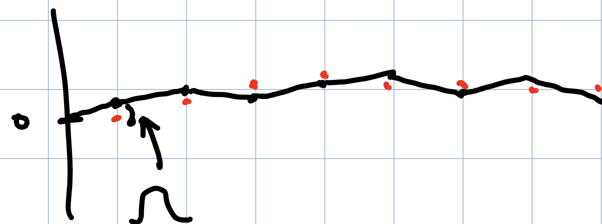
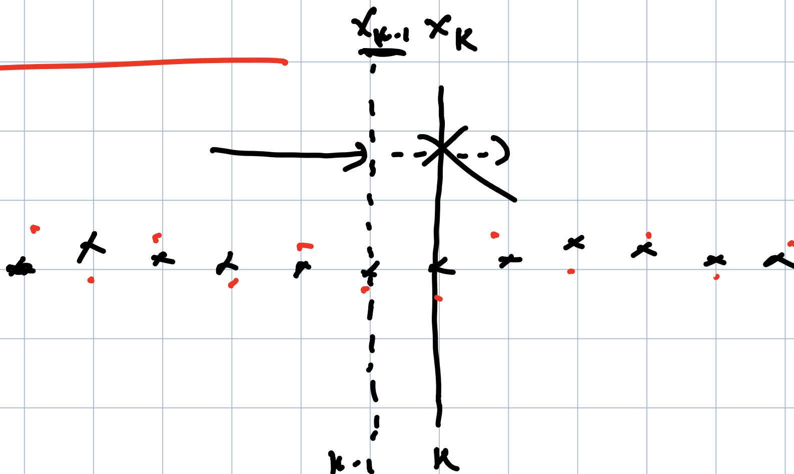
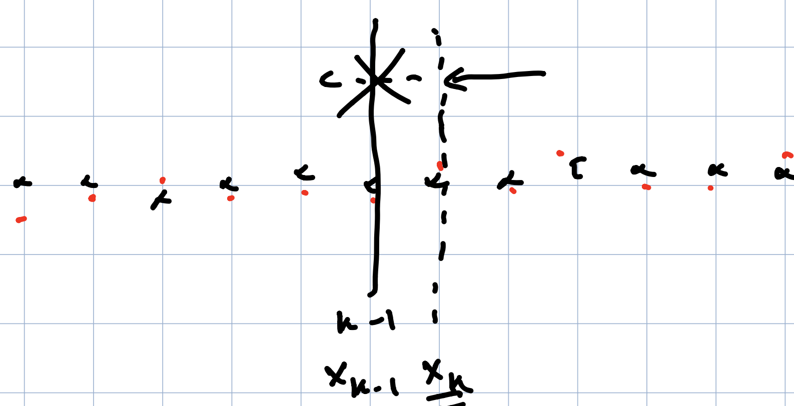
Gauss random walk







Known: $p(x_{k-1} | y_{1:k-1})$

we want: $p(x_k | y_{1:k-1})$

recall: $p(x_k | y_{1:k-1})$

$$= \int p(x_k, x_{k-1} | y_{1:k-1}) \, dx_{k-1}$$

Markovianity :

$$\begin{aligned} &p(x_k, x_{k-1} \mid y_{1:k-1}) \\ &= \underbrace{p(x_k \mid x_{k-1}, y_{1:k-1})}\, p(x_{k-1} \mid y_{1:k-1}) \\ &= p(x_k \mid x_{k-1})\, p(x_{k-1} \mid y_{1:k-1}) \end{aligned}$$

$$\left\{ \begin{aligned} &p(A, B) \\ &= p(A \mid B)\, p(B) \end{aligned} \right.$$

$$\to \quad p(x_k \mid y_{1:k-1}) = \int p(x_k \mid x_{k-1})\, p(x_{k-1} \mid y_{1:k-1})\, dx_{k-1}$$

Prediction step ↑

$$p(x_k \mid y_{1:k-1})$$

$$p(y_k \mid x_k)$$

$$p(x \mid y) = \frac{p(y \mid x)\, p(x)}{p(y)} = \frac{p(y \mid x)\, p(x)}{\underbrace{\int p(y \mid x)\, p(x)\, dx}_{Z}}$$

$$\propto p(y \mid x)\, p(x)$$

$$p(x_k \mid y_{1:k}) = p(x_k \mid y_k, y_{1:k-1})$$

$$\to \; = \frac{p(y_k \mid x_k, \cancel{y_{1:k-1}})\, p(x_k \mid y_{1:k-1})}{\int (\dots)\, dx_k}$$

$$= \frac{p(y_k \mid x_k)\, p(x_k \mid y_{1:k-1})}{\int p(y_k \mid x_k)\, p(x_k \mid y_{1:k-1})\, dx_k}$$

Update step ↑

KF :

$$x_k = A x_{k-1} + q_{k-1}, \quad q_k \sim N(0, Q)$$
$$y_k = H x_k + r_k, \quad r_k \sim N(0, R)$$

$$p(x_k | x_{k-1}) = N(x_k | A x_{k-1}, Q)$$
$$p(y_k | x_k) = N(y_k | H x_k, R)$$

$$p(x_k | y_{1:k-1})$$

strategy : 1. form $\begin{pmatrix} x_{k-1} \\ x_k \end{pmatrix} \Big| y_{1:k-1}$

2. integrate out x_{k-1}

3. form $\begin{pmatrix} x_k \\ y_k \end{pmatrix} \Big| y_{1:k-1}$

4. condition on y_k

$$\Rightarrow p(x_k | y_k, y_{1:k-1})$$
$$= p(x_k | y_{1:k})$$

step 1:

$$x_{k-1} \sim N(m_{k-1}, P_{k-1})$$
$$p(x_{k-1}) = N(x_{k-1} | m_{k-1}, P_{k-1})$$
$$p(x_k | x_{k-1}) = N(x_k | A x_{k-1}, Q)$$

$$p\left(\begin{pmatrix} x_{k-1} \\ x_k \end{pmatrix} \Big| y_{1:k-1}\right)$$

$$= N\left(\begin{pmatrix} x_{k-1} \\ x_k \end{pmatrix} \Big| \begin{pmatrix} m_{k-1} \\ A m_{k-1} \end{pmatrix}, \begin{pmatrix} P_{k-1} & P_{k-1} A^T \\ A P_{k-1} & A P_{k-1} A^T + Q \end{pmatrix}\right)$$

2.

$$p(x_k | y_{1:k-1}) = N(x_k | A m_{k-1}, A P_{k-1} A^T + Q)$$
$$= N(x_k | m_k^-, P_k^-)$$

3.
$$p(x_k) = N(x_k \mid m_k^-, P_k^-)$$
$$p(y_k \mid x_k) = N(y_k \mid H x_k, R)$$

$$p\left(\begin{pmatrix} x_k \\ y_k \end{pmatrix}\right) = N\left(\begin{pmatrix} x_k \\ y_k \end{pmatrix} \,\middle|\, \begin{pmatrix} m_k^- \\ H m_k^- \end{pmatrix}, \begin{pmatrix} P_k^- & P_k^- H^T \\ H P_k^- & H P_k^- H^T + R \end{pmatrix}\right)$$

4.
$$p(x_k \mid y_k, y_{1:k-1})$$
$$= N\Big(x_k \,\Big|\, m_k^- + P_k^- H^T (H P_k^- H^T + R)^{-1} (y_k - H m_k^-),$$
$$P_k^- - P_k^- H^T (H P_k^- H^T + R)^{-1} H P_k^- \Big)$$

$$m_k = m_k^- + P_k^- H^T \underbrace{(H P_k^- H^T + R)}_{S_k}{}^{-1} (y_k - H m_k^-)$$

$$= m_k^- + \underbrace{P_k^- H^T S_k^{-1}}_{K_k} (y_k - H m_k^-)$$

$$= m_k^- + K_k (y_k - H m_k^-)$$

$$K_k = P_k^- H^T S_k^{-1}$$
$$K_k^T = (P_k^- H^T S^{-1})^T$$
$$= (S_k^{-1})^T H (P_k^-)^T$$
$$= S_k^{-1} H P_k^-$$

$$P_k = P_k^- - P_k^- H^T (H P_k^- H^T + R)^{-1} H P_k^-$$
$$= P_k^- - P_k^- H^T S_k^{-1} H P_k^-$$
$$= P_k^- - K_k H P_k^-$$
$$= P_k^- - K_k S_k \underbrace{S_k^{-1} H P_k^-}_{K_k^T}$$
$$= P_k^- - K_k S_k K_k^T$$