Detecting and Repairing Arbitrage

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Why must we detect and repair it?

- Model calibration
- Smoothing and filtering
- Data repair

Assumptions

- $T^e = \{T_i\}_{1 \le i \le m}$
- $P^{T,K} = \left\{(T_i, K_j^i)\right\}_{1 \le i \le m, 1 \le j \le n_i}$
- C_j^i
- Deterministic interest and dividends
- Zero-coupon bonds and forward contracts
- Static arbitrage

Constraints

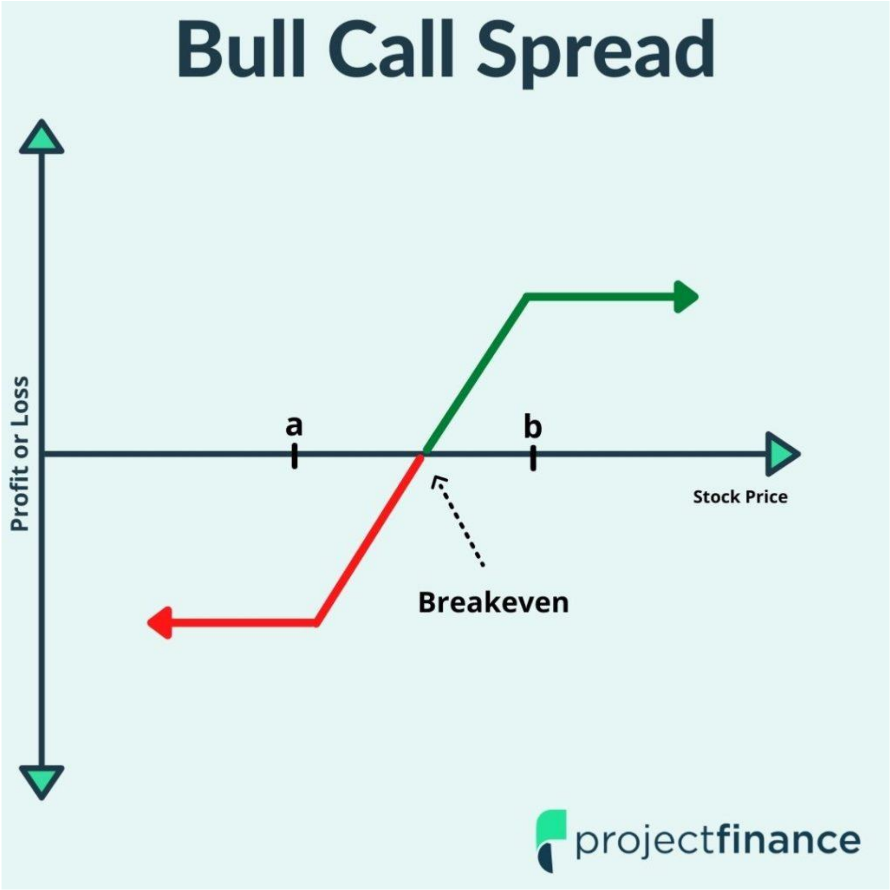
- First Fundamental Theorem of Asset Pricing
- Necessary and sufficient constraints
- Strategies

Constraints

- Vertical spread
- Calendar spread
- Calendar vertical spread
- Vertical butterfly
- Calendar butterfly

Vertical spread

- Long $\frac{1}{K^i_{\{j_1\}} - K^i_{\{j_2\}}}$ of $(i, j_1)th$ option
- Short $\frac{1}{K^i_{\{j_1\}} - K^i_{\{j_2\}}}$ of $(i, j_2)th$ option
- cost = $\frac{C^i_{\{j_2\}} - C^i_{\{j_1\}}}{K^i_{\{j_1\}} - K^i_{\{j_2\}}}$
- $0 \leq payoff \leq 1$



Reduction of constraints

Table 1. The reduced set of static arbitrage constraints.

Category	Constraints	Number
C1 Outright	$i \in [1, m], c^i_{n_i} \geq 0$	m
C2 Vertical spread	$i \in [1, m], j \in [1, n_i]$, $\mathrm{VS}^i_{j,j-1} \geq 0$ and $\mathrm{VS}^i_{1,0} \leq 1$	$N + m$
C3 Vertical butterfly	$i \in [1, m], j \in [1, n_i - 1], \mathrm{VB}^i_{j,j-1,j+1} \geq 0$	$N - m$
C4 Calendar spread	$1 \leq i_1 < i_2 \leq m, j_1 \in [0, n_{i_1}], j_2 \in [0, n_{i_2}]$, $\mathrm{CS}^{i_1,i_2}_{j_1,j_2} \geq 0$	$\mathcal{O}(mN)$
C5 Calendar vertical spread	$i^* \in [1, m], j^* \in [1, n_{i^*}]$, define $\mathcal{I} := \{i, j : T_i > T_{i^*}, k^{i^*}_{j^*-1} < k^i_j < k^{i^*}_{j^*}\}$, then $i, j \in \mathcal{I}, \mathrm{CVS}^{i^*,i}_{j^*,j} \geq 0$	$\mathcal{O}(mN)$
C6.1 Calendar butterfly I (Absolute location convexity)	$i^* \in [1, m], j^* \in [1, n_{i^*} - 1]$, define $\mathcal{I} := \{i, j : T_i > T_{i^*}, k^{i^*}_{j^*-1} < k^i_j < k^{i^*}_{j^*}\}$, then $i, j \in \mathcal{I}, \mathrm{CB}^{i^*,i,i^*}_{j^*,j,j^*+1} \geq 0$;	$\mathcal{O}(m^2N)$
	$i^* \in [1, m], j^* \in [2, n_{i^*}]$, define $\mathcal{I} := \{i, j : T_i > T_{i^*}, k^{i^*}_{j^*-1} < k^i_j < k^{i^*}_{j^*}\}$, then $i, j \in \mathcal{I}, \mathrm{CB}^{i^*,i^*,i}_{j^*-1,j^*-2,j} \geq 0$;	
	$i^* \in [1, m]$, define $\mathcal{I} := \{i, j : T_i > T_{i^*}, k^i_j > k^{i^*}_{n_{i^*}}\}$, then $i, j \in \mathcal{I}, \mathrm{CB}^{i^*,i^*,i}_{n_{i^*},n_{i^*}-1,j} \geq 0$	
C6.2 Calendar butterfly II (Relative location convexity)	$i^* \in [1, m], j^* \in [1, n_{i^*} - 1]$, define $\mathcal{I}_1 := \{i, j : T_i > T_{i^*}, k^{i^*}_{j^*-1} < k^i_j < k^{i^*}_{j^*}\}$, $\mathcal{I}_2 := \{i, j : T_i > T_{i^*}, k^{i^*}_{j^*} < k^i_j < k^{i^*}_{j^*+1}\}$, $i_1, j_1 \in \mathcal{I}, \quad i_2, j_2 \in \mathcal{I}_2, \mathrm{CB}^{i^*,i_1,i_2}_{j^*,j_1,j_2} \geq 0$;	$\mathcal{O}(m^2N)$
	$i^* \in [1, m]$, define $\mathcal{I}_1 := \{i, j : T_i > T_{i^*}, k^{i^*}_{n_{i^*}-1} < k^i_j < k^{i^*}_{n_{i^*}}\}$, $\mathcal{I}_2 := \{i, j : T_i > T_{i^*}, k^i_j > k^{i^*}_{n_{i^*}}\}$, $i_1, j_1 \in \mathcal{I}, \quad i_2, j_2 \in \mathcal{I}_2, \mathrm{CB}^{i^*,i_1,i_2}_{n_{i^*},j_1,j_2} \geq 0$	

Equivalent optimization problem

$$\boldsymbol{minimize} \quad \boldsymbol{f(\epsilon)} \ \boldsymbol{subject\ to} \quad \boldsymbol{A\epsilon \geq b - Ac}$$

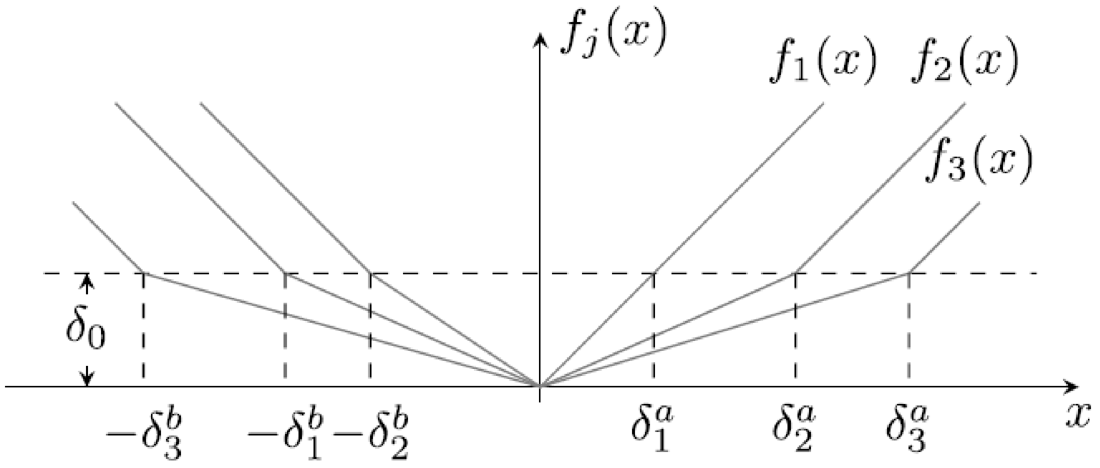
How to choose $\boldsymbol{f}$

- $l^2 - norm$
- $l^0 - norm$
- $l^1 - norm$
- $l_{BA} - norm$

$l^0 - norm\ \&\ l^1 - norm$

$l_{BA} - norm$

- $f(\epsilon) = \sum f_j(\epsilon_j)$
- $f_j(0) = \inf f_j(x) = 0$
- $f_j(x)$ *is increasing for* $x > 0$
- $f_j(-\delta_j^b) = f_j(\delta_j^a) = \delta_0$
- $\frac{df_j(x)}{d|x|} = 1 \; for \; x \neq (-\delta_j^b, \delta_j^a)$



$l_{BA} - norm$

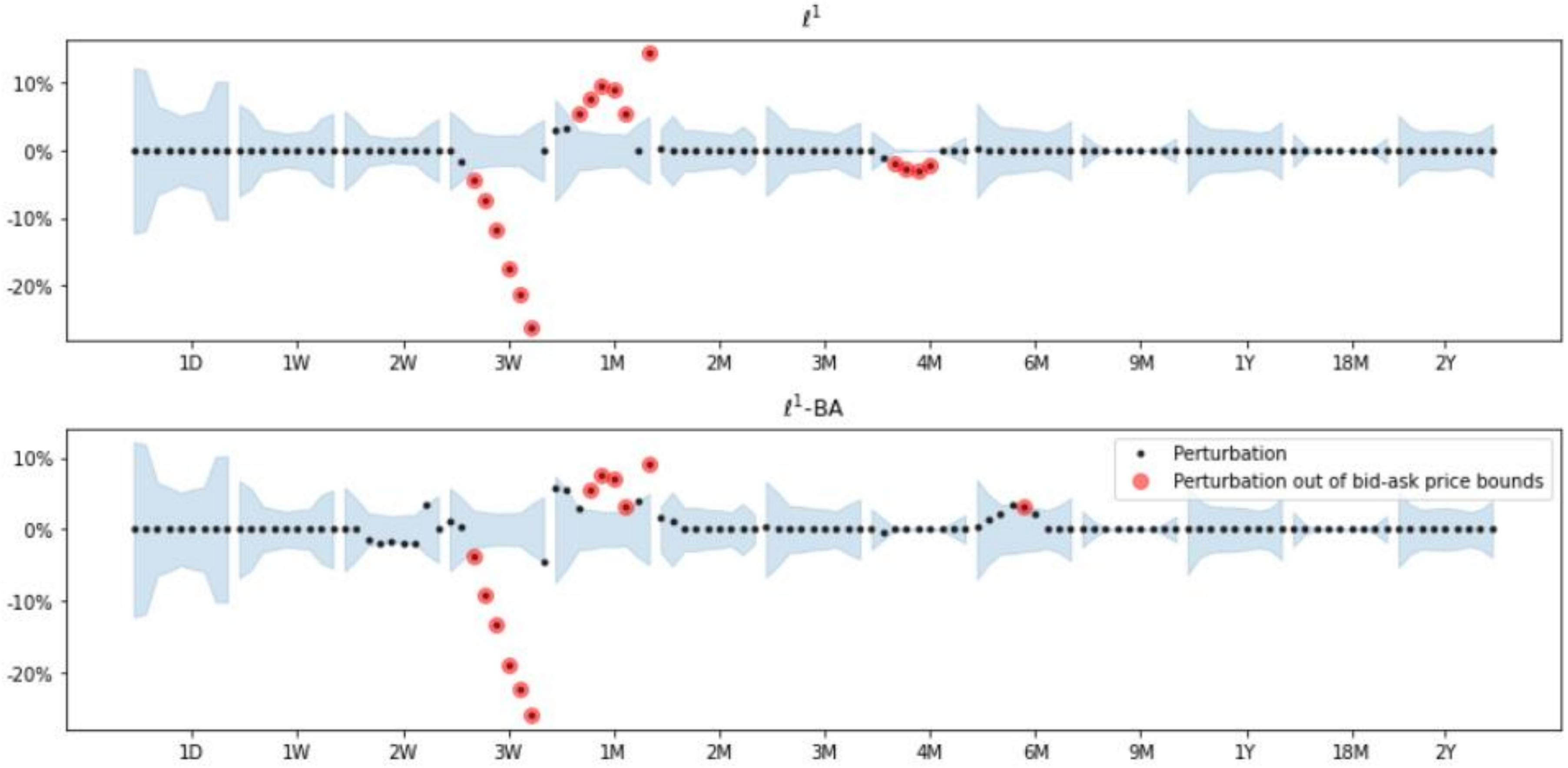
$$f_j(x) = max(-x - \delta_j^b + \delta_0, x - \delta_j^a + \delta_0, -\frac{\delta_0}{\delta_j^b}x, \frac{\delta_0}{\delta_j^a}x)$$

$$f(\varepsilon) = \sum_{j=1}^{N} \max\left(-\mathbf{e}_j^\top \varepsilon - \delta_j^b + \delta_0, -\frac{\delta_0}{\delta_j^b}\mathbf{e}_j^\top \varepsilon, \frac{\delta_0}{\delta_j^a}\mathbf{e}_j^\top \varepsilon, \mathbf{e}_j^\top \varepsilon - \delta_j^a + \delta_0\right)$$

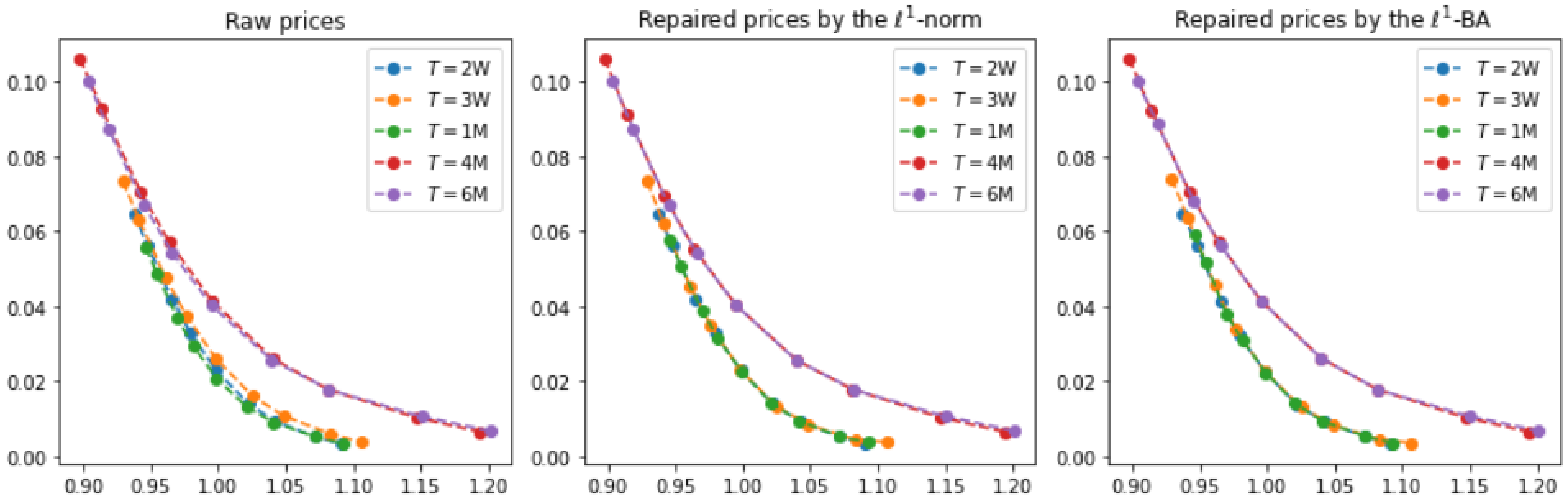
Programming using Python

- Requirements
- Constraints
- Repair
- Arbitrage_repair

Comparison


ℓ^1
10%
0%
-10%
-20%
1D
1W
2W
3W
1M
2M
3M
4M
6M
9M
1Y
18M
2Y
ℓ^1-BA
Perturbation
Perturbation out of bid-ask price bounds

Comparison



Raw prices
Repaired prices by the ℓ^1-norm
Repaired prices by the ℓ^1-BA
$T = 2W$
$T = 3W$
$T = 1M$
$T = 4M$
$T = 6M$
0.00
0.02
0.04
0.06
0.08
0.10
0.90
0.95
1.00
1.05
1.10
1.15
1.20