Aalto University
School of Electrical Engineering

Exercise Round 4

The deadline of this exercise round is **Wednesday February 5, 2020**. The solutions will be gone through during the exercise session in room T2 in Konemiehentie 2 (CS) on that day starting at 14:15.

The problems should be *solved before the exercise session*, and during the session those who have completed the exercises may be asked to present their solutions on the board/screen.

Exercise 1. (Extended Kalman Filter)

Consider the following non-linear state space model

$$\begin{aligned} x_k &= x_{k-1} - 0.01 \sin(x_{k-1}) + q_{k-1}, \\ y_k &= 0.5 \sin(2\, x_k) + r_k, \end{aligned} \tag{1}$$

where q_{k-1} has variance 0.01^2 and r_k has variance 0.02.

(a) Derive the required derivatives for EKF for this model and check the derivatives numerically (recall that $df(x)/dx \approx (f(x+h) - f(x))/h$ when h is small).

(b) Implement the EKF for the model. Simulate trajectories from the model, compute the RMSE values and plot the results.

Exercise 2. (Alternative Form of SLF)

In this exercise your task is to derive the derivative form of the statistically linearized filter (SLF).

(a) Prove using integration by parts the following identity for Gaussian random variable $\mathbf{x}$, a differentiable non-linear function $\mathbf{g}(\mathbf{x})$ and its Jacobian matrix $\mathbf{G}_x(\mathbf{x}) = \partial \mathbf{g}(\mathbf{x}) / \partial \mathbf{x}$:

$$\mathrm{E}[\mathbf{g}(\mathbf{x})\, (\mathbf{x} - \mathbf{m})^\mathsf{T}] = \mathrm{E}[\mathbf{G}_x(\mathbf{x})]\, \mathbf{P}, \tag{2}$$

where $\mathrm{E}[\cdot]$ denotes the expected value with respect to $\mathrm{N}(\mathbf{x} \mid \mathbf{m}, \mathbf{P})$. *Hint*: $\frac{\partial}{\partial \mathbf{x}} \mathrm{N}(\mathbf{x} \mid \mathbf{m}, \mathbf{P}) = -\mathbf{P}^{-1} (\mathbf{x} - \mathbf{m})\, \mathrm{N}(\mathbf{x} \mid \mathbf{m}, \mathbf{P})$.

(b) Prove the following: Let

$$\boldsymbol{\mu}(\mathbf{m}) = \mathrm{E}[\mathbf{g}(\mathbf{x})], \tag{3}$$

where $\mathrm{E}[\cdot]$ denotes the expected value with respect to $\mathrm{N}(\mathbf{x} \mid \mathbf{m}, \mathbf{P})$. Then

$$\frac{\partial \boldsymbol{\mu}(\mathbf{m})}{\partial \mathbf{m}} = \mathrm{E}[\mathbf{G}_x(\mathbf{x})]. \tag{4}$$

(c) Write down the additive form of the SLF equations in an alternative form, where you have eliminated all the cross terms of form $\mathrm{E}[\mathbf{f}(\mathbf{x}_{k-1})\, \delta\mathbf{x}_{k-1}^{\mathsf{T}}]$ and $\mathrm{E}[\mathbf{h}(\mathbf{x}_k)\, \delta\mathbf{x}_k^{\mathsf{T}}]^{\mathsf{T}}$ using the result in (a).

(d) How can you utilize the result (b) when using the alternative form of the SLF?

Exercise 3. (Bearings Only Target Tracking with EKF)

In this exercise we consider a classical bearings only target tracking problem which frequently arises in the context of passive sensor tracking. In this problem there is a single target in the scene and two angular sensors are used for tracking it. The scenario is illustrated in Figure 1.

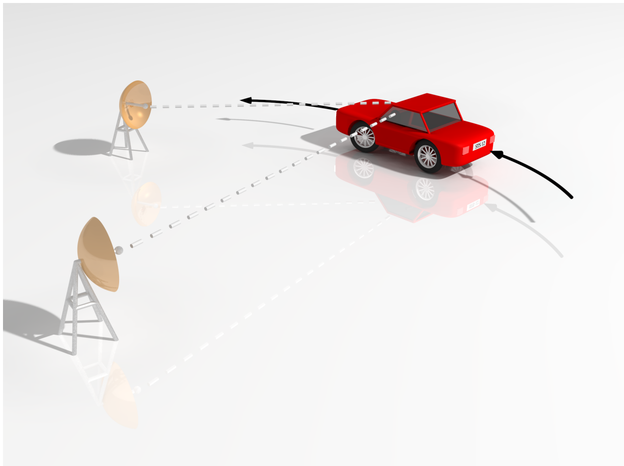

Figure 1: In a bearings only target tracking problem the sensors generate angle measurements of the target, and the purpose is to determine the target trajectory.

The state of the target at time step k consist of the position (x_k, y_k) and the velocity $(\dot{x}_k, \dot{y}_k)$. The dynamics of the state vector $\mathbf{x}_k = (x_k \; y_k \; \dot{x}_k \; \dot{y}_k)^\mathsf{T}$ are modeled with the discretized Wiener velocity model:

$$\begin{pmatrix} x_k \\ y_k \\ \dot{x}_k \\ \dot{y}_k \end{pmatrix} = \begin{pmatrix} 1 & 0 & \Delta t & 0 \\ 0 & 1 & 0 & \Delta t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_{k-1} \\ y_{k-1} \\ \dot{x}_{k-1} \\ \dot{y}_{k-1} \end{pmatrix} + \mathbf{q}_{k-1},$$

where $\mathbf{q}_k$ is a zero mean Gaussian process noise with covariance

$$\mathbf{Q} = \begin{pmatrix} q_1^c \, \Delta t^3/3 & 0 & q_1^c \, \Delta t^2/2 & 0 \\ 0 & q_2^c \, \Delta t^3/3 & 0 & q_2^c \, \Delta t^2/2 \\ q_1^c \, \Delta t^2/2 & 0 & q_1^c \, \Delta t & 0 \\ 0 & q_2^c \, \Delta t^2/2 & 0 & q_2^c \, \Delta t \end{pmatrix}.$$

In this scenario the diffusion coefficients are $q_1^c = q_2^c = 0.1$ and the sampling period is $\Delta t = 0.1$. The measurement model for sensor $i \in \{1, 2\}$ is the following:

$$\theta_k^i = \tan^{-1}\left(\frac{y_k - s_y^i}{x_k - s_x^i}\right) + r_k, \tag{5}$$

where (s_x^i, s_y^i) is the position of the sensor i and $r_k \sim \mathrm{N}(0, \sigma^2)$ is a Gaussian measurement noise with standard deviation of $\sigma = 0.05$ radians. At each sampling time, which occurs 10 times per second (*i.e.*, $\Delta t = 0.1$), both of the two sensors produce a measurement.

In the file `angle_ex.m` (in MyCourses) there is a baseline solution, which computes estimates of the position from the crossing of the measurements and estimates the velocity to be always zero. Your task is to implement an EKF for the problem and compare the results graphically and in RMSE sense.

(a) Implement an EKF for the bearings only target tracking problem, which uses the non-linear measurement model (5) as its measurement model function (not the crossings). *Hints:*

- Use the Matlab function `atan2` in the measurement model instead of `atan` to directly get an answer in the range $[-\pi, \pi]$.
- The two measurements at each measurement time can be processed one at a time, that is, you can simply perform two scalar updates instead of a single two dimensional measurement update.

- Start by computing the Jacobian matrix of the measurement model function with respect to the state components. Before implementing the filter, check by finite differences that the Jacobian matrix is correct.

(b) Compute the RMSE values and plot figures of the estimates.