King Fahd University For Petroleum and Minerals

PHYS441 Project

Examples of Symmetry in Particle Physics

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Abstract

Symmetry principles play an important role in developing our understanding of particle physics. This study investigates the various forms of symmetry in the realm of subatomic particles, like flavor symmetry, parity symmetry, and charge conjugation symmetry. The journey starts with giving an example of how symmetry is connected to conservation law or conservation quantity. Then, it discusses flavor symmetry and isospin conservation and how they are powerful in calculating relative decay rates and cross sections. After that, it discusses parity symmetry by trying to calculate the parity operator and gives an example of how QED interactions can be conserved under parity symmetry. Then it discusses charge conjugation symmetry and derives the form of charge conjugation in the Dirac equation. In addition, the study examines cases in which symmetries break down by exploring the tau-theta puzzle.

1 Introduction

Symmetry is an idea that arises in physics and mathematics. Generally speaking, symmetry appears as a feature when we apply a transformation to a quantity or a mathematical object such that the quantity or object remains the same or (as we like to call it) *invariant* under the action of that transformation. This concept of symmetry emerged first in the field of crystallography in which crystallographers were focused on the symmetry of the crystals shape [1], but before that, symmetry was not recognized explicitly and though some of the laws of conservation were a manifestation of this symmetry, physicists thought they were a consequence of the laws of dynamics and not due to the symmetry underlying such laws [2]. Symmetry was even implicit in Maxwell's equations in the sense that the fields calculated from potentials were gauge invariant. The first serious attempt to connect symmetry with physics was made by the mathematician Emmy Noether who proved a theorem in 1915 and published it in 1918 [3]. The theorem states the following: *To every continuous symmetry of the lagrangian there correspond a conservation law or a conserved quantity in time.* This theorem is now called *Noether's theorem.* One of the many consequences of this theorem is that the energy is conserved due to time translation symmetry. After this point in the 20th century the idea of symmetry became abundant and one can find examples of symmetry in special relativity, electrodynamics or quantum mechanics. Our concern in this paper is to treat symmetries that emerge in particle physics and these are flavour, parity and charge conjugation symmetries. These symmetries can be continuous meaning (informally) that the transformation is taken over a range of continuous values and maintaining the symmetry, or discrete and hence the symmetry occur only at discrete values. One final note, the mathematical arena in which symmetries occur is *group theory*, so we will make an occasional reference to groups like $U(n)$, $SU(n)$, $O(n)$ and $SO(n)$ which will be detailed later on.

2 Motivation

To see the power of symmetry in physical systems, we will present an example that will serve as an illustrative to the concept of symmetry. The example that we will discuss is that of the conservation of energy, though it is not related directly to our topic, it will be a good starting point to motivate the subject at hand and at the same time it will offer some details about Noether's theorem which we will refer to throughout the text. Consider the lagrangian

$$L(q_1, ..., q_n; \dot{q_1}, ..., \dot{q_n}, t) \tag{1}$$

differentiating with respect to time and using the chain rule we get

$$\frac{dL}{dt} = \sum_{j=1}^{n} \frac{\partial L}{\partial q_j}\frac{dq_j}{dt} + \sum_{j=1}^{n} \frac{\partial L}{\partial \dot{q}_j}\frac{d\dot{q}_j}{dt} + \frac{\partial L}{\partial t} \tag{2}$$

using Euler-Lagrange equation

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = 0 \tag{3}$$

Then we substitute

$$\frac{dL}{dt} = \sum_{j=1}^{n} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\dot{q}_j + \sum_{j=1}^{n} \frac{\partial L}{\partial \dot{q}_j}\frac{d\dot{q}_j}{dt} + \frac{\partial L}{\partial t} \tag{4}$$

This can be reduced to

$$\frac{dL}{dt} = \sum_{j=1}^{n} \frac{d}{dt}\left(\dot{q}_j\frac{\partial L}{\partial \dot{q}_j}\right) + \frac{\partial L}{\partial t} \tag{5}$$

and taking the total time derivative of the lagrangian to the other side

$$\frac{d}{dt}\left(\sum_{j=1}^{n} \dot{q}_j\frac{\partial L}{\partial \dot{q}_j} - L\right) + \frac{\partial L}{\partial t} = 0 \tag{6}$$

if the Lagrangian does not have an explicit time dependence, then $\frac{\partial L}{\partial t} = 0$ and what is inside the parenthesis is the Hamiltonian H so

$$\frac{dH}{dt} = 0 \tag{7}$$

Thus the Hamiltonian is conserved and under special circumstances the Hamiltonian equals the total energy and therefore energy is conserved or invariant under time translation operator, in this case $\frac{d}{dt}$ [4].

Take a moment of contemplation. This result is too powerful, because we knew little about the system we are studying yet we could relate its initial energy to its final energy without knowing how the particles are interacting or how many of them in our system. This gives physicists a lot of motive to search for such symmetries if they can have invaluable consequences.

3 Continuous symmetries

When we talk about continuous symmetry, we mean a symmetry that is due to an operator that can vary continuously while depending on one or more parameters which themselves vary over closed intervals. These operators leave an essential quantity in our system unchanged. In the context of classical mechanics that quantity is the Lagrangian, while in quantum mechanics it is the Hamiltonian where we require that $\hat{U}^\dagger \hat{H} \hat{U} = \hat{H}$ in which $\hat{U}$ is the operator under consideration. An example of continuous operator is the rotation operator which depend on the angle of rotation. This can be represented in matrix form as:

$$U(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{8}$$

In fact, these matrices form a group known as SO(2) which is abelian (this is evident, since going through an angle θ_1 and then θ_2 will be the same if done in reverse and the result would be the sum of the two angles). SO(2) is the group of 2×2 rotation matrices in the cartesian plane and "S" stands for special which means that all matrices have unit determinant and "O" stands for orthogonal which means $O^T O = I$ for all $O \in SO(2)$. You can notice that for any inner product

$\langle x, y\rangle$ of two vectors in $\mathbb{R}^2$ this product will be the same after the vectors is transformed and hence the inner product is conserved.

A key idea for discussing lie group elements' was advanced by the mathematician Sophus Lie which says that we can obtain every group element by successive small increments in the parameter φ which is multiplied by some operator S and added to the identity.

$$U(\delta\varphi) = I + i\delta\varphi S \tag{9}$$

where $\delta\varphi$ indicates that the parameter is small. We call the operator S the generator of the group. Notice that S is hermitian whenever U is unitary that is:

$$\begin{gathered} U(\delta\varphi)U(\delta\varphi)^\dagger = (I + i\delta\varphi S)(I + i\delta\varphi S)^\dagger = I - i\delta\varphi(S - S^\dagger) + (\delta\varphi)^2 S^\dagger = I \Rightarrow \\ I - i\delta\varphi(S - S^\dagger) = I \Rightarrow \\ S = S^\dagger \end{gathered} \tag{10}$$

Where we omitted the $(\delta\varphi)^2$ term, since it is small compared to other terms. When φ is large, it means that we are applying the operation U (which could be rotation or translation etc.) many times, say N times, with the small increment φ/N. This can be written as [5]:

$$U(\varphi) = \lim_{N\to\infty}\left(I + \frac{i\varphi S}{N}\right)^N = e^{i\varphi S} \tag{11}$$

If we differentiated the above equation, evaluated the derivative at zero and multiplied by $-i$ we can obtain the generators of the group:

$$-i\frac{dU}{d\varphi}\Big|_{\varphi=0} = S \tag{12}$$

3.1 Flavour symmetry

After the discovery of the neutron in 1932 by James Chadwick, the physicist Werner Heisenberg noticed the striking similarity between the proton and the neutron in terms of their masses. The mass of the proton is $938.28MeV/c^2$ and that of the neutron is $939.57MeV/c^2$. This led him to consider the possibility that these two particles are two different states of the same particle that he called the *nucleon*. Heisenberg thought that if the electromagnetic interaction were to disappear from our world, the proton and the neutron would be the same when ignoring charge difference (and also the small difference in mass) and hence would be indistinguishable and we can replace any proton with a neutron. Putting this idea to work we have the following:

$$p = \begin{pmatrix}1\\0\end{pmatrix} \qquad n = \begin{pmatrix}0\\1\end{pmatrix} \tag{13}$$

where p and n are the proton and neutron states respectively. If the nucleon is the linear combination of the state of the proton and the neutron, then we can write this as

$$N = \begin{pmatrix}\alpha\\\beta\end{pmatrix} = \alpha\begin{pmatrix}1\\0\end{pmatrix} + \beta\begin{pmatrix}0\\1\end{pmatrix} \qquad where\ |\alpha|^2 + |\beta|^2 = 1 \tag{14}$$

To account for this symmetry of the strong interaction, Heisenberg introduced the concept of isospin which, as the name suggest, is similar to spin in formalism. we deonte isospin and its third component by I and I_3 respectively. We write a ket as $|I, I_3\rangle$ in the same manner as in spin $|s, m_s\rangle$. We assign $I = 1/2$ for nucleons (which are either protons or neutrons or a combination) so we can write the state of p ad n in ket notation as

$$p = \left|\frac{1}{2}\frac{1}{2}\right\rangle \qquad n = \left|\frac{1}{2} - \frac{1}{2}\right\rangle \tag{15}$$

if we were to search for a symmetry in isospin space, then we have to find an operator $\hat{U}$ such that the state normalization is unchanged

$$\langle\psi|\psi\rangle = \langle\psi'|\psi'\rangle = \langle\hat{U}\psi|\hat{U}\psi\rangle = \langle\psi|\hat{U}^\dagger\hat{U}|\psi\rangle \tag{16}$$

which implies that our symmetry operator must be unitary $\hat{U}^{\dagger}\hat{U} = I$. So we should have:

$$\begin{pmatrix} p' \\ n' \end{pmatrix} = \hat{U}\begin{pmatrix} p \\ n \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} p \\ n \end{pmatrix} \tag{17}$$

for simplicity, we will assume that these unitary matrices have a unit determinant, then this means that:

$$\hat{U}^{\dagger} = \hat{U}^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} \tag{18}$$

where the star refer to the complex conjugate. equating the corresponding entries, we see that U have the following form:

$$U = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} = \begin{pmatrix} x_1 + ix_2 & x_3 + ix_4 \\ -x_3 + ix_4 & x_1 - ix_2 \end{pmatrix} \tag{19}$$

using eq(12) we can differentiate U with respect to each x_i where $i = 1, 2, 3, 4$ and find the generators:

$$-i\frac{dU}{dx_1}\Big|_{x_1=0} = -i\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad -i\frac{dU}{dx_2}\Big|_{x_2=0} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_3 \tag{20}$$

$$-i\frac{dU}{dx_3}\Big|_{x_3=0} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \sigma_2, \qquad -i\frac{dU}{dx_4}\Big|_{x_4=0} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1$$

We can notice that the last three matrices are the Pauli matrices while the first is related to a global transformation between different flavour states and hence we ignore it. The elements of the group that we were talking about belong to the group SU(2) of unitary 2×2 matrices with unit determinant. As you maybe noticed, since the *algebra* related to isospin is the same as that in spin angular momentum, then both shares the same matheamtical formalism. If we used eq(11) and replaced S by $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ and replaced φ with $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ making U depends on three paramters instead of one we get:

$$U = e^{(i/2)\boldsymbol{\alpha}\cdot\boldsymbol{\sigma}} \tag{21}$$

where we added a factor of 1/2 in the exponent. Now we can define the isospin operator [6]

$$\hat{\boldsymbol{T}} := \frac{1}{2}\boldsymbol{\sigma} \tag{22}$$

Now, if we want to show that our symmetry operator U commutes with the Hamiltonian $\hat{H}$ and hence commutes with the isospin operator $\hat{T}$ such that $[\hat{H}, \hat{T}] = 0$, we would require the specific form of $\hat{H}$ which is difficult to calculate and therefore difficult to prove the conservation of isospin. In spite of this, it is believed experimentally that in many strong interactions isospin is conserved. By eq(10) the isospin operator is hermitian and therefore it corresponds to an observable. As we stated before, the observables of interest are I and I_3.

We may ask, after all of this, what are the consequences of isospin conservation?. One important consequence is our ability to determine the relative decay rates and cross sections. This can be seen from the following formula [7]:

$$\mathcal{M}_{if} = \langle\psi_f|\, \hat{T}_{if}\, |\psi_i\rangle \tag{23}$$

Where $\mathcal{M}$ is the matrix element or the scattering amplitude, and ψ_i and ψ_f are the initial and final states respectively. $\hat{T}_{if}$ is an isospin operator connecting states of the same isospin (an isospin operator cannot connect states of different isospin). The cross section σ and the decay rate Γ are related to the matrix element by:

$$\Gamma, \sigma \propto |\mathcal{M}_{if}|^2 \tag{24}$$

To demonstrate this idea, we would like to calculate the relative decay rates of the delta baryon multiplet into pions and nucleons (protons and neutrons), but before we do this, we would like to state the rule for assigning isospin quantum numbers as we did in eq(15). The rule is as follows: count the number of particles in a multiplet and then equate it with $2I + 1$. This is because I_3 ranges from $-I$ to I by increasing 1 for each step. Notice that this is the same formula in spin angular momentum but we can apply it because as we said the formalism is the same. Then, the particle with the largest charge takes $I_3 = I$ while the next largest one takes $I_3 = I - 1$ and so until we reach the lowest charge which takes $I_3 = -I$

Now, let's consider the different decay processes that the delta baryons decay into [8]:

$$\begin{aligned}
&a) \quad \Delta^{++} \longrightarrow \pi^+ + p \\
&b) \quad \Delta^{+} \longrightarrow \pi^0 + p \\
&c) \quad \Delta^{+} \longrightarrow \pi^+ + n \\
&d) \quad \Delta^{0} \longrightarrow \pi^0 + n \\
&e) \quad \Delta^{0} \longrightarrow \pi^- + p \\
&f) \quad \Delta^{-} \longrightarrow \pi^- + n
\end{aligned} \tag{25}$$

To assign isospin numbers, we note that there are four delta baryons and hence $4 = 2I + 1$ yields $I = 3/2$ and I_3 will be assigned as stated before:

$$\Delta^{++} : \left|\frac{3}{2}\frac{3}{2}\right\rangle, \quad \Delta^{+} : \left|\frac{3}{2}\frac{1}{2}\right\rangle, \quad \Delta^{0} : \left|\frac{3}{2} - \frac{1}{2}\right\rangle, \quad \Delta^{-} : \left|\frac{3}{2} - \frac{3}{2}\right\rangle \tag{26}$$

We do the same for pions:

$$\pi^+ : |1\,1\rangle\,, \quad \pi^0 : |1\,0\rangle\,, \quad \pi^- : |1\ -1\rangle\,, \tag{27}$$

For protons and neutrons, the states are already written in eq(15). Since the final states of these interactions are composed of two particles, we decompose these states using Clebsch-Gordan coefficients as follows:

$$\begin{aligned}
&a) \quad \pi^+ p : |1\,1\rangle \left|\frac{1}{2}\frac{1}{2}\right\rangle = \left|\frac{3}{2}\frac{3}{2}\right\rangle \\
&b) \quad \pi^0 p : |1\,0\rangle \left|\frac{1}{2}\frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}}\left|\frac{3}{2}\frac{1}{2}\right\rangle - \frac{1}{\sqrt{3}}\left|\frac{1}{2}\frac{1}{2}\right\rangle \\
&c) \quad \pi^+ n : |1\,1\rangle \left|\frac{1}{2} - \frac{1}{2}\right\rangle = \frac{1}{\sqrt{3}}\left|\frac{3}{2}\frac{1}{2}\right\rangle + \sqrt{\frac{2}{3}}\left|\frac{1}{2}\frac{1}{2}\right\rangle \\
&d) \quad \pi^0 n : |1\,0\rangle \left|\frac{1}{2} - \frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}}\left|\frac{3}{2} - \frac{1}{2}\right\rangle + \frac{1}{\sqrt{3}}\left|\frac{1}{2} - \frac{1}{2}\right\rangle \\
&e) \quad \pi^- p : |1\ -1\rangle \left|\frac{1}{2}\frac{1}{2}\right\rangle = \frac{1}{\sqrt{3}}\left|\frac{3}{2} - \frac{1}{2}\right\rangle - \sqrt{\frac{2}{3}}\left|\frac{1}{2} - \frac{1}{2}\right\rangle \\
&f) \quad \pi^- n : |1\ -1\rangle \left|\frac{1}{2} - \frac{1}{2}\right\rangle = \left|\frac{3}{2} - \frac{3}{2}\right\rangle
\end{aligned} \tag{28}$$

States with $I = 3/2$ are related to the matrix element $\mathcal{M}_3$ while those related to $I = 1/2$ are related to the matrix element $\mathcal{M}_1$. Performing the inner product in eq(23), we get for each decay:

$$\begin{aligned}
&a)\ \mathcal{M}_3, \quad b)\ \sqrt{\frac{2}{3}}\mathcal{M}_3, \quad c)\ \frac{1}{\sqrt{3}}\mathcal{M}_3 \\
&d)\ \sqrt{\frac{2}{3}}\mathcal{M}_3, \quad e)\ \frac{1}{\sqrt{3}}\mathcal{M}_3, \quad f)\ \mathcal{M}_3
\end{aligned} \tag{29}$$

So now the relative decay rates are:

$$\Gamma_a : \Gamma_b : \Gamma_c : \Gamma_d : \Gamma_e : \Gamma_f := |\mathcal{M}_3|^2 : \frac{2}{3}|\mathcal{M}_3|^2 : \frac{1}{3}|\mathcal{M}_3|^2 : \frac{2}{3}|\mathcal{M}_3|^2 : \frac{1}{3}|\mathcal{M}_3|^2 : |\mathcal{M}_3|^2 \tag{30}$$

which reduces to

$$\Gamma_a : \Gamma_b : \Gamma_c : \Gamma_d : \Gamma_e : \Gamma_f := 3 : 2 : 1 : 2 : 1 : 3 \tag{31}$$

Note that this symmetry of SU(2) is restricted to the strong interaction Hamiltonian with disregard to the electromagnetic Hamiltonian and the difference in mass between the particles. In fact this symmetry is justifiable in light of the quark model. If we look at the quarks that constitute a proton and a neutron they are *uud* and *ddu* respectively, but up and down quarks have a very small difference in mass ($\approx 3MeV/c^2$) therefore the symmetry is good but approximate.

4 Discrete symmetries

In physics, discrete symmetry means that the laws and properties of physics do not change when the system goes through a specific discrete change. A common-sense example of discrete symmetry is when you take a square piece of paper and rotate it by an angle of 30. The paper will not be in the same orientation you started with, but when you rotate by an angle of 90, it will return to the original orientation. That means a square piece of paper has discrete rotational symmetry. Discrete symmetry plays a critical role in understanding our universe, and it is a crucial concept in various branches of physics. Some key examples of symmetries are Parity symmetry, Charge conjugation symmetry, and CP symmetry.

4.1 Parity symmetry

Parity symmetry, often referred to as "P-symmetry" or "parity", is a symmetry that happens when you change the sign of spatial components of a system and that system remains unchanged.

$$\hat{p}\psi(r) = \psi'(r) = \psi(-r) \tag{32}$$

The parity operator is usually denoted by $\hat{p}$. In the realm of particle physics, the strong and electromagnetic interactions respect parity symmetry, meaning the laws of physics remain unchanged under parity transformations. However, the weak force violates parity symmetry, a discovery made through the study of beta decay in certain nuclear processes, as we will see in the symmetry breaking section. Parity symmetry plays a significant role in particle physics such as QED and QCD interactions conserve parity. To understand why QED and QCD interactions conserve parity, we are going first to identify the corresponding parity operator which acts as solution to the Dirac equation.

let's assume that ψ is a solution for dirac equation and ψ' is the corresponding solution in the "parity mirror" obtained from the action of the parity operator $\hat{p}$ such that

$$\psi(x, y, z) \to \psi'(x, y, z) = \hat{p}\psi((x, y, z)) = \psi'(-x, -y, -z) \tag{33}$$

We have used the Cartesian coordinate. The form of the parity operator can be deduced by considering a wavefunction $\psi(x, y, z)$ which satisfies the free-particle Dirac equation[6]

$$(i\hbar\gamma^{\mu}\partial_{\mu} - mc)\psi = 0 \tag{34}$$

$$i\hbar\gamma^0\frac{\partial\psi}{\partial t} - i\hbar\gamma^1\frac{\partial\psi}{\partial x} - i\hbar\gamma^2\frac{\partial\psi}{\partial y} - i\hbar\gamma^3\frac{\partial\psi}{\partial z} - mc\psi = 0 \tag{35}$$

consider $c, \hbar = 1$

$$i\gamma^1\frac{\partial\psi}{\partial x} + i\gamma^2\frac{\partial\psi}{\partial y} + i\gamma^3\frac{\partial\psi}{\partial z} - m\psi = -i\gamma^0\frac{\partial\psi}{\partial t} \tag{36}$$

The parity transformed wavefunction $\psi'(x, y, z) = \hat{p}\psi(x, y, z, t)$ must satisfy the Dirac equation in the new coordinate system

$$i\gamma^1\frac{\partial\psi'}{\partial x'} + i\gamma^2\frac{\partial\psi'}{\partial y'} + i\gamma^3\frac{\partial\psi'}{\partial z'} - m\psi' = -i\gamma^0\frac{\partial\psi'}{\partial t'} \tag{37}$$

As we know the parity operator reflects the wavefunction around the origin. From this definition we conclude that in two successive reflections we return to the original wavefunction.

$$\hat{p}\psi'(x, y, z) = \hat{p}\hat{p}\psi((x, y, z)) = \psi(x, y, z) \tag{38}$$

Writing $\psi = \hat{p}\psi'$, equation (36) becomes

$$i\gamma^1\hat{p}\frac{\partial\psi'}{\partial x} + i\gamma^2\hat{p}\frac{\partial\psi'}{\partial y} + i\gamma^3\hat{p}\frac{\partial\psi'}{\partial z} - m\hat{p}\psi' = -i\gamma^0\hat{p}\frac{\partial\psi'}{\partial t} \tag{39}$$

multiplying by γ^0 and expressing the derivatives in terms of the new coordinate (which introduces minus signs for all the space-like coordinates) gives

$$-i\gamma^0\gamma^1\hat{p}\frac{\partial\psi'}{\partial x'} - i\gamma^0\gamma^2\hat{p}\frac{\partial\psi'}{\partial y'} - i\gamma^0\gamma^3\hat{p}\frac{\partial\psi'}{\partial z'} - m\gamma^0\hat{p}\psi' = -i\gamma^0\gamma^0\hat{p}\frac{\partial\psi'}{\partial t} \tag{40}$$

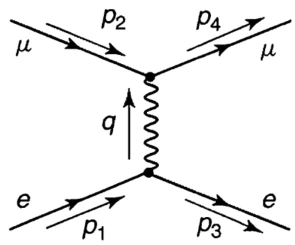

Figure 1: Electron-muon scattering [9]

using $\gamma^0\gamma^k = -\gamma^k\gamma^0$ equation(40) can be written

$$i\gamma^1\gamma^0\hat{p}\frac{\partial\psi'}{\partial x'} + i\gamma^2\gamma^0\hat{p}\frac{\partial\psi'}{\partial y'} + i\gamma^3\gamma^0\hat{p}\frac{\partial\psi'}{\partial z'} - m\gamma^0\hat{p}\psi' = -i\gamma^0\gamma^0\hat{p}\frac{\partial\psi'}{\partial t} \tag{41}$$

In order for eq(41) to reduce to eq(37), $\gamma^0\hat{p}$ should be proportional to the identity matrix.

$$\gamma^0\hat{p} \propto I \tag{42}$$

In addition, from eq(38) we can deduce that

$$\hat{p}\hat{p} = I \Rightarrow \hat{p}^2 = I \tag{43}$$

therefore the parity operator for Dirac spinors can be identified as

$$\hat{p} = \gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{44}$$

Now we come back again to show how QED and QCD are conserved under a parity transformation. In Feynman diagrams, particle interactions are represented by vertices where particles come together or split apart. Parity symmetry implies that the fundamental interactions at these vertices should remain invariant under parity transformations. To show how parity symmetry is conserved in QED and QCD, take, for example, electron-muon scattering as shown in Figure 1. the amplitude $\mathcal{M}$ is given by

$$\mathcal{M} = -\frac{(g_e^2)}{(p_1 - p_3)^2}[\bar{u}(3)\gamma^\mu u(1)][\bar{u}(4)\gamma_\mu u(2)] \tag{45}$$

can be written as the four-vector scalar product

$$\mathcal{M} = -\frac{(g_e^2)}{(p_1 - p_3)^2} j_i . j_f$$

where

$$j_i^u = \bar{u}(p3)\gamma^\mu u(p1) \tag{46}$$

$$j_f^v = \bar{u}(p4)\gamma^\mu u(p2) \tag{47}$$

To generate the equivalent matrix element for the process after a parity transformation, when the three-momenta of all particles are reversed, one can apply the parity operator to the spinors (–).

$$u \to u' = \hat{p}u = \gamma^0\bar{u} \tag{48}$$

the adjoint spinors transform as

$$\bar{u}' = (u^\dagger\gamma^0)' = (u'^\dagger\gamma^0) = ((\hat{P}U)^\dagger\gamma^0) = ((\gamma^0 U)^\dagger\gamma^0) = U^\dagger\gamma^0\gamma^0 = \bar{u}\gamma^0 \tag{49}$$

From (48) and (49), it can be seen that the four-vector j_i^u (46) becomes

$$(j_i^u)' = (\bar{u}(p3)\gamma^\mu u(p1))' = \bar{u}(p3)\gamma^0\gamma^\mu\gamma^0 u(p1)$$

time-like component of $(j_i^u)'$:

$$\bar{u}(p3)\gamma^0\gamma^0\gamma^0 u(p1) = \bar{u}(p3)\gamma^0 u(p1) = j_i^0$$

The time-like component is unchanged under parity transformation. The space-like components of j_i^u transform as

$$(j_i^k)' = \bar{u}(p3)\gamma^0\gamma^k\gamma^0 u(p1) = -\bar{u}(p3)\gamma^k\gamma^0\gamma^0 u(p1) = -(p3)\gamma^k u(p1) = -j_i^k$$

Hence, the parity operation changes the signs of the spatial components of the four-vector while keeping the time component unchanged. Therefore, the transformed four-vector scalar product in the QED matrix element $j_i \cdot\ j_f = j_i^0 j_f^0 - j_i^k j_f^k$ can be expressed as:

$$(j_i \cdot\ j_f)' = (j_i^0 j_f^0 - j_i^k j_f^k)' = j_i^0 j_f^0 - (-j_i^k)(-j_f^k) = j_i^0 j_f^0 - j_i^k j_f^k) = j_i \cdot\ j_f \tag{50}$$

It can be inferred that the QED matrix element remains unchanged when subjected to the parity operation. Therefore, the terms in the Hamiltonian that are associated with the QED interaction remain unchanged when undergoing parity transformations. This invariance indicates that the conservation of parity remains unchanged in QED.

4.1.1 Scalars, pseudoscalars, vectors and axial vectors

Physical quantities can be categorized based on their dimensionality and features related to parity symmetry. For example, scalar values with a single value, like mass and temperature, remain unchanged when subjected to rotational transformations. In the other hand, vector quantities, such as position and velocity, change its component when subjected to rotational transformations. $x \to -x$ and $p \to -p$. There is also a second type of vector quantity known as an axial vector, which is also known as a pseudovector. Axial vectors are produced by the cross product of two vector quantities that do not change sign when subjected to parity transformations. One example is angular momentum. $\boldsymbol{L} = \boldsymbol{x} \times \boldsymbol{p}$ Because both x and p change sign when parity is applied, the axial vector L remains unchanged. Other axial vectors include the magnetic moment and the magnetic flux density $\boldsymbol{B}$, which are related to the current density $\boldsymbol{j}$ by the Biot-Savart law. One way to create scalar values is to take the scalar product of two vectors or two axial vectors. The most basic example of this is the magnitude squared of the momentum vector $p^2 = \boldsymbol{p} \cdot \boldsymbol{p}$. A pseudoscalar is a type of second-class scalar quantity. Pseudoscalars are single-valued quantities that undergo a sign change during the parity operation. They are created by multiplying a vector by an axial vector. Helicity is a typical example of a pseudoscalar.

4.2 Charge conjugation symmetry

Charge conjugation is a transformation that interchanges particles with their corresponding antiparticles, reversing all charges, including not only electric charge but also the charges associated with other forces.

$$C\,|p\rangle = |\bar{p}\rangle \tag{51}$$

In classical dynamics, the motion of a charged particle in an electromagnetic $A^u = (\phi, \mathbf{A})$ field can be obtained by minimal substitution[6].

$$E \to E - q\phi \tag{52}$$

$$\mathbf{p} \to \mathbf{p} - q\mathbf{A} \tag{53}$$

ϕ and $\mathbf{A}$ represent the scalar and vector potentials of electromagnetism, respectively, whereas q denotes the charge of the particle. we can write eq (47) and eq (48) as

$$p_u = p_u - qA_U \tag{54}$$

replacing energy and momentum by the quantum operators

$$i\partial_u \to i\partial_u - qA_u \tag{55}$$

In the presence of an electromagnetic field, the Dirac equation for charged particles can be obtained by performing the minimal substitution of eq (55), which results in the following:

$$\gamma^u(\partial_u - ieA_u)\psi + im\psi = 0 \tag{56}$$

To derive the equivalent equation for the ant-particle, we must initially obtain the complex conjugate of (56), followed by a pre-multiplication by $-i\gamma^2$ operation that yields.

$$-i\gamma^2\gamma^u(\partial_u + ieA_u)\psi^* - m\gamma^2\psi^* = 0 \tag{57}$$

using $\gamma^2\gamma^k = -\gamma^k\gamma^2$ eq(57) can be written

$$\gamma^u(\partial_u + ieA_u)i\gamma^2\psi^* + im\gamma^2 i\psi^* = 0 \tag{58}$$

we can define $\psi' = i\gamma^2\psi^*$

$$\gamma^u(\partial_u + ieA_u)\psi' + im\gamma^2\psi' = 0 \tag{59}$$

The Dirac equation (59) for ψ' is analogous to the Dirac equation (56), except that the ieA_u term now has the opposite sign. As a result, ψ' is a wavefunction representing a particle with the same mass as the original particle but opposite charge; this wavefunction can be interpreted as the antiparticle wavefunction. we can now define the charge conjugation operator $\hat{C}$, which converts a particle's wavefunction into its antiparticle's wavefunction.

$$\psi' = \hat{C}\psi = i\gamma^2\psi^* \tag{60}$$

5 Symmetry breaking

A symmetry that holds in one type of interaction may not hold in another, thus we call this phenomenon *symmetry breaking.* We present two instances of broken symmetries.

5.1 Flavour symmetry as an approximate symmetry

As we saw at the end of *flavour symmetry* section that this symmetry is approximate if we ignored the mass difference between the quarks and disregarded the electromagnetic force. Therefore this symmetry of SU(2) in the strong interaction makes no distinction between an up quark and a down quark. This symmetry can be extended if we include the strange quark as a basis vector along with the up and down quark in the sense [6]

$$u = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad d = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad s = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{61}$$

If we proceeded as before to find rotation matrices to the linear combination of these states such that $U^\dagger U = I$. This means that we want a matrix 3×3 with complex entries and imposed on it the unitarity condition. This makes the matrix U has nine parameters, So we will have eight generators and one that represents a global transformation so we discard it. The eight generators are called the *Gell-Mann matrices* and the symmetry group is SU(3) (of course we want these matrices to have unit determinant as before). The symmetry of SU(3) provides us with a way to organize particles with similar properties in octets, nonets and decuplets. A diagram called the *eightfold way* was introduced by Murray Gell-Mann to accomplish this job. One representation of SU(3) is the baryon octet where its diagram is shown in Figure 2.

We can see that the symmetry of SU(3) is more broken than that of SU(2) since the mass of the strange quark differ from the up and down by $\approx 100MeV/c^2$. We can keep extending this idea by including the remaining quarks but then we would end-up with a totally broken useless symmetry.

5.2 Parity violation: The tau-theta puzzle and Wu's experiment

At the beginning of the 1950's, there was a problem that confounded many physicists called *the tau-theta puzzle.* It was about two particles, one is the τ (not to be confused with the lepton) and the other called θ. Now these two particles have the exact same properties like the charge, the mass

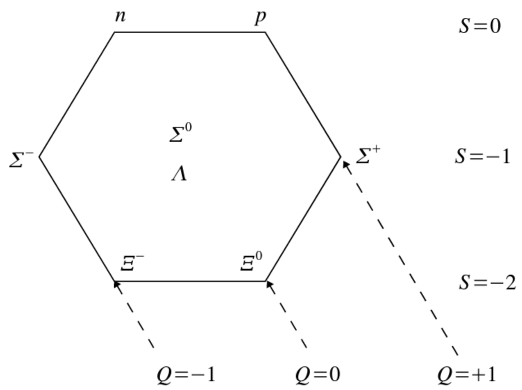

Figure 2: Particles on the same diagonal have the same charge while those on the same row have the same strangeness number [8]

and even the same lifetime (up to some experimental errors) but had two different decay modes. The θ decayed into two pions while the τ decayed into three pions [9]

$$\begin{aligned} \theta^+ &\longrightarrow \pi^+ + \pi^0 \\ \tau^+ &\longrightarrow \pi^+ + \pi^0 + \pi^0 \\ \tau^+ &\longrightarrow \pi^+ + \pi^- + \pi^- \end{aligned} \tag{62}$$

A particle with orbital angular momentum l and decay to two pions will have a parity $P = (-1)^l$ and a particle that decays into three pions will have a parity of $P = (-1)^{l+1} = -(-1)^l$ since the pion have an intrinsic parity of -1. These two particles will be the same if it was not for the fact that they have different intrinsic parity. From here, physicists began to be suspicious because this is a strange coincidence to have two particles that are the same in every aspect but differ in parity.

In 1956, Tsung Dao Lee and Chen Ning Yang who were Chinese physicists began to review the literature to see if parity was also conserved in weak interactions. There were ample evidence that parity was conserved in electromagnetic and strong interactions. What they noticed is that no experimental test have been done to verify that parity was also conserved in weak interactions. It seems that people have been taking parity symmetry in weak interactions for granted, after all it seems intuitive because why nature would prefer left over right. To test their hypothesis, Lee and Yang wrote a paper [10] in which they suggested some experiemntal tests that would reveal if parity was indeed symmetric in weak interactions. Out of these experimental ideas, there was one about beta decay of Co^{60} that suggested the following: when Co^{60} undergoes beta decay it emits the products:

$$^{60}_{27}\text{Co} \longrightarrow {}^{60}_{28}\text{Ni} + e^- + \bar{\nu}_e + 2\gamma \tag{63}$$

So Co^{60} emits a nickle atom and an electron and an electron antineutrino. The two gamma rays comes from the daughter nickle atom being excited and therefore emits two photons. Now we can use a magnetic field and polarize the cobalt nuclei by making them spin in the same direction as the magnetic field which we may choose to be the z-axis. Electrons will be emitted with a linear momenta that make an angle θ with the spin vector or the +z-axis. The mirror image of this event will be a cobalt atom that spins in the -z-axis but now the electrons' momenta make an angle of $\pi - \theta$. We may define the *helicity* to be the projection of the spin onto the linear momentum and hence for an electron with spin 1/2 parallel to the momentum we assign a helicity of $+1$ and call it right handed and when they are anti-parallel we assign -1 and call it left handed. Thus parity would be conserved if as many right handed particles would be emitted as the number of left handed particles. Have that not been the case then we can conclude that parity was violated and nature prefers one direction over the other. The situation is depicted in Figure 3

One day, Lee visited his colleague office, Chein-Shiung Wu at Pupin Physics lab. Wu was an expert in beta decay, so Lee asked her if an experiment similar to the one described previously was conducted but she could not provide him with any information but suggested that this can be carried

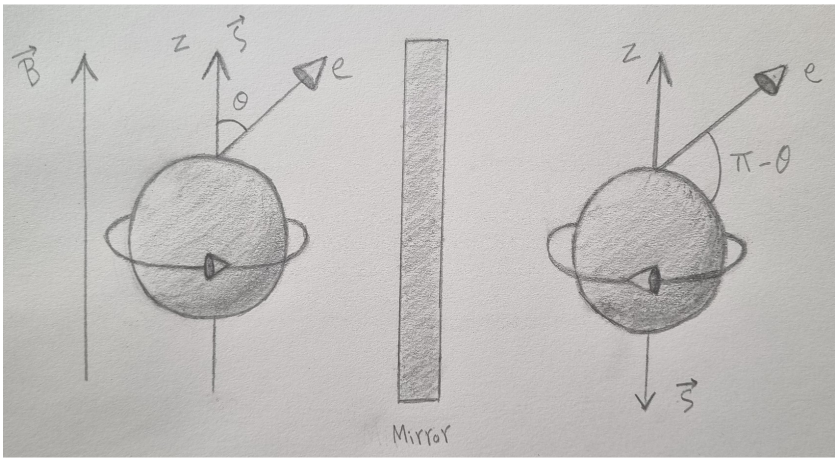

Figure 3: When a magnetic field is applied in the z-axis, the Co^{60} atom will be spinning in the +z-axis and the emitted electron will make an angle θ with the z-axis . In the mirror image on the right, the Co^{60} atom will be spinning in -z-axis making an angle of $\pi - \theta$ with the electron's momentum.

out using Co^{60} as a source of beta decay and polarize it by a process called *adiabatic demagnetization* [11]. At that time, Wu and her husband were about to travel to Europe but she saw an opportunity for a breakthrough and hence preferred to stay in the US.

Wu knew that this experiment must be conducted at low temperatures because if we want to align the magnetic moments of the cobalt nuclei with the applied magnetic field, thermal fluctuations must be eliminated so that the alignment remains for a reasonable observation time. To do this experiment, Wu contacted Ernest Ambler who was an expert in nuclear orientation and was working at the National Bureau of Standards now known as NIST. Going there, she met Ambler and other physicists namely, Hudson, Hayward and Hoppes who were then became part of her group members.

The apparatus used by Wu is shown in Figure 4. This is a cryostat inside of which a Co^{60} thin crystal layer is supported and shielded by a cerium magnesium nitrite (CMN) which is a paramagnetic salt. Without the CMN housing the specimen will warm up quickly and the polarization will disappear so to increase the observation time, the CMN housing was needed. In addition the adiabatic demagnetization lowered the specimen temperature to 0.003°K. Located above the specimen an anthracene crystal that scintillates upon interaction with beta decay. These scintillations are trasmitted through a lucite rod and then detected by a photomultiplier located at the end of the rod (it is not shown in the figure). There are also two NaI detectors that detect gamma rays, an equitorial detector 46cm from the specimen and a polar detector 41.5cm from the specimen.

The gamma rays detectors were used for two reasons [13], the first is to determine the degree of polarization of Co^{60} atoms since gamma rays are emitted isotropically but when the polarization occur we will have gamma anisotropy. The second reason is that parity symmetry holds in electromagnetism and so gamma rays could be used to compare with beta decay asymmetry and any huge deviation from gamma rays distribution would indicate parity nonconservation. The magnet that is used to polarize the specimen is a solenoid that is placed around the cryostat (not to be confused with the one used for adiabatic demagnetization).

The results of the experiment are displayed on the right side of Figure 4. Note that beta particles can only be detected in one direction because there is only one detector above the specimen. To detect beta particles in the opposite direction we have to make the setup upside down but that is equivalent to reversing the direction of the magnetic field which is easier to perform. In Figure 4 the top graph is the counts rate versus time of gamma rays with two curves a and b corresponding to the equatorial and polar detectors. The polarization time is about 8 minutes after which the gamma anisotropy disappears and return to be emitted isotropically. We can see in the graph in the bottom that beta particles are emitted more often in the direction opposite the magnetic field (upper curve) while those emitted in the same direction of the field are emitted less often (lower curve) and therefore we can conclude that in weak interactions, parity is not conserved or symmetric.

Shortly after this breakthrough by one year, Lee and Yang were awarded the 1957 Nobel prize

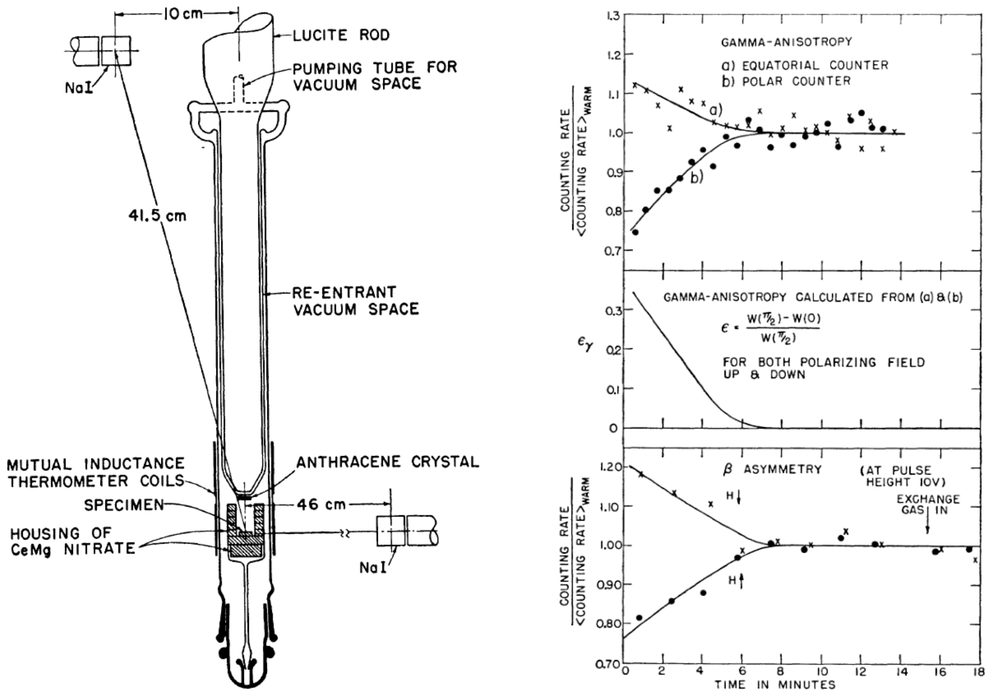

Figure 4: Left) the apparatus used by Wu which is a cryostat and inside it Co^{60} specimen which is shielded by CMN housing. Right) the results obtained from the detector. The top figure measures the anisotropy of gamma rays while the bottom one measures the beta asymmetry both with respect to time. Taken from Wu's paper [12].

in physics though Wu did not share it with them. Many criticized the Nobel prize committee for overlooking this significant contribution. Later on she went to claim the first Wolf prize in physics in 1978.

We would like to close this section by answering what particle was that in the tau-theta puzzle since we know now that they are the same particle? it turns out that this is the well known kaon or K meson particle which had different decay modes.

6 Conclusion

In conclusion, the exploration of symmetry in particle physics has provided valuable insights into the fundamental nature of the universe. This research focused on many types of symmetry in the realm of subatomic particles, such as flavour symmetry, parity symmetry, and charge conjugation symmetry. The journey began with an illustration of how symmetry is related to conservation law or conservation quantity. The paper then goes over flavor symmetry and isospin conservation and how they can be used to calculate relative decay rates and cross-sections. Following that, it examines parity symmetry by attempting to compute the parity operator and provides an example of how QED interactions can be conserved under parity symmetry. As demonstrated in this study, the exploration of flavor symmetry, isospin conservation, parity symmetry, and charge conjugation symmetry has yielded valuable insights into the conservation of quantities, decay rates, and cross-sections in particle interactions. Additionally, the examination of cases where symmetries break down like weak interaction by discussing the wu experiment. Deepening our understanding of the complexities within the subatomic realm.

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