1) To derive statistically linearised RTS smoother

$$p(x_k, x_{k+1} | y_{1:k}) = N\left(\begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix} \Big| m_1, P_1\right)$$

$$m_1 = \begin{pmatrix} m_k \\ E[f(x_k)] \end{pmatrix}$$

$$P_1 = \begin{pmatrix} P_k & E[f(x_k)\delta x_k^T]^T \\ E[f(x_k)\delta x_k^T] & E[f(x_k)\delta x_k^T] P_k^{-1} E[f(x_k)\delta x_k^T]^T + Q_k \end{pmatrix}$$

$$p(x_k | x_{k+1}, y_{1:T}) = p(x_k | x_{k+1}, y_{1:k})$$

$$= \sim N(x_k | m_2, P_2)$$

$$G_k = E[f(x_k)\delta x_k^T] \left[E[f(x_k)\delta x_k^T] P_k^{-1} E[f(x_k)\delta x_k^T]^T + Q_k\right]^{-1}$$

$$m_2 = m_k + G_k (x_{k+1} - E[f(x_k)])$$

$$P_2 = P_k - G_k \left(E[f(x_k)\delta x_k^T] P_k^{-1} E[f(x_k)\delta x_k^T]^T + Q\right) G_k^T$$

$$p(x_{k+1}, x_k | y_{1:T}) = p(x_k | x_{k+1}, y_{1:T})\, p(x_{k+1} | y_{1:T})$$

$$\sim N(x_k | m_2, P_2)\, N(x_{k+1} | m_{k+1}^s, P_{k+1}^s)$$

$$= N\left(\begin{bmatrix} x_{k+1} \\ x_k \end{bmatrix} \Big| m_3, P_3\right)$$

$$M_3 = \left(\quad \right)$$

$$M_3 = \begin{pmatrix} m_{k+1}^s \\ m_{k} + G_k \left(m_{k+1}^s - E[f(x_k)] \right) \end{pmatrix}.$$

$$P_3 = \begin{pmatrix} P_{k+1}^s & P_{k+1}^s G_k^T \\ G_k P_{k+1}^s & G_k P_{k+1}^s G_k^T + \cancel{Q_k} \end{pmatrix}$$

$$m_k^s = m_k + G_k \left(m_{k+1}^s - E[f(x_k)] \right)$$

$$P_k^s = P_k + G_k \left(P_{k+1}^s - E[f(x_k)\delta x_k^T]^T P_k E[f(x_k)\delta x_k]^T - Q_k \right) G_k^T$$

$$p(x_k | y_{1:T}) = N(x_k | m_k^s, P_k^s)$$

Backward smoothing equations :-

$$\bar{m}_{k+1} = E[f(x_k)]$$

$$\bar{P}_{k+1} = E[f(x_k)\delta x_k^T] P_k E[f(x_k)\delta x_k^T]^T + Q_k$$

$$G_k = P_k \, E[f(x_k)\delta x_k^T]^T [\bar{P}_{k+1}]^{-1}$$

$$m_k^s = m_k + G_k [m_{k+1}^s - E[f(x_k)]]$$

$$P_k^s = P_k + G_k \left(P_{k+1}^s - E[f(x_k)\delta x_k^T] P_k E[f(x_k)\delta x_k^T]^T \right) G_k^T$$

$$P_k^s = P_k + G_k \left(P_{k+1}^s - \bar{P}_{k+1} \right) G_k^T$$