King Fahd University For Petroleum and Minerals

MATH435 Project

Mathieu's Equation in Paul Traps

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1 Introduction

In this report we will discuss a differential equation that arises in physics. When we want to confine an ion into some region of space, we use a type of traps called *quadrupole ion trap* or *Paul trap*. In these types of traps an oscillating electric field is used to do the trapping. The device consist of the following components: two end-cap electrodes that make the ions confined in z-axis and a ring shaped electrode that provides an alternating potential to confine the ions in the xy-plane. Such a setup provides a quadrupole potential, hence the name of the device.

The potential in the xy-plane is

$$\Phi(x, y, t) = \frac{V_0}{2r_0^2}\cos(\Omega t)(x^2 - y^2) \tag{1}$$

where V_0 is the maximum voltage supplied and r_0 is the trap length, while Ω is the angular frequency of the potential. t,x,y are the time and position at which we calculate the potential respectively. we know that the electric field is

$$\mathbf{E} = -\nabla\Phi \tag{2}$$

the x component of the electric field is therefore

$$E_x = -\frac{\partial\Phi}{\partial x} = -\frac{V_0}{r_0^2}\cos(\Omega t)x \tag{3}$$

the force in the x-axis on the ion is $F_x = qE_x$ and by Newton's second law

$$m\ddot{x} = -\frac{qV_0}{r_0^2}\cos(\Omega t)x \tag{4}$$

where m is the mass of the ion and q is its charge. Therefore the equation of motion for this ion can be written as follows

$$\ddot{x} + \frac{qV_0}{mr_0^2}\cos(\Omega t)x = 0 \tag{5}$$

to make things cleaner for now, we will set $a = \frac{qV_0}{mr_0^2}$, $b = \Omega$

$$\ddot{x} + a\cos(bt)x = 0 \tag{6}$$

The equation that we arrived at is called the **Mathieu's equation**. throughout this report, we will analyze qualitatively this equation.

2 Transforming the equation into a system

Since any nth order differential equation can be transformed into a system of n first order equations, we can write this second order equation as a system of two equations of first order. Let $x_1 = x$, $\dot{x} = x_2$ so that

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a\cos(bt)x_1 \end{aligned} \tag{7}$$

which can be put into matrix form as

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -a\cos(bt) & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{8}$$

where

$$A(t) = \begin{pmatrix} 0 & 1 \\ -a\cos(bt) & 0 \end{pmatrix} \tag{9}$$

This is a homogeneous system of equations. We will consider the problem with the following initial conditions $x(0) = 0$, $\dot{x}(0) = 0$ or $x_1(0) = 0$, $x_2(0) = 0$.

3 Existence and Uniqueness

Consider the box $B = \{(t, x_1, x_2)|\ |t| \leq \alpha\ , |(x_1, x_2)| \leq \beta\}$. If we want to apply the existence and uniqueness theorem (or the Picard–Lindelöf theorem) then we have to check if the following conditions are met, but first let

$$\mathbf{f}(t, x_1, x_2) = \begin{pmatrix} x_2 \\ -a\cos(bt)x_1 \end{pmatrix} \tag{10}$$

Then we want to show that

$$|\mathbf{f}(t, x_1, x_2)| \leq M, \quad \Big|\frac{\partial \mathbf{f}}{\partial x_1}(t, x_1, x_2)\Big| \leq K_1, \quad \Big|\frac{\partial \mathbf{f}}{\partial x_2}(t, x_1, x_2)\Big| \leq K_2 \tag{11}$$

for some M,K_1,K_2. The derivatives are

$$\frac{\partial \mathbf{f}}{\partial x_1} = \begin{pmatrix} 0 \\ -a\cos(bt) \end{pmatrix}, \quad \frac{\partial \mathbf{f}}{\partial x_2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{12}$$

from which we can see that $\mathbf{f}$, $\frac{\partial \mathbf{f}}{\partial x_1}$, $\frac{\partial \mathbf{f}}{\partial x_2}$ are all continous on the entire $\mathbb{R}^3$. To establish the inequalities in 11 we do this as follows

$$\begin{aligned} &|\mathbf{f}(t, x_1, x_2)| = |x_2| + |a\cos(bt)||x_1| \leq \beta + |a|\beta = \beta(1 + |a|) \\ &\Big|\frac{\partial \mathbf{f}}{\partial x_1}(t, x_1, x_2)\Big| \leq |a\cos(bt)| \leq |a| \\ &\Big|\frac{\partial \mathbf{f}}{\partial x_2}(t, x_1, x_2)\Big| = 1 \end{aligned} \tag{13}$$

Therefore, the Picard-Lindelöf theorem says that there exist a unique solution on the interval $|t| \leq \gamma = \min\{\alpha, \frac{\beta}{\beta(1+|a|)}\} = \min\{\alpha, \frac{1}{1+|a|}\}$, but since the solutions are stable (for specific a and b as we show in the stability section) we can conclude boundedness of the solution for these specific cases and hence we can continue the whole solution for all $t \geq 0$.

4 Periodicity and Floquet Theory

By observing the matrix $A(t)$ we can see that it is periodic with period $\omega = 2\pi/b$. Therefore, we can conclude from Floquet theorem that for any fundamental matrix $\Phi(t)$, there exist a periodic function $P(t)$ with period ω that is invertible and a constant matrix R such that

$$\Phi(t) = P(t)e^{tR} \tag{14}$$

If we want to find the multipliers of the system (i.e. the eigenvalues of the matrix $e^{\omega R}$ then we have to have the solutions of the system. We will solve this system numerically by specifying the values of a and b. The solutions for this system are the well known *Mathieu functions* which do not have closed form expressions. These solutions can be periodic in which case we call them Mathieu functions of the first kind, and if they are non periodic, then we call them Mathieu functions of the second kind.

4.1 First case: $a = 0.5$, $b = 2$

Consider the Mathieu equation

$$\ddot{x} + 0.5\cos(2t)x = 0 \tag{15}$$

with $a = 0.5$ and $b = 2$. This means that $A(t)$ is of period $\omega = \pi$. We will solve this system numerically using Mathematica and keeping in mind that we want to calculate the Floquet multipliers.

In[1]:=

```
DSolve[x''[t]+0.5Cos[2t]x[t]==0,x[t],t]
```

Out[1]=

```
{{x[t]→c1 MathieuC[0.,-0.25,1. t]+c2  MathieuS[0.,-0.25,1. t]}}
```

Here, MathieuC and MathieuS denote Mathieu cosine like function and sine like function respectively (this is because they resemble cosine and sine in their oscillatory behavior and their evenness and oddness).Hence the fundamental solution matrix $B(t)$ is

In[2]:=

```
MatrixForm[B[t]]
```

Out[2]//MatrixForm=

```
MathieuC[0.,-0.25,1. t]            MathieuS[0.,-0.25,1. t]

MathieuCPrime[0.,-0.25,1. t]       MathieuSPrime[0.,-0.25,1. t]
```

MathieuCPrime and MathieuSPrime are the derivatives of MathieuC and MathieuS.

In[3]:=

```
B[t]/.t->0 //MatrixForm
```

Out[3]//MatrixForm=

```
(1.12548      0.

0.        0.153901)
```

So we can see that the fundamental matrix $B(0) \neq I$. So we need to define a new solution $Q(t) = B(t)B^{-1}(0)$ such that $Q(0) = I$ so that $Q(\omega) = e^{\omega R}$

In[4]:=

```
b=B[t]/.t->t+Pi
```

In[5]:=

```
s=Inverse[B[t]]/.t->0
```

Out[5]=

```
{{0.888507,0.},{0.,6.49768}}
```

In[6]:=

```
A[t_]:=B[t].s
```

In[7]:=

```
Q=MatrixForm[A[t]/.t->Pi]
```

Out[7]//MatrixForm=

```
(0.845986      3.89935
-0.0729116     0.845986)
```

In[8]:=

```
Eigenvalues[Q]
```

Out[8]=

```
{0.845986+0.533205 i,0.845986-0.533205 i}
```

In[10]:=

```
Abs[0.845986+0.533205 i]
```

Out[10]=

```
1.
```

This shows that the modulus of the Floquet multipliers are actually equal to 1 from which we can conclude that the system is stable and bounded. We now turn to calculating the Floquet exponents. This can be done using the formula

$$\rho_i = \frac{1}{\omega}\log(\lambda_i) \tag{16}$$

we get

$$\begin{aligned} \rho_1 &= 0.179013i \\ \rho_2 &= -0.179013i \end{aligned} \tag{17}$$

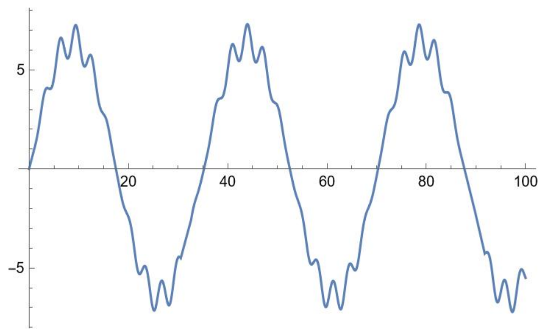

Figure 1: A plot of the solution for the initial conditions $x(0) = 0$, $\dot{x}(0) = 1$

4.2 Second case: $a = 3.79$, $b = 1$

We now want to examine another case in which $a = 3.79$, $b = 1$. We will proceed as in the previous case and we will compute the Floquet multipliers.

In[11]:=

```
a=3.78985
```

In[12]:=

```
DSolve[x''[t]+a*Cos[t]x[t]==0,x[t],t]
```

Out[12]=

$$\{\{x[t]\to c_1\ \ \mathrm{MathieuC}[0,-7.57969,\frac{t}{2}]+c_2\ \ \mathrm{MathieuS}[0,-7.57969,\frac{t}{2}]\}\}$$

In[13]:=

$$y_1[t\_]:=\mathrm{MathieuC}[0,-7.57969,\frac{t}{2}]$$

In[14]:=

$$y_2[t\_]:=\mathrm{MathieuS}[0,-7.57969,\frac{t}{2}]$$

In[15]:=

$$B[t\_]:=\{\{y_1[t],y_2[t]\},\{D[y_1[t],\ t],D[y_2[t],t]\}\}$$

In[16]:=

```
s=Inverse[B[t]]/.t->0;
```

In[17]:=

```
A[t_]:=B[t].s
```

In[20]:=

```
Q=N[A[t]/.t->2*Pi]
```

Out[20]=

```
{{1.,-6.95144*10^-13},{402.114,1.}}
```

In[21]:=

```
Eigenvalues[Q]
```

Out[21]=

```
{1.+0.0000167191 i,1.-0.0000167191 i}
```

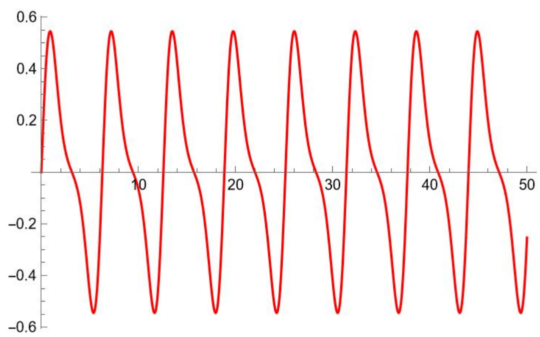

Figure 2: A plot of the solution for the initial conditions $x(0) = 0$, $\dot{x}(0) = 1$

As can be seen in the last line in the computations. These specific parameters would produce a solution that is periodic (since the Floquet multipliers are approximately equal to 1 when discarding the imaginary part since it is very small). Based on the calculations of the multipliers we can discuss questions of stability in the next section.

A final note before ending this section. The value that we chose for a which is 3.79 may seem arbitrary, but it was chosen so as to achieve a periodic solution. We solved the ODE symbolically without specifying a, then we computed the eigenvalues which were a function of a. This made us able to solve for the values of a that will make the eigenvalues equal to 1 and this is the reason why the eigenvalues of the monodromy matrix or the Floquet multipliers were not exactly equal to 1.

5 Stability

From Floquet theory, we can actually conclude some questions on the stability of these solutions.

5.1 First case: $a = 0.5$, $b = 2$

As we saw in this case, the modulus of each of the Floquet multipliers is 1. Therefore, we can conclude that the solution to this equation is actually stable and therefore bounded for all times, but since the multipliers have real parts that are positive, the solution is not asymptotically stable.

5.2 Second case: $a = 3.79$, $b = 1$

In this case the solution is periodic and stable and therefore bounded for all times. The same can be concluded as in the previous case, that is the solution is not asymptotically stable due to the multipliers being positive.