Homework 5, Fall 2024

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Problem 1: Vanilla Logistic Regression for multi-classification:

- k classes. - The input $X \in R^d$

- Prob. to each class is:

$$P(Y=k \mid X=x) = \frac{\exp(w_k^T x)}{1 + \sum_{i=1}^{k-1} \exp(w_i^T x)}, \text{ for } k = 1, 2, \ldots, k-1$$

$$P(Y=k \mid X=x) = \frac{1}{1 + \sum_{i=1}^{k-1} \exp(w_i^T x)}$$

If we define $w_k = 0$, Then:

$$P(Y=k \mid X=x) = \frac{\exp(w_k^T x)}{1 + \sum_{i=1}^{k-1} \exp(w_i^T x)}, \; k = 1, \ldots, k$$

Question 1 what and how many Para. are there to be opt. ?

1. Weight vectors (w_k)
   - → for each class k, there is $w_k \in R^d$.
   - → assuming $w_k = 0$.
   - → means: The # of weight Para.:
     $d \times (k-1)$
2. Bias Parameters: b_k
   - → one bias for each class: $(k-1)$

∴ Total Parameters: $d \times (k-1) + (k-1)$
$= (d+1)(k-1)$ #

Question 2 $L(w_1, \cdots, w_{k-1}) = \sum_{i=1}^{n} \ln P(Y = y_i \mid X = x_i)$

$$= \sum_{i=1}^{n} \ln \left(\frac{\exp(w_k^T x)}{1 + \sum_{l=1}^{k-1} \exp(w_l^T x)} \right)$$

$$= \sum_{i=1}^{n} \left(\ln(\exp(w_k^T x) - \ln\left(1 + \sum_{l=1}^{k-1} \exp(w_l^T x)\right) \right)$$

$$= \underbrace{\sum_{i=1}^{n} w_k^T x}_{I} - \underbrace{\ln\left(1 + \sum_{l=1}^{k-1} \exp(w_l^T x)\right)}_{II}$$

Question 3 The gradient of L w.r.t. w_k:

$$\frac{\partial L}{\partial w_k} = \sum_{i=1}$$

$$I: \frac{\partial I}{\partial w_k} = X$$

$$II: \frac{\partial II}{\partial w_k} = \frac{\exp(w_i^T x)\, X}{1 + \sum_{l=1}^{k-1} \exp(w_l^T x)}$$

$$\therefore I + II: \frac{\partial L}{\partial w_k} = \sum_{i=1}^{n} \left(X - \left(\frac{\exp(w_i^T x)\, X}{1 + \sum_{l=1}^{k-1} \exp(w_l^T x)} \right) \right)$$

$$= \sum_{i=1}^{n} \left(X - P(Y = k \mid X = x_i) X \right)$$

$$= \sum_{i=1}^{n} \left(\left(\mathbb{1}_{(y=k)} P(Y = k \mid X = x_i) \right) X \right)$$ (In terms of Probabi

Question 4 :- With adding the regularization term :-

$$f(w_1, \dots, w_{k-1}) = \underbrace{L(w_1, \dots, w_{k-1})}_{\text{I}} - \underbrace{\frac{\lambda}{2} \sum_{l=1}^{k-1} \|w_l\|_2^2}_{\text{II}}$$

$$\frac{\partial f}{\partial w_k} :-$$

$$\text{I} :- \quad \frac{\partial I}{\partial w_k} = \sum_{i=1}^{n} \left(\left(1 - P(Y = k \mid X = x_i) \right) X \right)$$

$$\text{II} :- \quad \frac{\partial II}{\partial w_k} = -\lambda w_k$$

$$\therefore \text{I} + \text{II} :- \quad \frac{\partial f}{\partial w_k} = \sum_{i=1}^{n} \left(\left(1 - P(Y = k \mid X = x_i) \right) X \right) - \lambda w_k$$

Question 5 :- USPS Handwritten recognition digit dataset.

- Image size: 16 x 16
- For each digit (i.e. 0, 1, ..., 9) there are 600 train samples. 500 testing ones.

(a)

Continue in the next pages.