King Fahd University For Petroleum and Minerals

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Applications to Holstein-Primakoff Transformations

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Abstract

In this report, we discuss magnetic systems and list applications of them. We first discuss the mathematical formalism of second quantization, which we use throughout the report. In addition, we examine one example of mapping the Hubbard model to the Heisenberg model using second-order degenerate perturbation theory. We then move on to discuss the so-called Holstein-Primakoff transformations and show some sample applications. Here, we use them in our computations to map the spin operators to a description in terms of creation and annihilation operators which enable us (after applying Fourier transform) to write the Hamiltonian in a matrix format that is ready to be diagonalized. In the last section, we propose an idea for future work which talks about examining the honeycomb Kitaev model for a flat lattice placed or mapped onto a sphere.

1 Introduction

In recent years, strongly correlated systems had diverse applications in technology. Strongly correlated systems can be seen in many fields like spintronics, quantum computing and superconductivity. Certain magnetic systems are strongly correlated and can be divided to two parts. The first is magnetically ordered systems. These systems have long range order of the spins comprising the system like the spins of a ferromagnetic material where all the spins are pointing in the same direction or in the case of antiferromagnets where the net magnetization is zero and exhibits long range order. The second type of magnetic systems are not magnetically ordered systems. Our focus is on strongly correlated magnetically ordered systems, which we discuss at the end, but we will also give examples of strongly correlated systems that are not necessarily magnetically ordered like superconductivity and topological quantum computers.

Strongly correlated electrons play a significant role in superconductivity. An interesting phenomenon that appears in superconductivity that is related to magnetism is the *Meissner effect*. Some materials, when cooled to a low, so-called critical temperature, have their resistivity drop to zero, the material exhibits superconductivity. [1].When a material becomes superconducting, magnetic fields are expelled from it. Moreover, in a superconducting state, pairs of electrons are formed, called *cooper pairs*. These cooper pairs counteract the applied magnetic field by creating surface currents that oppose the applied field and give rise to the Meissner effect. The Meissner effect is the principle on which magnetic levitation is obtained. Magnetic levitation could be used for transportation in maglev trains that are less noisy and potentially faster than conventional trains. In maglev trains, superconducting magnets are attached to the sides of the train, and permanent magnets are placed on a guideway and the repulsion between the two makes the train levitate [2].

topological quantum computing (TQC) is another research area where strongly correlated systems make a frequent appearance[3]. In quantum computing, errors in the computations can appear because of the fragility of qubits.. Any interaction with the environment can possibly make the system decohere and produce computational errors. The job of TQC is to make fault-tolerant computations by exploiting some types of quasiparticles called *anyons*. These particles can only exist

in two spatial dimensions and have statistics that are different from fermions and bosons and acquire a complex phase upon particle exchange for the so-called *abelian anyons* or more complicated unitary transformation for *non-abelian anyons*. This exchange of non-abelian anyons in TQC is called *braiding* and it is the reason why TQC are insensitive to small disturbances in the system [4]. Non-abelian anyons are expected to appear in systems with topological order like the so-called *fractional quantum hall effect* (FQHE) [5]. FQHE can only be achieved under low temperature and strong magnetic field for an electron gas in two dimensions [6].

As we want to focus on strongly correlated magnetically-ordered systems in this report, we give an application of magnetically-ordered systems. These systems can be found in the area of spin electronics or spintronics (which exploit the spin and electric charge of the electrons). After the discovery of the giant magnetoresistance (GMR) effect by Fert and Grünberg [7, 8], spintronics has become a popular research area. GMR is seen in multilayered systems where we have a non-ferromagnetic layer that is inserted between two ferromagnetic layers. When incoming electrons pass through the material, they experience less scattering when their spin is aligned with the magnetization of the ferromagnet layer than when their spin is anti-aligned with the magnetization of the ferromagnet and thus will feel a lower resistivity. GMR found many applications in technology. For instance, GMR sensors are being used in read heads of hard disk drives because of their high sensitivity to changes in magnetic fields [9].

In this report we will discuss the mathematical formulation of the so-called *Second quantization*. In addition we will discuss how can we apply second-order degenerate perturbation theory to the Hubbard model and map it to the so-called Heisenberg model. Lastly, we will discuss some sample applications of bosonic magnetic systems with spin Hamiltonains and apply the Holstein-Primakoff transformation on them to map them to a description in terms of creation and annihilation operators.

2 Second Quantization

For identical particles, we can write the wavefunction for a system of two fermions or bosons as

$$\psi_{\pm}(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{\sqrt{2}}[\psi_a(\mathbf{r}_1)\psi_b(\mathbf{r}_2) \pm \psi_b(\mathbf{r}_1)\psi_a(\mathbf{r}_2)] \tag{1}$$

Where bosons take the positive sign, and fermions take the negative sign. Notice that we are building the wavefunction of a many-body system from a combination of single-body states. For a system with n-particles we will need to use the *slater determinant* and the *permanent* to write the wavefunction for fermions and bosons respectively. As one can see, this formalism is cumbersome and only deals with systems with fixed number of particles. To improve, we will introduce the language of second quantization.

In second quantization, a state with multi-particles can be written as [10]

$$|n_1, n_2, ..., n_k\rangle \tag{2}$$

where n_i represents the number of particles at position i The state with no physical particles is of much significance and it is called the **quantum vacuum**

$$|0, 0, ..., 0\rangle = |\mathbf{0}\rangle \tag{3}$$

We insist that the vacuum state is normalized i.e. $\langle \mathbf{0}|\mathbf{0}\rangle = 1$. Now we want to define an operator $a_i^\dagger$ that creates a particle at position i by the following relation

$$a_i^\dagger |n_1, n_2, ..., n_k\rangle \propto |n_1, n_2, ..., n_i + 1, ..., n_k\rangle \tag{4}$$

Where we can get rid of the proportionality by normalization. In addition we need an operator that destroys a particle at position i so we define

$$a_i |n_1, n_2, ..., n_k\rangle \propto |n_1, n_2, ..., n_i - 1, ..., n_k\rangle \tag{5}$$

To specify the action of these operators completely, we will need to postulate two more relations

$$a_i |\mathbf{0}\rangle = 0 \qquad \langle \mathbf{0}| a_i^\dagger = 0 \tag{6}$$

The requirement of symmetrization or anti-symmetrization can be expressed in the following way. If we want to create a particle at position i with state $|k_i\rangle$ and then create another at position j with state $|k_j\rangle$ then if we want to permute the order we can write this as

$$a_i^\dagger a_j^\dagger |\mathbf{0}\rangle = \pm a_j^\dagger a_i^\dagger |\mathbf{0}\rangle \tag{7}$$

From which we can extract the following commutation (anti-commutation relations) $[a_i^\dagger, a_j^\dagger] = 0$ for bosons and $\{a_i^\dagger, a_j^\dagger\} = 0$ for fermions. Other relations can be extracted in similar way and are thus tabulated.

Fermions	Bosons
$\{c_i, c_j^\dagger\} = \delta_{ij}$	$[a_i, a_j^\dagger] = \delta_{ij}$
$\{c_i, c_j\} = 0$	$[a_i, a_j] = 0$
$\{c_i^\dagger, c_j^\dagger\} = 0$	$[a_i^\dagger, a_j^\dagger] = 0$

Table 1: commutation (anti-commutation) relations for bosons (fermions)

.

Where in Table 1 we denoted fermions by the letter c and bosons with a. Note that These relations in the table together with equation 6 and the fact that the vacuum is normalized is enough to specify and create any state from the vacuum. From the third anti-commutation relation we can deduce the Pauli exclusion principle which is ${c_i^\dagger}^2 = 0$ so that ${c_i^\dagger}^2 |\mathbf{0}\rangle = 0$ Compare the ease with which now we can write Equation 1 for the case of fermions $\psi_-(\mathbf{r}_1, \mathbf{r}_2) = c_{r_1}^\dagger c_{r_2}^\dagger |\mathbf{0}\rangle$.

as we have seen in this section, the language of second quantization allows us to write the wavefunction for many-body systems in more compact form than in first quantization.

3 Mapping to spin Hamiltoniansn

In this section, we will discuss how to map from the Hubbard model to the Heisenberg model. First, we will define the Hubbard model H, which describes a system of fermions.

$$H = -t \sum_{i,j,\sigma} N_{ij} C_{i\sigma}^\dagger C_{j\sigma} + U \sum_i n_{i\uparrow} n_{i\downarrow} \tag{8}$$

Where N_{ij} is one when i and j are neighboring sites and zero otherwise. $n_{i,\sigma} = C_{i\sigma}^\dagger C_{j\sigma}$ is the number operator that gives the expectation value of the number of particles in site i with spin state σ. For fermions, $n_{i,\sigma}$ can be either 0 or 1. We are going to split our Hamiltonian into two parts, a kinetic energy term and a potential energy term.

$$T = -t \sum_{i,j,\sigma} N_{ij} C_{i\sigma}^\dagger C_{j\sigma} \tag{9}$$

$$V = U \sum_i n_{i\uparrow} n_{i\downarrow} \tag{10}$$

We then multiply each term in the equation 9 on the left by $n_{i,\overline{\sigma}} + h_{i,\overline{\sigma}} = 1$, and on the right by $n_{j,\overline{\sigma}} + h_{j,\overline{\sigma}} = 1$

$$T = -t \sum_{i,j,\sigma} (n_{i,\overline{\sigma}} + h_{i,\overline{\sigma}}) N_{ij} C_{i\sigma}^\dagger C_{j\sigma} (n_{j,\overline{\sigma}} + h_{j,\overline{\sigma}}) \tag{11}$$

$$T = -t \sum_{i,j,\sigma} N_{ij} n_{i,\overline{\sigma}} C_{i\sigma}^\dagger C_{j\sigma} n_{j,\overline{\sigma}} + N_{ij} n_{i,\overline{\sigma}} C_{i\sigma}^\dagger C_{j\sigma} h_{j,\overline{\sigma}} + N_{ij} h_{i,\overline{\sigma}} C_{i\sigma}^\dagger C_{j\sigma} n_{j,\overline{\sigma}} + N_{ij} h_{i,\overline{\sigma}} C_{i\sigma}^\dagger C_{j\sigma} h_{j,\overline{\sigma}} \tag{12}$$

We group the terms into three components and write the kinetic energy as $T = T_0 + T_1 + T_{-1}$ where

$$T_0 = -t \sum_{ij\sigma} N_{ij} \left(n_{i\overline{\sigma}} C_{i\sigma}^\dagger C_{j\sigma} n_{j\overline{\sigma}} + h_{i\overline{\sigma}} C_{i\sigma}^\dagger C_{j\sigma} h_{j\overline{\sigma}} \right) \tag{13}$$

$$T_1 = -t \sum_{ij\sigma} N_{ij} n_{i\overline{\sigma}} C^{\dagger}_{i\sigma} C_{j\sigma} h_{j\overline{\sigma}} \tag{14}$$

$$T_{-1} = -t \sum_{ij\sigma} N_{ij} h_{i\overline{\sigma}} C^{\dagger}_{i\sigma} C_{j\sigma} n_{j\overline{\sigma}} \tag{15}$$

Now, we want to see what happens if we apply T_0, T_1, and T_{-1} operators to a quantum state. Let's take T_1 for example, $h_{j,\overline{\sigma}}$ gives one if site j does not contain an electron with spin $\overline{\sigma}$ and zero otherwise. $C_{j\sigma}$ gives zero if site j does not contain an electron with spin σ. We can conclude that the site j must contain an electron only with spin σ which means that site j is singly occupied. Similarly, i we notice $n_{i\overline{\sigma}}$ is one if site i contains an electron with spin $\overline{\sigma}$. Meaning that state i is also singly occupied with spin $\overline{\sigma}$. Now applying the operator to the state, we move the electron on site j to site i . This makes site i doubly occupied state and site j empty. Hence, T_1 increases the number of states with double occupancy by 1 and T_{-1} decreases it by 1. T_0 leaving the number of double occupancy state unchanged.

Now, we want to eliminate terms in the Hamiltonian that couple different energy levels or subspaces, such as T_1 and T_{-1} in our untransformed Hamiltonian. This will reduce the problem to an effective Hamiltonian that describes the dynamics within a low-energy subspace. We can decouple the states by applying a unitary transformation to our Hamiltonian.

$$H' = e^{iS} H e^{-iS} \tag{16}$$

Introducing a unitary transformation e^{-iS}, where S is an anti-Hermitian operator ($S^{\dagger} = -S$). Now we can expand e^{iS}.

$$H' = UHU^{\dagger} = e^{iS} H e^{-iS} \approx (1 + iS - \frac{1}{2}S^2) H (1 - iS - \frac{1}{2}S^2) \tag{17}$$

$$H' \approx H + i(SH - HS) + \frac{i^2}{2}(S^2 H + HS^2 - 2SHS) \tag{18}$$

$$H' \approx H + [iS, H] + \frac{1}{2}[iS, [iS, H]] \tag{19}$$

We can rewrite our untransformed Hamiltonian as follows:

$$H = V + T_0 + T_1 + T_{-1} \tag{20}$$

The last two terms T_1 and T_{-1} can be eliminated by choosing

$$is = U^{-1}(T_1 - T_{-1}) \tag{21}$$

put equations 21 and 20 in equation 19

$$\begin{aligned} H' \approx & (V + T_0 + T_1 + T_{-1}) \\ & + \left[U^{-1}(T_1 - T_{-1}), (V + T_0 + T_1 + T_{-1})\right] \\ & + \frac{1}{2}\left[U^{-1}(T_1 - T_{-1}), \left[U^{-1}(T_1 - T_{-1}), (V + T_0 + T_1 + T_{-1})\right]\right]. \end{aligned} \tag{22}$$

Let's calculate $\left[U^{-1}(T_1 - T_{-1}), (V + T_0 + T_1 + T_{-1})\right]$ using the properties of the commutator

$$[\hat{A} + \hat{B}, \hat{C}] = [\hat{A}, \hat{C}] + [\hat{B}, \hat{C}] \tag{23}$$

$$[C \cdot \hat{A}, \hat{B}] = C[\hat{A}, \hat{B}] \tag{24}$$

$$\begin{aligned} [U^{-1}(T_1 - T_{-1}), (V + T_0 + T_1 + T_{-1})] = U^{-1}[&[T_1, V] + [T_1, T_0] + [T_1, T_1] + [T_1, T_{-1}] \\ &-[T_{-1}, V] - [T_{-1}, T_0] - [T_{-1}, T_1] - [T_{-1}, T_{-1}]] \end{aligned} \tag{25}$$

We can use these relations to simplify the Hamiltonian [11].

$$[T_m, T_m] = 0 \tag{26}$$
$$[T_1, T_{-1}] = -[T_{-1}, T_1] \tag{27}$$
$$[T_m, V] = mUT_m \tag{28}$$

$$[U^{-1}(T_1 - T_{-1}), (V + T_0 + T_1 + T_{-1})] = -T_1 - T_{-1} - U^{-1}[+[T_1, T_0] + 2[T_1, T_{-1}] + [T_0, T_{-1}]] \tag{29}$$

Put equation 29 in equation 22

$$H'^{(2)} \equiv e^{iS^{(1)}} H e^{iS^{(1)}} = V + T_0 + U^{-1}\big([T_1, T_{-1}] + [T_0, T_{-1}] + [T_1, T_0]\big) \tag{30}$$

Now, substitute equations 13 , 14, and 15 in equation 30

$$H^{(2)} = -\frac{t^2}{U} \sum_{i_1, j_1, i_2, j_2} N_{i_1 j_1} N_{i_2 j_2} (h_{i_2 \bar{\sigma}_2} C^\dagger_{i_2 \sigma_2} C_{j_2 \sigma_2} n_{j_2 \bar{\sigma}_2})(n_{i_1 \bar{\sigma}_1} C^\dagger_{i_1 \sigma_1} C_{j_1 \sigma_1} h_{j_1 \bar{\sigma}_1}). \tag{31}$$

You can notice from equation 31 that when $(h_{i_2 \bar{\sigma}_2} C^\dagger_{i_2 \sigma_2} C_{j_2 \sigma_2} n_{j_2 \bar{\sigma}_2})$ act on a quantum state, it produces zero unless site j_2 is doubly occupied. On the other hand, $(n_{i_1 \bar{\sigma}_1} C^\dagger_{i_1 \sigma_1} C_{j_1 \sigma_1} h_{j_1 \bar{\sigma}_1})$ is zero unless site i contains an electron with spin $\bar{\sigma}$ and site j contains an electron with spin σ as shown in the figure. If we replace $i_2 = j_1$ and $j_2 = i_1$ we will get the same Hamiltonian that does that same job.

$$H^{(2)} = -\frac{t^2}{U} \sum_{i_1, j_1} N^2_{i_1 j_1} (h_{j_1 \bar{\sigma}_1} C^\dagger_{j_1 \sigma_2} C_{i_1 \sigma_2} n_{i_1 \bar{\sigma}_2})(n_{i_1 \bar{\sigma}_1} C^\dagger_{i_1 \sigma_1} C_{j_1 \sigma_1} h_{j_1 \bar{\sigma}_1}). \tag{32}$$

Using the Jordan-Schwinger map to express result in terms of spin operators

$$S_i^+ = c^\dagger_{i\uparrow} c_{i\downarrow}; \quad S_i^- = c^\dagger_{i\downarrow} c_{i\uparrow}, \quad S_i^z = \frac{1}{2}(n_{i\uparrow} - n_{i\downarrow}); \tag{33}$$
$$S_i^x = \frac{1}{2}\left(S_i^+ + S_i^-\right); \quad S_i^y = -\frac{i}{2}\left(S_i^+ - S_i^-\right) \tag{34}$$

In the $\langle V \rangle = 0$ subspace, S_i^z can be simplified because in this subspace, each site is occupied by exactly one fermion, either with spin-up or spin-down. This restriction allows the number operators to be written in $n_{i\uparrow} + n_{i\downarrow} = 1$. Using this condition, we can write S_z in the two following ways:

$$S_z = \frac{1}{2} - c^\dagger_\downarrow c_\downarrow \, ; S_z = c^\dagger_\uparrow c_\uparrow - \frac{1}{2} \tag{35}$$

we can write equation 34 in terms of S_i^+ and S_i^-

$$S_i^+ = S_i^x + iS_i^y; S_i^- = S_i^x - iS_i^y \tag{36}$$

S_i^x, S_i^y ,and S_i^z can be defined using Pauli matrices.

$$\hat{S}_i = \frac{\hbar}{2}\sigma_i, \quad \text{for } i = x, y, z, \tag{37}$$

Where,

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{38}$$

Using Mathematica to substitute $S_i^+ = c^\dagger_{i\uparrow} c_{i\downarrow}$, $S_i^- = c^\dagger_{i\downarrow} c_{i\uparrow}$, $S_z = \frac{1}{2} - c^\dagger_\downarrow c_\downarrow$, and $S_z = c^\dagger_\uparrow c_\uparrow - \frac{1}{2}$ into equation 32 and simplify the expression we get:

$$H_2^{(2)} = \frac{-t^2}{2U} \sum_{i,j} N_{i,j} \left(1 - \boldsymbol{\sigma}^{(i)} \cdot \boldsymbol{\sigma}^{(j)}\right) \tag{39}$$

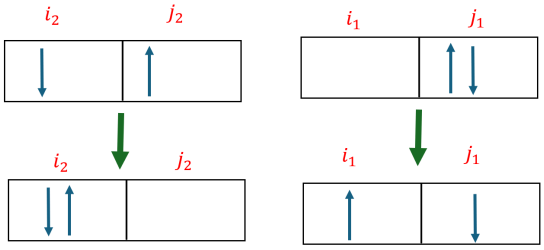

Figure 1: When $H^{(2)}$ acts on a quantum state

4 Holstein Primakoff transformation

The following relations will be used to map the spin components into raising and lowering spin operators:

$$S_i^x = \frac{1}{2}\left(S_i^+ + S_i^-\right) \tag{40}$$

$$S_i^y = -\frac{i}{2}\left(S_i^+ - S_i^-\right) \tag{41}$$

$$S_i^z = S - \hat{n}_i = S - \hat{a}_i^\dagger \hat{a}_i \tag{42}$$

and Holstein Primakoff approximation:

$$S_i^+ = \sqrt{2S - \hat{n}_i}\hat{a}_i \approx \sqrt{2S}a_i \tag{43}$$

$$S_i^- = \hat{a}_i^\dagger\sqrt{2S - \hat{n}_i} \approx a_i^\dagger\sqrt{2S} \tag{44}$$

$$S_i^+ = \sqrt{2s - \hat{n}_i}\hat{a}_i \approx \sqrt{2s}a_i \tag{45}$$

and the bosonic commutation relation:

$$a_j a_i^\dagger = \delta_{ij} + a_i^\dagger a_j \tag{46}$$

And Fourier transformation:

$$\hat{b}j^\dagger = \frac{1}{\sqrt{N}}\sum \mathbf{k} e^{-i\mathbf{k}\cdot\mathbf{r}j}\hat{b}\mathbf{k}^\dagger \tag{47}$$

$$\hat{b}j = \frac{1}{\sqrt{N}}\sum \mathbf{k} e^{i\mathbf{k}\cdot\mathbf{r}j}\hat{b}\mathbf{k} \tag{48}$$

$$\sum_j e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}j} = N\delta\mathbf{k}\mathbf{k}' \tag{49}$$

$$\sum_{\mathbf{k}} e^{i\mathbf{k}\cdot(\mathbf{r}i-\mathbf{r}_j)} = N\delta ij \tag{50}$$

5 Sample applications

5.1 Application I

Here we will give examples of systems where we use the Holstein-Primakoff transformations to write a Hamiltonian in a way that is easy to diagonalize and find the eigen-energies. Consider the following Hamiltonian with staggered magnetic field:

$$H = \sum_i [\mathbf{S}_i \cdot \mathbf{S}_{i+1} + (-1)^i S_i^z] \tag{51}$$

Our goal is to find the spin orientation that will provide the lowest classical ground state. Note

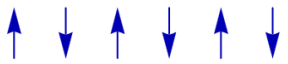

Figure 2: This configuration represents the lowest possible classical ground state.

here that we are considering infinitely long chain of particles. To develop some intuition let's consider two state first where we have $H = \mathbf{S}_1 \cdot \mathbf{S}_2 - S_1^z$ Notice that we can achieve the lowest possible energy when the two spins are anti-parallel (and let's assume they lie on the z-axis) that is $H = -S^2 - S$. Again for three particles we have $H = \mathbf{S}_1 \cdot \mathbf{S}_2 - S_1^z + \mathbf{S}_2 \cdot \mathbf{S}_3 + S_2^z = -2S^2 - 2S$ which again can be guessed to be all anti-parallel with their neighbors. Continuing this way, we see that for n-sites the lowest classical ground state is $H = -nS^2 - nS$ That is when each particle have a spin anti-parallel with its neighbor, this actually represents an *anti-ferromagnet* configuration. The situation is pictured in Figure 2. The next task we will do is to find rotation matrices $R_i(\theta)$ for each site that can align all the foregoing spins in the positive z-axis. Two matrices can rotate a vector to the positive z-axis; the first matrix rotates the vector 180 around the x-axis. the second one rotates a vector 180° along the y-axis.

$$R_x(180) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 180^\circ & -\sin 180^\circ \\ 0 & \sin 180^\circ & \cos 180^\circ \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{52}$$

$$R_y(180^\circ) = \begin{pmatrix} \cos 180^\circ & 0 & \sin 180^\circ \\ 0 & 1 & 0 \\ -\sin 180^\circ & 0 & \cos 180^\circ \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{53}$$

we will work with R_x. we can reflect the resultant vector in the yz plane by a matrix:

$$R = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{54}$$

The Resultant improper rotation matrix:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{55}$$

When we apply this matrix to a general vector

$$R(\theta)\vec{\tilde{S}}_i = \begin{pmatrix} -\tilde{S}_x \\ -\tilde{S}_y \\ -\tilde{S}_z \end{pmatrix} = \begin{pmatrix} S_x \\ S_y \\ S_z \end{pmatrix}$$

we assume in our system that sites with odd values $1, 2, 3 \ldots$ point in the positive direction while even sites $2, 4, 6, \ldots$ point in the negative direction. splitting our summation into odd and even terms and applying a unitary transformation to each component that points in the negative direction of our Hamiltonian

$$\begin{aligned} H = &\sum_{\text{odd}} \left((-\tilde{S}_x^i - \tilde{S}_y^i - \tilde{S}_z^i)(S_x^{i+1} + S_y^{i+1} + S_z^{i+1}) - \frac{S_i^2}{2} \right) + \\ &\sum_{\text{even}} \left((S_x^i + S_y^i + S_z^i)(-\tilde{S}_x^{i+1} - \tilde{S}_y^{i+1} - \tilde{S}_z^{i+1}) - \frac{S_i^2}{2} \right) \end{aligned} \tag{56}$$

$$H = \sum_i \left(-\tilde{S}_x^i \tilde{S}_x^{i+1} - \tilde{S}_y^i \tilde{S}_y^{i+1} - \tilde{S}_z^i \tilde{S}_z^{i+1} - \frac{\tilde{S}^i}{2} \right) = - \left[\sum_i \vec{S}_i \cdot \vec{S}_{i+1} + S_i^z \right] \tag{57}$$

using the equations now using raising and lowering spin operators and Holstein Primakoff approximation, The Hamiltonian can be expressed as follows:

$$= - \left[\sum_i S \left[-a_{i+1}^\dagger a_{i+1} - a_i^\dagger a_i + a_{i+1}^\dagger a_i + a_i^+ a_{i+1} \right] + a_i^\dagger a_{i+1}^\dagger a_i a_{i+1} - a_i^\dagger a_i + S^2 + S \right] \tag{58}$$

and after simplification, we get:

$$= -\left[\sum_i -s\left(a_i^\dagger - a_{i+1}^\dagger\right)(a_i - a_{i+1}) + \hat{n}_i\hat{n}_{i+1} - \hat{n}_i + s + s^2\right] \tag{59}$$

now we can neglect terms that are in zeroth order of S since in the ground state the Hamiltonian is either aligned or anti-aligned with the Z axis:

$$= -\sum_i -s\left(a_i^\dagger - a_{i+1}^\dagger\right)(a_i - a_{i+1}) + s + s^2 \tag{60}$$

going to K-space using Fourier transformation using Fourier transformation and substituting r_i by $a \cdot i$ since the system is only 1D:

$$H = \sum_K [S(2 - cos(ka))a_k^\dagger a_k + S + S^2] \tag{61}$$

5.2 Application II

Consider the Hamiltonian

$$\begin{aligned} H &= -\left[\sum_i (\mathbf{S}_i \cdot \mathbf{S}_{i+1} + S_i^z) + \gamma S_i^x S_{i+1}^x\right] \\ &= -\left[\sum_i (S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + S_i^z S_{i+1}^z + S_i^z) + \gamma S_i^x S_{i+1}^x\right] \end{aligned} \tag{62}$$

We will substitute the following

$$S_i^x = \frac{S_i^+ + S_i^-}{2}, \quad S_i^y = i\frac{S_i^- - S_i^+}{2} \tag{63}$$

Therefore, we will have

$$\begin{aligned} H =& -\{\sum_i \left[\frac{1}{2}(S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+) + S_i^z S_{i+1}^z + S_i^z\right] \\ &+ \frac{\gamma}{4}(S_i^+ S_{i+1}^+ + S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+ + S_i^- S_{i+1}^-)\} \end{aligned} \tag{64}$$

We will then apply the Holstein-Primakoff transformations to first order approximation to map the spin operators to the creation and annihilation operators (this approximation is only valid when the spins are oriented along the z-axis)

$$S_i^+ \approx \sqrt{2s}\, b_i, \quad S_i^- \approx \sqrt{2s}\, b_i^\dagger, \quad S_i^z = s - n_i \tag{65}$$

$$\begin{aligned} \implies H =& -\{\sum_i [s(b_i b_{i+1}^\dagger + b_i^\dagger b_{i+1}) + (s - n_i)(s - n_{i+1}) + (s - n_i)] \\ &+ \frac{s\gamma}{2}(b_i b_{i+1} + b_i b_{i+1}^\dagger + b_i^\dagger b_{i+1} + b_i^\dagger b_{i+1}^\dagger)\} \end{aligned} \tag{66}$$

We will expand the parentheses

$$\begin{aligned} H =& -\{\sum_i [(s^2 - s n_{i+1} - s n_i + n_i n_{i+1} + s - n_i + s b_i b_{i+1}^\dagger + s b_i^\dagger b_{i+1}) \\ &+ \frac{s\gamma}{2} b_i b_{i+1} + \frac{s\gamma}{2} b_i b_{i+1}^\dagger + \frac{s\gamma}{2} b_i^\dagger b_{i+1} + \frac{s\gamma}{2} b_i^\dagger b_{i+1}^\dagger]\} \end{aligned} \tag{67}$$

Now we want to exploit the following fact since we are summing over an infinitely long chain

$$\sum_{-\infty}^{\infty} n_i = \sum_{-\infty}^{\infty} n_{i+1}, \quad \sum_{-\infty}^{\infty} b_i = \sum_{-\infty}^{\infty} b_{i+1}, \quad \sum_{-\infty}^{\infty} b_i^\dagger = \sum_{-\infty}^{\infty} b_{i+1}^\dagger \tag{68}$$

So the last form we get is

$$H = -\sum_i [s^2 - 2sn_i + s + s\left(1+\frac{\gamma}{2}\right)(b_i b_{i+1}^\dagger + b_i^\dagger b_{i+1}) + \frac{s\gamma}{2}(b_i b_{i+1} + b_i^\dagger b_{i+1}^\dagger)] \tag{69}$$

To go from the real space into the k-space or momentum space we use the following Fourier transformations and the related orthogonality conditions

$$\hat{b}_j^\dagger = \frac{1}{\sqrt{N}}\sum \mathbf{k} e^{-i\mathbf{k}\cdot\mathbf{r}_j}\hat{b}_{\mathbf{k}}^\dagger, \quad \hat{b}_j = \frac{1}{\sqrt{N}}\sum \mathbf{k} e^{i\mathbf{k}\cdot\mathbf{r}_j}\hat{b}_{\mathbf{k}} \tag{70}$$

$$\sum_j e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}_j} = N\delta_{\mathbf{k}\mathbf{k}'} \quad , \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot(\mathbf{r}_i-\mathbf{r}_j)} = N\delta_{ij} \tag{71}$$

We will apply these transformations to each term containing a combination of creation and annihilation operators

$$\begin{aligned}
\sum_j b_j^\dagger b_{j+1}^\dagger =& \frac{1}{N}\sum_j \sum_{k,k'} e^{ikj} e^{ik'(j+1)} b_k^\dagger b_{k'}^\dagger \\
=& \frac{1}{N}\sum_{k,k'}\sum_j e^{i(k+k')j} e^{ik'} b_k^\dagger b_{k'}^\dagger \\
=& \sum_{k,k'} \delta_{k,-k'} e^{ik'} b_k^\dagger b_{k'}^\dagger \\
=& \sum_k e^{-ik} b_k^\dagger b_{-k}^\dagger
\end{aligned} \tag{72}$$

Notice here that we changed the summation index to j so that it does not get confused with the imaginary unit i. Moreover, the bold notation was dropped from the transformation because we are only working in 1-dimension. Also notice that we can get the other term $b_i b_{i+1}$ just by taking the hermitian conjugate of 72. We list the transformation for other terms

$$\begin{aligned}
\sum_j b_j b_{j+1}^\dagger &= \sum_k e^{ik} b_k b_k^\dagger \\
\sum_j b_j^\dagger b_{j+1} &= \sum_k e^{ik} b_k^\dagger b_k \\
\sum_j b_j b_{j+1} &= \sum_k e^{ik} b_k b_{-k}
\end{aligned} \tag{73}$$

putting these togther and using the fact that $\cos(k) = (e^{-ik} + e^{ik})/2$ we arrive at

$$\begin{aligned}
H =& -\{\sum_k s^2 + s - s[(2\cos(k)\left(1+\frac{\gamma}{2}\right) + 2) b_k^\dagger b_k] \\
&+ \frac{s\gamma}{2}[e^{ik} b_k b_{-k} + e^{-ik} b_k^\dagger b_{-k}^\dagger]\}
\end{aligned} \tag{74}$$

Now we will split the sum into two parts, one over positive values of k and one over the negative values.

$$\begin{aligned}
H = -\sum_k (s^2 + s) &- \sum_{k>0}\left\{-s\left[2\cos(k)\left(1+\frac{\gamma}{2}\right)+2\right] b_k^\dagger b_k + \frac{s\gamma}{2}[e^{ik} b_k b_{-k} + e^{-ik} b_k^\dagger b_{-k}^\dagger]\right\} \\
&- \sum_{k<0}\left\{-s\left[2\cos(k)\left(1+\frac{\gamma}{2}\right)+2\right] b_k^\dagger b_k + \frac{s\gamma}{2}[e^{ik} b_k b_{-k} + e^{-ik} b_k^\dagger b_{-k}^\dagger]\right\}
\end{aligned} \tag{75}$$

let $E_0 = -\sum_k (s^2 + s)$ and now we will turn the sum over $k < 0$ to a sum over $k > 0$ by performing the replacement $k \to -k$ and $-k \to k$. By doing this, we will be able to combine the sum again but it will be now over $k > 0$ only

$$H = E_0 - \sum_{k>0} \left\{ -s \left[2\cos(k) \left(1 + \frac{\gamma}{2} \right) + 2 \right] (b_k^\dagger b_k + b_{-k}^\dagger b_{-k}) + s\gamma[\cos(k) b_k b_{-k} + \cos(k) b_k^\dagger b_{-k}^\dagger] \right\} \quad (76)$$

This equation can be turned into matrix form in the following way

$$H = E_0 + \sum_{k>0} \begin{pmatrix} b_k^\dagger & b_{-k} \end{pmatrix} \begin{pmatrix} s\left[2\cos(k)\left(1+\frac{\gamma}{2}\right)+2\right] & -s\gamma\cos(k) \\ -s\gamma\cos(k) & s\left[2\cos(k)\left(1+\frac{\gamma}{2}\right)+2\right] \end{pmatrix} \begin{pmatrix} b_k \\ b_{-k}^\dagger \end{pmatrix} \quad (77)$$

This is a Bogoliubov type bosonic Hamiltonian. We cannot diagonalize such a matrix using unitary transformation since these transformations would produce operators that do not satisfy the commutation relations. To circumvent this issue, we will use another tranformation called *paraunitary* P such that it satisfies the following relation.

$$P^\dagger \Gamma P = \Gamma \quad (78)$$

Where Γ is

$$\Gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (79)$$

let h be the square matrix in 77 and H_b be the product of the matrices inside the sum, then we can write H_b as

$$H_b = \begin{pmatrix} b_k^\dagger & b_{-k} \end{pmatrix} P^\dagger \Gamma P \Gamma h P^{-1} P \begin{pmatrix} b_k \\ b_{-k}^\dagger \end{pmatrix} \quad (80)$$

Notice that $P^\dagger \Gamma P \Gamma = \Gamma^2 = I$ and $P^{-1}P = I$ and therefore nothing has been changed, we only multiplied by identity matrices. Moreover, we can rewrite the Bogoliubov Hamiltonian as

$$H_b = \begin{pmatrix} \beta_k^\dagger & \beta_k \end{pmatrix} D \begin{pmatrix} \beta_k \\ \beta_{-k}^\dagger \end{pmatrix} \quad (81)$$

Where $D = \Gamma P \Gamma h P^{-1}$ and

$$\begin{pmatrix} \beta_k^\dagger & \beta_k \end{pmatrix} = \begin{pmatrix} b_k^\dagger & b_{-k} \end{pmatrix} P^\dagger, \quad \begin{pmatrix} \beta_k \\ \beta_{-k}^\dagger \end{pmatrix} = P \begin{pmatrix} b_k \\ b_{-k}^\dagger \end{pmatrix} \quad (82)$$

6 Proposal

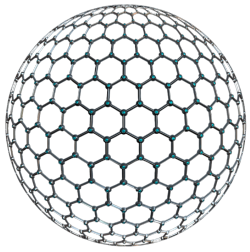

Figure 3: We want to study the Kitaev model on a spherical lattice

In this section, we want to examine an idea to be investigated in a future work. We propose to study the honeycomb Kitaev model both in flat space and when covering a sphere, where it involves curvature. The Hamiltonian on a flat surface is

$$H = -J_x \sum_{\text{x-links}} \sigma_j^x \sigma_k^x - J_y \sum_{\text{y-links}} \sigma_j^y \sigma_k^y - J_z \sum_{\text{z-links}} \sigma_j^z \sigma_k^z \quad (83)$$

We want to make a comparison between them to see how the effect of introducing curvature produces new physics. We will examine how the band structure and the density of states will be

affected on a curved lattice where the length of the bonds and the angle between them will change, and also the exchange energy J may change when curvature is included. In addition, defects could play a role in this type of geometry since a sphere cannot be formed from hexagonal honeycomb lattice and therefore pentagonal shapes maybe introduced. Another interesting thing to be considered in this curved lattice is the change in geometric frustration which might favor the formation of spin liquids.

7 Conclusion

We examined in this report some applications of the Holstein-Primakoff transformations in an attempt of diagonalizing a Hamiltonian. The first example was a spin chain with a staggered magnetic field along the z-axis. After we applied the Holstein-Primakoff transformations, we then transformed the Hamiltonian from the real space to the momentum space using a Fourier transform and it turns out that the Hamiltonian was already diagonalized. In the second example, we added an anisotropy term in the x direction and we did the same thing as in the first example, but this time we needed to use a para-unitary matrix to diagonalize the Hamiltonian to preserve the commutation relation at the same time.

8 Contribution Statement

Section 3 was written entirely by Mohammed Hammadi, section 4 was written by Husam Noorwali and section 5.1 was jointly written by Mohammed Alsadah, Mohammed Hammadi and Husam Noorwali.

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