

Chapter 11

- 11.1 Gibbs sampler
- 11.2 Metropolis and Metropolis-Hastings
- 11.3 Using Gibbs and Metropolis as building blocks
- 11.4 Inference and assessing convergence (important)
 - potential scale reduction \hat{R}
- 11.5 Effective number of simulation draws (important)
 - effective sample size (ESS)
- 11.6 Example: hierarchical normal model (quick glance)

Chapter 11 demos

- demo11_1: Gibbs sampling
- demo11_2: Metropolis sampling
- demo11_3: Convergence of Markov chain
- demo11_4: split- \hat{R} and effective sample size (ESS or S_{eff})

It's all about expectations

$$E_{p(\theta|y)}[f(\theta)] = \int f(\theta) p(\theta|y) d\theta,$$

$$\text{where } p(\theta|y) = \frac{p(y|\theta)p(\theta)}{\int p(y|\theta)p(\theta)d\theta}$$

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- Monte Carlo methods which can sample from $p(\theta^{(s)}|y)$ using only $q(\theta^{(s)}|y)$

$$E_{p(\theta|y)}[f(\theta)] \approx \frac{1}{S} \sum_{s=1}^S f(\theta^{(s)})$$

Monte Carlo

- Monte Carlo methods we have discussed so far
 - Inverse CDF works for 1D
 - Analytic transformations work for only certain distributions
 - Factorization works only for certain joint distributions
 - Grid evaluation and sampling works in less than a few dimensions
 - Rejection sampling works mostly in 1D (truncation is a special case)
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- What to do in high dimensions?
 - Markov chain Monte Carlo (Ch 11-12)
 - Laplace, Variational*, EP* (Ch 4,13*)

Markov chain

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- The probability of each event depends only on the state attained in the previous event (or finite number of previous events)
- Markov's one example was the sequence of letters in Pushkin's novel "Yevgeniy Onegin"

Markov chain

- Example of a simple Markov chain

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 - + central limit theorem holds for expectations
 - draws are dependent
 - construction of efficient Markov chains is not always easy

Markov chain

- Set of random variables $\theta^1, \theta^2, \dots$, so that with all values of t , θ^t depends only on the previous $\theta^{(t-1)}$

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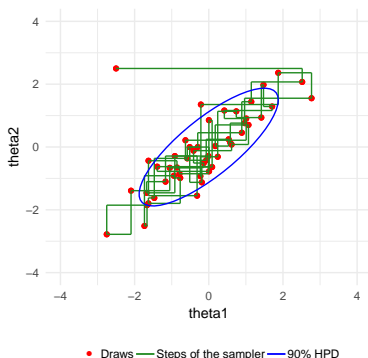
- Chain has to be initialized with some starting point θ^0
- Transition distribution $T_t(\theta^t | \theta^{t-1})$ (may depend on t)
 - by choosing a suitable transition distribution, the stationary distribution of Markov chain is $p(\theta | y)$

Gibbs sampling

- Alternate sampling from 1D conditional distributions
 - 1D is easy even if no conjugate prior and analytic posterior

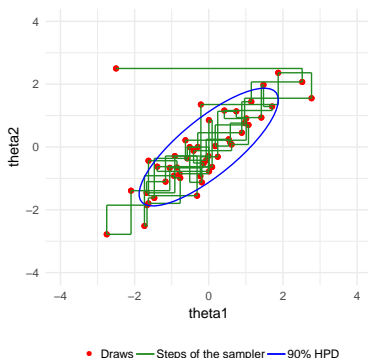
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- Basic algorithm

sample θ_j^t from $p(\theta_j | \theta_{-j}^{t-1}, y),$

where $\theta_{-j}^{t-1} = (\theta_1^t, \dots, \theta_{j-1}^t, \theta_{j+1}^{t-1}, \dots, \theta_d^{t-1})$

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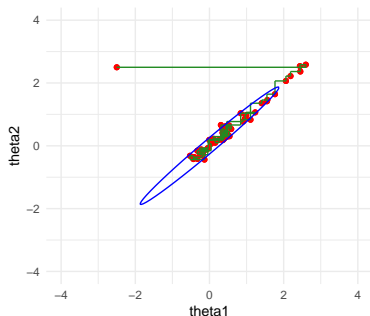
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- Slow if parameters are highly dependent in the posterior
 - demo11_1 continues



Conditional vs joint

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Conditional vs joint

- How about sampling θ jointly?
 - e.g. it is easy to sample from multivariate normal
- Can we use that to form a Markov chain?

Metropolis algorithm

- Algorithm
 1. starting point θ^0
 2. $t = 1, 2, \dots$
 - (a) pick a proposal θ^* from the proposal distribution $J_t(\theta^*|\theta^{t-1})$.
Proposal distribution has to be symmetric, i.e.
 $J_t(\theta_a|\theta_b) = J_t(\theta_b|\theta_a)$, for all θ_a, θ_b

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$$\theta^t = \begin{cases} \theta^* & \text{with probability } \min(r, 1) \\ \theta^{t-1} & \text{otherwise} \end{cases}$$

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ie, if $p(\theta^*|y) > p(\theta^{t-1}|y)$ accept the proposal always
and otherwise accept the proposal with probability r

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- step c is executed by generating a random number from $U(0, 1)$
- $p(\theta^*|y)$ and $p(\theta^{t-1}|y)$ have the same normalization terms, and thus instead of $p(\cdot|y)$, unnormalized $q(\cdot|y)$ can be used, as the normalization terms cancel out!

Metropolis algorithm

- Example: one bivariate observation (y_1, y_2)
 - bivariate normal distribution with unknown mean and known covariance

$$\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} \Big| y \sim N \left(\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \right)$$

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- More examples <https://chi-feng.github.io/mcmc-demo/>

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- Theoretically
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 2. Prove that this stationary distribution is the desired target distribution

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 - = probability to return to a state i is 1
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- since their joint distribution is symmetric, θ^t and θ^{t-1} have the same marginal distributions, and so $p(\theta|y)$ is the stationary distribution of the Markov chain of θ

Metropolis-Hastings algorithm

- Generalization of Metropolis algorithm for non-symmetric proposal distributions
 - acceptance ratio includes ratio of proposal distributions

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 - small scale
 - many steps accepted, but the chain moves slowly due to small steps
 - big scale
 - long steps proposed, but many of those rejected and again chain moves slowly

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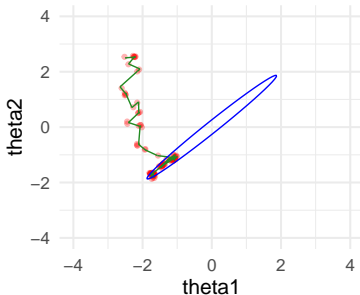
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- Generic rule for rejection rate is 60-90% (but depends on dimensionality and a specific algorithm variation)

Gibbs sampling

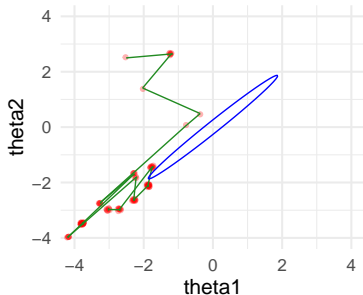
- Specific case of Metropolis-Hastings algorithm
 - single updated (or blocked)
 - proposal distribution is the conditional distribution
 - proposal and target distributions are same
 - acceptance probability is 1

Metropolis

- Usually doesn't scale well to high dimensions
 - if the shape doesn't match the whole distribution, the efficiency drops
 - demo11_2



• Draws — Steps of the sampler — 90% HPD



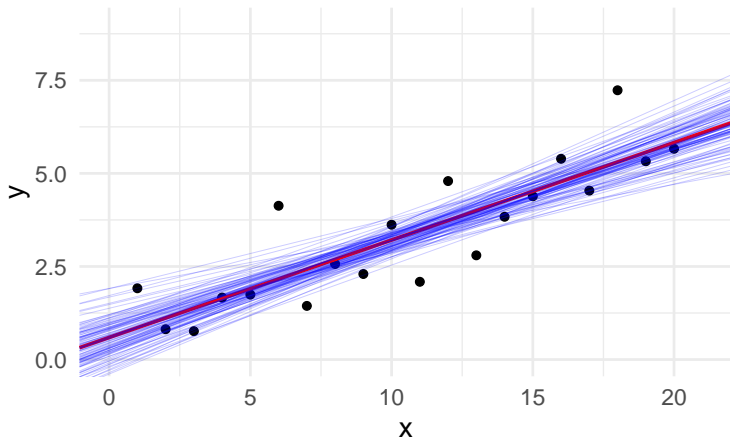
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Dynamic Hamiltonian Monte Carlo and NUTS

- Chapter 12 presents some more advanced methods
 - Chapter 12 includes Hamiltonian Monte Carlo and NUTS, which is one of the most efficient methods
 - uses gradient information
 - Hamiltonian dynamic simulation reduces random walk
 - state-of-the-art MCMC used by most modern probabilistic programming frameworks
- More next week

Example of uncertainty in modeling

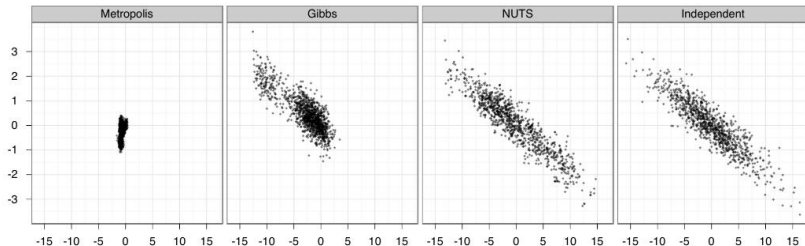
Posterior draws



HMC / NUTS

Comparison of algorithms on **highly correlated** 250-dimensional Gaussian distribution

- Do **1,000,000** draws with both Random Walk Metropolis and Gibbs, thinning by 1000
- Do **1,000** draws using Stan's NUTS algorithm (no thinning)
- Do 1,000 independent draws (we can do this for multivariate normal)

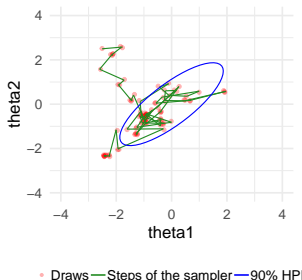


Warm-up and convergence diagnostics

- Asymptotically chain spends the $\alpha\%$ of time where $\alpha\%$ posterior mass is

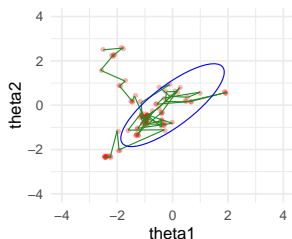
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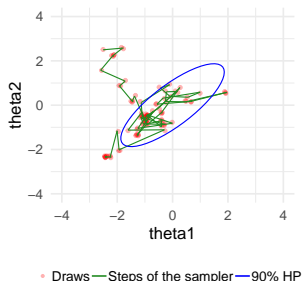


• Draws — Steps of the sampler — 90% HPD

- Warm-up = remove draws from the beginning of the chain
 - warm-up may include also phase for adapting algorithm parameters

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- Warm-up = remove draws from the beginning of the chain
 - warm-up may include also phase for adapting algorithm parameters
- Convergence diagnostics
 - Do we get samples from the target distribution?

MCMC draws are dependent

- Monte Carlo estimates still valid (central limit theorem holds)

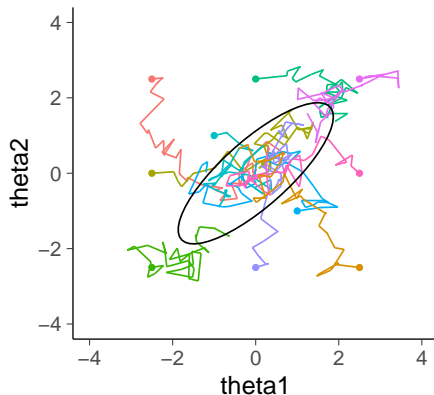
$$E_{p(\theta|y)}[f(\theta)] \approx \frac{1}{S} \sum_{s=1}^S f(\theta^{(s)})$$

- Estimation of Monte Carlo error is more difficult
 - evaluation of *effective* sample size

Several chains

- Use of several chains make convergence diagnostics easier
- Start chains from different starting points – preferably overdispersed

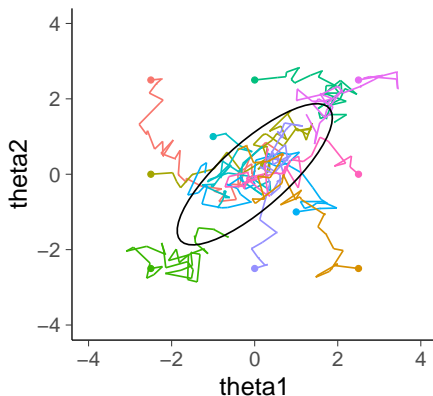
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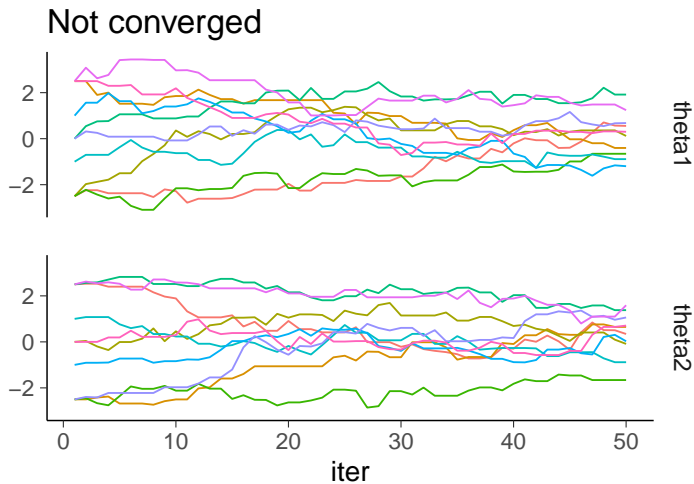
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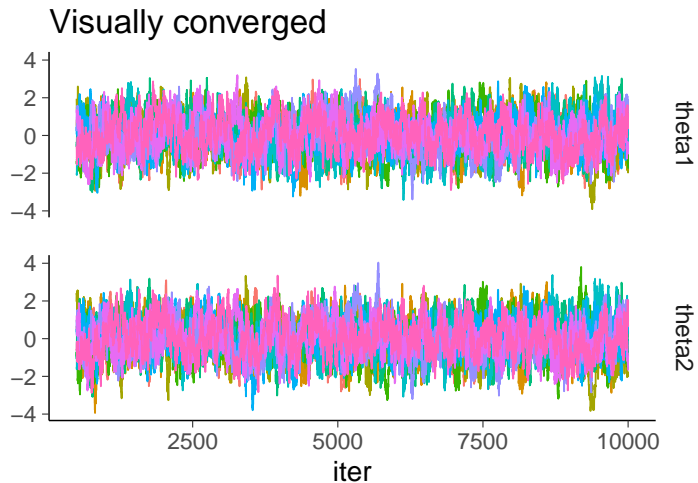


- Remove draws from the beginning of the chains and run chains long enough so that it is not possible to distinguish where each chain started and the chains are well mixed

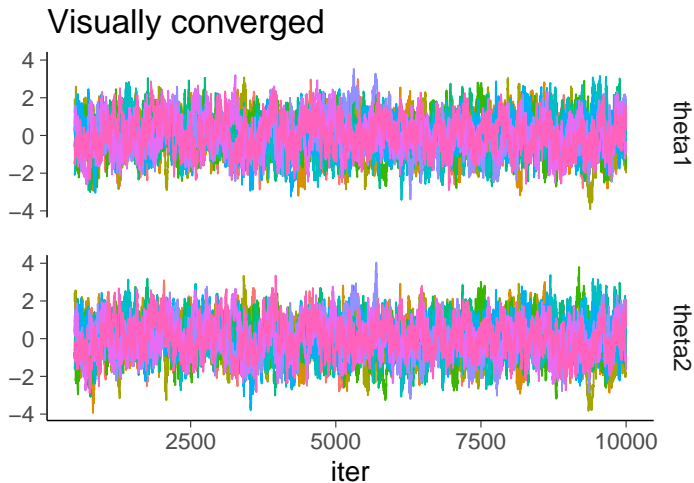
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Visual convergence check is not sufficient

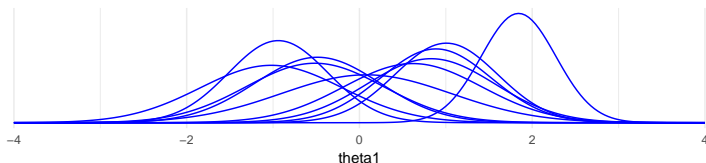
\hat{R} : comparison of within and between variances of the chains

- BDA3: \hat{R} aka *potential scale reduction factor* (PSRF)
- Compare means and variances of the chains

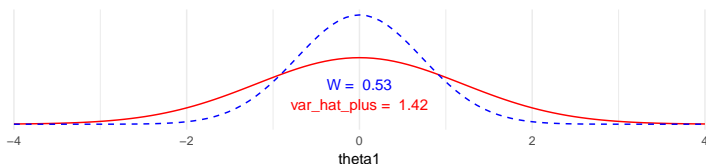
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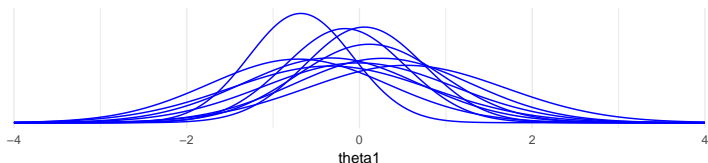
Rhat = 1.64



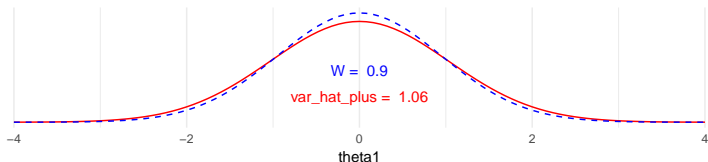
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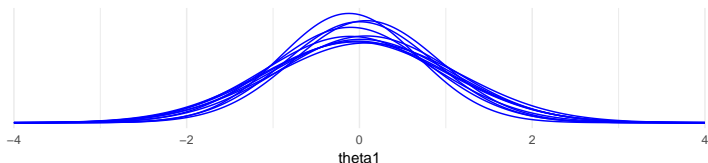
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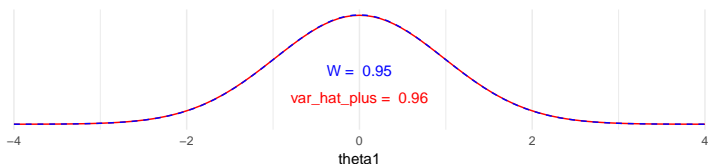
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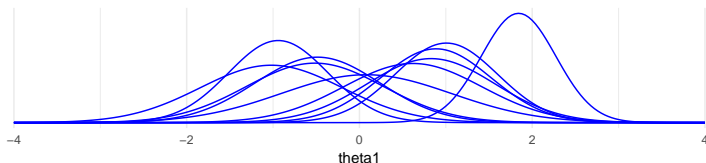
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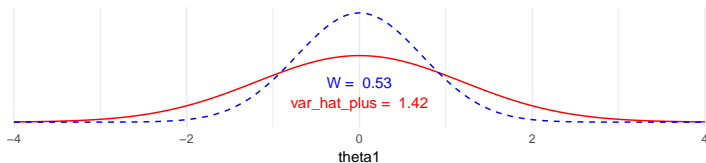
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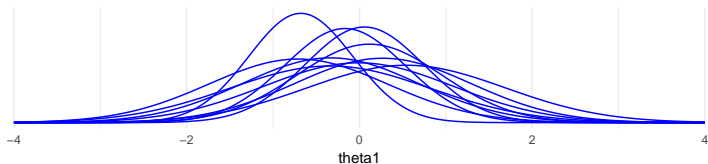


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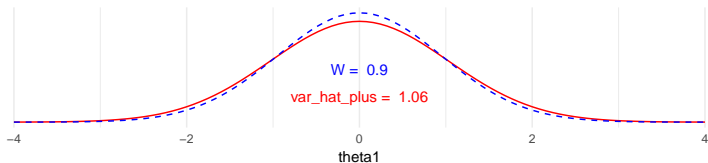


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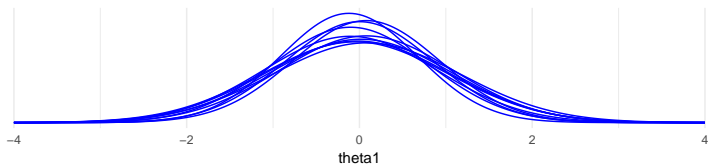


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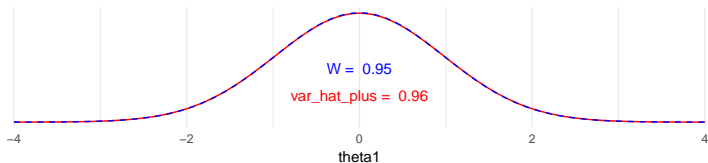


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- Estimates how much the scale of ψ could reduce if $N \rightarrow \infty$
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- If \hat{R} close to 1, it is still possible that chains have not converged
 - if starting points were not overdispersed
 - distribution far from normal (especially if infinite variance)
 - just by chance when N is finite

Split- \hat{R}

- BDA3: split- \hat{R}
- Examines *mixing* and *stationarity* of chains
- To examine stationarity chains are split to two parts
 - after splitting, we have M chains, each having N draws
 - scalar draws θ_{nm} ($n = 1, \dots, N; m = 1, \dots, M$)
 - compare means and variances of the split chains

Rank normalized \hat{R}

- Original \hat{R} requires that the target distribution has finite mean and variance

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Rank-normalization, folding, and localization: An improved R-hat
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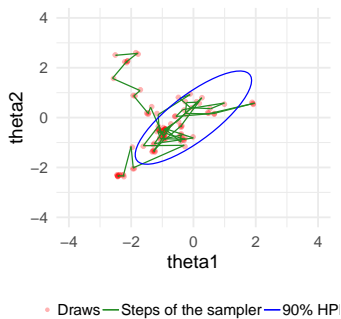
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- Notation updated compared to BDA3

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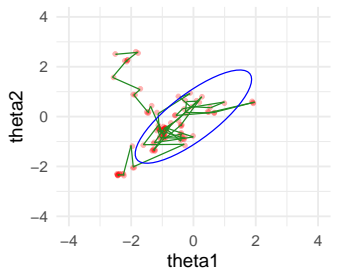
Time series analysis

- Autocorrelation function
 - describes the correlation given a certain lag
 - can be used to compare efficiency of MCMC algorithms and parameterizations

Autocorrelation

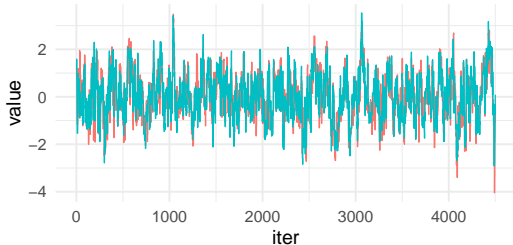


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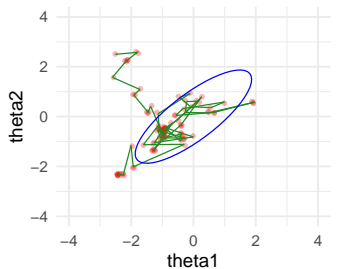


• Draws — Steps of the sampler — 90% HP

Trends

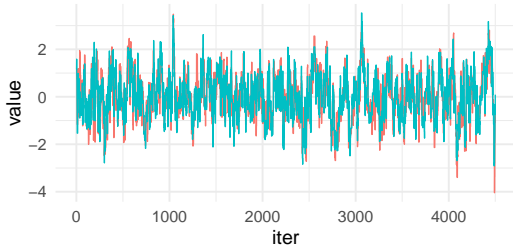


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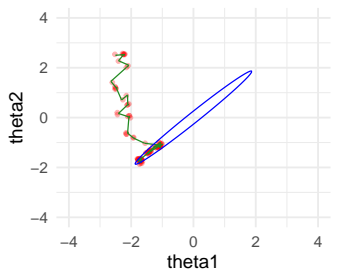


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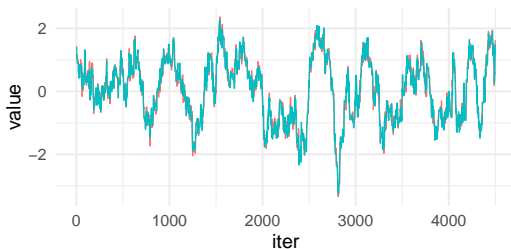


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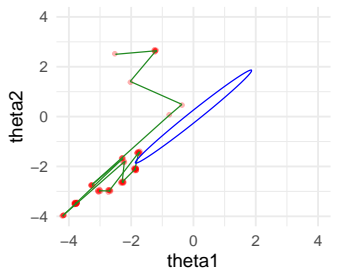


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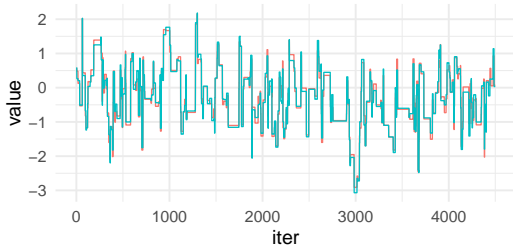


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- Time series analysis can be used to estimate Monte Carlo error in case of MCMC
- For expectation $\bar{\theta}$

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- new R-hat paper $S = NM$ (in BDA3 $N = nm$ and $n_{\text{eff}} = N/\tau$)
- BDA3 focuses on S_{eff} and not the Monte Carlo error directly
new R-hat paper discusses more about MCSEs for different quantities

Time series analysis

- Estimation of the autocorrelation using several chains

$$\hat{\rho}_n = 1 - \frac{W - \frac{1}{M} \sum_{m=1}^M \hat{\rho}_{n,m}}{2\widehat{\text{var}}^+}$$

where $\hat{\rho}_{n,m}$ is autocorrelation at lag n for chain m

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- BDA3 has slightly different and less accurate equation. The above equation is used in Stan 2.18+
- Compared to a method which computes the autocorrelation from a single chain, the multi-chain estimate has smaller variance

Time series analysis

- Estimation of τ

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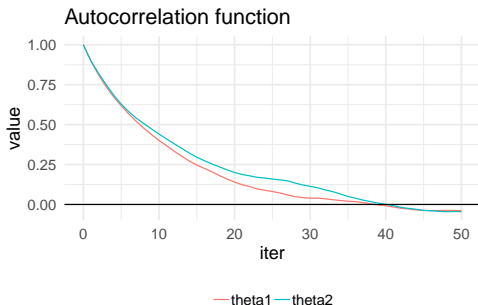
- As τ is estimated from a finite number of draws, it's expectation is overoptimistic
 - if $\hat{\tau} > MN/20$ then the estimate is unreliable

Geyer's adaptive window estimator

- Truncation can be decided adaptively
 - for stationary, irreducible, recurrent Markov chain
 - let $\Gamma_m = \rho_{2m} + \rho_{2m+1}$, which is sum of two consequent autocorrelations
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 - Γ_m is positive, decreasing and convex function of m
- Initial positive sequence estimator (Geyer's IPSE)
 - Choose the largest m so, that all values of the sequence $\hat{\Gamma}_1, \dots, \hat{\Gamma}_m$ are positive

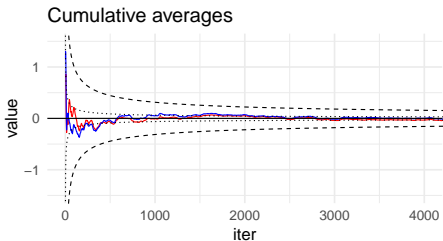
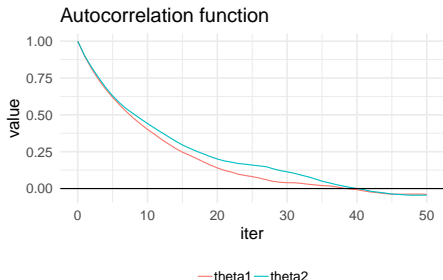
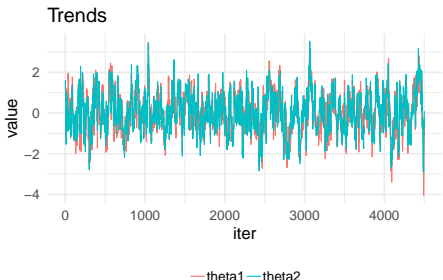


Effective sample size

Effective sample size $ESS = S_{\text{eff}} \approx S/\hat{\tau}$

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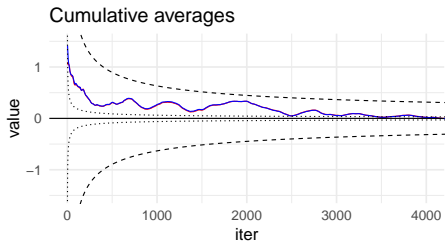
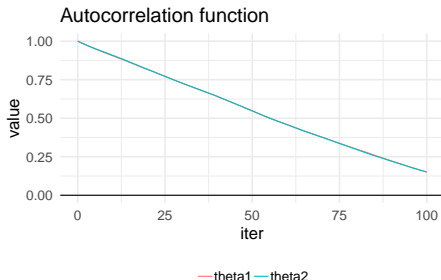
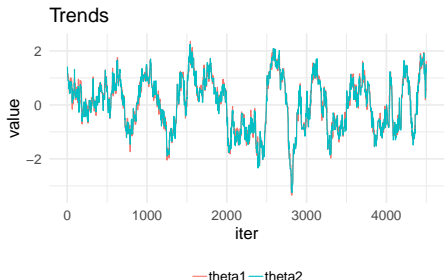
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$$\hat{\tau} = 1 + 2 \sum_{t=1}^T \hat{\rho}_t$$
$$\approx 24$$

Effective sample size

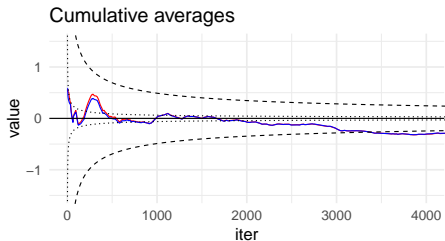
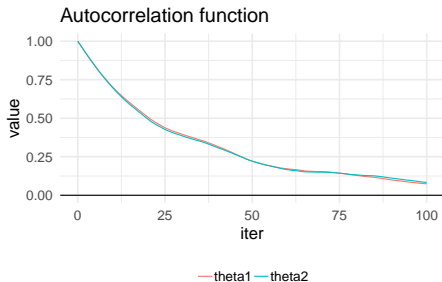
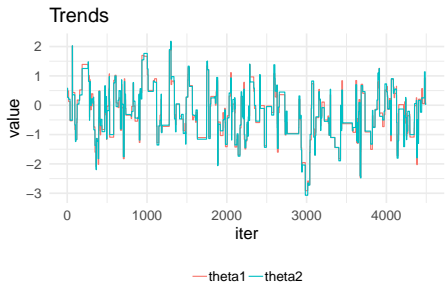
$$\text{Effective sample size ESS} = S_{\text{eff}} \approx S/\hat{\tau}$$



$$\hat{\tau} = 1 + 2 \sum_{t=1}^T \hat{\rho}_t$$
$$\approx 104$$

Effective sample size

Effective sample size $ESS = S_{\text{eff}} \approx S/\hat{\tau}$



$$\hat{\tau} = 1 + 2 \sum_{t=1}^T \hat{\rho}_t$$

$$\approx 63$$

Problematic distributions

- Nonlinear dependencies
 - optimal proposal depends on location

Problematic distributions

- Nonlinear dependencies
 - optimal proposal depends on location
- Funnels
 - optimal proposal depends on location

Problematic distributions

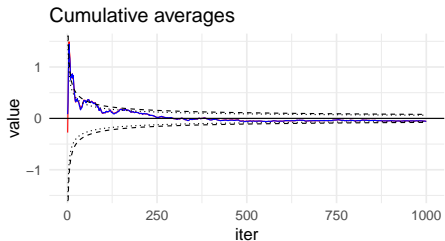
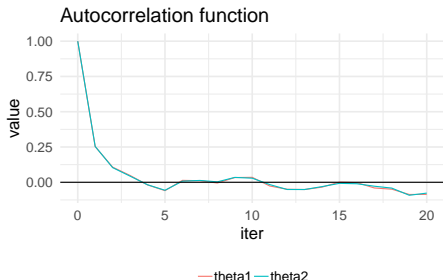
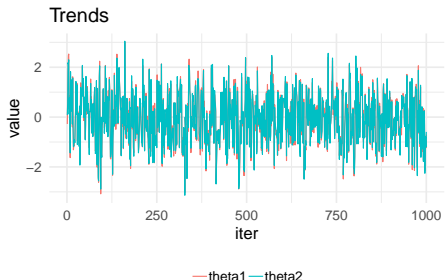
- Nonlinear dependencies
 - optimal proposal depends on location
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 - difficult to move from one mode to another

Problematic distributions

- Nonlinear dependencies
 - optimal proposal depends on location
- Funnels
 - optimal proposal depends on location
- Multimodal
 - difficult to move from one mode to another
- Long-tailed with non-finite variance and mean
 - central limit theorem for expectations does not hold

Next week: HMC, NUTS, and dynamic HMC

Effective sample size $ESS = S_{\text{eff}} \approx S/\hat{\tau}$



$$\hat{\tau} = 1 + 2 \sum_{t=1}^T \hat{\rho}_t$$
$$\approx 1.6$$

Further diagnostics

- Dynamic HMC/NUTS has additional diagnostics
 - divergences
 - tree depth exceedences
 - fraction of missing information