1.) $y_k = a_1 s_k + a_2 \sin(s_k) + b + \varepsilon_k.$

$\varepsilon_k \sim N(0, R),$

$a_1 \sim N(0, \sigma_1^2)$

$a_2 \sim N(0, \sigma_2^2)$

$b \sim N(0, \sigma_b^2)$

$x_k = \{a_1 \quad a_2 \quad b\}$

$H_k = \{s_k \quad \sin(s_k) \quad 1\}.$

$$\boxed{\begin{aligned} x_k &= x_{k-1}. \\ y_k &= H_k x_k + \varepsilon_k. \end{aligned}}$$

is the estimation problem as a Linear Gaussian-SS model.

2.)

a.)

$$p(\theta \mid y_{1:T}) \propto \exp\left(-\frac{1}{2}\sum_{k=1}^{T}(y_k - \theta_1 x_k - \theta_2)^2\right) \exp\left(\frac{-1}{2\sigma^2}\theta_1^2\right) \exp\left(\frac{-1}{2\sigma^2}\theta_2^2\right)$$

$$\propto \exp\left(-\frac{1}{2}\sum_{k=1}^{T}(y_k - \theta_1 x_k - \theta_2)^2\right). \exp\left(\frac{-1}{2\sigma^2}(\theta_1^2 + \theta_2^2)\right)$$

$$\propto \exp\left(-\frac{1}{2}(Y - X\theta)^T(Y - X\theta)\right). \exp\left(-\frac{1}{2\sigma^2}\theta^T\theta\right).$$

$$X = \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & 1 \\ x_T & 1 \end{bmatrix} \quad Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_T \end{bmatrix} \quad \theta = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}.$$

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b) ~~Find~~ To find the mean (Posterior)

i.e.

argmax. $\exp(E)$.

where

$$E = -\frac{1}{2}\left[(Y-X\theta)^T(Y-X\theta)\right] - \frac{1}{2\sigma^2}\theta^T\theta$$

$$\nabla E = \frac{d}{d\theta}\left[-\frac{1}{2}\left[Y^TY - 2\theta^TX^TY + \theta^TX^TX\theta\right] - \frac{1}{2\sigma^2}\theta^T\theta\right]$$

$\nabla E = 0$

$$0 = -\frac{1}{2}\left[-2X^TY + 2X^TX\theta\right] - \frac{1}{2\sigma^2}2\theta$$

$$X^TY - X^TX\theta = \frac{\theta}{\sigma^2}$$

$$X^TY = \frac{\theta}{\sigma^2} + X^TX\theta.$$

$$\text{mean}_\theta = \frac{X^TY}{\left(X^TX + \frac{I}{\sigma^2}\right)} = \left(X^TX + \frac{I}{\sigma^2}\right)^{-1}X^TY$$

$$\cancel{= X^TY\left(X^TX + \frac{I}{\sigma^2}\right)^{-1}}$$

c.) To find the covariance & Hessian matrix

$$H = \nabla^2 E = -X^TX - \frac{I}{\sigma^2}.$$

Given

$$P = -H^{-1} = \left(X^TX + \frac{I}{\sigma^2}\right)^{-1}.$$

d.) The final Posterior distribution would be.

$\theta \sim N(M, P)$

when the variance $\sigma^2 = 0$ the mean m of θ will be same as Linear state estimator.

when $\sigma^2 = 0$

~~8~~) i.e. $M(\theta) = (X^T X)^{-1} X^T Y$

3.)

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$\lambda \to 0.$

$\theta \sim N(\mu_n, \Sigma_n)$

$f(x) = \frac{1}{2\pi\sigma^2} e^{\left(\frac{x-m}{\sigma}\right)^2}$

3.) $P = -H^{-1}$

$\begin{pmatrix} x - m \\ -1 - Hm \end{pmatrix}^T$

$a = x$

$b = y$

$P(x) = N(x \mid m, P)$

$P(y \mid x) = N(y \mid Hx, R)$

To prove

$P(x, y) = P\left(\frac{y}{x}\right) . P(x)$

$P(x) = \frac{1}{(2\pi)^{n_x/2} \sqrt{\det P}} \exp\left(\cancel{x}\frac{-1}{2} (x - m)^T P^{-1} (x - m)\right)$

$P\left(\frac{y}{x}\right) = \frac{1}{(2\pi)^{n_b/2} \sqrt{\det R}} \exp\left(-\frac{1}{2} (y - Hx)^T R^{-1} (y - Hx)\right)$

$P(x, y) = P(x) . P\left(\frac{y}{x}\right)$

$= \frac{1}{(2\pi)^{\frac{n_a + n_b}{2}} \sqrt{\det(P) \det R}}$

$\exp\left(-\frac{1}{2} (x - m)^T P^{-1} (x - m) - \frac{1}{2} (y - Hx)^T R^{-1} (y - Hx)\right)$

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$x = m + e$

$y - H(m+e)$ $\quad m + e$

$y - Hm - He$ $\quad b - He$

$\underbrace{y - Hm}_{b} - Hx$

$\boxed{e = x - m}$

$\boxed{b = y - Hm}$

$$E = (x-m)^T P^{-1} (x-m) + (y - Hx)^T R^{-1} (y - Hx)$$

$$= e^T P^{-1} e$$

$$= e^T P^{-1} e + (b - He)^T R^{-1} (b - He)$$

$\boxed{x = e + m}$

$y - Hm - He$

$$= e^T P^{-1} e + (b^T - e^T H^T) R^{-1} (b - He)$$

$$= e^T P^{-1} e + (b^T b - b^T He - e^T H^T b + e^T H^T He) R^{-1}$$

$$= e^T P^{-1} e + b^T R^{-1} b - b^T R^{-1} He - e^T H^T R^{-1} b + e^T H^T R^{-1} He$$

$$= e^T \left[P^{-1} + H^T R^{-1} H \right] e - e^T H^T R^{-1} b - b^T R^{-1} He + b^T R^{-1} b$$

$$= \begin{pmatrix} e \\ b \end{pmatrix}^T \begin{pmatrix} P^{-1} + H^T R^{-1} H & -H^T R^{-1} \\ -R^{-1} H & R^{-1} \end{pmatrix} \begin{pmatrix} e \\ b \end{pmatrix}$$

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$$M^{-1} = \begin{pmatrix} P^{-1} + H^T R^{-1} H & -H^T R^{-1} \\ -R^{-1} H & R^{-1} \end{pmatrix}$$

$$M = \frac{1}{\det M} (\operatorname{adj} M.)$$

$$= \frac{1}{(\det M)} \begin{pmatrix} R^{-1} & + H^T R^{-1} \\ R^{-1} H & P^{-1} + H^T R^{-1} H \end{pmatrix}$$

$$\frac{1}{\det M} = \det M^{-1} = R^{-1} (P^{-1} + H^T R^{-1} H) - H^T R^{-1} R^{-1} H.$$

$$= R^{-1} P^{-1}.$$

$$= \begin{pmatrix} \frac{R^{-1}}{R^{-1} P^{-1}} & \frac{H^T R^{-1}}{R^{-1} P^{-1}} \\ \frac{R^{-1} H}{R^{-1} P^{-1}} & \frac{P^{-1}}{R^{-1} P^{-1}} + \frac{H^T R^{-1} H}{R^{-1} P^{-1}} \end{pmatrix}$$

$$= \begin{pmatrix} P & H^T P \\ HP & R + HPH^T \end{pmatrix}$$

$\therefore$ Substituting

$$P(x,y) = \frac{1}{2\pi^{\frac{n_a + n_b}{2}} \sqrt{\det(P) \det(R)}} \exp\left(\frac{-1}{2} \begin{pmatrix} e \\ f \end{pmatrix}^T \begin{pmatrix} P^{-1} + H^T R^{-1} H & -H^T R^{-1} \\ -R^{-1} H & R^{-1} \end{pmatrix} \begin{pmatrix} e \\ f \end{pmatrix} \right)$$

$= \alpha \, f^m$

$$= \frac{1}{(2\pi)^{\frac{n_a+n_b}{2}} \sqrt{\det(P)\det(R)}} \exp\left(-\frac{1}{2}\begin{pmatrix} x-m \\ y-Hm \end{pmatrix}^T \begin{pmatrix} P^{-1}+H^TR^{-1}H & -H^TR^{-1} \\ -R^{-1}H & R^{-1} \end{pmatrix} \begin{pmatrix} x-m \\ y-Hm \end{pmatrix}\right)$$

This is obtained by inverse of (M)

$$= N\left(\begin{pmatrix} m \\ Hm \end{pmatrix}, \begin{pmatrix} P & PH^T \\ HP & HPH^T+R \end{pmatrix}\right).$$

Marginal density

$$p(y) = \int p(x,y)\, dx.$$

$$= \int p\left(\frac{y}{x}\right) \cdot p(x) \cdot dx$$

$$= \int N(x|m,P)\; N(y|Hx,R)\; dx.$$

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E.)

$$X = \begin{pmatrix} x \\ y \end{pmatrix}$$

Given.

$$\mu = \text{mean} = \begin{pmatrix} \mu_x \\ \mu_y \end{pmatrix} \qquad \phi = \Sigma^{-1} = \begin{pmatrix} \Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{xy}^{T} & \Sigma_{yy} \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} A & B \\ B^T & D \end{pmatrix}$$

$$p(x,y) = \frac{1}{(2\pi)^{n_x/2}\sqrt{\det(\Sigma)}} \left(\exp\left(-\frac{1}{2} (X-\mu)^T \Sigma^{-1} (X-\mu) \right) \right)$$

$$p(y) = \frac{1}{\sqrt{\det(\Sigma_{yy})}} \exp\left(-\frac{1}{2} (y-\mu_y)^T \Sigma_{yy}^{-1} (y-\mu_y) \right)$$

$$p\left(\frac{x}{y}\right) = \frac{p(x,y)}{p(y)}$$

$$= \frac{\sqrt{\det(\Sigma_{yy})}}{(2\pi)^{\frac{n_x}{2}} \sqrt{\det \Sigma}} \exp(E)$$

$$E = -\frac{1}{2} \left[(X-\mu)^T \Sigma^{-1} (X-\mu) + (y-\mu_y)^T \Sigma_{yy}^{-1} (y-\mu_y) \right]$$

from schur component

$$\det \Sigma = \det \Sigma_{yy} \det(\Sigma_{xx} - \Sigma_{xy} \Sigma_{yy}^{-1} \Sigma_{yx})$$

on substituting

we get

$$= \frac{\exp(E)}{(2\pi)^{\frac{n_x}{2}} \sqrt{\det(\Sigma_{xx} - \Sigma_{xy} \Sigma_{yy}^{-1} \Sigma_{yx})}}$$

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$$E = -\frac{1}{2}\left[(x-\mu)^{T} \Sigma^{-1} (x-\mu)\right] - \frac{1}{2}\left[(y-\mu_y)^{T} \Sigma_{yy}^{-1} (y-\mu_y)\right]$$

Substituting $\Sigma^{-1} = \phi = \begin{pmatrix} A & B \\ B^T & D \end{pmatrix}$

and expanding we get

$$= -\frac{1}{2}(x-\mu_x)^T A (x-\mu_x) - \frac{1}{2}(x-\mu_x)^T B (y-\mu_y)$$

$$-\frac{1}{2}(y-\mu_y)^T B^T (x-\mu_x) - \frac{1}{2}(y-\mu_y)^T (D-\Sigma_{yy}^{-1})(y-\mu_y)$$

we solve.

$$= -\frac{1}{2} x^T A x + x^T (A\mu_x - B(y-\mu_y))$$

$$-\frac{1}{2}\mu_x^T A \mu_x + \mu_x^T B(y-\mu_y) - \frac{1}{2}(y-\mu_y)^T (D-\Sigma_{yy}^{-1})(y-\mu_y)$$

using block inversion matrix,

$$\Sigma_{yy}^{-1} = D - B^T A^{-1} B.$$

substituting in above, we get

$$= -\frac{1}{2} x^T A x + x^T (A\mu_x - B(y-\mu_y)) - \frac{1}{2}\mu_x^T A \mu_x$$

$$+ \mu_x^T B(y-\mu_y) - \frac{1}{2}(y-\mu_y)^T B^T A^{-1} B(y-\mu_y)$$

$$= -\frac{1}{2}\left(x - (A\mu_x - B(y-\mu_y))\right)^T A \left(x - (A\mu_x - B(y-\mu_y))\right)$$

$$= -\frac{1}{2}\left[\left(x - \left(\mu_x + \overset{\Sigma_{xy}\Sigma_{yy}^{-1}}{\cancel{B}}(y-\mu_y)\right)\right)^T \frac{A}{A^2} \left(x - \left(\mu_x - \overset{\Sigma_{xy}\Sigma_{yy}^{-1}}{\cancel{B}}(y-\mu_y)\right)\right)\right]$$

$$\boxed{E = -\frac{1}{2}\left[\left(x - \left(\mu_x + \overset{\Sigma_{xy}\Sigma_{yy}^{-1}}{\cancel{B}}(y-\mu_y)\right)\right)^T A^{-1} \left(x - \mu_x - \overset{\Sigma_{xy}\Sigma_{yy}^{-1}}{\cancel{B}}(y-\mu_y)\right)\right]}$$

$$A^{-1} = \Sigma_{xx} - \Sigma_{xy}\Sigma_{yy}^{-1}\Sigma_{xy}^T$$

$$B = -A\,\Sigma_{xy}\Sigma_{yy}^{-1}$$

$$E = -\frac{1}{2}\left[\left(x - \left(\mu_x + \Sigma_{xy}(y-\mu_y)\right)\right)^T \Sigma_{xx} - \Sigma_{xy}\Sigma_{yy}^{-1}\Sigma_{xy}^T \left(x - \left(\mu_x + \Sigma_{xy}(y-\mu_y)\right)\right)\right] \quad \text{— (A)}$$

$$p\left(\frac{x}{y}\right) = \frac{1}{(2\pi)^{\frac{n}{2}}} \cdot \frac{\exp(E)}{\sqrt{\det\left(\Sigma_{xx} - \Sigma_{xy}\Sigma_{yy}^{-1}\Sigma_{yx}\right)}}$$

where E is present in (A).