Chapter 5

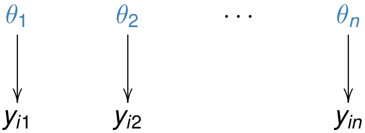
- 5.1 Lead-in to hierarchical models
- 5.2 Exchangeability (useful concept)
- 5.3 Bayesian analysis of hierarchical models (but we use Stan for computation)
- 5.4 Hierarchical normal model (but we use Stan for computation)
- 5.5 Example: parallel experiments in eight schools (uses hierarchical normal model, part of assignment, but we use Stan for computation)
- 5.6 Meta-analysis (can be skipped)
- 5.7 Weakly informative priors for hierarchical variance parameters

Hierarchical model

- In simple model: posterior for the parameters
- In hierarchical model: posterior for the prior parameters

Hierarchical model

- Example: CVD treatment effectiveness
  - in hospital j the survival probability is θ_j
  - observations y_{ij} tell whether patient i survived in hospital j



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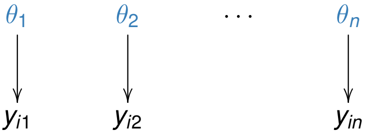
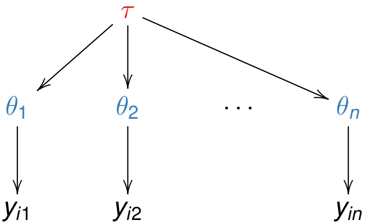


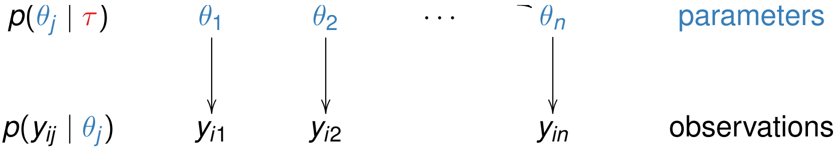
  - sensible to assume that θ_j are similar



  - natural to think that θ_j have common population distribution
  - θ_j is not directly observed and the population distribution is unknown

Hierarchical model: terms

Level 1: observations given parameters $p(y_{ij} \mid \theta_j)$



$p(\theta_j \mid \tau)$ θ_1 θ_2 $\cdots$ θ_n parameters

$p(y_{ij} \mid \theta_j)$ y_{i1} y_{i2} y_{in} observations

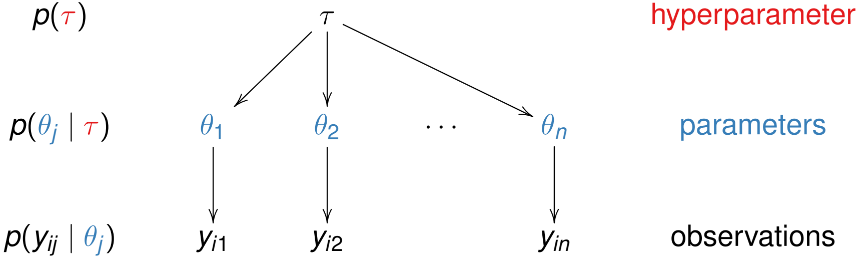
Joint posterior

$$\begin{aligned} p(\theta, \tau \mid y) &\propto p(y \mid \theta, \tau) p(\theta, \tau) \\ &\propto p(y \mid \theta) p(\theta \mid \tau) p(\tau) \end{aligned}$$

Hierarchical model: terms

Level 1: observations given parameters $p(y_{ij} \mid \theta_j)$

Level 2: parameters given hyperparameters $p(\theta_j \mid \tau)$



$p(\tau)$ τ hyperparameter

$p(\theta_j \mid \tau)$ θ_1 θ_2 $\cdots$ θ_n parameters

$p(y_{ij} \mid \theta_j)$ y_{i1} y_{i2} y_{in} observations

Joint posterior

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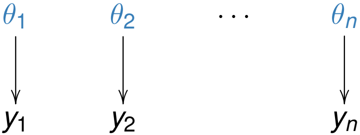
\end{aligned}
$$

Compare

- ”Separate model” (model with separate/independent effects)



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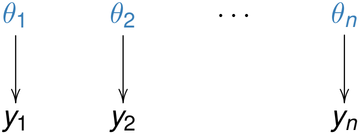
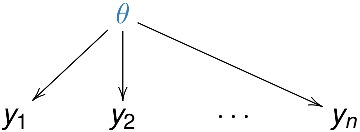


- ”Joint model” (model with a common effect / pooled model)



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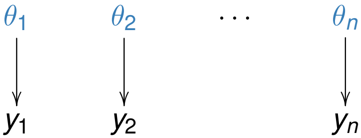

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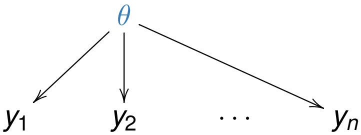
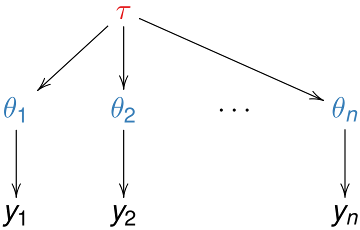


- Hierarchical model



Hierarchical binomial model: rats

- Medicine testing
- Type F344 female rats in control group given placebo
  - count how many get endometrial stromal polyps
  - familiar binomial model example

Hierarchical binomial model: rats

- Medicine testing
- Type F344 female rats in control group given placebo
  - count how many get endometrial stromal polyps
  - familiar binomial model example
- Experiment has been repeated 71 times

0/20	0/20	0/20	0/20	0/20	0/20	0/20	0/19	0/19	0/19
0/19	0/18	0/18	0/17	1/20	1/20	1/20	1/20	1/19	1/19
1/18	1/18	2/25	2/24	2/23	2/20	2/20	2/20	2/20	2/20
2/20	1/10	5/49	2/19	5/46	3/27	2/17	7/49	7/47	3/20
3/20	2/13	9/48	10/50	4/20	4/20	4/20	4/20	4/20	4/20
4/20	10/48	4/19	4/19	4/19	5/22	11/46	12/49	5/20	5/20
6/23	5/19	6/22	6/20	6/20	6/20	16/52	15/46	15/47	9/24
4/14									

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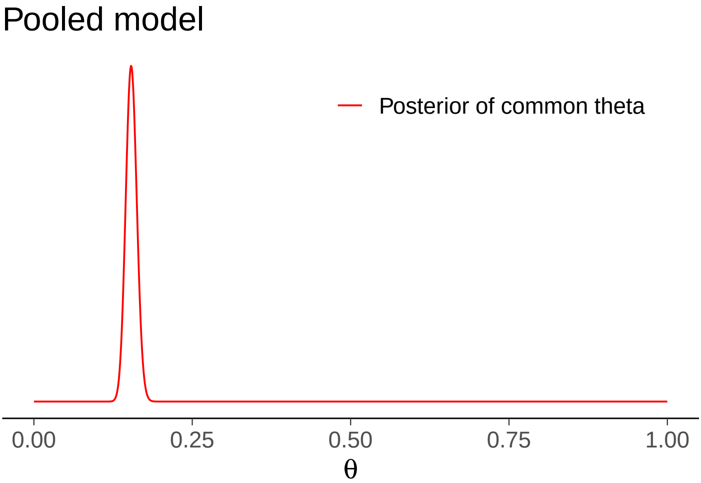

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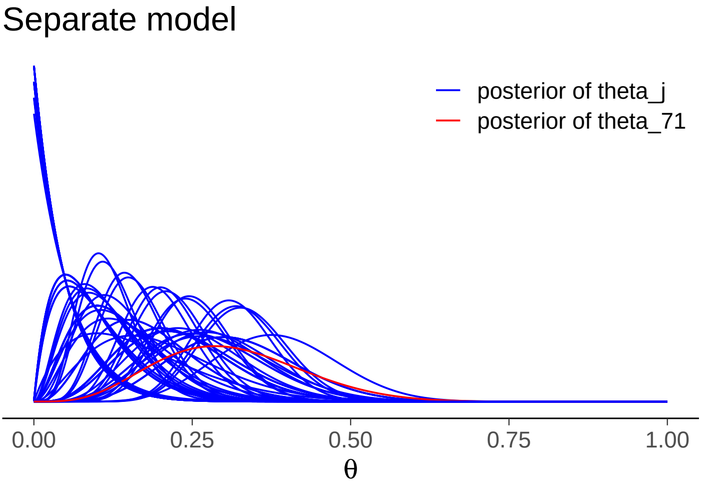
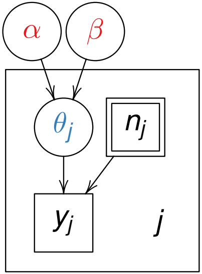


Hierarchical binomial model: rats

- Hierarchical binomial model for rats
  prior parameters α and β are unknown

$$\theta_j \mid \alpha, \beta \sim \mathrm{Beta}(\theta_j \mid \alpha, \beta)$$

$$y_j \mid n_j, \theta_j \sim \mathrm{Bin}(y_j \mid n_j, \theta_j)$$



- Joint posterior $p(\theta_1, \ldots, \theta_J, \alpha, \beta \mid y)$
  - multiple parameters

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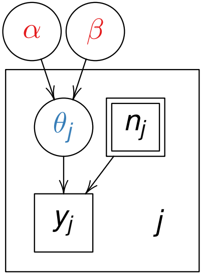
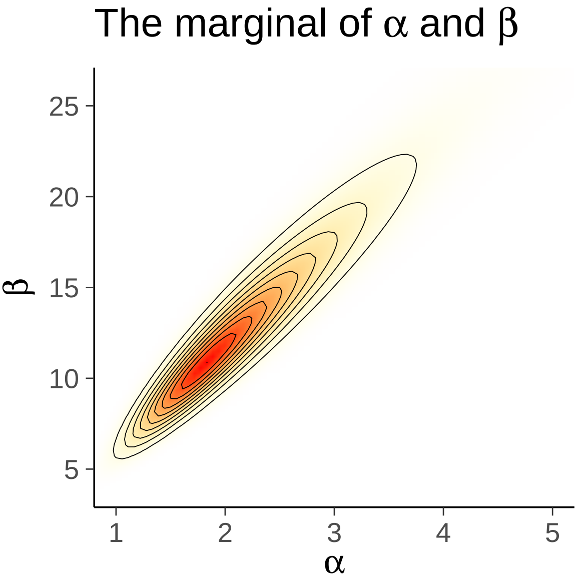


- Joint posterior $p(\theta_1, \ldots, \theta_J, \alpha, \beta \mid y)$
  - multiple parameters
  - factorize $\prod_{j=1}^{J} p(\theta_j \mid \alpha, \beta, y) p(\alpha, \beta \mid y)$

Hierarchical binomial model: rats

- Population prior $\mathrm{Beta}(\theta_j \mid \alpha, \beta)$
- Hyperprior $p(\alpha, \beta)$?
  - α, β both affect the location and scale
  - BDA3 has $p(\alpha, \beta) \propto (\alpha + \beta)^{-5/2}$
    - diffuse prior for location and scale (BDA3 p. 110)
- demo5_1

Hierarchical binomial model: rats



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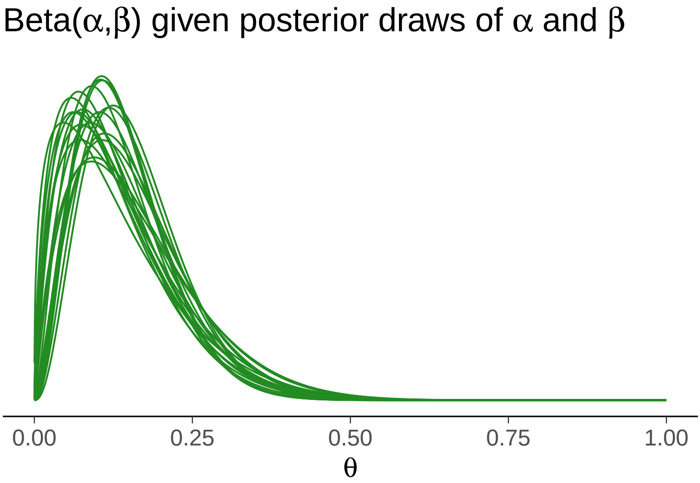
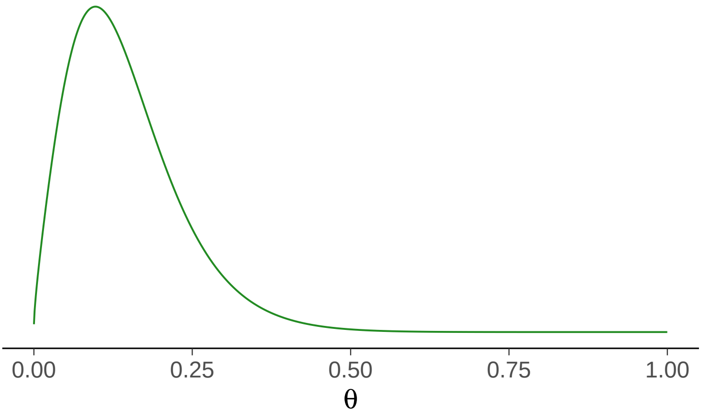


Hierarchical binomial model: rats

Population distribution (prior) for θ_j



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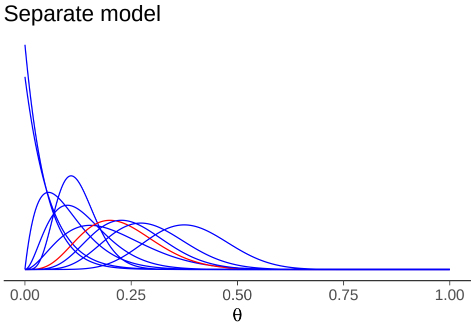
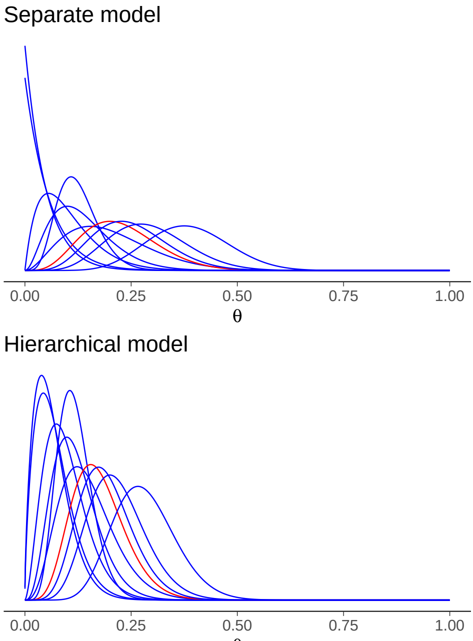


Separate model

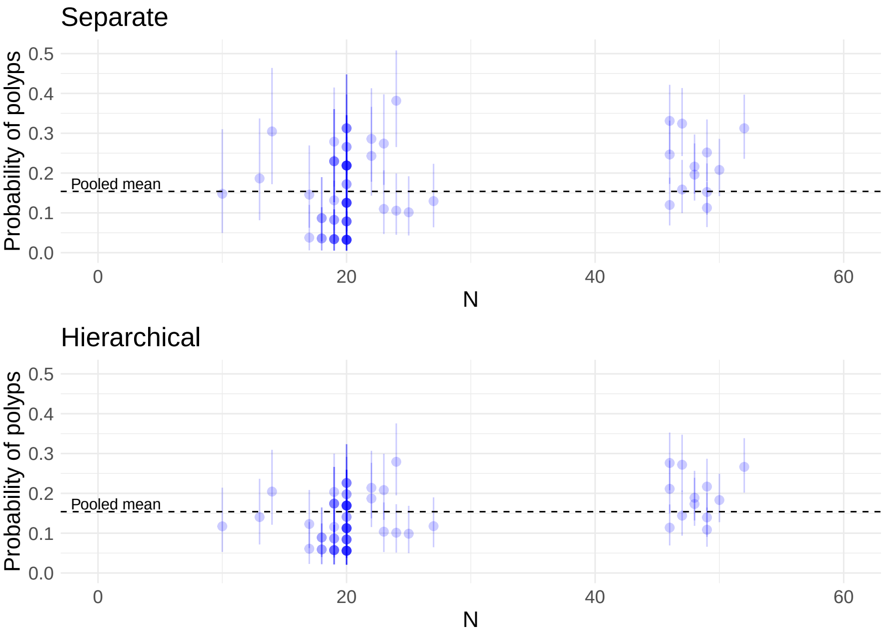
Hierarchical binomial model: rats



Separate model

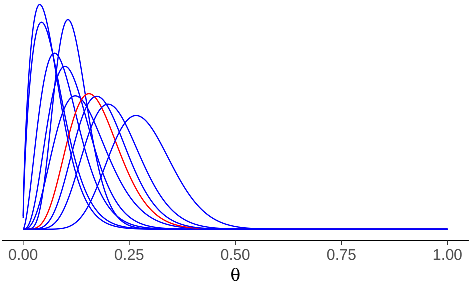
Hierarchical model

Hierarchical model and group size: Rats



Hierarchical binomial model: rats

Hierarchical model



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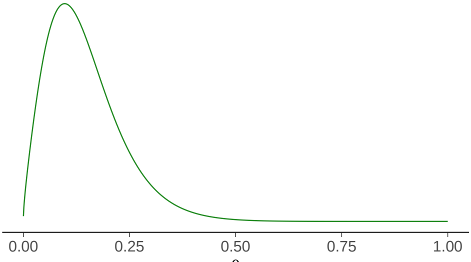
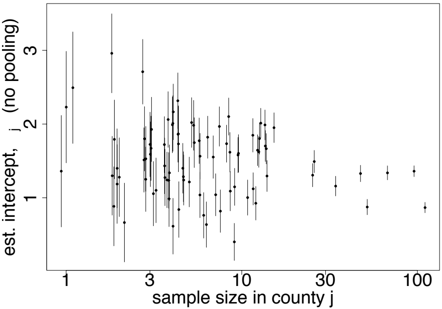


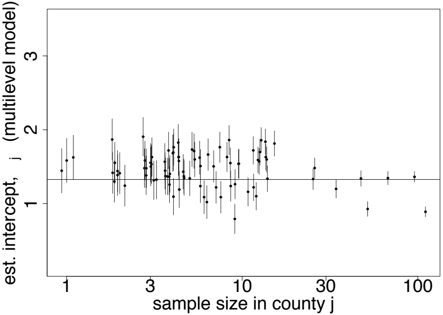
Hierarchical model and group size: Radon

919 home radon levels in 85 counties in Minnesota:

Separate



Hierarchical

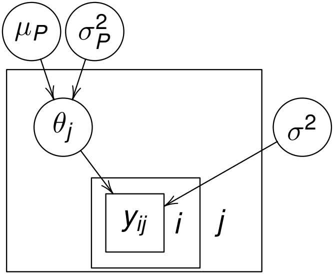


Hierarchical normal model: factory

- Factory has 6 machines which quality is evaluated
- Assume hierarchical model
  - each machine has its own (average) quality θ_j and common variance σ^2

$$\theta_j \mid \mu_P, \sigma_P^2 \sim \mathrm{N}(\mu_P, \sigma_P^2)$$

$$y_{ij} \mid \theta_j \sim \mathrm{N}(\theta_j, \sigma^2)$$



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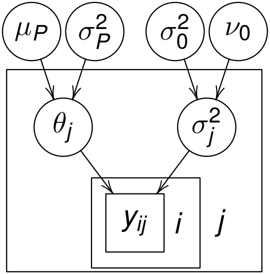

$\sigma_j^2 \mid \sigma_0^2, \nu_0 \sim \text{Inv-}\chi^2(\sigma_0^2, \nu_0)$

- Can be used to predict the future quality produced by each machine and quality produced by a new similar machine

Hierarchical normal model: 8 schools

- Example: SAT coaching effectiveness
  - in USA commonly used Scholastic Aptitude Test (SAT) is designed so that short term practice should not improve the results significantly
  - schools have anyway coaching courses
  - test the effectiveness of the coaching courses

Hierarchical normal model: 8 schools

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    - schools have anyway coaching courses
    - test the effectiveness of the coaching courses
- SAT
    - standardized multiple choice test
    - mean about 500 and standard deviation about 100
    - most scores between 200 and 800
    - different topics, e.g., V=Verbal, M=Mathematics
    - pre-test PSAT

Hierarchical normal model: 8 schools

- Effectiveness of the SAT coaching
  - students had made pre-tests PSAT-M and PSAT-V
  - part of students were coached
  - linear regression was used to estimate the coaching effect y_j for the school j (could be denoted with $\bar{y}_{.j}$, too) and variances σ_j^2
  - y_j approximately normally distributed, with variances assumed to be known based on about 30 students per school
  - data is group means and variances (not personal results)

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- Data:

School	A	B	C	D	E	F	G	H
y_j	28	8	-3	7	-1	1	18	12
σ_j	15	10	16	11	9	22	20	28

Hierarchical normal model for group means

- J experiments, unknown θ_j and known σ^2

$$y_{ij} \mid \theta_j \sim \mathrm{N}(\theta_j, \sigma^2), \quad i = 1, \ldots, n_j; \quad j = 1, \ldots, J$$

- Group j sample mean and sample variance

$$\begin{aligned} \bar{y}_{.j} &= \frac{1}{n_j} \sum_{i=1}^{n_j} y_{ij} \\ \sigma_j^2 &= \frac{\sigma^2}{n_j} \end{aligned}$$

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- Use model

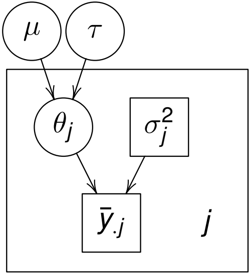
$$\bar{y}_{.j} \mid \theta_j \sim \mathrm{N}(\theta_j, \sigma_j^2)$$

this model can be generalized so that, σ_j^2 can be different from each other for other reasons than n_j

Hierarchical normal model for group means

$$\theta_j \mid \mu, \tau \sim \mathrm{N}(\mu, \tau^2)$$

$$\bar{y}_{.j} \mid \theta_j \sim \mathrm{N}(\theta_j, \sigma_j^2)$$



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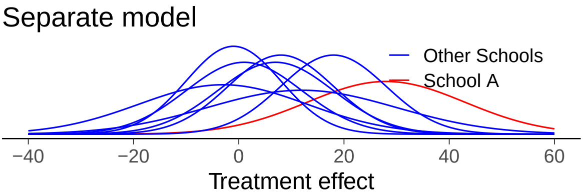
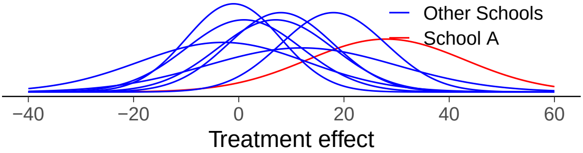
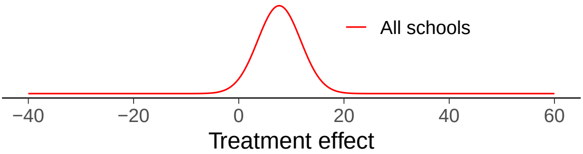


Hierarchical normal model: 8 schools

Separate model



Pooled model



Hierarchical normal model: 8 schools

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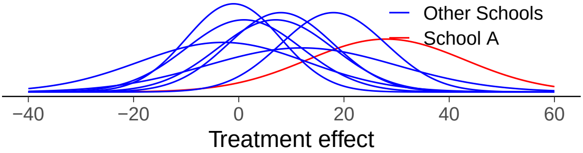

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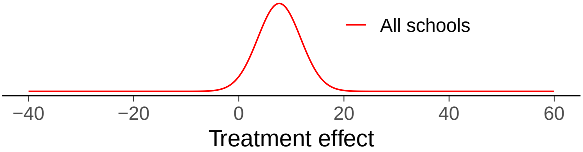

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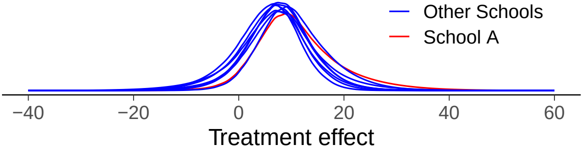

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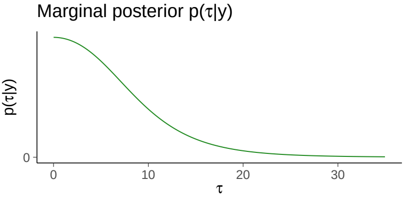
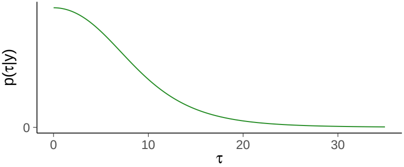
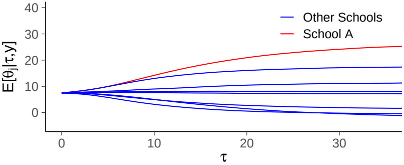


Hierarchical normal model: 8 schools

Marginal posterior $p(\tau|y)$



Conditional means $E[\theta_j|\tau,y]$



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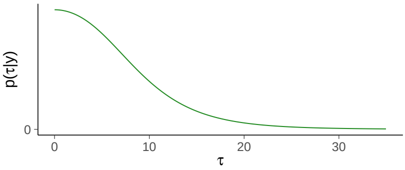

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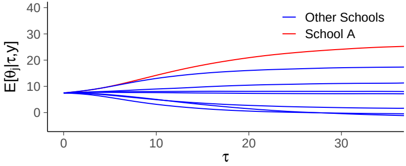

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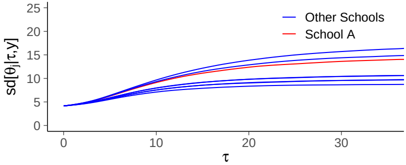

Conditional standard deviations $sd[\theta_j|\tau,y]$

Exchangeability

- Justifies why we can use
  - a joint model for data
  - a joint prior for a set of parameters
- Less strict than independence

Exchangeability

- *Exchangeability*: Parameters $\theta_1, \ldots, \theta_J$ (or observations $y_1, \ldots, y_J$) are exchangeable if the joint distribution p is invariant to the permutation of indices $(1, \ldots, J)$
- e.g.
$$p(\theta_1, \theta_2, \theta_3) = p(\theta_2, \theta_3, \theta_1)$$
- Exchangeability implies symmetry: If there is no information which can be used *a priori* to separate θ_j form each other, we can assume exchangeability. (”Ignorance implies exchangeability”)

Exchangeability

- Exchangeability does not mean that the results of the experiments could not be different
  - e.g. if we know that the experiments have been in two different laboratories, and we know that the other laboratory has better conditions for the rats, but we do not know which experiments have been made in which laboratory
  - a priori experiments are exchangeable
  - model could have unknown parameter for the laboratory with a conditional prior for rats assumed to come form the same place (clustering model)

Exchangeability and additional information

- Example: bioassay
  - y_i number of dead animals are not exchangeable alone

Exchangeability and additional information

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Exchangeability and additional information

- Example: bioassay
  - y_i number of dead animals are not exchangeable alone
  - x_i dose is additional information
  - (x_i, y_i) exchangeable and logistic regression was used

$$p(\alpha, \beta \mid y, n, x) \propto \prod_{i=1}^{n} p(y_i \mid \alpha, \beta, n_i, x_i) p(\alpha, \beta)$$

Hierarchical exchangeability

- Example: hierarchical rats example
    - all rats not exchangeable

Hierarchical exchangeability

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  - all rats not exchangeable
  - in a single laboratory rats exchangeable

Hierarchical exchangeability

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  - laboratories exchangeable

Hierarchical exchangeability

- Example: hierarchical rats example
  - all rats not exchangeable
  - in a single laboratory rats exchangeable
  - laboratories exchangeable
  - $\rightarrow$ hierarchical model

Partial or conditional exchangeability

- Conditional exchangeability
  - if y_i is connected to an additional information x_i, so that y_i are not exchangeable, but (y_i, x_i) exchangeable use joint model or conditional model $(y_i \mid x_i)$.

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- Partial exchangeability
  - if the observations can be grouped (a priori), then use hierarchical model

Exchangeability

- The simplest form of the exchangeability (but not the only one) for the parameters θ conditional independence

$$p(x_1, \ldots, x_J \mid \theta) = \prod_{j=1}^{J} p(x_j \mid \theta)$$

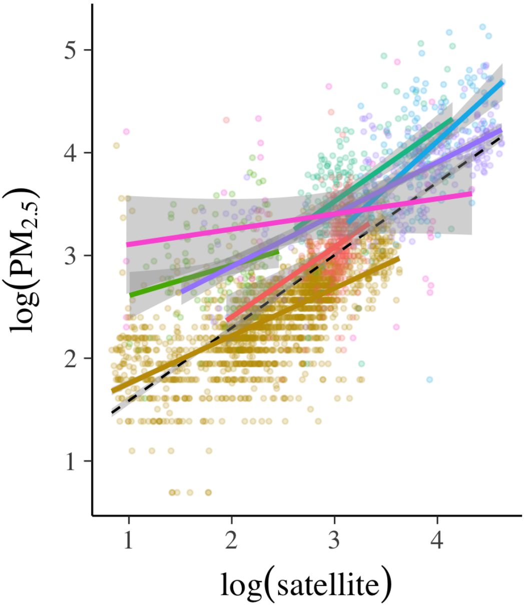
Exchangeability - Counter example

- A six sided die with probabilities $\theta_1, \ldots, \theta_6$
  - without additional knowledge $\theta_1, \ldots, \theta_6$ exchangeable
  - due to the constraint $\sum_{j=1}^{6} \theta_j$, parameters are not independent and thus joint distribution can not be presented as iid

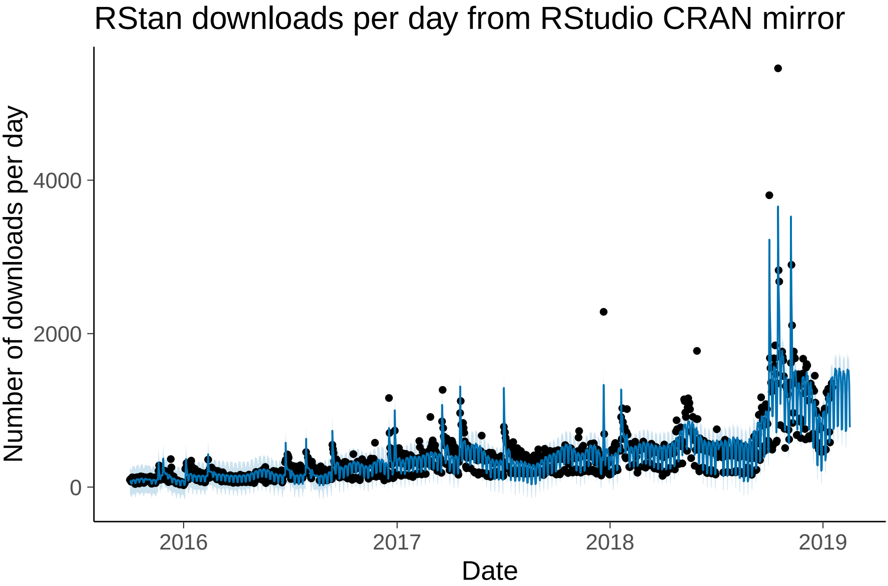
Exchangeability

- See more examples in the BDA_notes_ch5.pdf

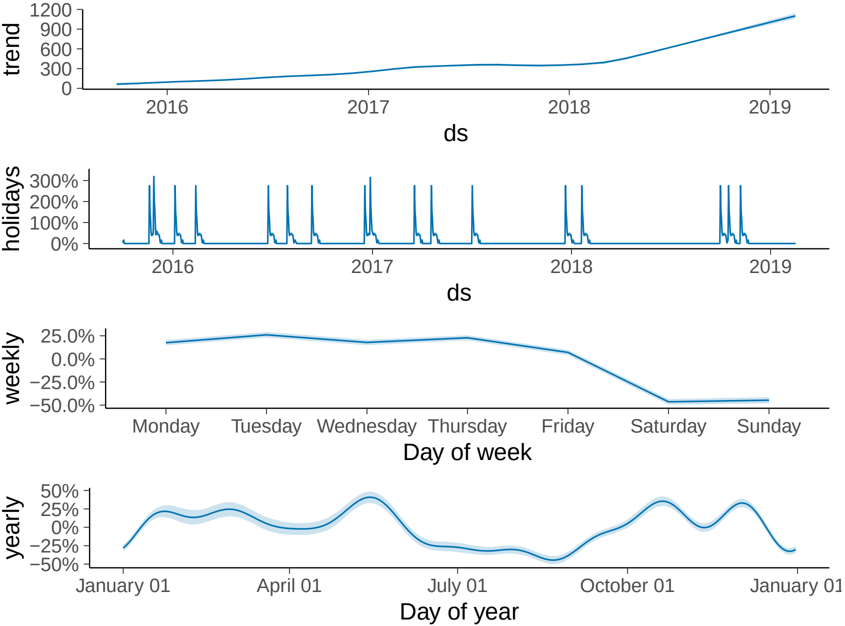
Hierarchical linear model



Hierarchical models and smooths



Hierarchical models and smooths



Hierarchical models and Bayesian deep learning

- Prior scale on each weight layer is a hyperparameter
  - connections between priors and drop-out / stochastic optimization
  - connections between posterior draws and big networks / ensembles