Exercise Round 3

The deadline of this exercise round is **Wednesday January 29, 2019**. The solutions will be gone through during the exercise session in room T2 in Konemiehentie 2 (CS) on that day starting at 14:15.

The problems should be *solved before the exercise session*, and during the session those who have completed the exercises may be asked to present their solutions on the board/screen.

Exercise 1. (Kalman Filter with Non-Zero Mean Noises)

Derive the Kalman filter equations for the following linear-Gaussian filtering model with non-zero-mean noises:

$$\begin{aligned} \mathbf{x}_k &= \mathbf{A}\,\mathbf{x}_{k-1} + \mathbf{q}_{k-1}, \\ \mathbf{y}_k &= \mathbf{H}\,\mathbf{x}_k + \mathbf{r}_k, \end{aligned} \tag{1}$$

where $\mathbf{q}_{k-1} \sim \mathrm{N}(\mathbf{m}_q, \mathbf{Q})$ and $\mathbf{r}_k \sim \mathrm{N}(\mathbf{m}_r, \mathbf{R})$.

Exercise 2. (Filtering of Finite-State HMMs)

Write down the Bayesian filtering equations for finite-state hidden Markov models (HMM). That is, write down the prediction and update equations for the system where $x_k \in \{1, \ldots, N_x\}$, the dynamic model is defined by the discrete probability distribution

$$p(x_k = i \mid x_{k-1} = j), \; i, j \in \{1, \ldots, N_x\}, \tag{2}$$

and the measurement model is

$$p(y_k \mid x_k = i), \; i \in \{1, \ldots, N_x\}. \tag{3}$$

Exercise 3. (Kalman Filter for Noisy Resonator)

Consider the following dynamic model:

$$\begin{aligned}
\mathbf{x}_k &= \begin{pmatrix} \cos\omega & \frac{\sin(\omega)}{\omega} \\ -\omega\,\sin(\omega) & \cos(\omega) \end{pmatrix} \mathbf{x}_{k-1} + \mathbf{q}_{k-1} \\
y_k &= \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x}_k + r_k,
\end{aligned}$$

where $\mathbf{x}_k \in \mathbb{R}^2$ is the state, y_k is the measurement, $r_k \sim \mathrm{N}(0, 0.1)$ is a white Gaussian measurement noise, and $\mathbf{q}_k \sim \mathrm{N}(\mathbf{0}, \mathbf{Q})$, where

$$\mathbf{Q} = \begin{pmatrix} \frac{q^c\,\omega - q^c\,\cos(\omega)\,\sin(\omega)}{2\omega^3} & \frac{q^c\,\sin^2(\omega)}{2\omega^2} \\ \frac{q^c\,\sin^2(\omega)}{2\omega^2} & \frac{q^c\,\omega + q^c\,\cos(\omega)\,\sin(\omega)}{2\omega} \end{pmatrix}.$$

The angular velocity is $\omega = 1/2$ and the spectral density is $q^c = 0.01$. The model is a discretized version of noisy resonator model with a given angular velocity ω.

In the file `kf_ex.m` (in MyCourses) there is a simulation of the dynamic model together with a baseline solution, where the measurement is directly used as the estimate of the state component x_1 and the second component x_2 is computed as a weighted average of the measurement differences. *If you prefer to use, e.g., Python or R for implementation, you can also store the simulated data from the Matlab/Octave script and implement the actual filter in another language.*

(a) Implement the Kalman filter for the model and compare its performance (in RMSE sense) to the baseline solution. Plot figures of the solutions.

(b) Compute (numerically) the stationary Kalman filter corresponding to the model. Test this stationary filter against the baseline and Kalman filter solutions. Plot the results and report the RMSE values for the solutions. What is the practical difference in the stationary and non-stationary Kalman filter solutions?