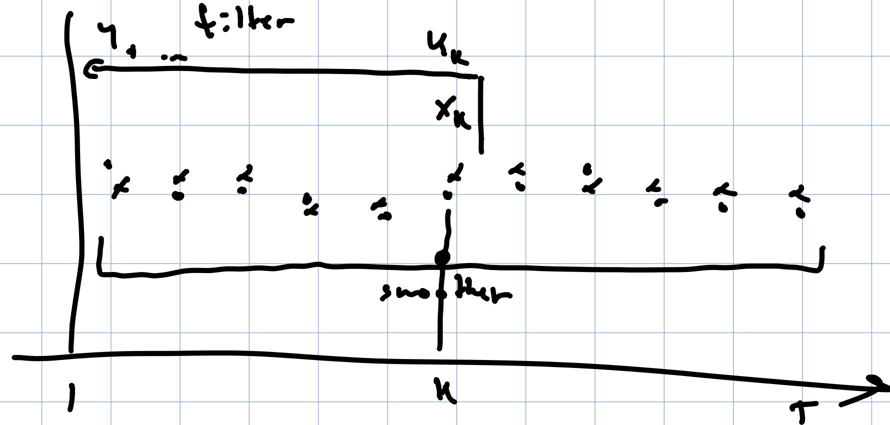


filter : $p(x_k \mid y_1, \dots, y_k)$

smoother : $p(x_k \mid \underbrace{y_1, \dots, y_k, y_{k+1}, \dots, y_T}_{y_1, \dots, y_T \,=\, y_{1:T}})$, $T > k$

---

we know $p(x_{k+1} \mid y_{1:T})$

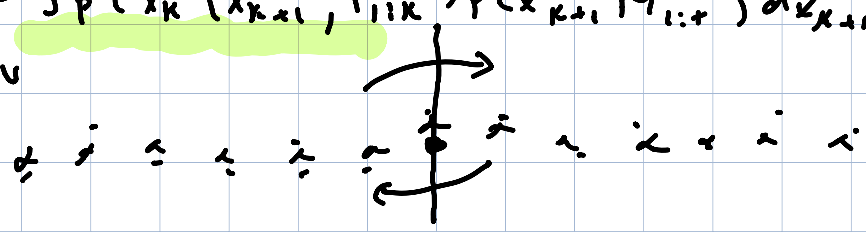
compute $p(x_k \mid y_{1:T})$

derivation:

- assume that we know $p(x_{k+1} \mid y_{1:T})$

$$
\begin{aligned}
p(x_k \mid y_{1:T}) &= \int p(x_k, x_{k+1} \mid y_{1:T})\, dx_{k+1} \\
&= \int p(x_k \mid x_{k+1}, y_{1:T})\, p(x_{k+1} \mid y_{1:T})\, dx_{k+1} \\
&= \int p(x_k \mid x_{k+1}, y_{1:k})\, p(x_{k+1} \mid y_{1:T})\, dx_{k+1}
\end{aligned}
$$

Markov



x_{k+1}

$$p(x_k \mid x_{k+1}, y_{1:k})$$
$$= \frac{p(x_k, x_{k+1} \mid y_{1:k})}{p(x_{k+1} \mid y_{1:k})}$$
$$= \frac{p(x_{k+1} \mid x_k, \cancel{y_{1:k}})\, p(x_k \mid y_{1:k})}{p(x_{k+1} \mid y_{1:k})}$$
$$= \frac{p(x_{k+1} \mid x_k)\, p(x_k \mid y_{1:k})}{p(x_{k+1} \mid y_{1:k})}$$

$$\underline{p(x_k \mid y_{1:T})} = \int \frac{p(x_{k+1} \mid x_k)\, p(x_k \mid y_{1:k})}{p(x_{k+1} \mid y_{1:k})} \cdot p(x_{k+1} \mid y_{1:T}) \cdot dx_{k+1}$$
$$= p(x_k \mid y_{1:k}) \int \frac{p(x_{k+1} \mid x_k)\, p(x_{k+1} \mid y_{1:T})}{p(x_{k+1} \mid y_{1:k})} \cdot dx_{k+1}$$

$$p(x_{k+1} \mid y_{1:T}) = N(x_{k+1} \mid m^s_{k+1}, P^s_{k+1})$$
$$p(x) = N(x \mid m, P)$$
$$p(y \mid x) = N(y \mid Hx + u, R)$$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} \sim N\left(\begin{pmatrix} m \\ Hm + u \end{pmatrix}, \begin{pmatrix} P & PH^T \\ HP & HPH^T + R \end{pmatrix} \right)$$
$$y \sim N(Hm + u, HPH^T + R)$$

$$p(x_k, x_{k+1} \mid y_{1:k})$$
$$= p(x_{k+1} \mid x_k, \cancel{y_{1:k}})\, p(x_k \mid y_{1:k})$$

$$= p(x_{k+1} \mid x_k)\, p(x_k \mid y_{1:k})$$

"y" = "x_{k+1}"
"x" = "x_k"

$$N(x_{k+1} \mid A x_k, Q)$$

$$N(x_k \mid \underline{m_k}, P_k)$$

$$p(x_k, x_{k+1} \mid y_{1:k})$$
$$= N\left(\begin{pmatrix} x_k \\ x_{k+1} \end{pmatrix} \middle| \begin{pmatrix} m_k \\ A m_k \end{pmatrix}, \begin{pmatrix} P_k & P_k A^T \\ A P_k & A P_k A^T + Q \end{pmatrix}\right)$$

$$p(x_k \mid x_{k+1}, y_{1:k})$$
$$= N\Big(x_k \,\Big|\, m_k + \overbrace{P_k A^T (A P_k A^T + Q)^{-1}}^{G_k} (x_{k+1} - A m_k),$$
$$P_k - P_k A^T (A P_k A^T + Q)^{-1} A P_k\Big)$$
$$= p(x_k \mid x_{k+1}, y_{1:T})$$

$$\int p(x_k \mid x_{k+1}, y_{1:k})\, p(x_{k+1} \mid y_{1:T})\, dx_{k+1}$$
$$= \int p(x_k \mid x_{k+1}, y_{1:T})\, p(x_{k+1} \mid y_{1:T})\, dx_{k+1}$$
$$= \int p(x_k, x_{k+1} \mid y_{1:T})\, dx_{k+1}$$

$$p(x_k \mid x_{k+1}, y_{1:T})$$
$$= N\big(x_k \mid \overbrace{m_k - G_k A m_k}^{u} + \overset{M}{G_k}\, x_{k+1},$$
$$P_k - G_k (A P_k A^T + Q) G_k^T\big)$$

$$p(x_{k+1} \mid y_{1:T})$$
$$= N(x_{k+1} \mid m^s_{k+1}, P^s_{k+1})$$

$$p(x_k, x_{k+1} \mid y_{1:T})$$

$$= N\left(\begin{pmatrix} x_{k+1} \\ x_k \end{pmatrix} \,\middle|\, \begin{pmatrix} m^s_{k+1} \\ G_k m^s_{k+1} + m_k - G_k A m_k \end{pmatrix}, \begin{pmatrix} P^s_{k+1} & P^s_{k+1} G_k^T \\ G_k P^s_{k+1} & G_k P^s_{k+1} G_k^T + P_k - G_k (A P_k A^T + Q) G_k^T \end{pmatrix} \right)$$

$$p(x_k \mid y_{1:T})$$

$$= N\left(x_k \mid m_k + G_k (m^s_{k+1} - A m_k), \; P_k + G_k (P^s_{k+1} - A P_k A^T - Q) G_k^T \right)$$

$$G_k = P_k A^T (A P_k A^T + Q)^{-1}$$