Problem 1

Consider two point masses m_1, m_2 connected by massless rods of lengths l_1, l_2 in a vertical plane.

Coordinates

$$x_1 = l_1 \sin\phi_1, \qquad y_1 = -l_1\cos\phi_1, \tag{1}$$
$$x_2 = l_1 \sin\phi_1 + l_2\sin\phi_2, \qquad y_2 = -l_1\cos\phi_1 - l_2\cos\phi_2 \tag{2}$$

Velocities

$$\dot{x}_1 = l_1\dot{\phi}_1\cos\phi_1, \qquad \dot{y}_1 = l_1\dot{\phi}_1\sin\phi_1, \tag{3}$$
$$\dot{x}_2 = l_1\dot{\phi}_1\cos\phi_1 + l_2\dot{\phi}_2\cos\phi_2, \qquad \dot{y}_2 = l_1\dot{\phi}_1\sin\phi_1 + l_2\dot{\phi}_2\sin\phi_2. \tag{4}$$

Kinetic Energy

$$\begin{aligned} T &= \frac{1}{2}m_1(\dot{x}_1^2+\dot{y}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2+\dot{y}_2^2) && (5)\\ &= \frac{1}{2}(m_1+m_2)l_1^2\dot{\phi}_1^2 + \frac{1}{2}m_2 l_2^2\dot{\phi}_2^2 + m_2 l_1 l_2\dot{\phi}_1\dot{\phi}_2\cos(\phi_1-\phi_2). && (6)\end{aligned}$$

Potential Energy

$$V = -(m_1+m_2)gl_1\cos\phi_1 - m_2 g l_2\cos\phi_2. \tag{7}$$

Lagrangian

$$\begin{aligned} L = {}& \frac{1}{2}(m_1+m_2)\,l_1^2\,\dot{\phi}_1^2 + \frac{1}{2}m_2\,l_2^2\,\dot{\phi}_2^2 + m_2\,l_1 l_2\,\dot{\phi}_1\dot{\phi}_2\cos(\phi_1-\phi_2)\\ &+ (m_1+m_2)\,g\,l_1\cos\phi_1 + m_2\,g\,l_2\cos\phi_2. \end{aligned} \tag{8}$$

Problem 2

A mass m_2 is attached to a pendulum of length l and the pendulum is attached to a massless moving pivot moves horizontally with coordinate $x(t)$.

Coordinates

$$
\begin{aligned}
x_2 &= x + l\sin\phi, && (9)\\
y_2 &= -l\cos\phi. && (10)
\end{aligned}
$$

Velocity

$$
\dot{x}_2^2 + \dot{y}_2^2 = \dot{x}^2 + l^2\dot{\phi}^2 + 2\dot{x}l\dot{\phi}\cos\phi. \tag{11}
$$

Lagrangian

$$
\begin{aligned}
L = \frac{1}{2}(m_1 + m_2)\,\dot{x}^2 + \frac{1}{2}m_2\,l^2\dot{\phi}^2\cos^2\phi + \frac{1}{2}m_2\,l^2\dot{\phi}^2\sin^2\phi \\
+ m_2\,\dot{x}\,l\,\dot{\phi}\cos\phi + m_2 gl\cos\phi.
\end{aligned} \tag{12}
$$

After simplifying:

$$
L = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\left(\dot{x}^2 + l^2\dot{\phi}^2 + 2\dot{x}l\dot{\phi}\cos\phi\right) + m_2 gl\cos\phi. \tag{13}
$$

Problem 3

A simple pendulum of mass m and length l whose point of support undergoes prescribed motion.

General Coordinates

Let $\phi(t)$ be the angular displacement from the vertical. The coordinates of the pendulum bob are

$$x = x_0(t) + l\sin\phi, \tag{14}$$
$$y = y_0(t) - l\cos\phi. \tag{15}$$

The velocity squared is

$$v^2 = [X^2 + Y^2] = (\dot{x}_0 + l\dot{\phi}\cos\phi)^2 + (\dot{y}_0 + l\dot{\phi}\sin\phi)^2. \tag{16}$$

The Lagrangian is

$$L = \frac{1}{2}m[X^2 + Y^2] - mgY. \tag{17}$$

(a) Support moving uniformly on a vertical circle

The point of support moves on a vertical circle of radius a with constant angular frequency γ:

$$x_0 = a\cos(\gamma t), \qquad y_0 = a\sin(\gamma t). \tag{18}$$

After substitution and simplification, neglecting total time derivatives, the Lagrangian becomes

$$L = \frac{1}{2}ml^2\dot{\phi}^2 + mal\gamma^2\cos(\phi - \gamma t) + mgl\cos\phi. \tag{19}$$

(b) Support oscillating horizontally

The support oscillates horizontally according to

$$x_0 = a\cos(\gamma t), \qquad y_0 = 0. \tag{20}$$

The Lagrangian is

$$L = \frac{1}{2}ml^2\dot{\phi}^2 + mal\gamma^2\cos(\gamma t)\cos\phi + mgl\cos\phi. \tag{21}$$

(c) Support oscillating vertically

The support oscillates vertically according to

$$y_0 = a\cos(\gamma t), \qquad x_0 = 0. \tag{22}$$

The Lagrangian becomes

$$L = \frac{1}{2}ml^2\dot{\phi}^2 + mgl\cos\phi - mal\gamma^2\cos(\gamma t)\cos\phi. \tag{23}$$