Chapter 9

Public Key Cryptography and RSA

Outline

- Public-key cryptosystems
- Applications for public-key cryptosystems
- Requirements for public-key cryptography
- Public-key cryptanalysis
- The RSA algorithm
  - Description of the algorithm
  - Computational aspects
  - Security of RSA

Terminology Related to Asymmetric Encryption

Asymmetric Keys
Two related keys, a public key and a private key that are used to perform complementary operations, such as encryption and decryption or signature generation and signature verification.

Public Key Certificate
A digital document issued and digitally signed by the private key of a Certification Authority that binds the name of a subscriber to a public key. The certificate indicates that the subscriber identified in the certificate has sole control and access to the corresponding private key.

Public Key (Asymmetric) Cryptographic Algorithm
A cryptographic algorithm that uses two related keys, a public key and a private key. The two keys have the property that deriving the private key from the public key is computationally infeasible.

Public Key Infrastructure (PKI)
A set of policies, processes, server platforms, software and workstations used for the purpose of administering certificates and public-private key pairs, including the ability to issue, maintain, and revoke public key certificates.

Source: *Glossary of Key Information Security Terms*, NIST IR 7298 [KISS06]

Misconceptions Concerning Public-Key Encryption

- Public-key encryption is more secure from cryptanalysis than symmetric encryption
  - Security of any encryption scheme depends on:
    - the length of the **key**
    - the **computational** work involved in breaking a cipher
- Public-key encryption is a general-purpose technique that has made symmetric encryption obsolete
  - Complexity of public-key limits its rule to **key-management** and **signature** application.

Principles of Public-Key Cryptosystems

- Public-key cryptography addresses two of the most difficult problems associated with symmetric encryption:

Key distribution

- How to have secure communications in general without having to trust a key distribution center (KDC) with your key

Digital signatures

- How to verify that a message comes intact from the claimed sender

- Whitfield Diffie and Martin Hellman from Stanford University achieved a breakthrough in 1976 by coming up with a method that addressed both problems and was radically different from all previous approaches to cryptography

Public-Key Cryptosystems

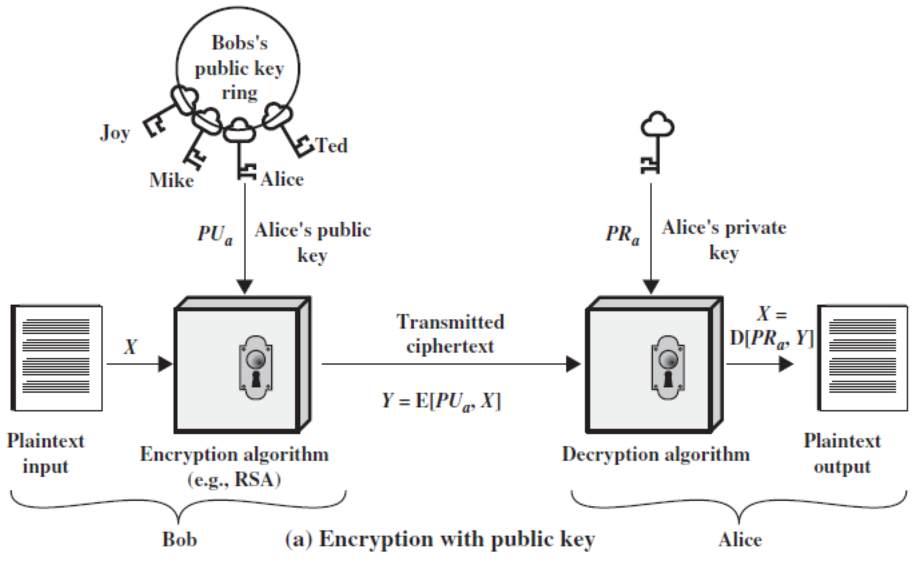
- A public-key encryption scheme has six ingredients:

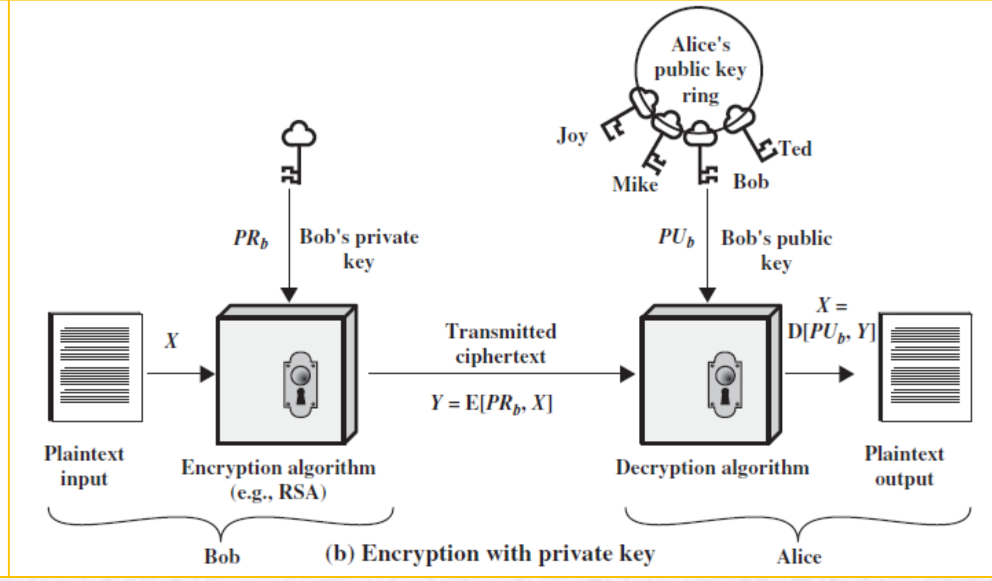
Plaintext	Encryption algorithm	Public key	Private key	Ciphertext	Decryption algorithm
The readable message or data that is fed into the algorithm as input	Performs various transforma-tions on the plaintext	Used for encryption or decryption	Used for encryption or decryption	The scrambled message produced as output	Accepts the ciphertext and the matching key and produces the original plaintext

Public-Key (Asymmetric) Algorithms Overview

- Public-Key (Asymmetric) algorithms is based on **two related keys**: one for encryption, another for decryption.
- These algorithms have the following important characteristic:
  - It is **computationally infeasible** to determine the decryption key given only knowledge of the cryptographic algorithm and the encryption key.
  - **Either** of the two related keys can be used for encryption, with the other used for decryption.

Public-Key Cryptography





Confidentiality

Authentication

The essential steps are the following.

1. Each user generates a pair of keys (public, private) to be used for the encryption and decryption of messages.
2. Each user places one of the two keys in a public register or other accessible file. This is the public key. The companion key is kept private. Each user maintains a collection of public keys obtained from others.

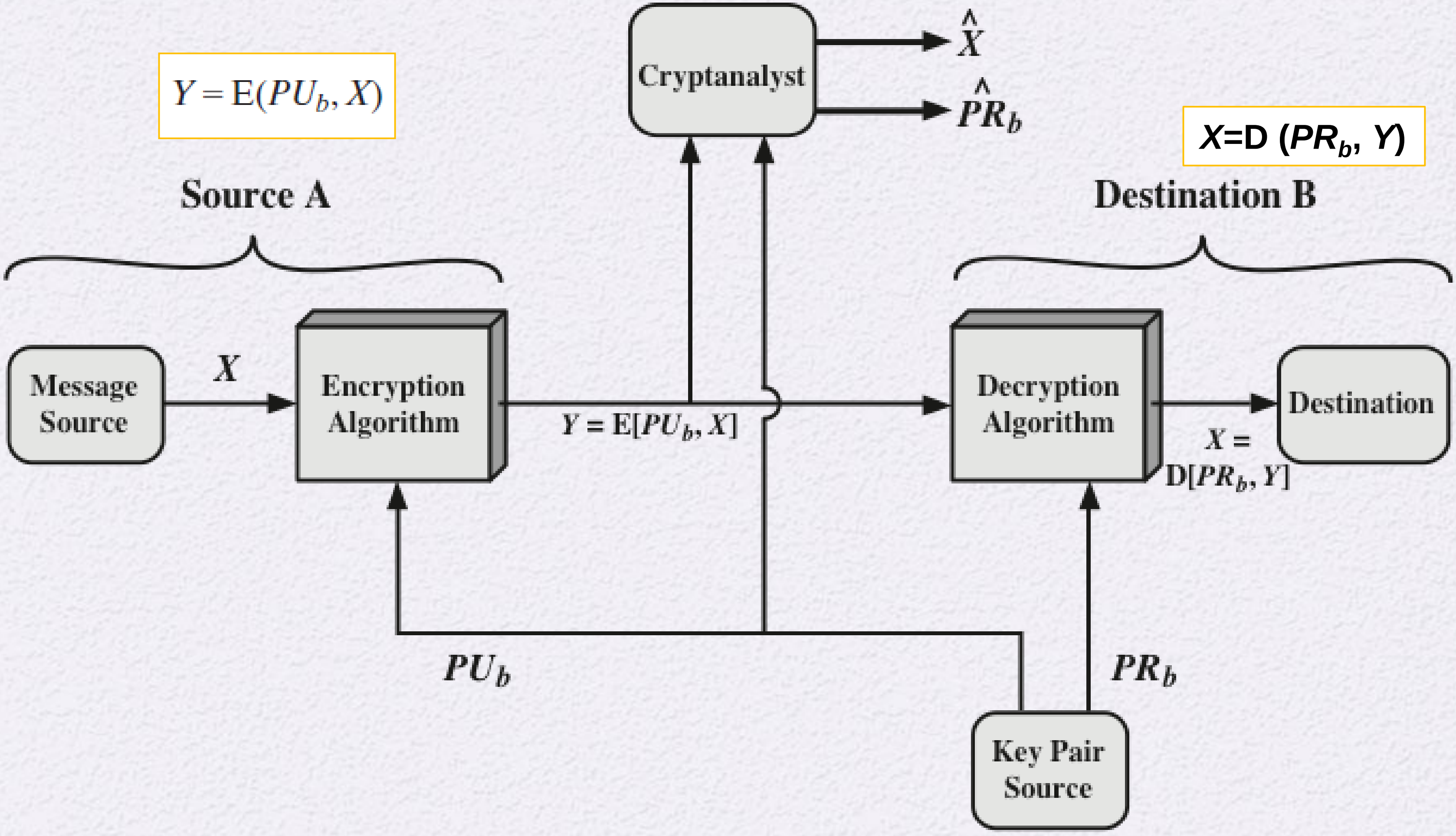
Public-Key Cryptography: essential steps, cont.

- At any time, a system can change its private key and publish the companion public key to replace its old public key.
- Notation:
  - Y = E(**PU_b**, X) : Y is the encryption of input X using **pubic key for "b"**
  - Z = E(**PR_a**, Y) : Z is the encryption of input X using **private key for "a"**

Conventional (symmetric) vs. Public-Key (asymmetric) Encryption

Conventional Encryption	Public-Key Encryption
Needed to Work: 1. The same algorithm with the same key is used for encryption and decryption. 2. The sender and receiver must share the algorithm and the key. *Needed for Security:* 1. The key must be kept secret. 2. It must be impossible or at least impractical to decipher a message if the key is kept secret. 3. Knowledge of the algorithm plus samples of ciphertext must be insufficient to determine the key.	*Needed to Work:* 1. One algorithm is used for encryption and a related algorithm for decryption with a pair of keys, one for encryption and one for decryption. 2. The sender and receiver must each have one of the matched pair of keys (not the same one). *Needed for Security:* 1. One of the two keys must be kept secret. 2. It must be impossible or at least impractical to decipher a message if one of the keys is kept secret. 3. Knowledge of the algorithm plus one of the keys plus samples of ciphertext must be insufficient to determine the other key.

Public-Key Cryptosystem: Secrecy

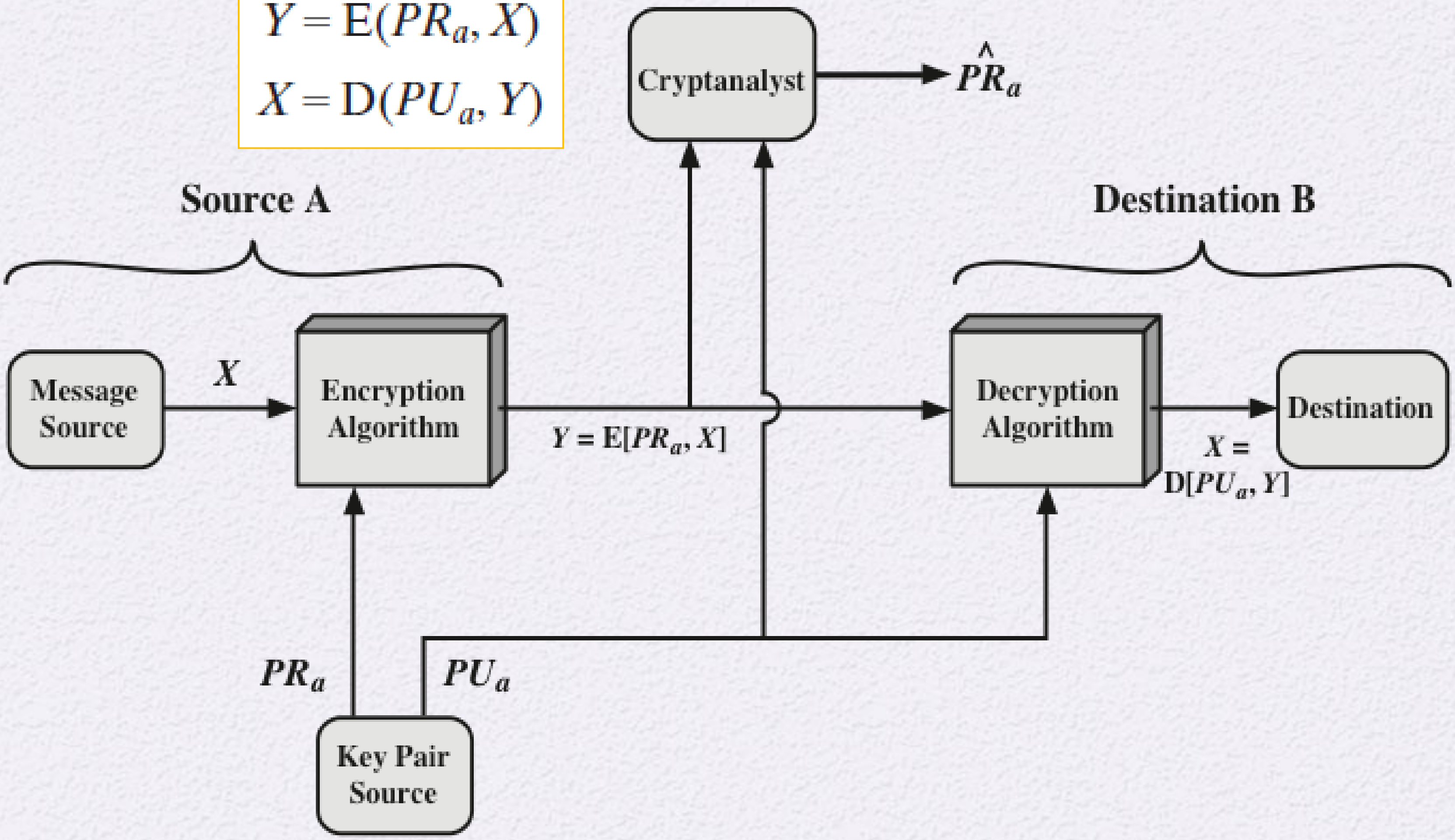


If **A** wishes to send a confidential message to **B**:

1. **A** encrypts the message using **B**'s public key.
2. When **B** receives the message, it decrypts message using **B**'s private key. No other recipient can decrypt the message because only **B** knows **B**'s private key.

Public-Key Cryptosystem: Authentication

$$Y = E(PR_a, X)$$
$$X = D(PU_a, Y)$$



Need to verify that message received by B is from A:

1. **A** encrypts the message using **A**'s private key.
2. When **B** receives the message, it decrypts message using his **A**'s public key.
   Any one has **A**'s public key can decrypt message. No confidentiality is provided.

Public-Key Cryptosystem: Authentication and Secrecy

$$Z = \mathrm{E}(PU_b, \mathrm{E}(PR_a, X))$$

$$X = \mathrm{D}(PU_a, \mathrm{D}(PR_b, Z))$$

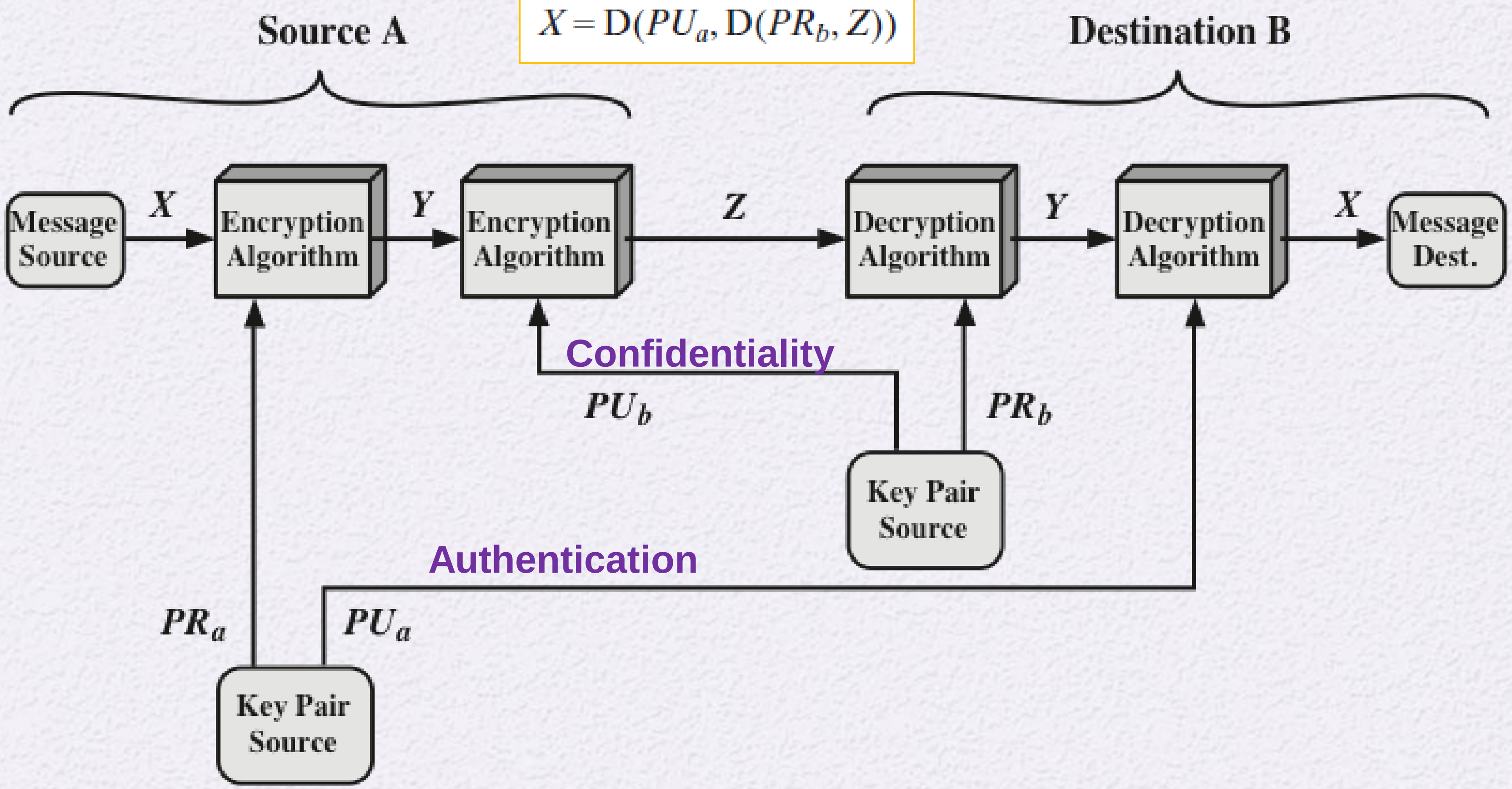


Figure 9.4 Public-Key Cryptosystem: Authentication and Secrecy

Applications for Public-Key Cryptosystems

- Public-key cryptosystems can be classified into three categories:

Category	How Keys are used
Encryption/decryption (confidentiality)	The sender encrypts a message with the recipient's public key
Digital signature (authentication)	The sender "signs" a message with its private key
Key exchange	Two sides cooperate to exchange a session key. Several different approaches are possible, involving the private key(s) of one or both parties.

- Some algorithms are suitable for all three applications, whereas others can be used only for one or two

Public-Key Algorithms and Applications

Algorithm	Encryption/Decryption	Digital Signature	Key Exchange
RSA	Yes	Yes	Yes
Elliptic Curve	Yes	Yes	Yes
Diffie-Hellman	No	No	Yes
DSS	No	Yes	No

RSA: acronym stands for Rivest, Shamir, and Adelman, the inventors of the technique.
DSS: Digital Signature Standard.

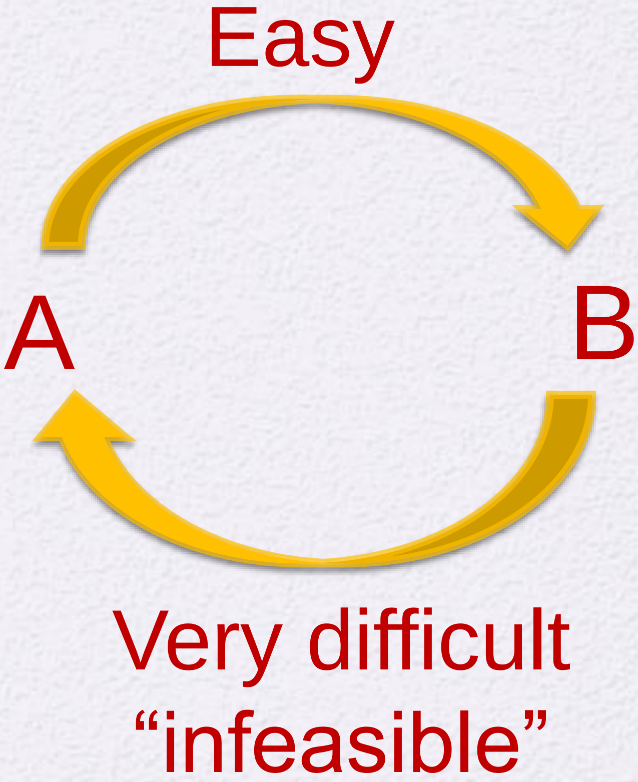
Public-Key Requirements

- Conditions that these algorithms must fulfill:
  - **Computationally easy** to do the followings:
    - It is computationally easy for a party B to **generate a pair of keys** (public-key PU_b, private key PR_b)
    - It is computationally easy for a sender A, knowing the public key and the message to be **encrypted**, to generate the corresponding ciphertext
    - It is computationally easy for the receiver B to **decrypt** the resulting ciphertext using the private key to recover the original message
  - **Computationally infeasible** to do the followings
    - It is computationally infeasible for an adversary, knowing the public key, to **determine the private key**
    - It is computationally infeasible for an adversary, knowing the public key and a ciphertext, to **recover the original message**
  - The two keys can be applied in either order

Trapdoor Function

In RSA:

- A→B
  - A : start with p, q
  - B :
    - $n = p \times q$
    - Calculate: e, d
- B →A
  - Given: e, n
  - Very hard to determine p, q and d.



Public-Key Requirements

- Need a trap-door one-way function
  - A one-way function is one that maps a domain into a range such that every function value has a unique inverse, with the condition that the calculation of the function is easy, whereas **the calculation of the inverse is infeasible**
    - $Y = f(X)$ easy
    - $X = f^{-1}(Y)$ infeasible
- A trap-door one-way function is a family of invertible functions f_k, such that
  - $Y = f_k(X)$ easy, if k and X are known
  - $X = f_k^{-1}(Y)$ easy, if k and Y are known
  - $X = f_k^{-1}(Y)$ infeasible, if Y known but k not known
- A practical public-key scheme depends on a suitable trap-door one-way function

Public-Key Cryptanalysis (read)

- A public-key encryption scheme is vulnerable to a brute-force attack
  - Countermeasure: use large keys
  - Key size must be small enough for practical encryption and decryption
  - Key sizes that have been proposed result in encryption/decryption speeds that are too slow for general-purpose use
  - Public-key encryption is currently confined to key management and signature applications
- Another form of attack is to find some way to compute the private key given the public key
  - To date it has not been mathematically proven that this form of attack is infeasible for a particular public-key algorithm
- Finally, there is a probable-message attack
  - This attack can be thwarted by appending some random bits to simple messages

Rivest-Shamir-Adleman (RSA) Algorithm

- Developed in 1977 at MIT by Ron Rivest, Adi Shamir & Len Adleman
- Most widely used general-purpose approach to public-key encryption
- Is a cipher in which the plaintext and ciphertext are integers between 0 and $n - 1$ for some n
  - A typical size for n is 1024 bits, or 309 decimal digits. ($2^{1024} \approx 10^{309}$)

RSA Algorithm

- RSA makes use of an expression with exponentials
- Plaintext is encrypted in blocks with each block having a binary value less than some number ***n***
  - Block size is ***i*** bits, where $2^i < n \le 2^{i+1}$.
- Encryption and decryption are of the following form, for some plaintext block ***M*** and ciphertext block **C**

  Encryption: $C = M^e \bmod n$

  Decryption: $M = C^d \bmod n = (M^e)^d \bmod n = M^{ed} \bmod n$
- Both sender and receiver must know the value of ***n***
- The sender knows the value of ***e***

  ➔ **public key of *PU={e,n}***
- Only the receiver knows the value of ***d***

  ➔ **private key of *PR={d,n}***

Algorithm Requirements

- For this algorithm to be satisfactory for public-key encryption, the following requirements must be met:

1. It is possible to find values of *e*, *d*, *n* such that $M^{ed} \bmod n = M$ for all $M < n$
2. It is relatively easy to calculate $M^e \bmod n$ and $C^d \bmod n$ for all values of $M < n$
3. It is infeasible to determine *d* given *e* and *n*

Public-Key Requirements

Key Generation by Alice

Select p, q	p and q both prime, $p \neq q$
Calculate $n = p \times q$	
Calculate $\phi(n) = (p-1)(q-1)$	
Select integer e	$\gcd(\phi(n), e) = 1;\ 1 < e < \phi(n)$
Calculate d	$d \equiv e^{-1} \pmod{\phi(n)}$
Public key	$PU = \{e, n\}$
Private key	$PR = \{d, n\}$

Encryption by Bob with Alice's Public Key

Plaintext:	$M < n$
Ciphertext:	$C = M^e \bmod n$

Decryption by Alice with Alice's Private Key

Ciphertext:	C
Plaintext:	$M = C^d \bmod n$

RSA: $M = C^d = M^{ed}$

➔ Prove: $M = M^{ed}$

$e \times d \equiv 1 \pmod{\phi(n)}$

$\phi(n) = (p-1)(q-1)$

$e \times d = k \times (p-1)(q-1) + 1$

<u>If we can show that:</u>

$M^{ed} \bmod \mathbf{p} = M$ *and*

$M^{ed} \bmod \mathbf{q} = M$

then by Chinese remainder theorem:

$M^{ed} \bmod \mathbf{n} = M$

$M = M^{ed} \bmod \mathbf{p} = M^{k \times (p-1)(q-1) + 1} \bmod \mathbf{p}$

$= M^{k \times (p-1)(q-1)} \times M \bmod \mathbf{p}$

From Fermat's theorem:

$M^{(p-1)} \bmod \mathbf{p} = 1$

$M = M^{ed} = (1)^{k(q-1)} \times M \bmod n$

$= (1) \times M \bmod n$

$= M$

Similarly,

$M = M^{ed} \bmod \mathbf{q} = M^{k \times (p-1)(q-1) + 1} \bmod \mathbf{q}$

$= M$

➔ $M^{ed} \bmod \mathbf{n} = M$

RSA Algorithm

RSA Proof

Relation between: *e* and *d*

$$M^{ed} \bmod n = M$$

The preceding relationship holds if e and d are multiplicative inverses modulo $\phi(n)$, where $\phi(n)$ is the Euler totient function. It is shown in Chapter 8 that for p, q prime, $\phi(pq) = (p - 1)(q - 1)$. The relationship between e and d can be expressed as

1

$$ed \bmod \phi(n) = 1 \tag{9.1}$$

2

This is equivalent to saying

$$ed \equiv 1 \bmod \phi(n)$$
$$d \equiv e^{-1} \bmod \phi(n)$$

That is, e and d are multiplicative inverses mod $\phi(n)$. Note that, according to the rules of modular arithmetic, this is true only if d (and therefore e) is relatively prime to $\phi(n)$. Equivalently, $\gcd(\phi(n), d) = 1$. See Appendix R for a proof that

RSA Numerical Exam

1. Select two prime numbers, $p = 17$ and $q = 11$.
2. Calculate $n = pq = 17 \times 11 = 187$.
3. Calculate $\phi(n) = (p-1)(q-1) = 16 \times 10 = 160$.
4. Select e such that e is relatively prime to $\phi(n) = 160$ and less than $\phi(n)$; we choose $e = 7$.
5. Determine d such that $de \equiv 1 \pmod{160}$ and $d < 160$. The correct value is $d = 23$, because $23 \times 7 = 161 = (1 \times 160) + 1$; d can be calculated using the extended Euclid's algorithm (Chapter 4).

The resulting keys are public key $PU = \{7, 187\}$ and private key $PR = \{23, 187\}$. The example shows the use of these keys for a plaintext input of $M = 88$. For encryption, we need to calculate $C = 88^7 \bmod 187$. Exploiting the properties of modular arithmetic, we can do this as follows.

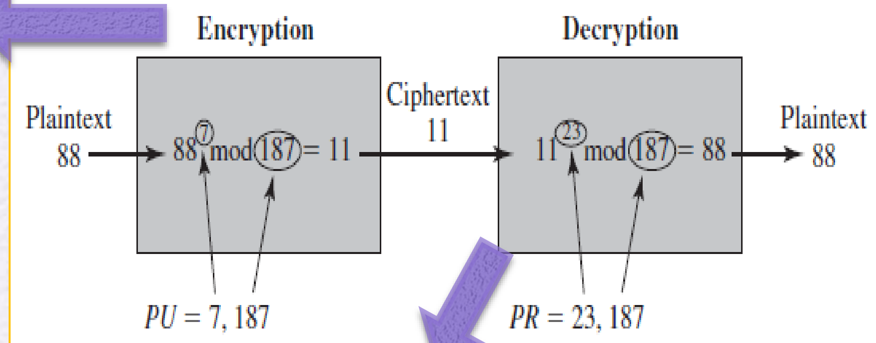
$$88^7 \bmod 187 = [(88^4 \bmod 187) \times (88^2 \bmod 187) \times (88^1 \bmod 187)] \bmod 187$$

$$88^1 \bmod 187 = 88$$

$$88^2 \bmod 187 = 7744 \bmod 187 = 77$$

$$88^4 \bmod 187 = 59{,}969{,}536 \bmod 187 = 132$$

$$88^7 \bmod 187 = (88 \times 77 \times 132) \bmod 187 = 894{,}432 \bmod 187 = 11$$



For decryption, we calculate $M = 11^{23} \bmod 187$:

$$11^{23} \bmod 187 = [(11^1 \bmod 187) \times (11^2 \bmod 187) \times (11^4 \bmod 187) \times (11^8 \bmod 187) \times (11^8 \bmod 187)] \bmod 187$$

$$11^1 \bmod 187 = 11$$

$$11^2 \bmod 187 = 121$$

$$11^4 \bmod 187 = 14{,}641 \bmod 187 = 55$$

$$11^8 \bmod 187 = 214{,}358{,}881 \bmod 187 = 33$$

$$11^{23} \bmod 187 = (11 \times 121 \times 55 \times 33 \times 33) \bmod 187 = 79{,}720{,}245 \bmod 187 = 88$$

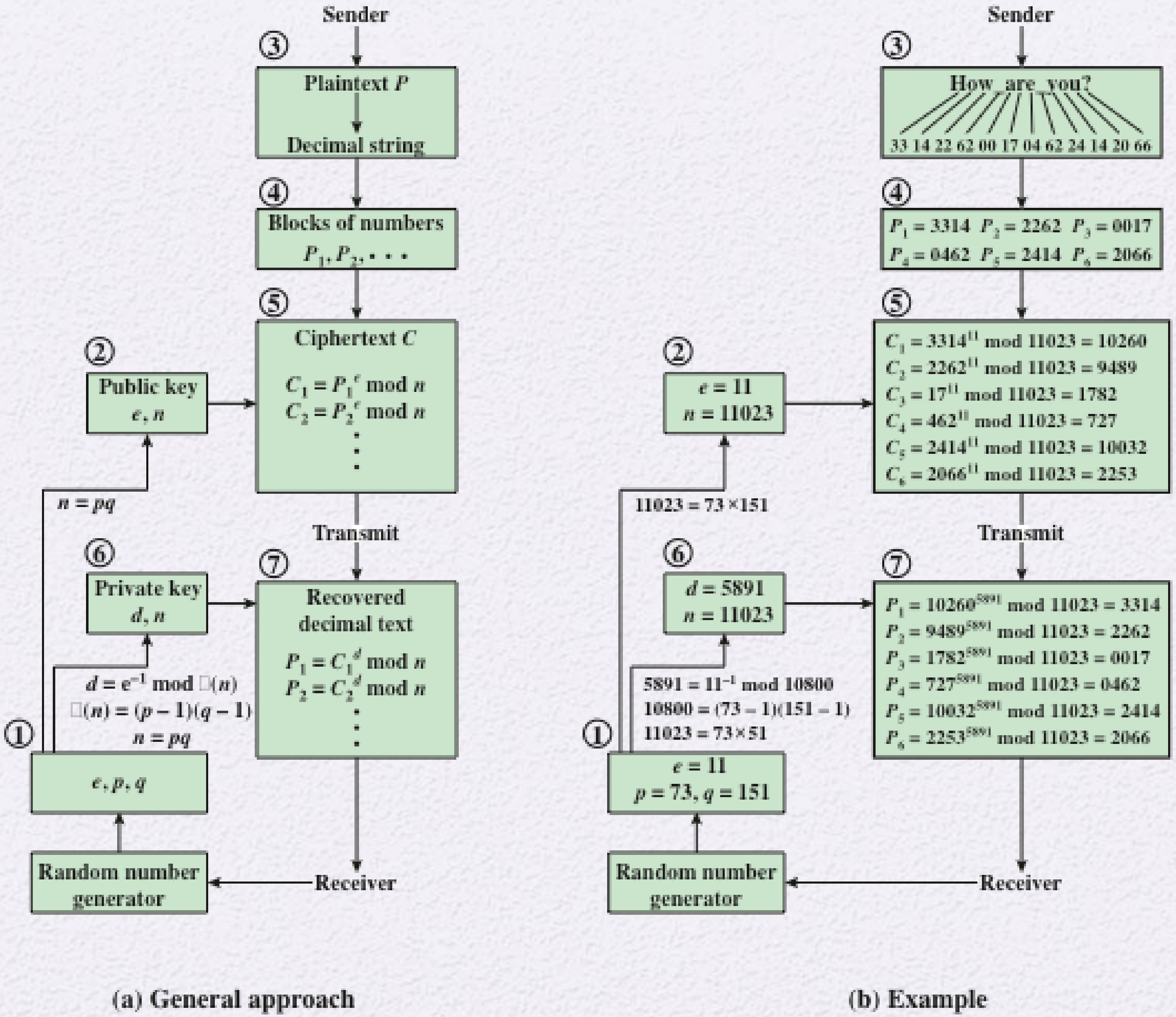


Figure 9.7 RSA Processing of Multiple Blocks

Exponentiation in Modular Arithmetic

- Both encryption and decryption in RSA involve raising an integer to an integer power, mod *n*
- Can make use of a property of modular arithmetic:

  $[(a \bmod n) \times (b \bmod n)] \bmod n = (a \times b) \bmod n$

- With RSA you are dealing with potentially large exponents so efficiency of exponentiation is a consideration

```
c ← 0; f ← 1
for i  ← k downto 0
      do   c ← 2 × c
           f ← (f × f) mod n
      if   b_i = 1
           then c ←  c + 1
                f ← (f × a) mod n
return f
```

Note: The integer b is expressed as a binary number $b_k b_{k-1} \ldots b_0$

Also, note: C is not needed, it is just added for clariy.

Figure 9.8 Algorithm for Computing a^b mod n

Example to Compute : 7^{560} mod 561

Result of the Fast Modular Exponentiation Algorithm for a^b mod n, where $a = 7$, $b = 560 = 1000110000$, and $n = 561$

```
c ← 0; f ← 1
for i  ← k downto 0
      do   c ← 2 × c
           f ← (f × f) mod n
      if   bi = 1
           then c ←  c + 1
                f ← (f × a) mod n
return f
```

b_i	1	0	0	0	1	1	0	0	0	0
f	$(1)^2 \equiv 1$ $1 \times 7 \equiv 7$	$(7)^2 \equiv 49$	$(49)^2 \equiv$ 157	$(157)^2 \equiv$ 526	$(526)^2 \equiv$ 103 $103 \times 7 \equiv$ 160	$(160)^2 \equiv$ 355 $355 \times 7 \equiv$ 241	$(241)^2 \equiv$ 298	$(298)^2 \equiv$ 166	$(166)^2 \equiv$ 67	$(67)^2 \equiv 1$

Efficient Operation Using the Public Key

- To speed up the operation of the RSA algorithm using the public key, a specific choice of *e* is usually made
- The most common choice is 65537 ($2^{16} + 1$)
  - Two other popular choices are e=3 and e=17
  - Each of these choices has only two 1 bits, so the number of multiplications required to perform exponentiation is minimized
  - With a very small public key, such as $e = 3$, RSA becomes vulnerable to a simple attack

Efficient Operation Using the Private Key

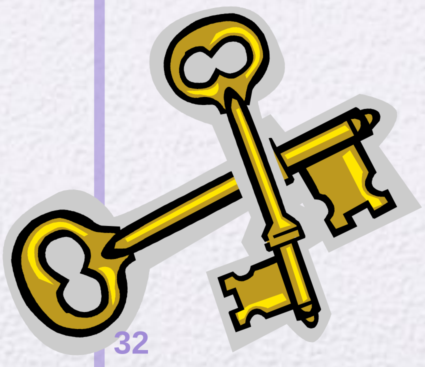
- Decryption uses exponentiation to power *d*
  - A small value of *d* is vulnerable to a brute-force attack and to other forms of cryptanalysis

- Can use the Chinese Remainder Theorem (CRT) to speed up computation
  - The quantities *d* mod (p – 1) and *d* mod (q – 1) can be precalculated
  - End result is that the calculation is approximately four times as fast as evaluating $M = C^d \bmod n$ directly

Key Generation

- Before the application of the public-key cryptosystem each participant must generate a pair of keys:
  - Determine two prime numbers *p* and *q*
  - Select either *e* or *d* and calculate the other

- Because the value of $n = pq$ will be known to any potential adversary, primes must be chosen from a sufficiently large set
  - The method used for finding large primes must be reasonably efficient



Procedure for Picking a Prime Number

- Pick an odd integer *n* at random
- Pick an integer *a* < *n* at random
- Perform the probabilistic primality test with *a* as a parameter. If *n* fails the test, reject the value *n* and go to step 1
- If *n* has passed a sufficient number of tests, accept *n*; otherwise, go to step 2

The Security of RSA

Five possible approaches to attacking RSA are:

Brute force

- Involves trying all possible private keys

Mathematical attacks

- There are several approaches, all equivalent in effort to factoring the product of two primes

Timing attacks

- These depend on the running time of the decryption algorithm

Hardware fault-based attack

- This involves inducing hardware faults in the processor that is generating digital signatures

Chosen ciphertext attacks

- This type of attack exploits properties of the RSA algorithm

Rest is Reading Material

Factoring Problem

- We can identify three approaches to attacking RSA mathematically:
  - Factor n into its two prime factors. This enables calculation of $ø(n) = (p - 1) \times (q - 1)$, which in turn enables determination of $d = e^{-1} \ (mod\ ø(n))$
  - Determine ø(n) directly without first determining p and q. Again this enables determination of $d = e^{-1} \ (mod\ ø(n))$
  - Determine d directly without first determining ø(n)

Number of Decimal Digits	Number of Bits	Date Achieved
100	332	April 1991
110	365	April 1992
120	398	June 1993
129	428	April 1994
130	431	April 1996
140	465	February 1999
155	512	August 1999
160	530	April 2003
174	576	December 2003
200	663	May 2005
193	640	November 2005
232	768	December 2009

Table 9.5 Progress in RSA Factorization

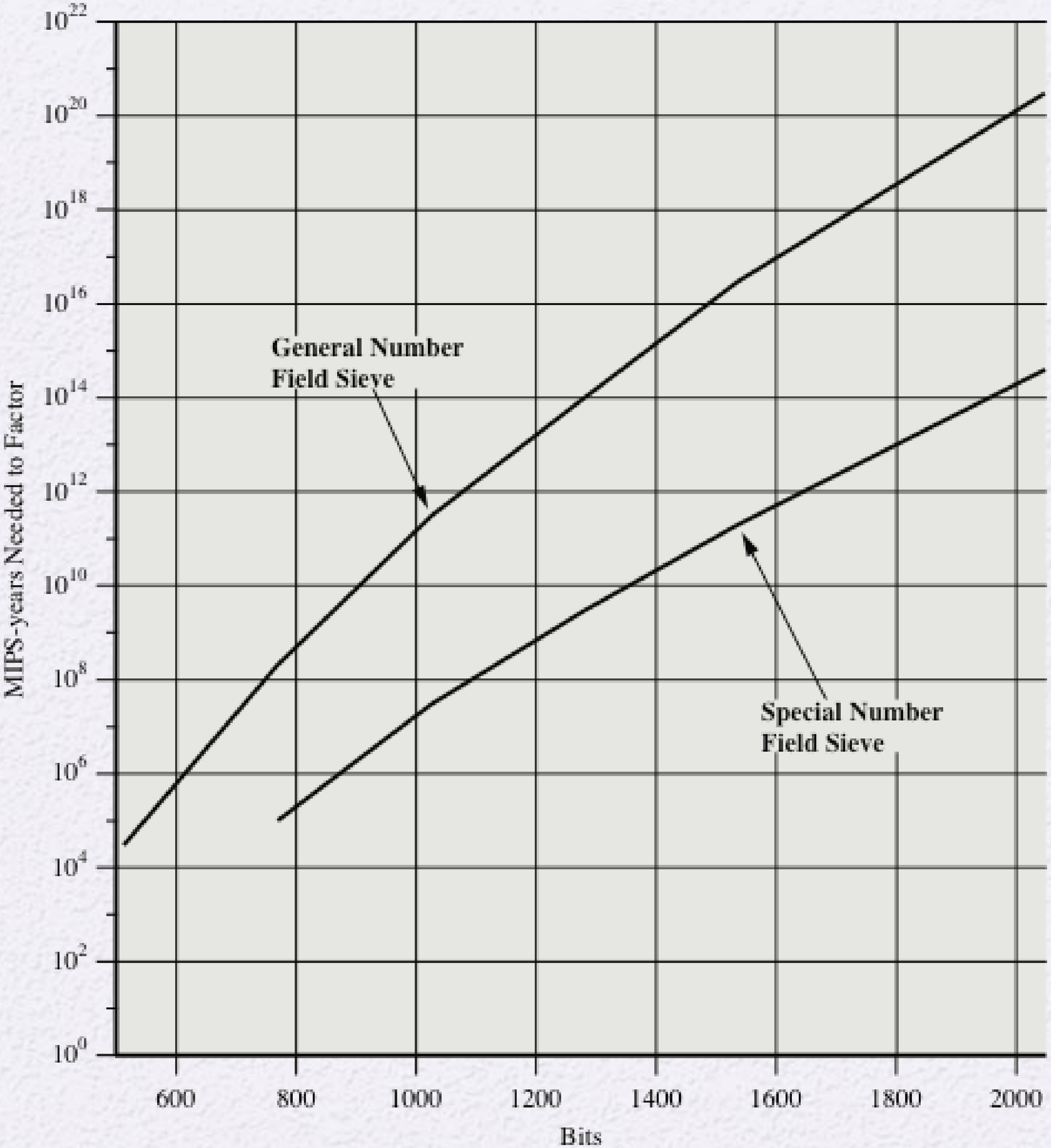


Figure 9.9 MIPS-years Needed to Factor

Timing Attacks

- Paul Kocher, a cryptographic consultant, demonstrated that a snooper can determine a private key by keeping track of how long a computer takes to decipher messages
- Are applicable not just to RSA but to other public-key cryptography systems
- Are alarming for two reasons:
  - It comes from a completely unexpected direction
  - It is a ciphertext-only attack

Countermeasures

Constant exponentiation time

- Ensure that all exponentiations take the same amount of time before returning a result; this is a simple fix but does degrade performance

Random delay

- Better performance could be achieved by adding a random delay to the exponentiation algorithm to confuse the timing attack

Blinding

- Multiply the ciphertext by a random number before performing exponentiation; this process prevents the attacker from knowing what ciphertext bits are being processed inside the computer and therefore prevents the bit-by-bit analysis essential to the timing attack

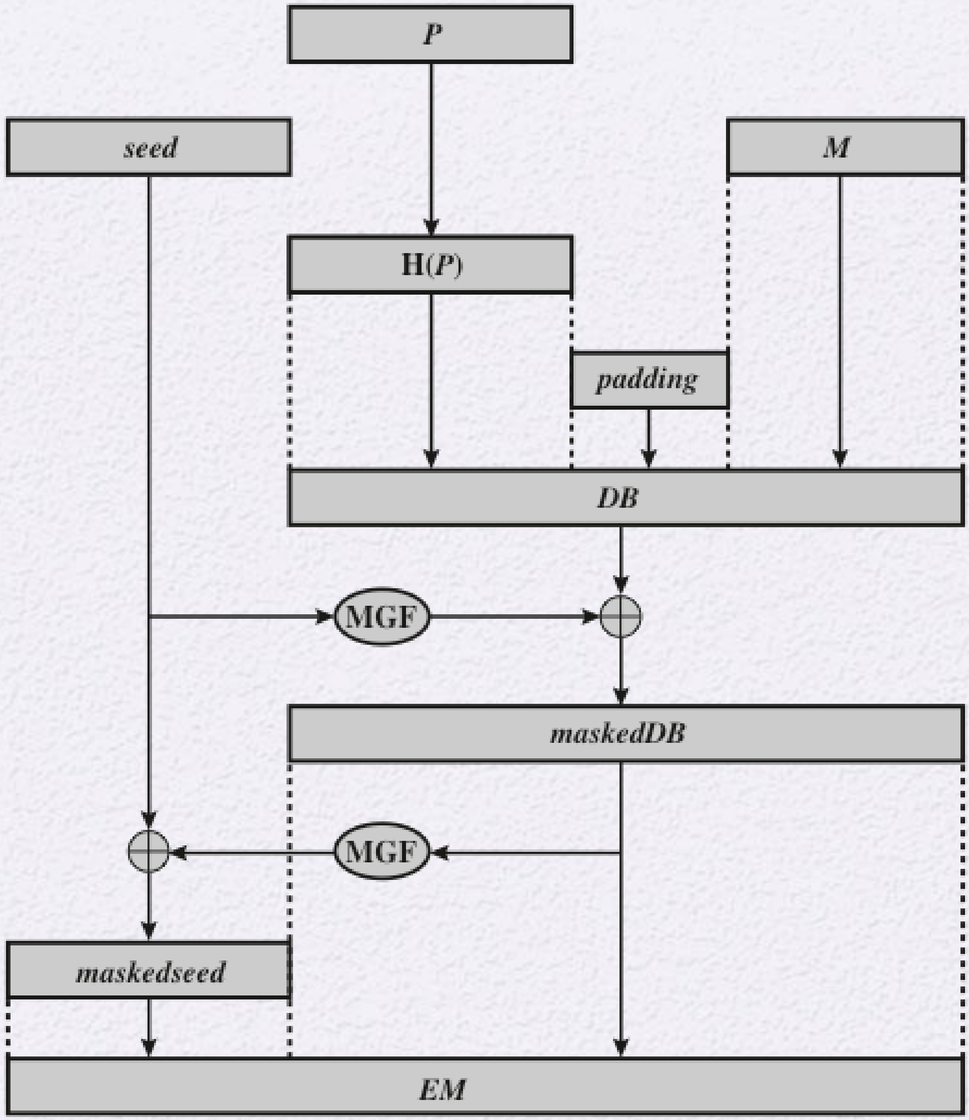
Fault-Based Attack

- An attack on a processor that is generating RSA digital signatures
    - Induces faults in the signature computation by reducing the power to the processor
    - The faults cause the software to produce invalid signatures which can then be analyzed by the attacker to recover the private key
- The attack algorithm involves inducing single-bit errors and observing the results
- While worthy of consideration, this attack does not appear to be a serious threat to RSA
    - It requires that the attacker have physical access to the target machine and is able to directly control the input power to the processor

Chosen Ciphertext Attack (CCA)

- The adversary chooses a number of ciphertexts and is then given the corresponding plaintexts, decrypted with the target's private key
  - Thus the adversary could select a plaintext, encrypt it with the target's public key, and then be able to get the plaintext back by having it decrypted with the private key
  - The adversary exploits properties of RSA and selects blocks of data that, when processed using the target's private key, yield information needed for cryptanalysis
- To counter such attacks, RSA Security Inc. recommends modifying the plaintext using a procedure known as *optimal asymmetric encryption padding* (OAEP)

Optimal Asymmetric Encryption Padding (OAEP)



P = encoding parameters
M = message to be encoded
H = hash function
DB = data block
MGF = mask generating function
EM = encoded message

Figure 9.10 Encryption Using Optimal Asymmetric Encryption Padding (OAEP)

Summary

- Public-key cryptosystems
- Applications for public-key cryptosystems
- Requirements for public-key cryptography
- Public-key cryptanalysis

- The RSA algorithm
  - Description of the algorithm
  - Computational aspects
  - Security of RSA