

Chapter 7

- 7.1 Measures of predictive accuracy
- 7.2 Information criteria and cross-validation
 - Instead of 7.2, read:
Vehtari, A., Gelman, A., Gabry, J. (2017). Practical Bayesian model evaluation using leave-one-out cross-validation and WAIC. *Statistics and Computing*. 27(5):1413–1432. [arXiv preprint](#).
- 7.3 Model comparison based on predictive performance
- 7.4 Model comparison using Bayes factors
- 7.5 Continuous model expansion / sensitivity analysis
- 7.5 Example (may be skipped)

Model assessment, selection and inference after selection

- Extra material at <https://avehtari.github.io/modelselection/>
 - Videos, Slides, Notebooks, References
- CV-FAQ
<https://avehtari.github.io/modelselection/CV-FAQ.html>

Predicting concrete quality



Predicting cancer recurrence

GIST Risk calculator

Tumor size (cm)

Mitotic count (per 50 HPFs*)

Tumor site

Tumor rupture

CALCULATE!

*HPF = high-power field of the microscope

[Show risk tables](#)

Made by

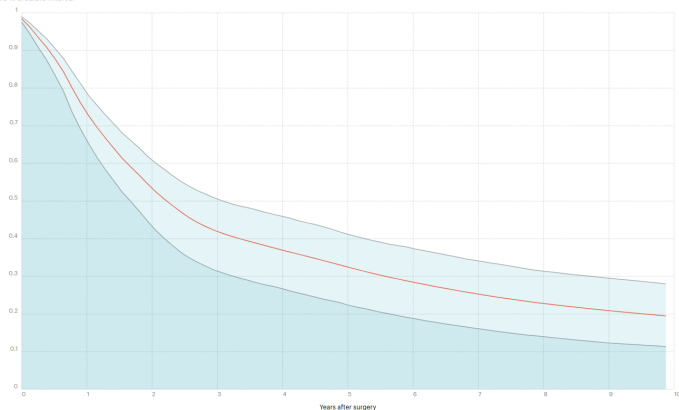
kaiku

Online platform for the future of data-driven
and personalized cancer care

Reaktor

Patients alive without recurrence [Show hazard](#)
90 % credible interval

10 year risk of GIST recurrence: 80%



Predictive performance

- ▶ True predictive performance is found out by using it to make predictions and comparing predictions to true observations
 - ▶ external validation

Predictive performance

- ▶ True predictive performance is found out by using it to make predictions and comparing predictions to true observations
 - ▶ external validation
- ▶ Expected predictive performance
 - ▶ approximates the external validation

Predictive performance

- ▶ We need to choose the utility/cost function
- ▶ Application specific utility/cost functions are important
 - ▶ eg. money, life years, quality adjusted life years, etc.

Predictive performance

- ▶ We need to choose the utility/cost function
- ▶ Application specific utility/cost functions are important
 - ▶ eg. money, life years, quality adjusted life years, etc.
- ▶ If are interested overall in the goodness of the predictive distribution, or we don't know (yet) the application specific utility, then good information theoretically justified choice is log-score

$$\log p(y^{\text{rep}} \mid y, M),$$

- What is cross-validation
 - Leave-one-out cross-validation (elpd_loo, p_loo)
 - Uncertainty in LOO (SE)
- When is cross-validation applicable?
 - data generating mechanisms and prediction tasks
 - leave-many-out cross-validation
- Fast cross-validation
 - PSIS and diagnostics in loo package (Pareto k, n_eff, Monte Carlo SE)
 - K-fold cross-validation
- Related methods (WAIC, *IC, BF)
- Model comparison and selection (elpd_diff, se)
- Model averaging with Bayesian stacking

Stan and loo package

Computed from 4000 by 20 log-likelihood matrix

	Estimate	SE
elpd_loo	-29.5	3.3
p_loo	2.7	1.0

Monte Carlo SE of elpd_loo is 0.1.

Pareto k diagnostic values:

		Count	Pct.	Min.	n_eff
(-Inf, 0.5]	(good)	18	90.0%	899	
(0.5, 0.7]	(ok)	2	10.0%	459	
(0.7, 1]	(bad)	0	0.0%	<NA>	
(1, Inf)	(very bad)	0	0.0%	<NA>	

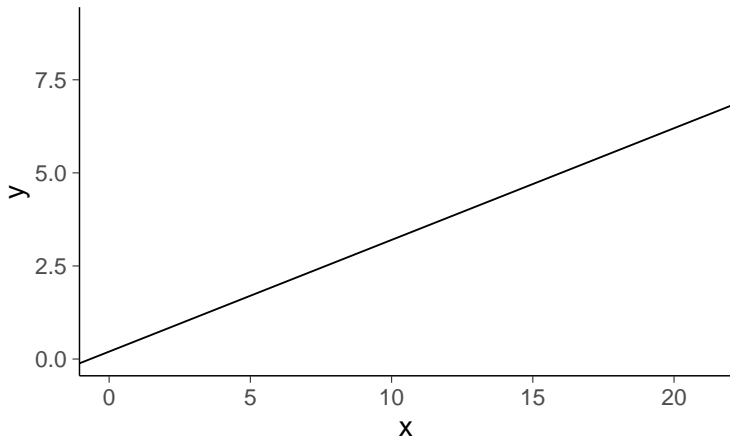
All Pareto k estimates are ok ($k < 0.7$).
See `help('pareto-k-diagnostic')` for details.

Model comparison:

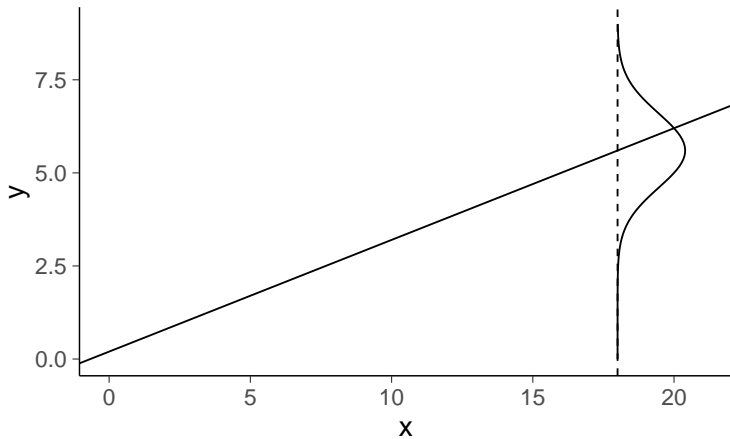
(negative 'elpd_diff' favors 1st model, positive favors 2nd)

elpd_diff	se
-0.2	0.1

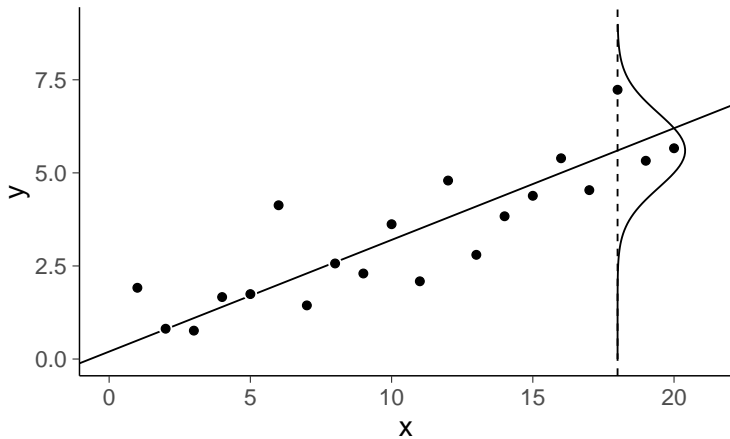
True mean $y = a + bx$



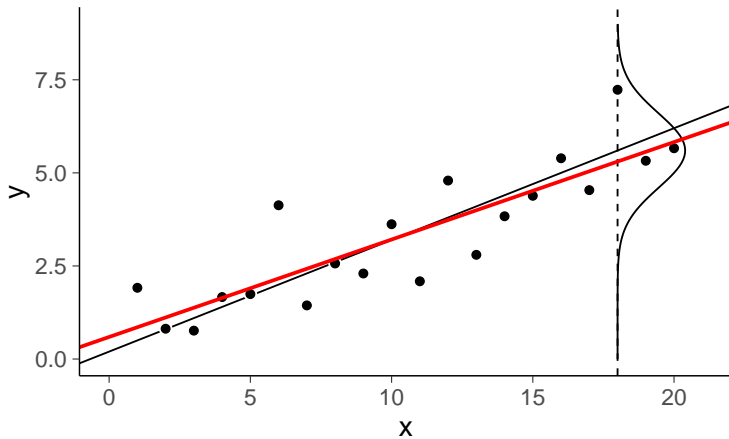
True mean and sigma



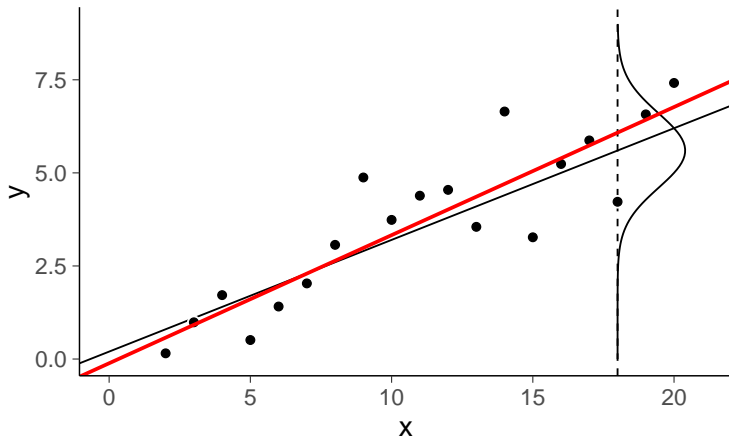
Data



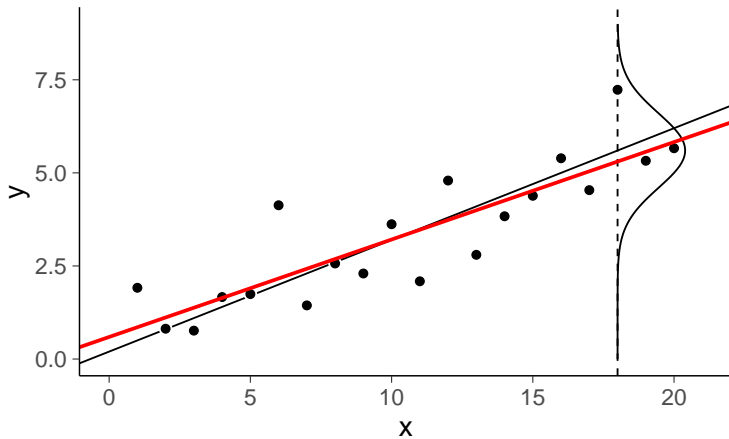
Posterior mean



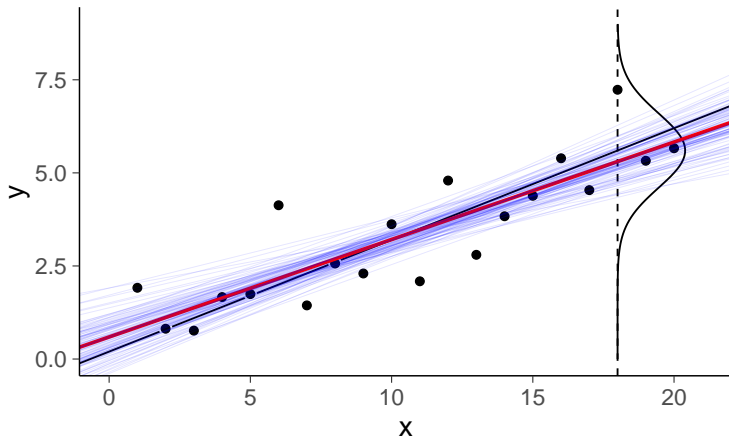
Posterior mean, alternative data realisation



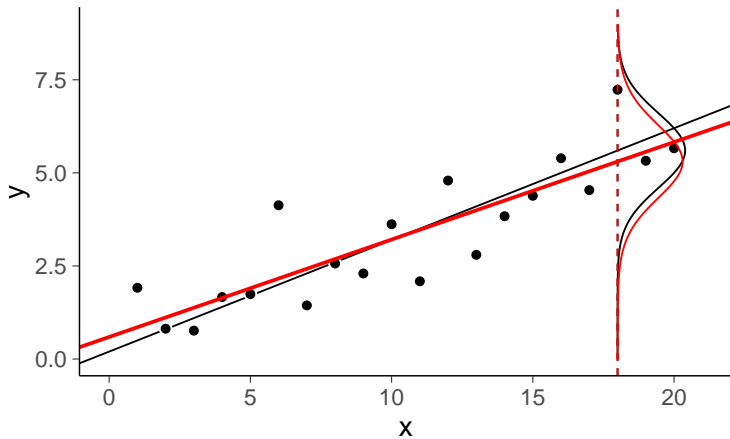
Posterior mean



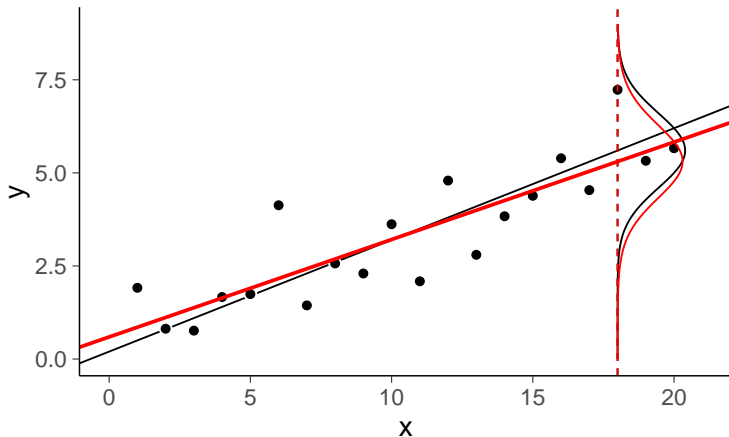
Posterior draws



Posterior predictive distribution

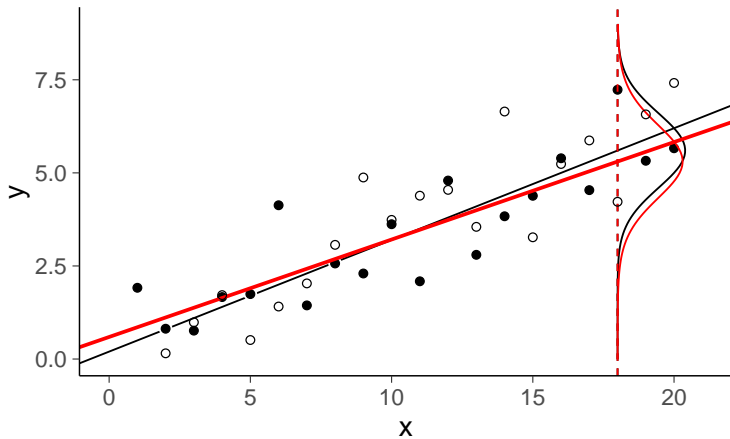


Posterior predictive distribution

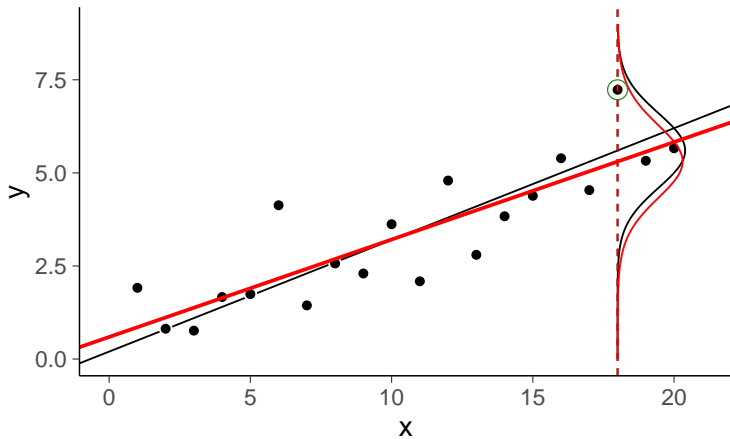


$$p(\tilde{y} \mid \tilde{x} = 18, x, y) = \int p(\tilde{y} \mid \tilde{x} = 18, \theta) p(\theta \mid x, y) d\theta$$

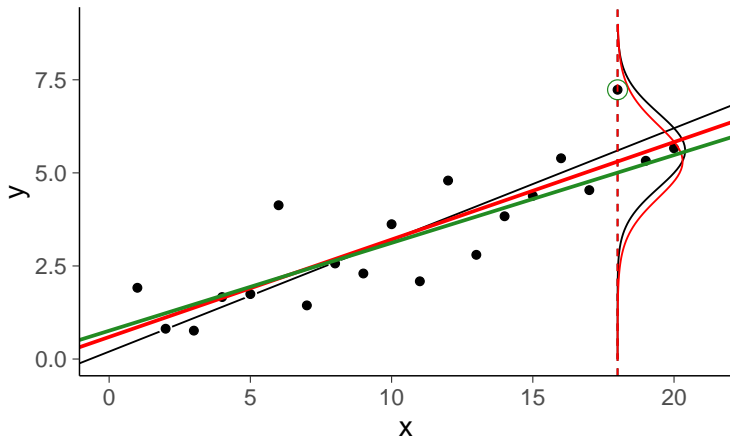
New data



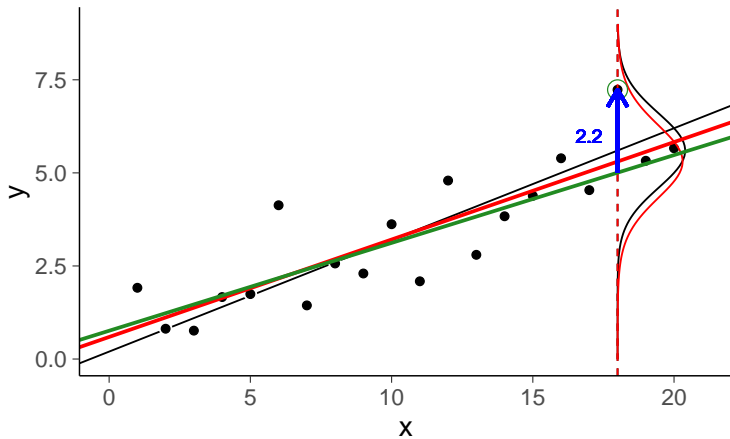
Posterior predictive distribution



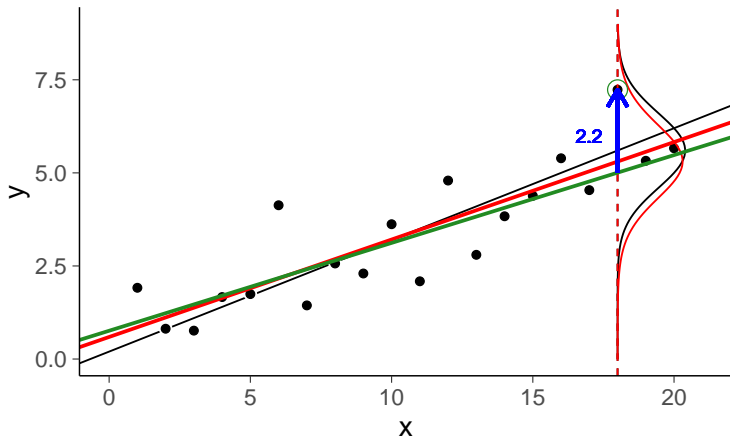
Leave-one-out mean



Leave-one-out residual

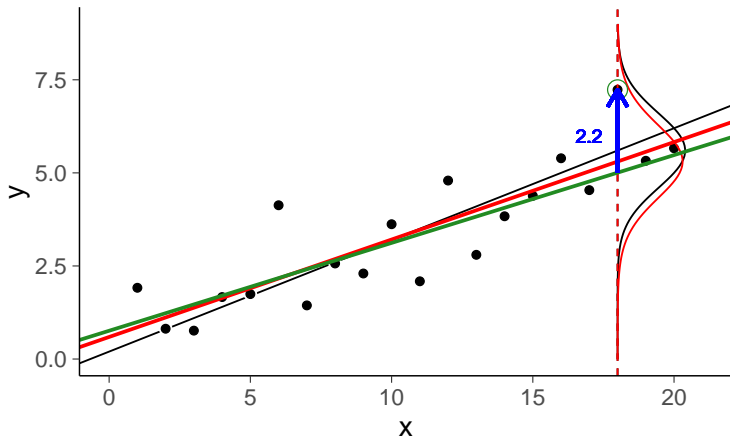


Leave-one-out residual



$$y_{18} - E[p(\tilde{y} \mid \tilde{x} = 18, x_{-18}, y_{-18})]$$

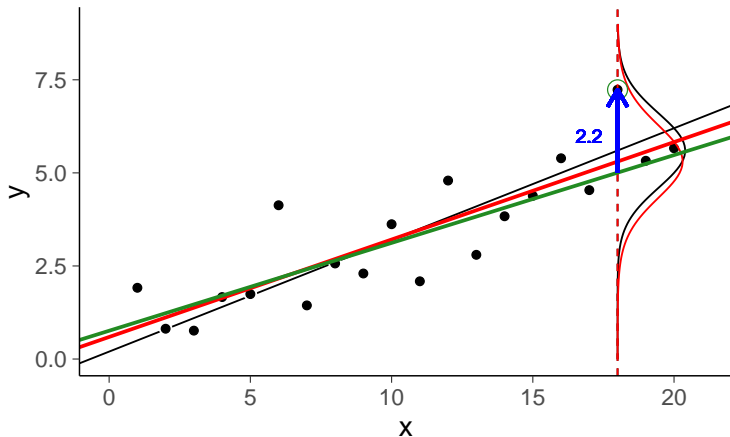
Leave-one-out residual



$$y_{18} - E[p(\tilde{y} | \tilde{x} = 18, x_{-18}, y_{-18})]$$

Can be used to compute, e.g., RMSE, R^2 , 90% error

Leave-one-out residual

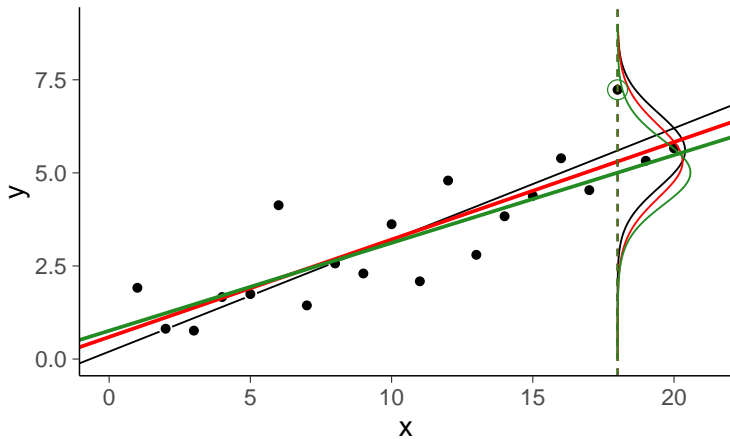


$$y_{18} - E[p(\tilde{y} \mid \tilde{x} = 18, x_{-18}, y_{-18})]$$

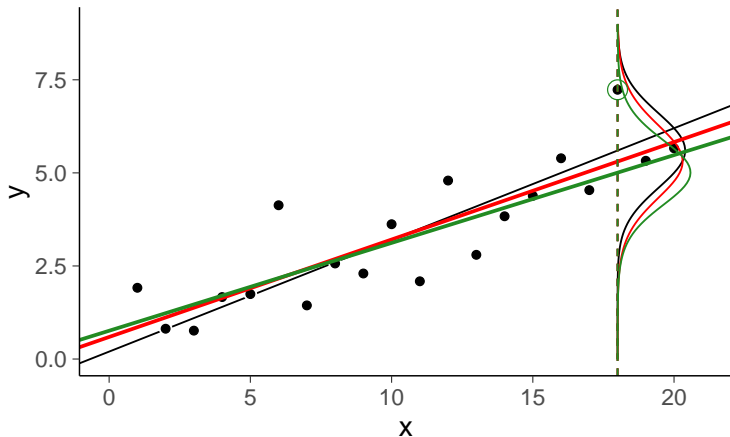
Can be use to compute, e.g., RMSE, R^2 , 90% error

See LOO- R^2 at avehtari.github.io/bayes_R2/bayes_R2.html

Leave-one-out predictive distribution

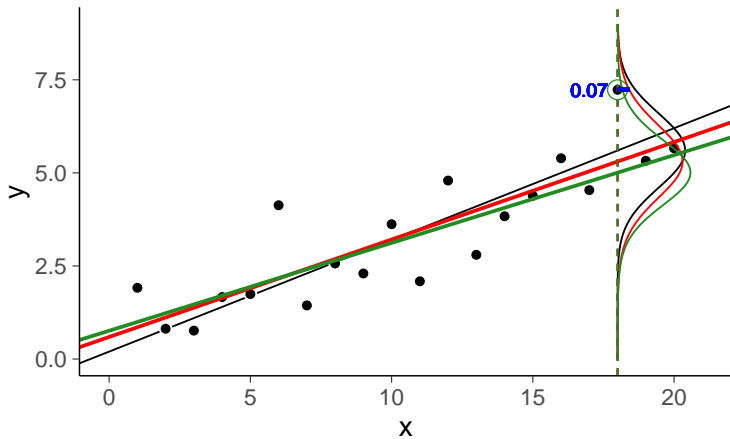


Leave-one-out predictive distribution

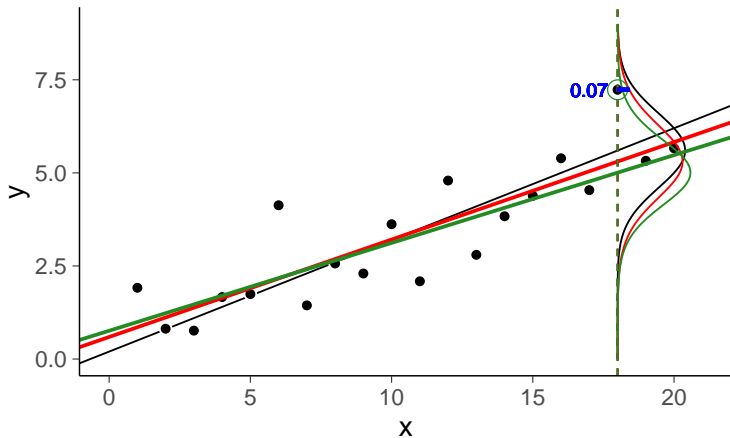


$$p(\tilde{y} \mid \tilde{x} = 18, x_{-18}, y_{-18}) = \int p(\tilde{y} \mid \tilde{x} = 18, \theta) p(\theta \mid x_{-18}, y_{-18}) d\theta$$

Posterior predictive density

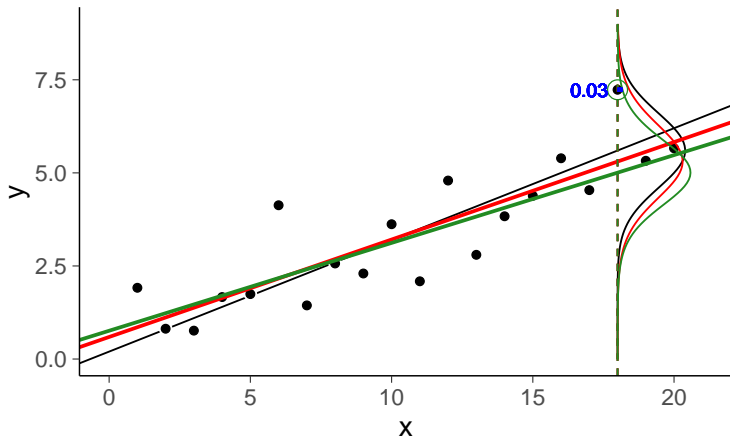


Posterior predictive density



$$p(\tilde{y} = y_{18} \mid \tilde{x} = 18, x, y) \approx 0.07$$

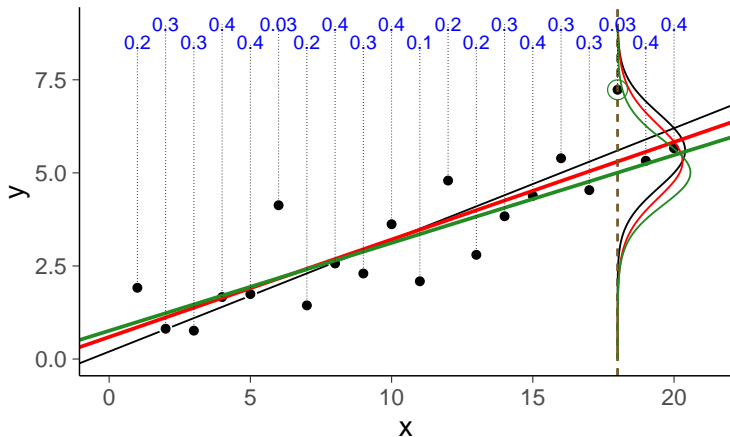
Leave-one-out predictive density



$$p(\tilde{y} = y_{18} \mid \tilde{x} = 18, x, y) \approx 0.07$$

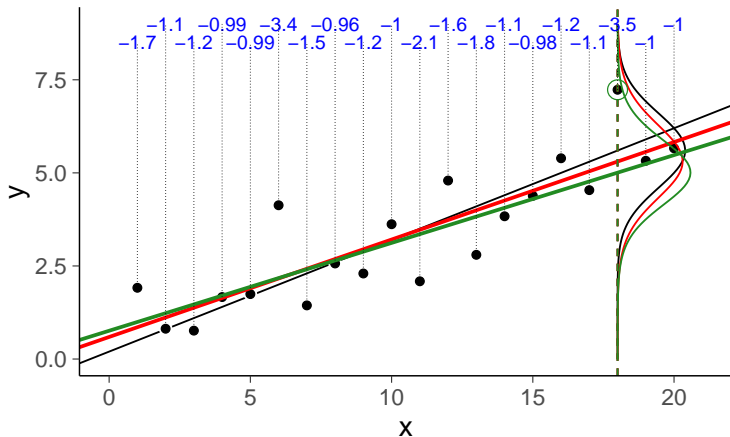
$$p(\tilde{y} = y_{18} \mid \tilde{x} = 18, x_{-18}, y_{-18}) \approx 0.03$$

Leave-one-out predictive densities



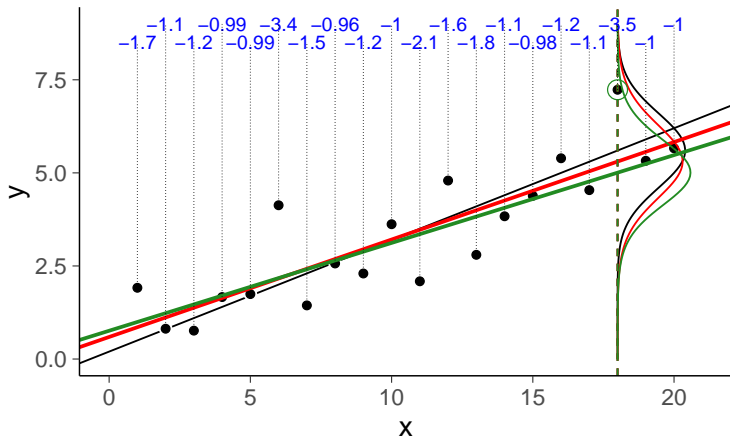
$$p(y_i | x_i, x_{-i}, y_{-i}), \quad i = 1, \dots, 20$$

Leave-one-out log predictive densities



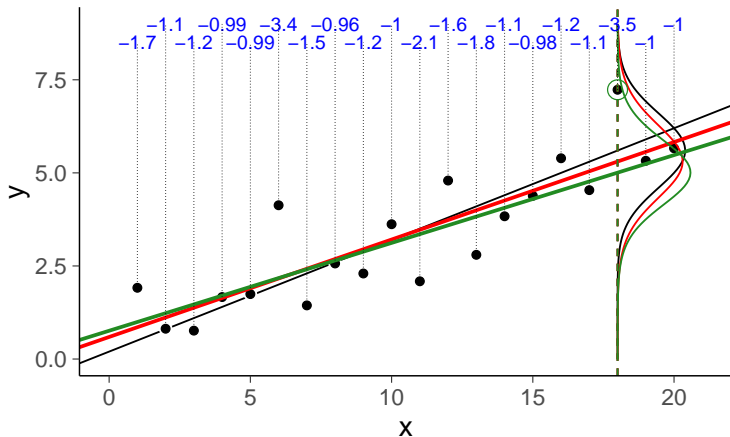
$$\log p(y_i | x_i, x_{-i}, y_{-i}), \quad i = 1, \dots, 20$$

Leave-one-out log predictive densities



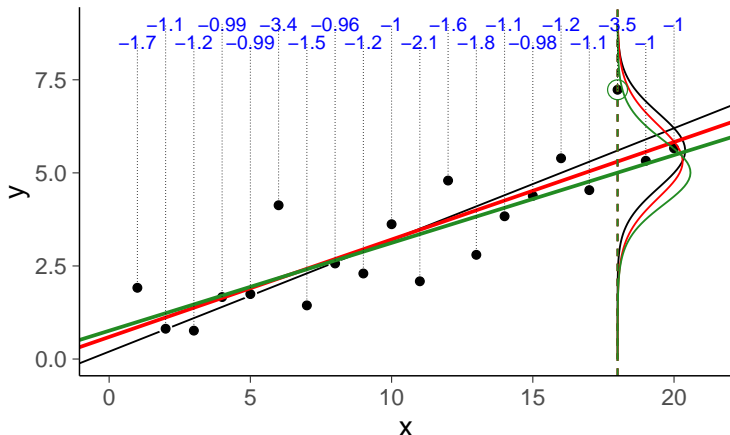
$$\sum_{i=1}^{20} \log p(y_i | x_i, x_{-i}, y_{-i}) \approx -29.5$$

Leave-one-out log predictive densities



$$\text{elpd_loo} = \sum_{i=1}^{20} \log p(y_i | x_i, x_{-i}, y_{-i}) \approx -29.5$$

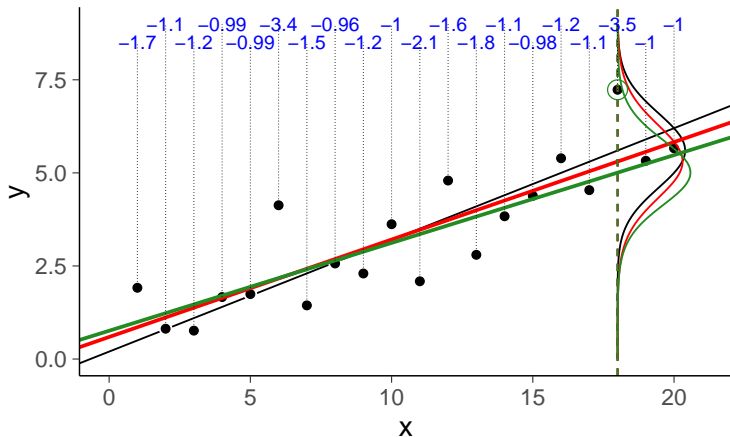
Leave-one-out log predictive densities



$$\text{elpd_loo} = \sum_{i=1}^{20} \log p(y_i | x_i, x_{-i}, y_{-i}) \approx -29.5$$

an estimate of log posterior pred. density for new data

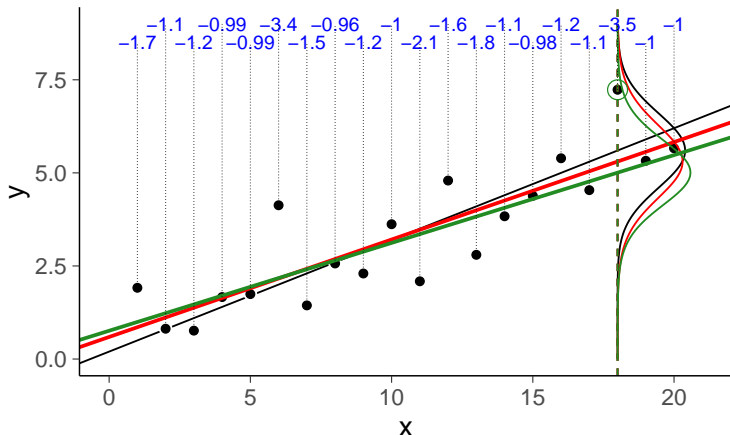
Leave-one-out log predictive densities



$$\text{elpd_loo} = \sum_{i=1}^{20} \log p(y_i | x_i, x_{-i}, y_{-i}) \approx -29.5$$

$$\text{lpd} = \sum_{i=1}^{20} \log p(y_i | x_i, x, y) \approx -26.8$$

Leave-one-out log predictive densities

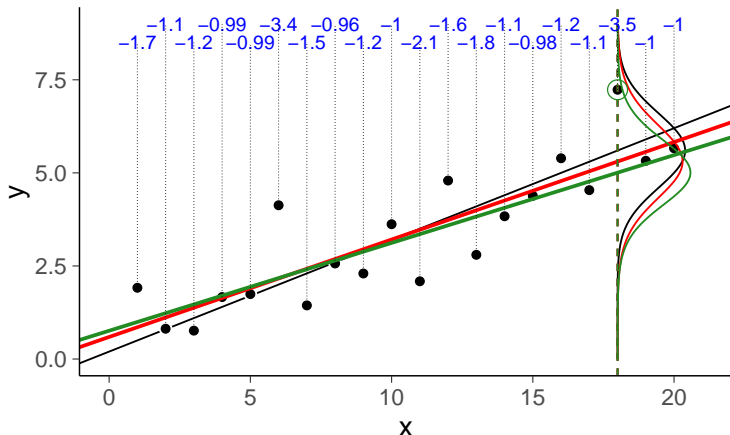


$$\text{elpd_loo} = \sum_{i=1}^{20} \log p(y_i | x_i, x_{-i}, y_{-i}) \approx -29.5$$

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$$\text{p_loo} = \text{lpd} - \text{elpd_loo} \approx 2.7$$

Leave-one-out log predictive densities

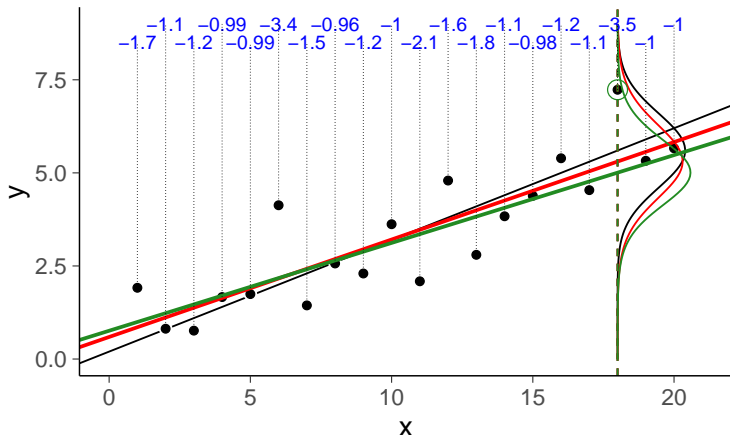


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asymptotically approaches p in case of regular faithful model

Leave-one-out log predictive densities

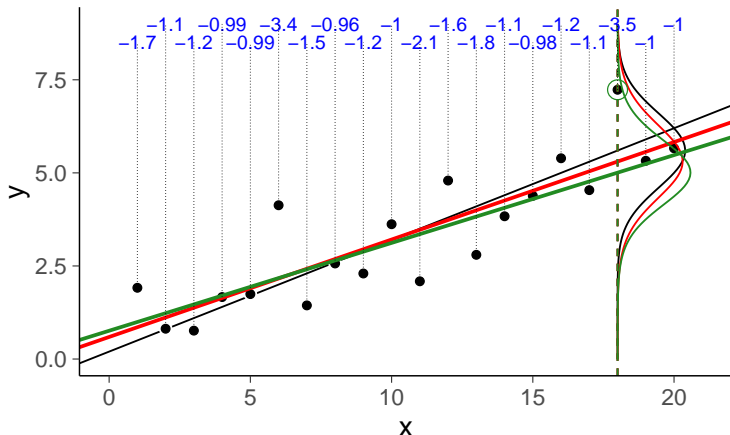


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asymptotically approaches p in case of regular faithful model

Leave-one-out log predictive densities

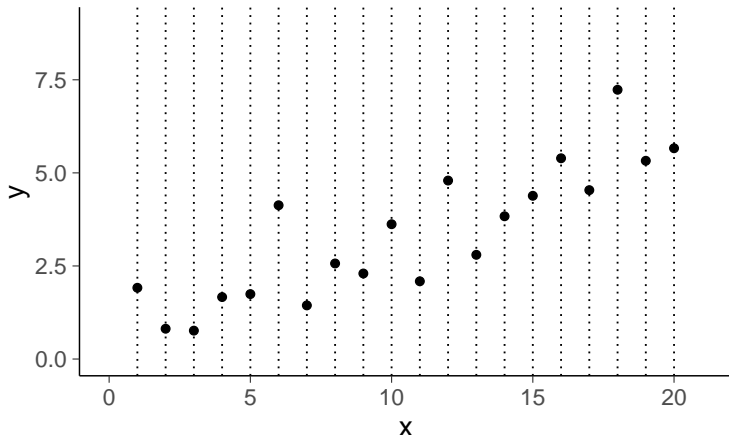


$$\text{elpd_loo} = \sum_{i=1}^{20} \log p(y_i | x_i, x_{-i}, y_{-i}) \approx -29.5$$

$$\text{SE} = \text{sd}(\log p(y_i | x_i, x_{-i}, y_{-i})) \cdot \sqrt{20} \approx 3.3$$

see Vehtari, Gelman & Gabry (2017a) and Vehtari & Ojanen (2012) for more

Fixed / designed x

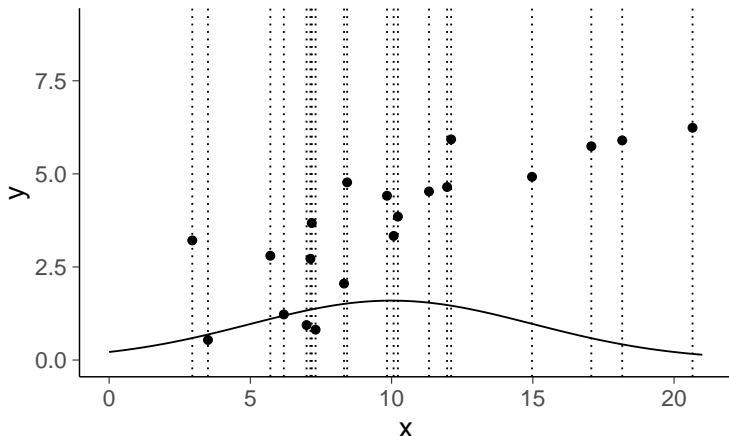


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LOO is ok for fixed / designed x . SE is uncertainty about $y | x$.

Distribution for x

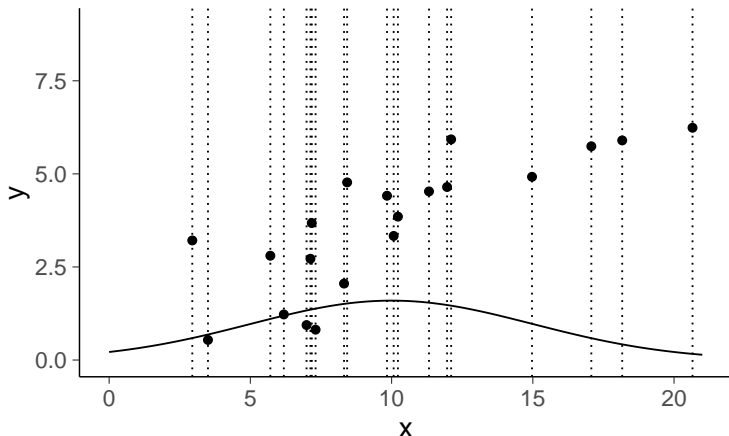


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LOO is ok for random x. SE is uncertainty about $y | x$ and x .

Distribution for x



$$\text{elpd_loo} = \sum_{i=1}^{20} \log p(y_i | x_i, x_{-i}, y_{-i}) \approx -29.5$$

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LOO is ok for random x. SE is uncertainty about $y | x$ and x .
Covariate shift can be handled with importance weighting or modelling

loo package

Computed from 4000 by 20 log-likelihood matrix

	Estimate	SE
elpd_loo	-29.5	3.3
p_loo	2.7	1.0

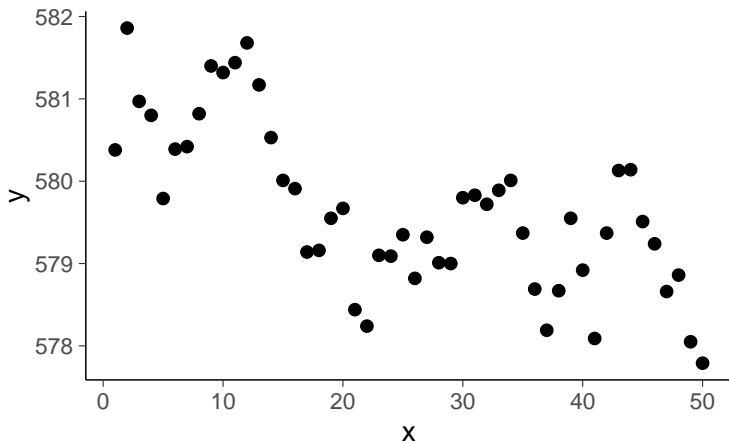
Monte Carlo SE of elpd_loo is 0.1.

Pareto k diagnostic values:

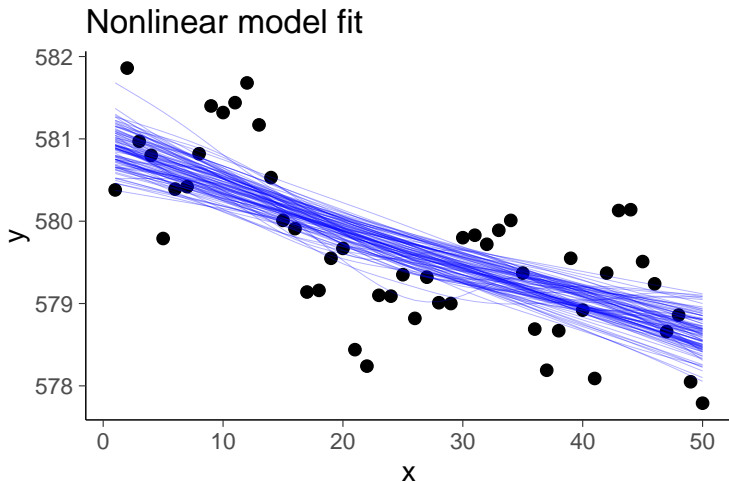
		Count	Pct.	Min.	n_eff
(-Inf, 0.5]	(good)	18	90.0%	899	
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(1, Inf)	(very bad)	0	0.0%	<NA>	

All Pareto k estimates are ok ($k < 0.7$).
See `help('pareto-k-diagnostic')` for details.

Interpolation vs extrapolation

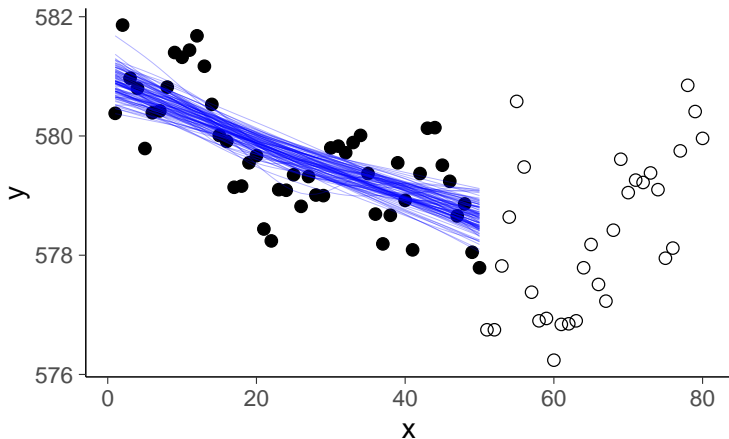


Interpolation vs extrapolation



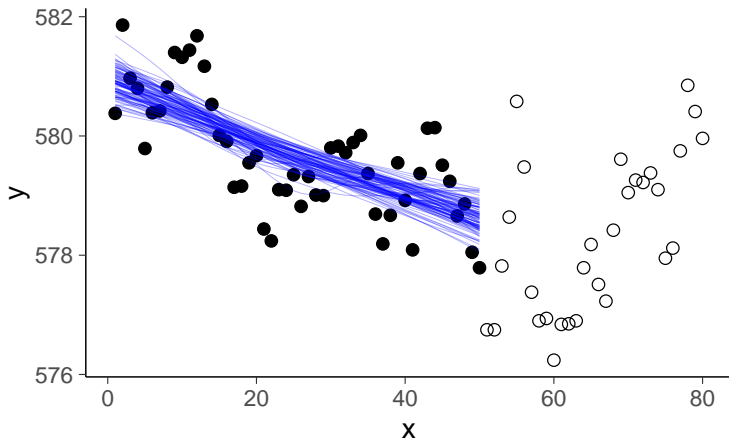
Interpolation vs extrapolation

Nonlinear model fit + new data



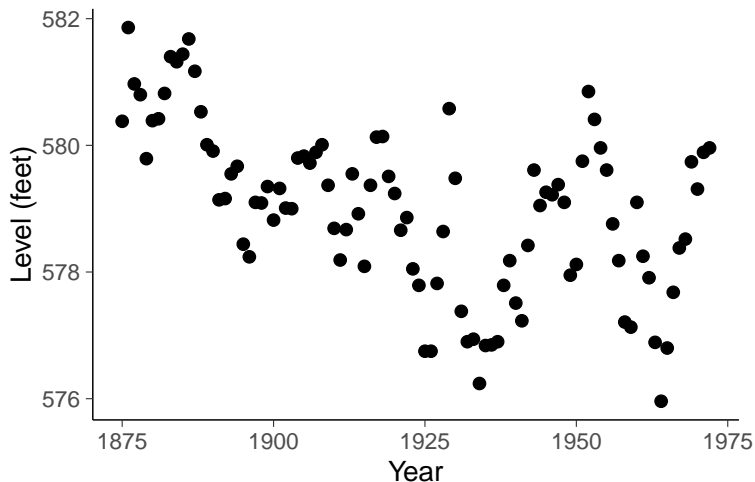
Interpolation vs extrapolation

Nonlinear model fit + new data



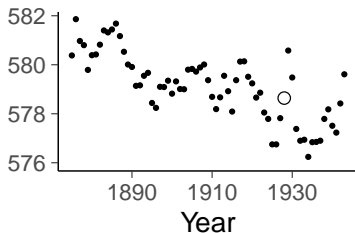
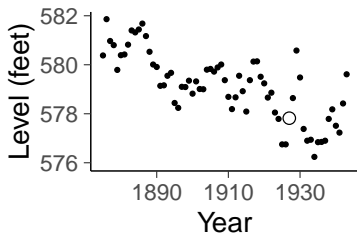
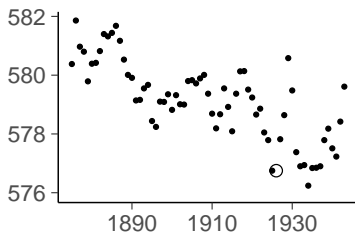
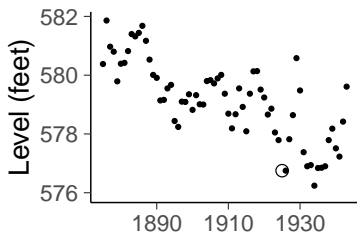
Extrapolation is more difficult

Cross-validation for time series?



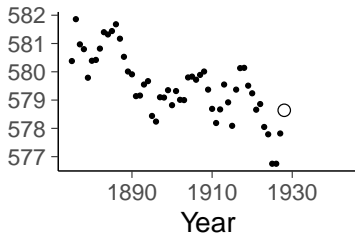
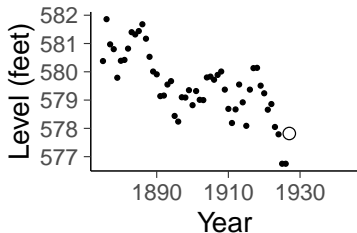
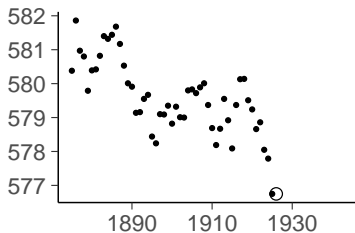
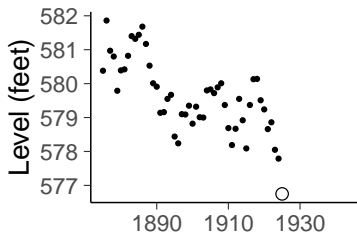
Can LOO or other cross-validation be used with time series?

Cross-validation for time series



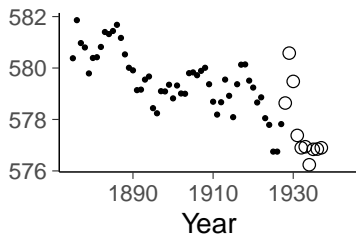
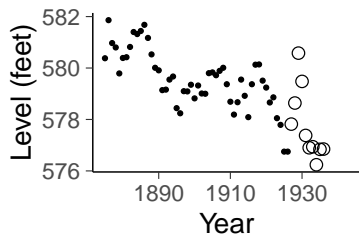
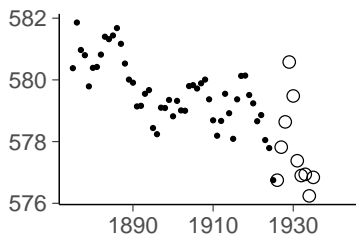
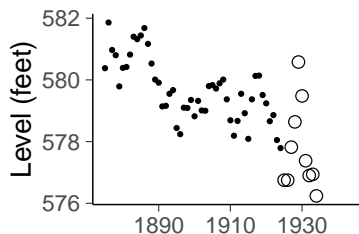
Leave-one-out cross-validation is ok for assessing conditional model

Cross-validation for time series



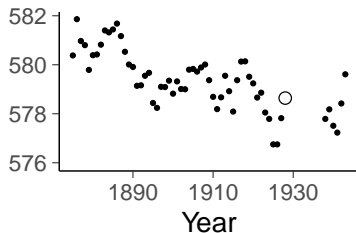
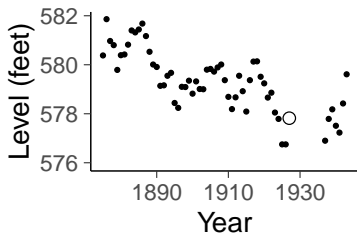
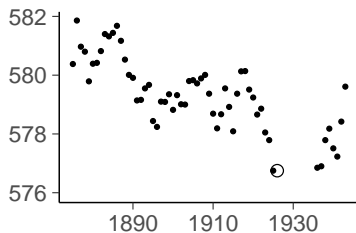
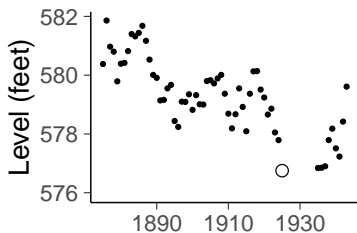
Leave-future-out cross-validation is better for predicting future

Cross-validation for time series



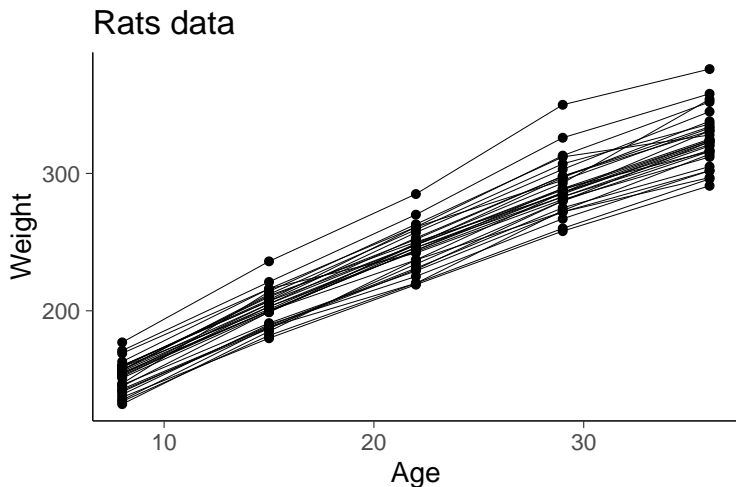
m-step-ahead cross-validation is better for predicting further future

Cross-validation for time series



m-step-ahead leave-a-block-out cross-validation

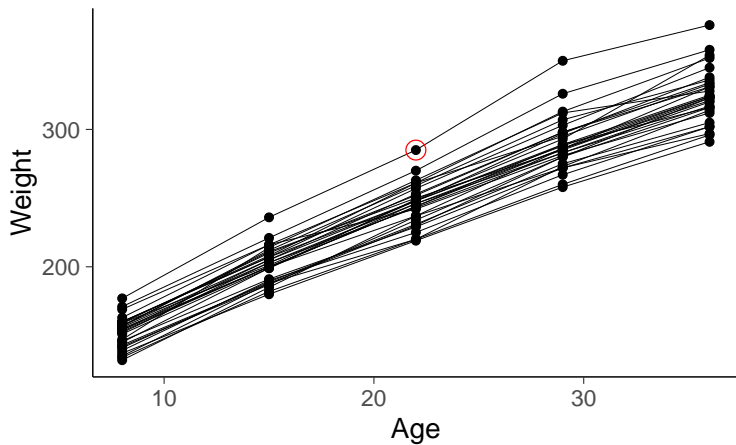
Cross-validation for hierarchical data



Can LOO or other cross-validation be used with hierarchical data?

Cross-validation for hierarchical data

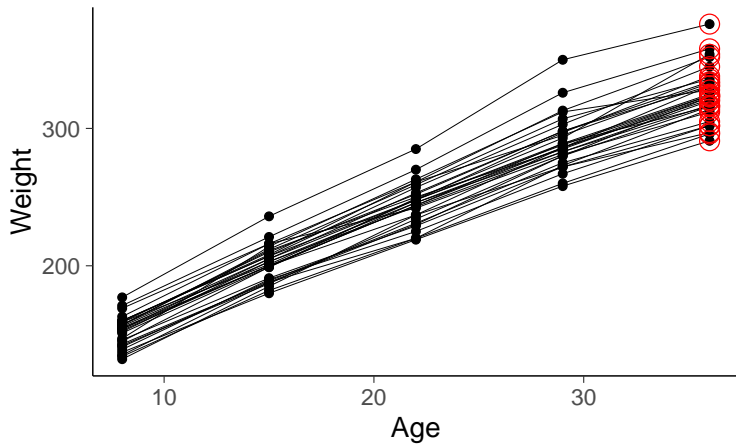
Leave-one-out?



Yes!

Cross-validation for hierarchical data

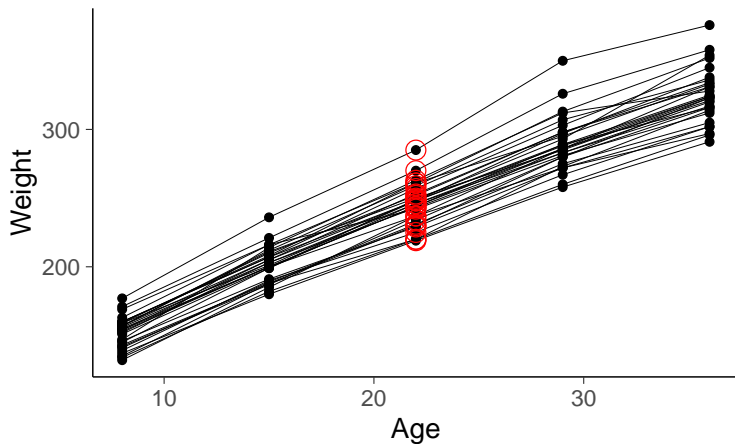
1-step-ahead?



Yes!

Cross-validation for hierarchical data

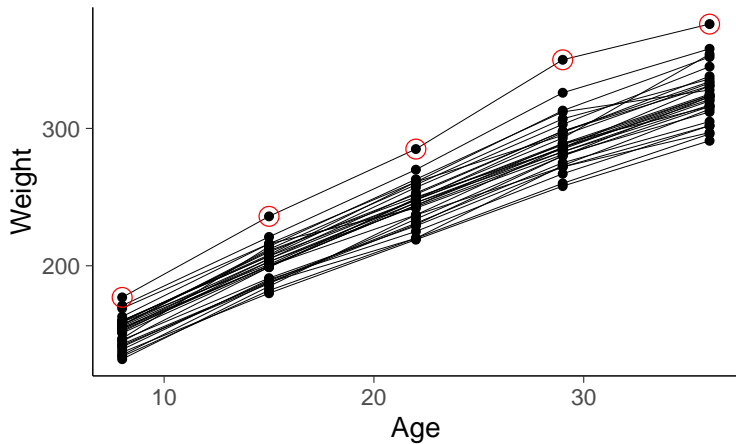
Leave-one-time-point-out?



Yes!

Cross-validation for hierarchical data

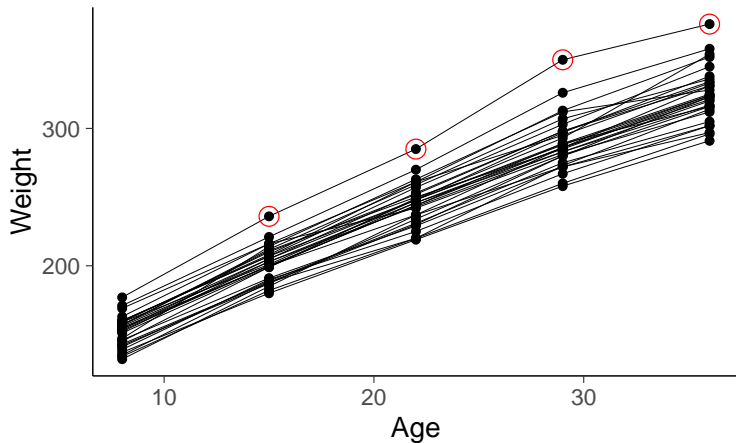
Leave-one-rat-out?



Yes!

Cross-validation for hierarchical data

Predict given initial weight?



Yes!

Summary of data generating mechanisms and prediction tasks

- You have to make some assumptions on data generating mechanism
- Use the knowledge of the prediction task if available
- Cross-validation can be used to analyse different parts, even if there is no clear prediction task

see [Vehtari & Ojanen \(2012\)](#) and [CV-FAQ](#)

Fast cross-validation

- Pareto smoothed importance sampling LOO (PSIS-LOO)
- K-fold cross-validation

see [Vehtari, Gelman & Gabry \(2017a\)](#) and mc-stan.org/loo/

Importance sampling leave-one-out cross-validation

- We want to compute

$$p(y_i | x_{-i}, y_{-i}) = \int p(y_i | x_i, \theta) p(\theta | x_{-i}, y_{-i}) d\theta$$

Importance sampling leave-one-out cross-validation

- We want to compute

$$p(y_i | x_{-i}, y_{-i}) = \int p(y_i | x_i, \theta) p(\theta | x_{-i}, y_{-i}) d\theta$$

- Proposal distribution is full posterior $\theta^{(s)} \sim p(\theta | x, y)$
- Target distribution is LOO-posterior $p(\theta | x_{-i}, y_{-i})$

Importance sampling leave-one-out cross-validation

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- Target distribution is LOO-posterior $p(\theta | x_{-i}, y_{-i})$
- Importance ratio

$$w_i^{(s)} = \frac{p(\theta^{(s)} | x_{-i}, y_{-i})}{p(\theta^{(s)} | x, y)} \propto \frac{1}{p(y_i | \theta^{(s)})}$$

Importance sampling leave-one-out cross-validation

- We want to compute

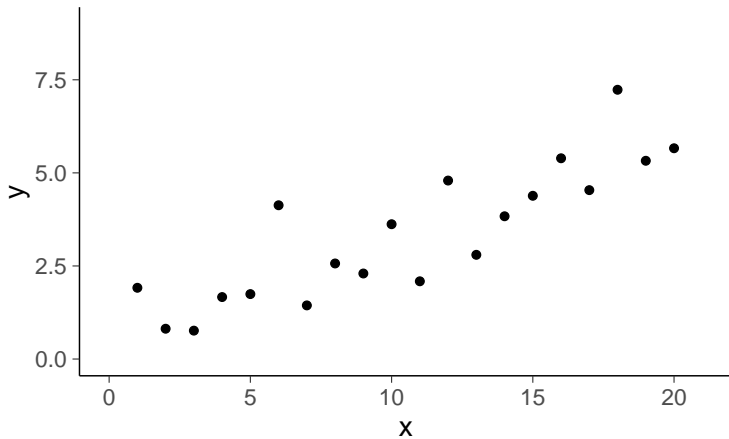
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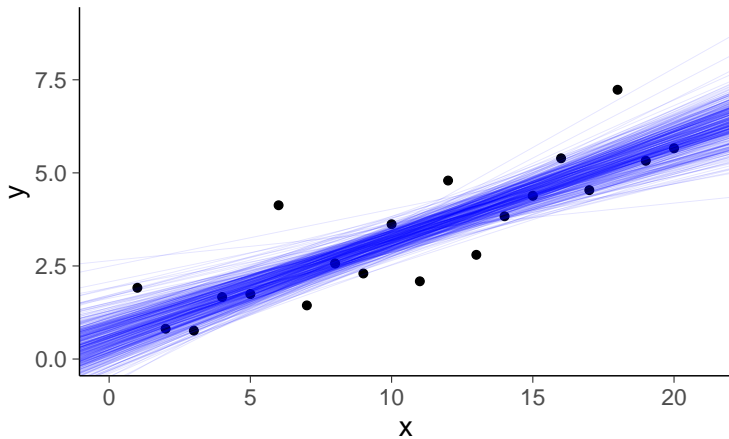
$$w_i^{(s)} = \frac{p(\theta^{(s)} | x_{-i}, y_{-i})}{p(\theta^{(s)} | x, y)} \propto \frac{1}{p(y_i | \theta^{(s)})}$$

$$\tilde{w}_i^{(s)} = \frac{w_i^{(s)}}{\sum_{s'=1}^S w_i^{(s')}}$$

Data

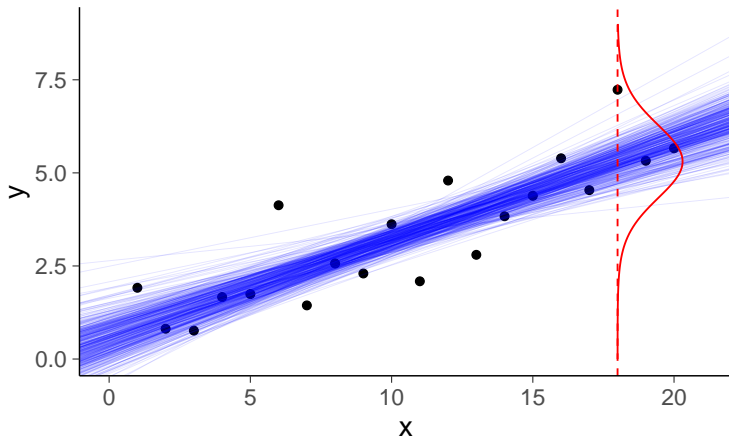


Posterior draws



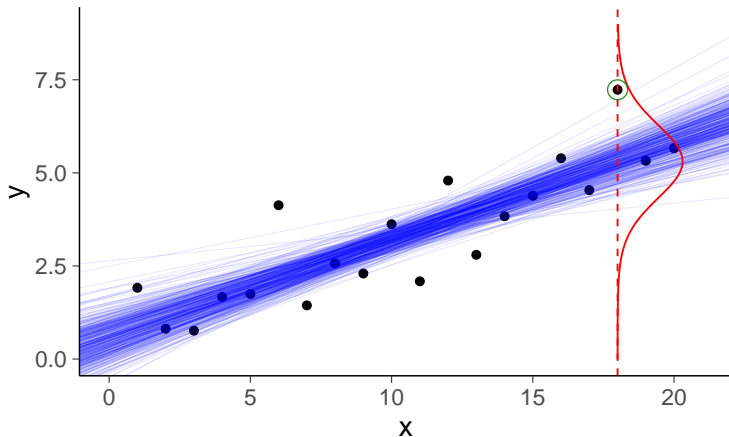
$$\theta^{(s)} \sim p(\theta \mid x, y)$$

Posterior predictive distribution



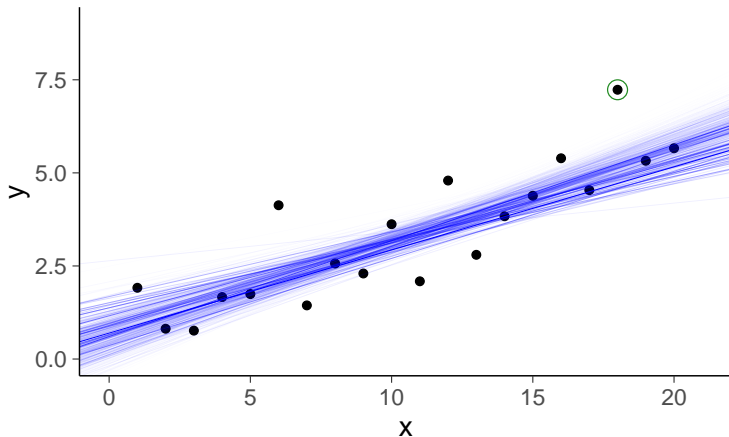
$$\theta^{(s)} \sim p(\theta \mid x, y), \quad p(\tilde{y} \mid \tilde{x}, x, y) \approx \frac{1}{S} \sum_{s=1}^S p(\tilde{y} \mid \tilde{x}, \theta^{(s)})$$

Posterior predictive distribution



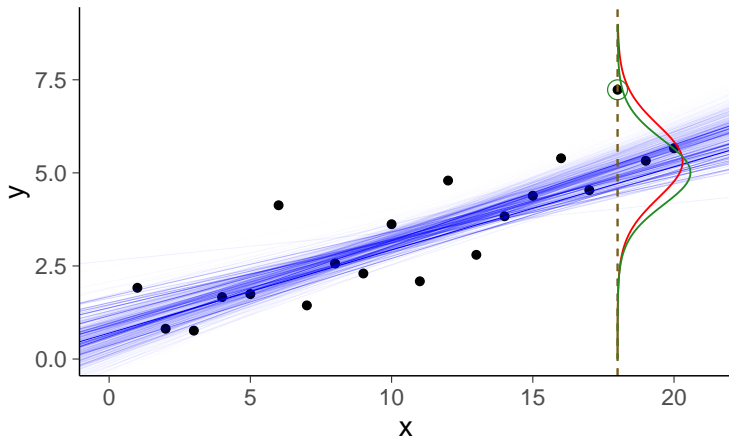
$$\theta^{(s)} \sim p(\theta \mid x, y), \quad p(\tilde{y} \mid \tilde{x}, x, y) \approx \frac{1}{S} \sum_{s=1}^S p(\tilde{y} \mid \tilde{x}, \theta^{(s)})$$

PSIS-LOO weighted draws



$$\theta^{(s)} \sim p(\theta \mid x, y), \quad w_i^{(s)} = p(\theta^{(s)} \mid x_{-i}, y_{-i}) / p(\theta^{(s)} \mid x, y)$$

PSIS-LOO weighted predictive distribution



$$\theta^{(s)} \sim p(\theta \mid x, y), \quad \tilde{w}_i^{(s)} = p(\theta^{(s)} \mid x_{-i}, y_{-i}) / p(\theta^{(s)} \mid x, y)$$

$$p(y_i \mid x_i, x_{-i}, y_{-i}) \approx \sum_{s=1}^S [\tilde{w}_i^{(s)} p(y_i \mid x_i, \theta^{(s)})]$$

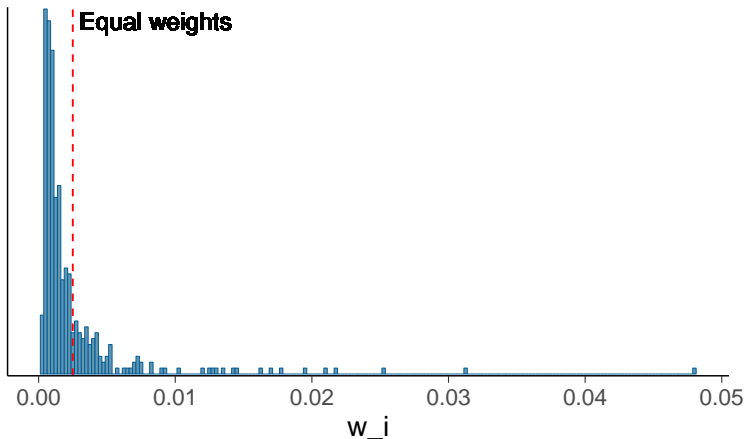
Pareto smoothed importance sampling LOO

- We want to compute
$$p(y_i | x_{-i}, y_{-i}) = \int p(y_i | x_i, \theta) p(\theta | x_{-i}, y_{-i}) d\theta$$
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- Importance ratio

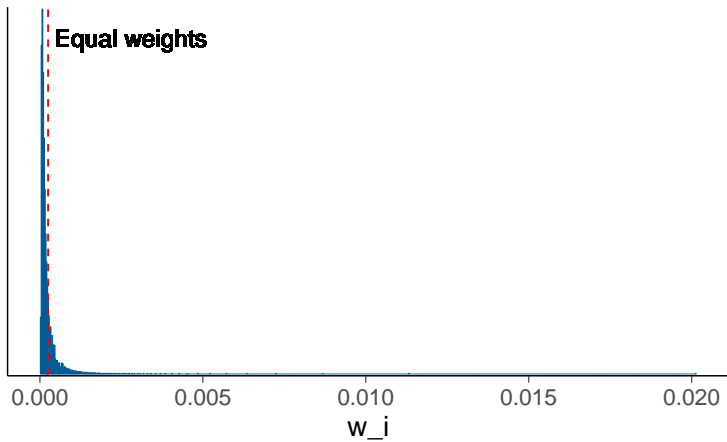
$$w_i^{(s)} = \frac{p(\theta^{(s)} | x_{-i}, y_{-i})}{p(\theta^{(s)} | x, y)} \propto \frac{1}{p(y_i | \theta^{(s)})}$$
$$\tilde{w}_i^{(s)} = \frac{w_i^{(s)}}{\sum_{s'=1}^S w_i^{(s')}}$$

- The variability of importance weights matter
 - Pareto- k diagnostic
 - Pareto smoothed importance sampling LOO (PSIS-LOO)

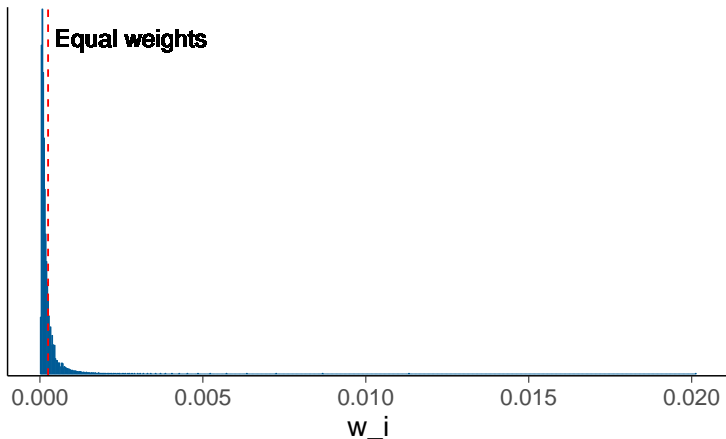
400 importance weights for leave-18th-out



4000 importance weights for leave-18th-out



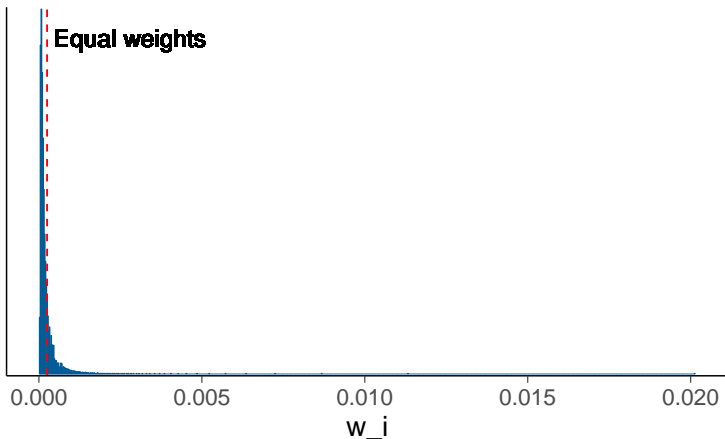
4000 importance weights for leave-18th-out



$$\text{ESS} \approx 1 / \sum_{s=1}^S (\tilde{w}^{(s)})^2 \approx 459$$

see [Vehtari, Gelman & Gabry \(2017b\)](#)

4000 importance weights for leave-18th-out



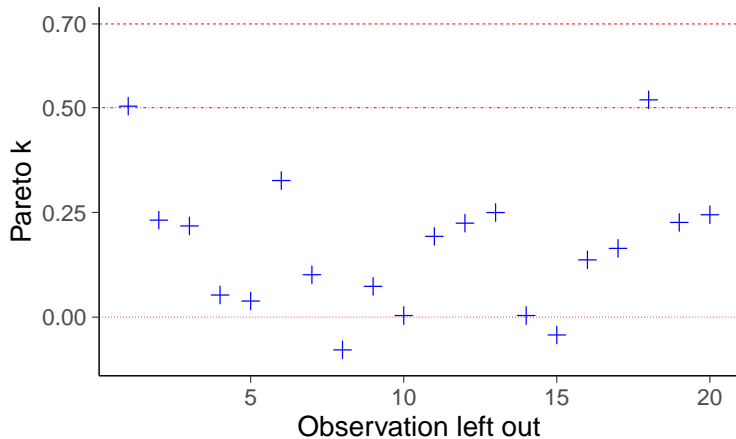
$$\text{ESS} \approx 1 / \sum_{s=1}^S (\tilde{w}^{(s)})^2 \approx 459$$

$$\text{Pareto } \hat{k} \approx 0.52$$

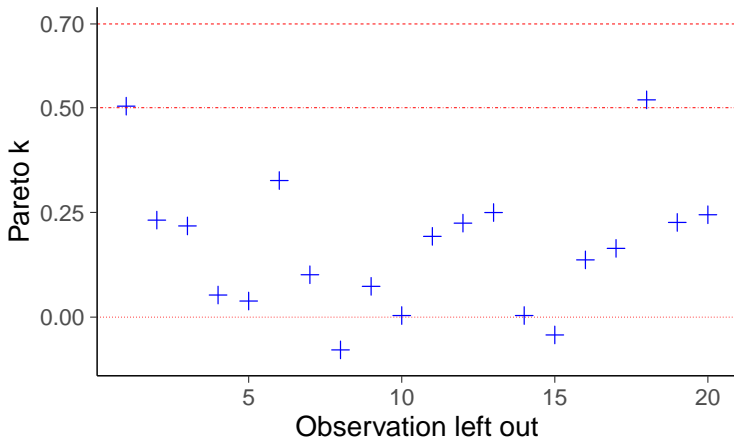
- Pareto \hat{k} estimates the tail shape which determines the convergence rate of PSIS. Less than 0.7 is ok.

see [Vehtari, Gelman & Gabry \(2017b\)](#)

PSIS-LOO diagnostics



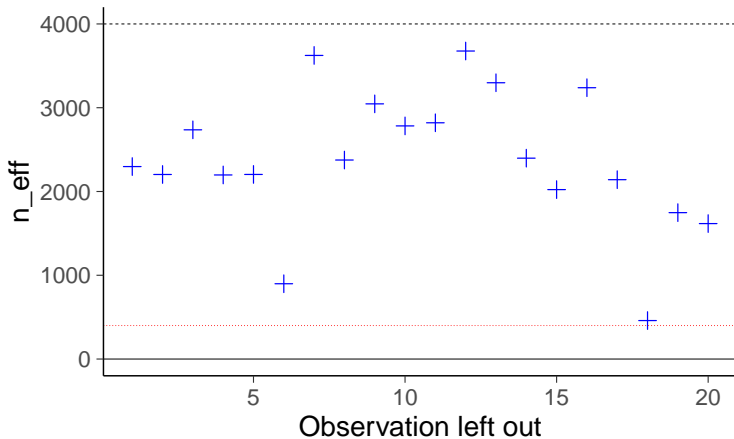
PSIS-LOO diagnostics



Pareto k diagnostic values:

		Count	Pct.	Min. n_eff
(-Inf, 0.5]	(good)	18	90.0%	899
(0.5, 0.7]	(ok)	2	10.0%	459
(0.7, 1]	(bad)	0	0.0%	<NA>
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PSIS-LOO diagnostics



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loo package

Computed from 4000 by 20 log-likelihood matrix

	Estimate	SE
elpd_loo	-29.5	3.3
p_loo	2.7	1.0

Monte Carlo SE of elpd_loo is 0.1.

Pareto k diagnostic values:

		Count	Pct.	Min.	n_eff
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All Pareto k estimates are ok ($k < 0.7$).
See `help('pareto-k-diagnostic')` for details.

see more in [Vehtari, Gelman & Gabry \(2017b\)](#)

Pareto smoothed importance sampling (PSIS)

- Replace the largest weights with ordered statistics of the fitted Pareto distribution
 - equivalent to using model to filter the noise out of the weights

See more in [Vehtari, Simpson, Gelman, Yao & Gabry \(2021\)](#)

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 - equivalent to using model to filter the noise out of the weights
- Reduced variability compared to the plain IS
- Reduced bias compared to the truncated IS
- Asymptotically consistent under some mild conditions

See more in [Vehtari, Simpson, Gelman, Yao & Gabry \(2021\)](#)

Stan code

$$\log(w_i^{(s)}) = \log(1/p(y_i | x_i, \theta^{(s)})) = \text{--log_lik}[i]$$

Stan code

$$\log(w_i^{(s)}) = \log(1/p(y_i | x_i, \theta^{(s)})) = -\text{log_lik}[i]$$

```
...  
model {  
  alpha ~ normal(pmualpha, psalpha);  
  beta ~ normal(pmubeta, psbeta);  
  y ~ normal(mu, sigma);  
}  
generated quantities {  
  vector[N] log_lik;  
  for (i in 1:N)  
    log_lik[i] = normal_lpdf(y[i] | mu[i], sigma);  
}
```

Stan code

$$\log(w_i^{(s)}) = \log(1/p(y_i | x_i, \theta^{(s)})) = -\text{log_lik}[i]$$

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}
```

- RStanARM and brms compute `log_lik` by default

loo()

- RStan (`log_lik` in gen. quantities), RStanARM, brms
`loo(fit)`

loo()

- RStan (`log_lik` in gen. quantities), RStanARM, brms
`loo(fit)`

- CmdStanR (eventually will work as others)

```
loocmd <- function(fit, ...) {  
  loo(fit$draws("log_lik"),  
    r_eff=relative_eff(fit$draws("log_lik"))),  
  ...  
}  
loocmd(fit)
```

What if many high Pareto- \hat{k} 's

- `rstan::loo(..., moment_match = TRUE)`
`brms::loo(..., moment_match = TRUE)`
support implicitly adaptive importance sampling with
moment matching algorithm by Paananen et al. (2021). See
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- `rstanarm::loo(..., k_threshold = TRUE)`
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runs MCMC for the folds with \hat{k} above the threshold
- Use K-fold-CV (more about this soon)

Pareto smoothed importance sampling CV variants

- PSIS-LOO for hierarchical models
 - leave-one-group out is challenging for PSIS-LOO
 - see Merkel, Furr and Rabe-Hesketh (2018) for an approach using quadrature integration
 - Stan demo of the challenges and integrated LOO at <https://avehtari.github.io/modelselection/roaches.html>

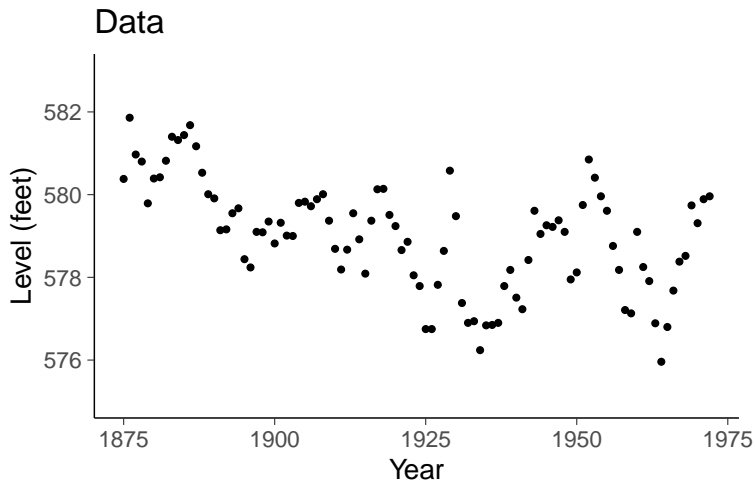
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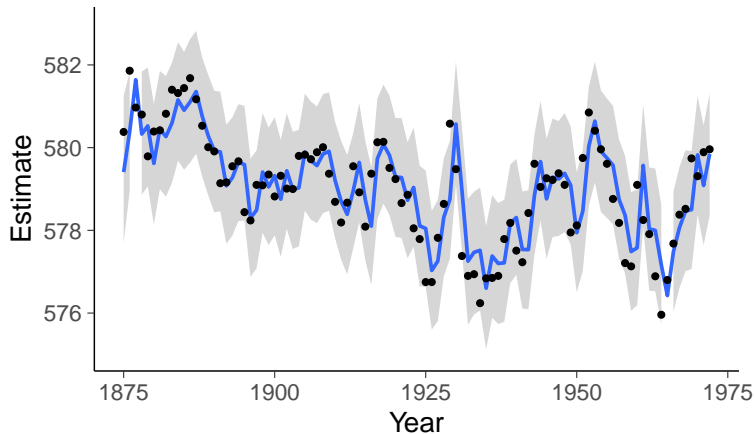
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- PSIS-LOO for non-factorizable models
 - mc-stan.org/loo/articles/loo2-non-factorizable.html
- PSIS-LOO for time series
 - Approximate leave-future-out cross-validation (LFO-CV)
mc-stan.org/loo/articles/loo2-lfo.html

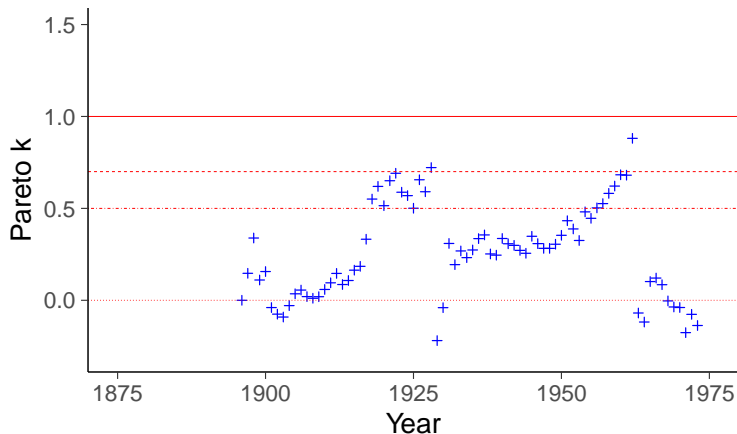
LFO-CV



AR-4 prediction with 95% interval



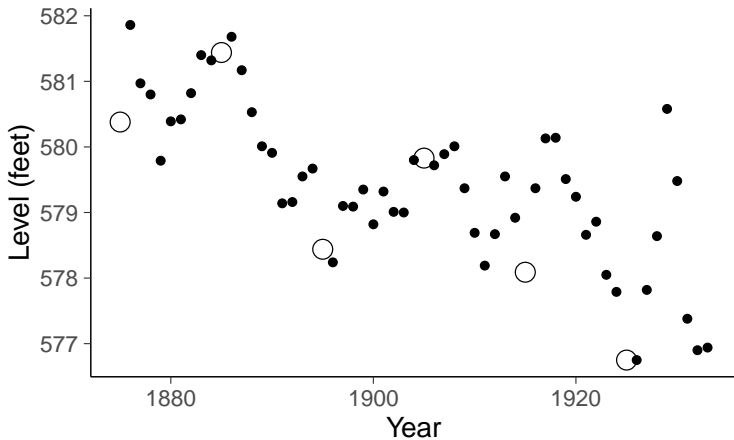
PSIS-1-step-ahead with refits



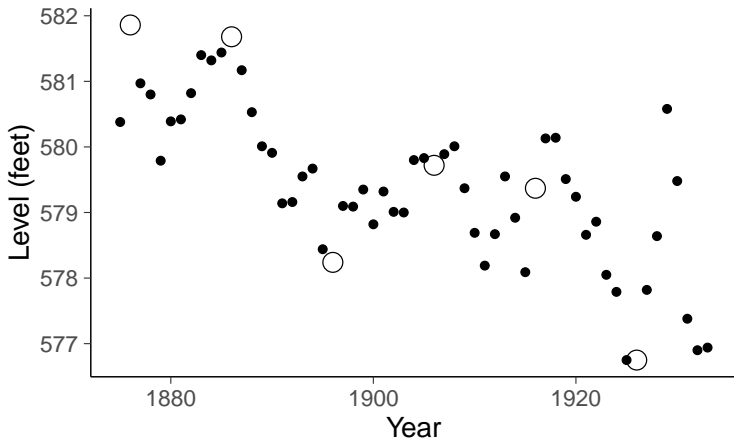
K-fold cross-validation

- K-fold cross-validation can approximate LOO
 - the same use cases as with LOO
- K-fold cross-validation can be used for hierarchical models
 - good for leave-one-group-out
- K-fold cross-validation can be used for time series
 - with leave-block-out

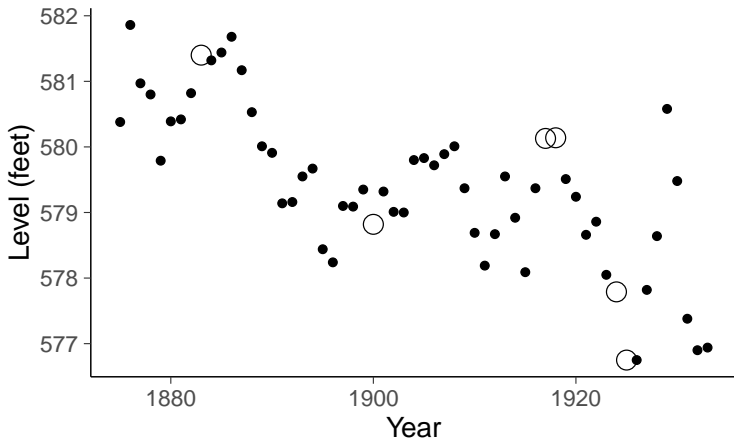
Balance k-fold approximation of LOO



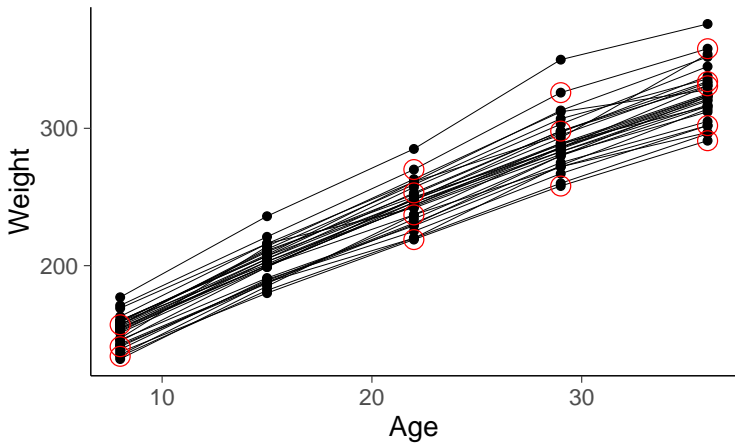
Balance k-fold approximation of LOO



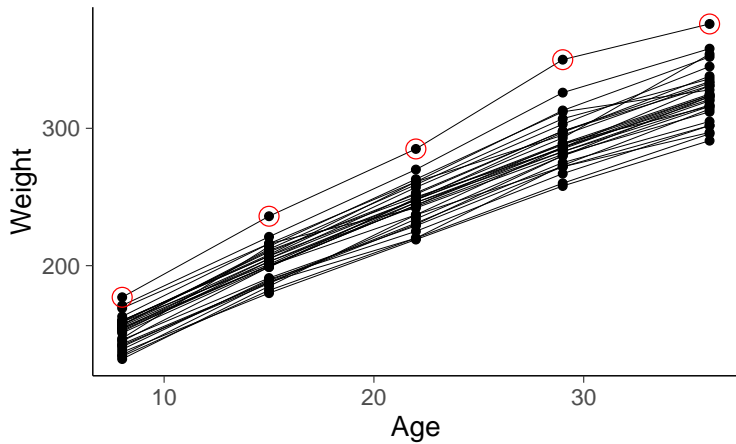
Random k-fold approximation of LOO



Random kfold approximation of LOO



Leave-one-rat-out



K-fold-CV code

- RStan, CmdStanR
See vignette <http://mc-stan.org/loo/articles/loo2-elpd.html>
- RStanARM, brms
`kfold(fit)`
- Alternative data divisions
`kfold_split_random()`
`kfold_split_balanced()`
`kfold_split_stratified()`

WAIC vs PSIS-LOO

see [Vehtari, Gelman & Gabry \(2017a\)](#)

WAIC vs PSIS-LOO

- WAIC has same assumptions as LOO

see [Vehtari, Gelman & Gabry \(2017a\)](#)

WAIC vs PSIS-LOO

- WAIC has same assumptions as LOO
- PSIS-LOO is more accurate

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WAIC vs PSIS-LOO

- WAIC has same assumptions as LOO
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- LOO makes the prediction assumption more clear, which helps if K-fold-CV is needed instead

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WAIC vs PSIS-LOO

- WAIC has same assumptions as LOO
- PSIS-LOO is more accurate
- PSIS-LOO has much better diagnostics
- LOO makes the prediction assumption more clear, which helps if K-fold-CV is needed instead
- Multiplying by -2 doesn't give any benefit (Watanabe didn't multiply by -2)

see [Vehtari, Gelman & Gabry \(2017a\)](#)

*IC

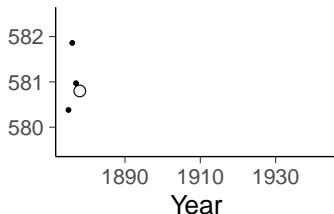
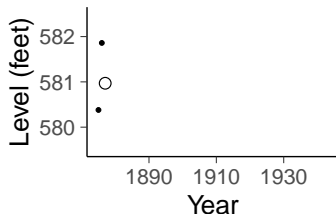
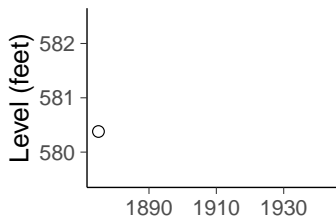
- AIC uses maximum likelihood estimate for prediction
- DIC uses posterior mean for prediction
- BIC is an approximation for marginal likelihood
- TIC, NIC, RIC, PIC, BPIC, QIC, AICc, ...

Marginal likelihood / Bayes factor

- Like leave-future-out 1-step-ahead cross-validation but starting with 0 observations

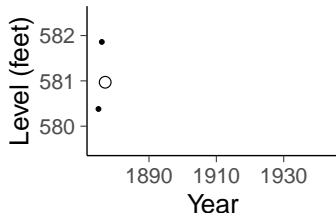
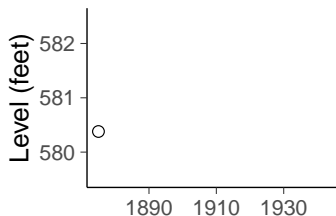
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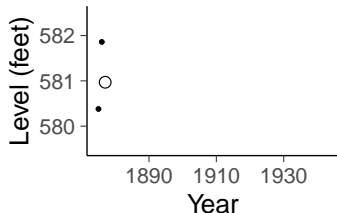
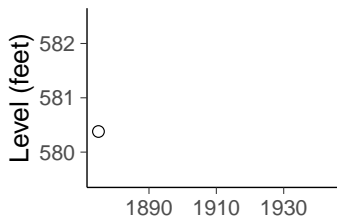
Marginal likelihood / Bayes factor

- Like leave-future-out 1-step-ahead cross-validation but starting with 0 observations
 - which makes it very sensitive to prior



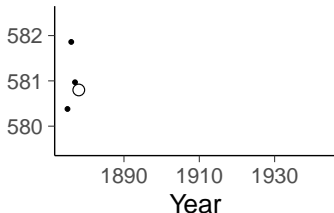
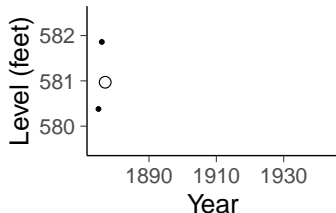
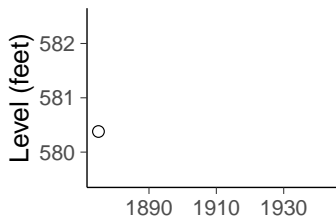
Marginal likelihood / Bayes factor

- Like leave-future-out 1-step-ahead cross-validation but starting with 0 observations
 - which makes it very sensitive to prior and
 - unstable in case of misspecified models



Marginal likelihood / Bayes factor

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 - which makes it very sensitive to prior and
 - unstable in case of misspecified models also asymptotically



Cross-validation for model assessment

- CV is good for model assessment when application specific utility/cost functions are used
 - e.g. 90% absolute error

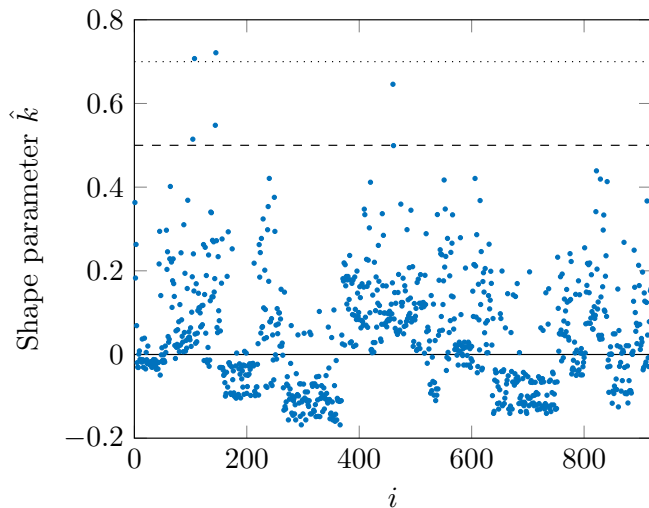
Cross-validation for model assessment

- CV is good for model assessment when application specific utility/cost functions are used
 - e.g. 90% absolute error
- Also useful in model checking in similar way as posterior predictive checking (PPC)
 - model misspecification diagnostics (e.g. Pareto- k and p_{loo})
 - checking calibration of leave-one-out predictive posteriors (`ppc_loo_pit` in `bayesplot`)

see demos avehtari.github.io/modelselection/

Radon example

PSIS-LOO diagnostics

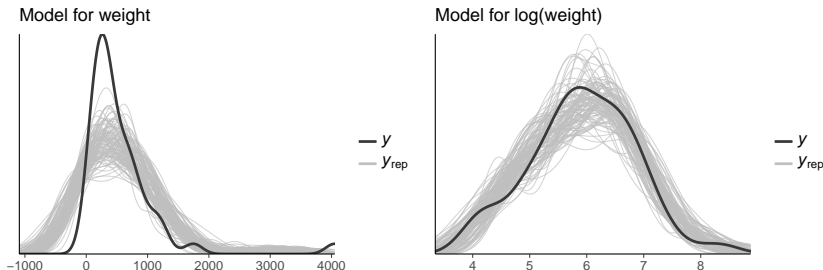


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Sometimes cross-validation is not needed

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- Posterior predictive checking is often sufficient

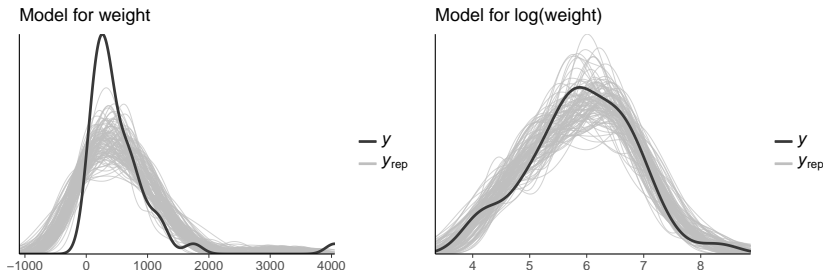


Predicting the yields of mesquite bushes.

Gelman, Hill & Vehtari (2020): Regression and Other Stories, Chapter 11.

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Predicting the yields of mesquite bushes.

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- BDA3, Chapter 6
- Gabry, Simpson, Vehtari, Betancourt, Gelman (2019). Visualization in Bayesian workflow. JRSS A, <https://doi.org/10.1111/rssa.12378>
- mc-stan.org/bayesplot/articles/graphical-ppcs.html

Sometimes cross-validation is not needed

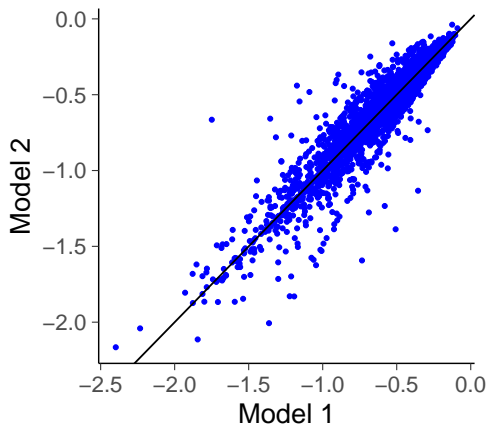
- With good priors that keep the prior on predictive space consistent, there is no need to do model selection to avoid overfitting

Arsenic well example – Model comparison

- Logistic regression for predicting probability of switching well with high arsenic level in rural Bangladesh
 - Model 1:
 $\log(\text{arsenic}) + \text{distance}$
 - Model 2:
 $\log(\text{arsenic}) + \text{distance} + \text{education level}$

Arsenic well example – Model comparison

Model 1 vs Model 2

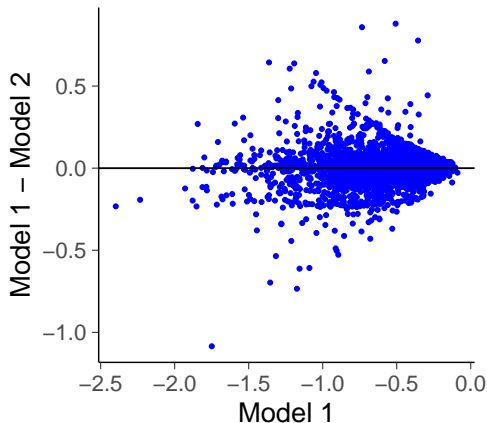


Model 1: $\widehat{\text{elpd}}_{\text{LOO}}(\mathbf{M}_a | \mathbf{y}^{\text{obs}}) \approx -1952, \text{SE}=16$

Model 2: $\widehat{\text{elpd}}_{\text{LOO}}(\mathbf{M}_b | \mathbf{y}^{\text{obs}}) \approx -1938, \text{SE}=17$

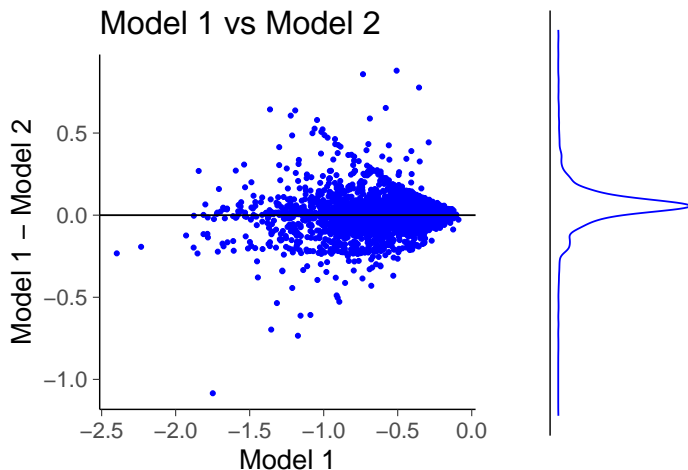
Arsenic well example – Model comparison

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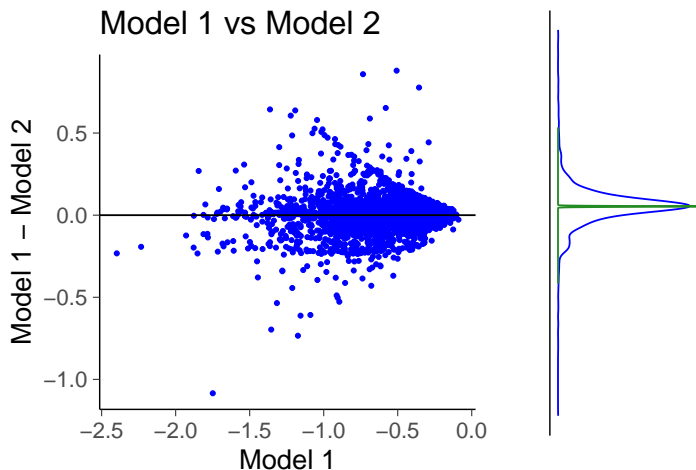
Difference: $\widehat{\text{elpd}}_{\text{LOO}}(M_a, M_b \mid y^{\text{obs}}) \approx -14.4, \text{SE} = 6.1$

Arsenic well example – Model comparison



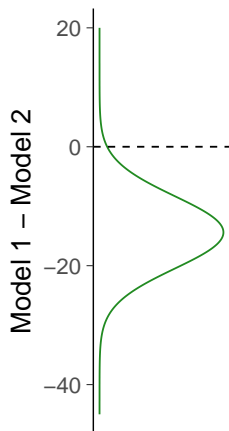
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Arsenic well example – Model comparison



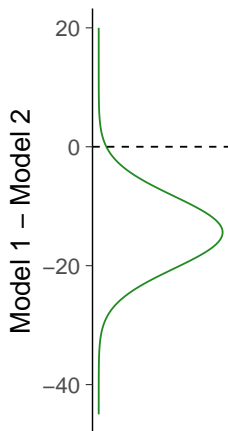
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Arsenic well example – Model comparison



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Arsenic well example – Model comparison



```
> loo_compare(model1, model2)
```

	elpd_diff	se_diff
model2	0.0	0.0
model1	-14.4	6.1

see Vehtari, Gelman & Gabry (2017a)

8 schools – Model comparison

```
> loo_compare(pooled, hierarchical)
              elpd_diff se_diff
pooled          0.0       0.0
hierarchical -0.3       0.7
```

No difference between pooled and hierarchical for predicting the future observations for a new school (exchangeable with the schools in the data).

Poisson vs Hurdle-Poisson example

	elpd_diff	se_diff
Hurdle-Poisson	0.0	0.0
Poisson	-215.9	22.1

Clear difference (which was also obvious in posterior predictive checks)

LOO difference uncertainty estimate reliability

1. The models make very similar predictions
2. The models are misspecified with outliers in the data
3. The number of observations is small

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 - in nested case the skewness favors the simpler model
 - any inference with small n is difficult
 - if $|\widehat{\text{elpd}}_{\text{LOO}}(\mathbf{M}_a, \mathbf{M}_b \mid \mathbf{y}^{\text{obs}})| > 4$, model is well specified, and $n > 100$ then the normal approximation is good

Sometimes cross-validation is not needed

- In nested case, often easier and more accurate to analyse posterior distribution of more complex model directly
avehtari.github.io/modelselection/betablockers.html

- instead of comparing

- Model 1: $y \sim \text{normal}(\alpha, \sigma)$

- vs

- Model 2: $y \sim \text{normal}(\alpha + \beta x, \sigma)$

- look at the posterior of β directly

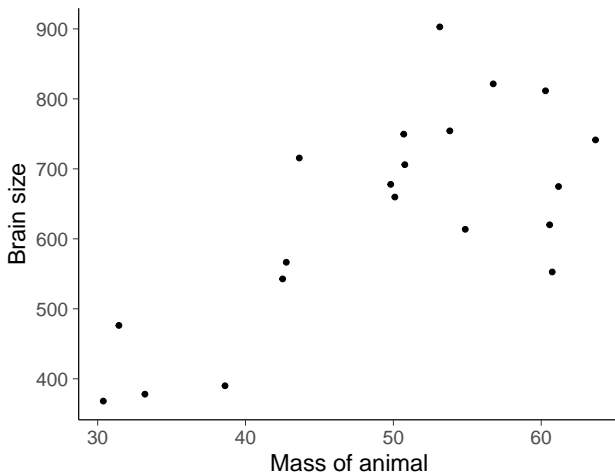
Model selection needed to avoid overfitting?

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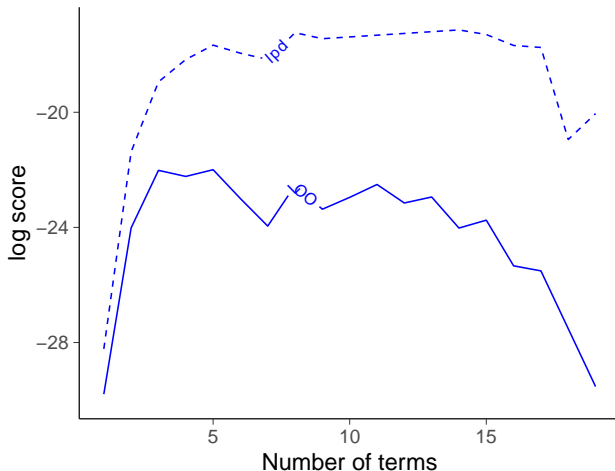
Simulated data by Richard McElreath



Model selection needed to avoid overfitting?

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Polynomial basis functions



Model selection needed to avoid overfitting?

- Gaussian process can be used as a prior on function space
 - GP can be approximated with basis functions

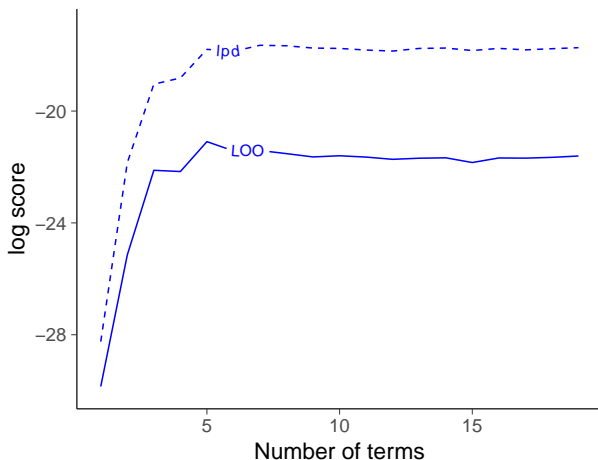
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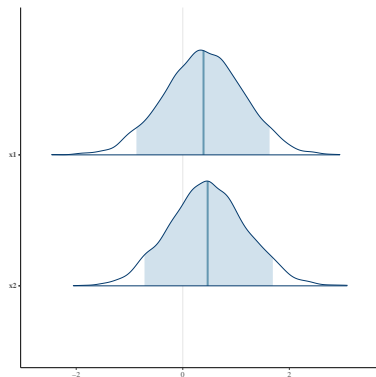
Gaussian process basis functions



Model is not needed to avoid overfitting

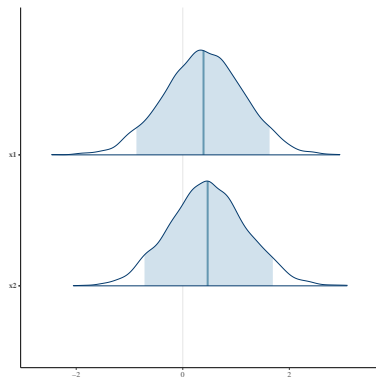
- No overfitting when using good priors that keep the prior on the predictive space approximately constant when more components are added, e.g.
 - Gaussian processes
 - (regularized) Horseshoe for sparsity
 - R2-D2 and R2-D2-M2 for prior on R^2

Sometimes predictive model comparison can be useful

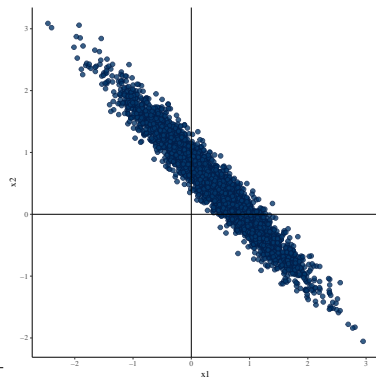


Marginal posterior intervals

Sometimes predictive model comparison can be useful



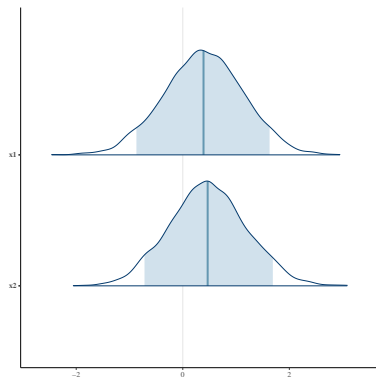
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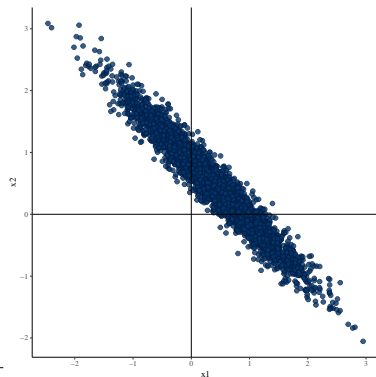
Joint posterior density

`rstanarm + bayesplot`

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Marginal posterior intervals



Joint posterior density

`rstanarm` + `bayesplot`

see also [Collinear demo](#)

What if one is not clearly better than others?

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- Continuous expansion including all models?
 - and then analyse the posterior distribution directly
avehtari.github.io/modelselection/betablockers.html
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Model averaging

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Model averaging

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- If needed integrate over the model space = model averaging
- Bayesian stacking may work better than BMA in case of misspecified models or small data
 - Yao, Vehtari, Simpson, & Gelman (2018)

Cross-validation and model selection

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- Overfitting in selection process is not unique for cross-validation

Selection induced bias and overfitting

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 - same data is used to assess the performance and make the selection
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 - the CV estimate for the selected model is biased
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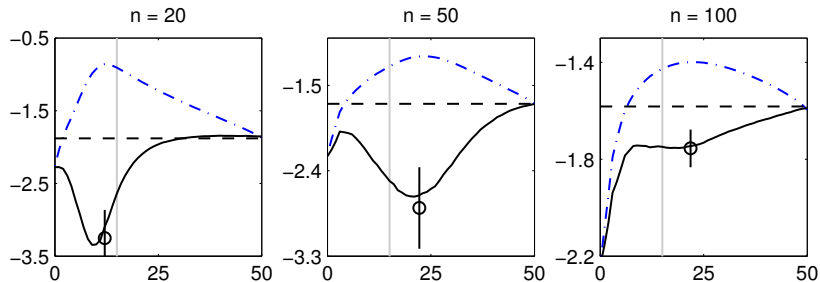
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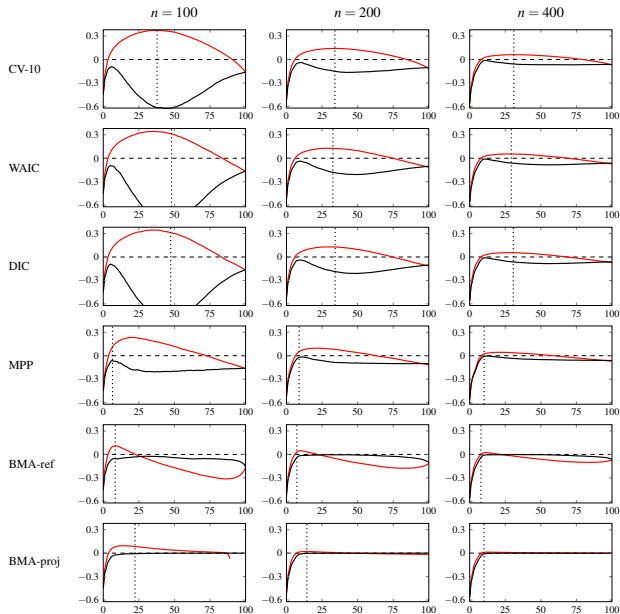
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- Bigger problem if there is a large number of models as in covariate selection

Selection induced bias in variable selection



Selection induced bias in variable selection



Take-home messages

- It's good to think predictions of observables, because observables are the only ones we can observe
- Cross-validation can simulate predicting and observing new data
- Cross-validation is good if you don't trust your model
- Different variants of cross-validation are useful in different scenarios
- Cross-validation has high variance, and **if** you trust your model you can beat cross-validation in accuracy

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