Mathieu's Equation in Paul Traps

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Setting up the equation

When we want to confine an ion into some region of space, we use a type of traps called *quadrupole ion trap* or *Paul trap*
The potential in the xy-plane is

$$\Phi(x, y, t) = \frac{V_0}{2r_0^2} \cos(\Omega t)(x^2 - y^2) \tag{1}$$

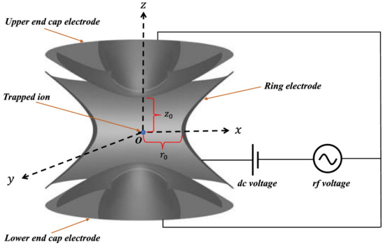

Figure: Paul Trap

Setting up the equation

we know that the electric field is

$$\mathbf{E} = -\nabla\Phi \tag{2}$$

the x component of the electric field is therefore

$$E_x = -\frac{\partial\Phi}{\partial x} = -\frac{V_0}{r_0^2}\cos(\Omega t)x \tag{3}$$

the force in the x-axis on the ion is $F_x = qE_x$ and by Newton's second law

$$m\ddot{x} = -\frac{qV_0}{r_0^2}\cos(\Omega t)x \tag{4}$$

Therefore the equation of motion for this ion can be written as follows

$$\ddot{x} + \frac{qV_0}{mr_0^2}\cos(\Omega t)x = 0 \tag{5}$$

Setting up the equation

to make things cleaner for now, we will set $a = \frac{qV_0}{mr_0^2}$, $b = \Omega$

$$\ddot{x} + a\cos(bt)x = 0 \tag{6}$$

The equation that we arrived at is called the **Mathieu's equation**.

Transforming the equation into a system

Let $x_1 = x$, $\dot{x} = x_2$ so that

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a\cos(bt)x_1 \end{aligned} \tag{7}$$

which can be put into matrix form as

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -a\cos(bt) & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{8}$$

where

$$A(t) = \begin{pmatrix} 0 & 1 \\ -a\cos(bt) & 0 \end{pmatrix} \tag{9}$$

This is a homogeneous system of equations. We will consider the problem with the following initial conditions $x(0) = 0$, $\dot{x}(0) = 0$ or $x_1(0) = 0$, $x_2(0) = 0$.

Existence and Uniqueness

Consider the box $B = \{(t, x_1, x_2) \mid |t| \leq \alpha \,, |(x_1, x_2)| \leq \beta\}$
let

$$\mathbf{f}(t, x_1, x_2) = \begin{pmatrix} x_2 \\ -a\cos(bt)x_1 \end{pmatrix} \tag{10}$$

Then we want to show that

$$|\mathbf{f}(t, x_1, x_2)| \leq M, \quad \left|\frac{\partial \mathbf{f}}{\partial x_1}(t, x_1, x_2)\right| \leq K_1, \quad \left|\frac{\partial \mathbf{f}}{\partial x_2}(t, x_1, x_2)\right| \leq K_2 \tag{11}$$

for some M,K_1,K_2. The derivatives are

$$\frac{\partial \mathbf{f}}{\partial x_1} = \begin{pmatrix} 0 \\ -a\cos(bt) \end{pmatrix}, \quad \frac{\partial \mathbf{f}}{\partial x_2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{12}$$

Existence and Uniqueness

To establish the inequalities in 11 we do this as follows

$$\begin{aligned}
&|\mathbf{f}(t, x_1, x_2)| = |x_2| + |a\cos(bt)||x_1| \leq \beta + |a|\beta = \beta(1 + |a|) \\
&\left|\frac{\partial \mathbf{f}}{\partial x_1}(t, x_1, x_2)\right| \leq |a\cos(bt)| \leq |a| \\
&\left|\frac{\partial \mathbf{f}}{\partial x_2}(t, x_1, x_2)\right| = 1
\end{aligned} \tag{13}$$

Therefore, the Picard-Lindelöf theorem says that there exists a unique solution on the interval $|t| \leq \gamma = \min\{\alpha, \frac{\beta}{\beta(1+|a|)}\} = \min\{\alpha, \frac{1}{1+|a|}\}$. In fact the solutions are bounded and therefore can be continued for $t \geq 0$.

Periodicity and Floquet Theory

From Floquet theory, there exist a periodic function $P(t)$ with period ω that is invertible and a constant matrix R such that

$$\Phi(t) = P(t)e^{tR} \tag{14}$$

If we want to find the multipliers of the system (i.e. the eigenvalues of the matrix $e^{\omega R}$ then we have to have the solutions of the system.
The solutions for this system are the well known *Mathieu functions* which do not have closed form expressions. We will solve numerically for specific a and b.

Periodicity and Floquet Theory

First case: a=0.5 , b=2

```
In[471]:= DSolve[x''[t] + 0.5 Cos[2 t] x[t] == 0, x[t], t]

Out[471]= {{x[t] → C1 MathieuC[0., -0.25, 1. t] + C2 MathieuS[0., -0.25, 1. t]}}

In[472]:= B[t_] := {{MathieuC[0.`, -0.25`, 1.` t], MathieuS[0.`, -0.25`, 1.` t]}, {D[MathieuC[0.`, -0.25`, 1.` t], t], D[MathieuS[0.`, -0.25`, 1.` t], t]}}

In[473]:= B[t] // MatrixForm

Out[473]//MatrixForm=
(      MathieuC[0., -0.25, 1. t]              MathieuS[0., -0.25, 1. t]      )
( 1. MathieuCPrime[0., -0.25, 1. t]   1. MathieuSPrime[0., -0.25, 1. t] )

In[474]:= B[t] /. t → 0 // MatrixForm

Out[474]//MatrixForm=
( 1.12548     0.       )
(   0.      0.153901   )

In[100]:= s = Inverse[B[t]] /. t → 0

Out[100]= {{0.888507, 0.}, {0., 6.49768}}

In[101]:= A[t_] := B[t].s

In[480]:= T = A[t] /. t → Pi

Out[480]= {{0.845986, 3.89935}, {-0.0729116, 0.845986}}

In[481]:= Eigenvalues[T]

Out[481]= {0.845986 + 0.533205 i, 0.845986 - 0.533205 i}

In[484]:= Abs[0.845986026912081` + 0.5332050658700754` i]

Out[484]= 1.
```

Periodicity and Floquet Theory

We now turn to calculating the Floquet exponents. This can be done using the formula

$$\rho_i = \frac{1}{\omega} \log(\lambda_i) \tag{15}$$

we get

$$\begin{aligned} \rho_1 &= 0.179013i \\ \rho_2 &= -0.179013i \end{aligned} \tag{16}$$

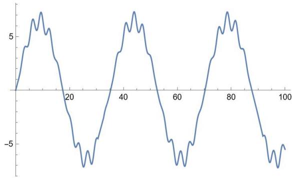

Figure: A plot of the solution for the initial conditions $x(0) = 0$, $\dot{x}(0) = 1$

Second case: $a = 3.79$, $b = 1$

```
In[488]:= a = 3.789845186866189`
Out[488]= 3.78985

In[489]:= DSolve[x''[t] + a * Cos[t] x[t] == 0, x[t], t]
Out[489]= {{x[t] → C[1] MathieuC[0, -7.57969, t/2] + C[2] MathieuS[0, -7.57969, t/2]}}

In[490]:= y1[t_] := MathieuC[0, -7.579690373732378`, t/2]

In[491]:= y2[t_] := MathieuS[0, -7.579690373732378`, t/2]

In[492]:= B[t_] := {{y1[t], y2[t]}, {D[y1[t], t], D[y2[t], t]}}

In[493]:= s = Inverse[B[t]] /. t → 0;

In[494]:= A[t_] := B[t].s

In[495]:= Q = N[A[t] /. t → 2 * Pi]
Out[495]= {{1., -6.95144 × 10^-13}, {402.114, 1.}}

In[496]:= Eigenvalues[Q]
Out[496]= {1. + 0.0000167191 i, 1. - 0.0000167191 i}
```

Periodicity and Floquet Theory

These specific parameters would produce a solution that is periodic.

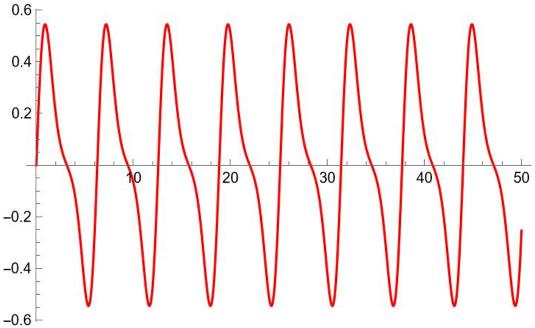

Figure: A plot of the solution for the initial conditions $x(0) = 0$, $\dot{x}(0) = 1$

Stability

First case: a =0.5, b =2

As we saw in this case, the modulus of each of the Floquet multipliers is 1. Therefore, we can conclude that the solution to this equation is actually stable and therefore bounded for all times, but since the multipliers have real parts that are positive, the solution is not asymptotically stable.

Second case: a = 3.79, b = 1

In this case the solution is periodic and stable and therefore bounded for all times. The same can be concluded as in the previous case, that is the solution is not asymptotically stable due to the multipliers being positive.