Implement a Rao–Blackwellized particle filter using the optimal proposal and compare against your Rao–Blackwellized bootstrap filter from the previous exercise.

Hint: Recall the law of total probability:

$$\begin{aligned} p(y_k \mid u_{1:k}, y_{1:k-1}) &= \int p(y_k \mid x_k, u_{1:k}, y_{1:k-1}) p(x_k \mid u_{1:k}, y_{1:k-1}) \mathrm{d}x_k \\ &= \int p(y_k \mid x_k, u_k) p(x_k \mid u_{1:k}, y_{1:k-1}) \mathrm{d}x_k, \end{aligned} \tag{8}$$

and $p(x_k \mid u_{1:k}, y_{1:k-1}) = \mathrm{N}(x_k \mid m_k^-, P_k^-)$.