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On The Darboux Integral and its Equivalence to the Riemann Integral

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abstract

In integration theory, many definitions of the integral arises depending on the field of mathematics they are used in. Some of the definitions are generalizations of others like the Riemann-Stieltjes integral and the Lebesgue integral that extend the definition of the Riemann integral to include functions that are not Riemann integrable. Another example is the Darboux integral which is not a generalization but rather a different approach to integrals which can be easier than the Riemann integral. In this paper we focus on the Darboux integral and its equivalence to the Riemann integral and we conduct a comparison between them.

1 Introduction

In this article we want to establish some theorems in integral calculus using the two definitions of the integral, the Riemann integral and the Darboux integral so that by comparison we can see what advantages does the Darboux integral has over the Riemann integral and what things does it lack. Both approach are equivalent to each other meaning that if a function is Darboux integrable then it is Riemann integrable and the value of the two integrals are equal. the two definitions are attributed to two mathematicians, Bernhard Riemann who is German and the french mathematician Gaston Darboux. Now before we start investigating both formulations one may ask why the Darboux integral is not popular while it could have some advantages over the Riemannian approach? the answer is that most textbooks use the Darboux approach very often but implicitly by putting it under the Riemann integral title. First thing we do is to introduce some concepts that will be needed when working with both definitions.

2 Partitions

let $I = [a, b]$ be a closed bounded interval, then we say that $\mathcal{P}$ is a partition of I such that $\mathcal{P}$ is the sequence of real numbers with $\mathcal{P} = (x_0, x_1, x_2, ..., x_n)$ and

$$a = x_0 < x_1 < x_2 < ... < x_n = b$$

so that the sequence is monotonically increasing. we can also put it another way with $\mathcal{P}$ being the set of subintervals such that $\mathcal{P} = \{[x_0, x_1], [x_1, x_2], ..., [x_{n-1}, x_n]\}$ which is what we will use throughout the paper.

Now we define another concept called the *norm* of a partition to be $||\mathcal{P}|| = max\{x_1 - x_0, x_2 - x_1, ..., x_n - x_{n-1}\}$ which means that the norm of a partition is just a number with the greatest length of subinterval between all other subintervals. we give an example of a set and a partition of it and its norm. let $I = [-1, 1]$ then a possible partition of I could be $\mathcal{P}_1 = \{[-1, -1/2], [-1/2, 0], [0, 1/2], [1/2, 1]\}$ with norm $||\mathcal{P}_1|| = 1/2$. a partition need not be of equal length so we can also have $\mathcal{P}_2 = \{[-1, -2/3], [-2/3, -1/5], [-1/5, 1]\}$ with norm $||\mathcal{P}_2|| = 6/5$. We can see that there are infinitely many ways to choose our partitioning set.

another useful concept is the ***tags*** of subintervals in the partition. to tag a partition is to associate with every subinterval a sample point from that interval hence creating a tagged partiton. for instance we can choose t_i to be the tag of the interval $[x_{i-1}, x_i]$ such that $t_i \in [x_{i-1}, x_i]$ and the tagged partition is the set of ordered pairs of subintervals together with their tags $\dot{\mathcal{P}} = \{([x_0, x_1], t_1), ([x_1, x_2], t_2), ..., ([x_{n-1}, x_n], t_n)\}$. As an example consider the previous partition $\mathcal{P}_1$ if we tag it we have $\dot{\mathcal{P}}_1 = \{([-1, -1/2], -1), ([-1/2, 0], -1/4), ([0, 1/2], 1/2), ([1/2, 1], 3/4)\}$. two things to note here, first is that the tag could be any point in the interval including the endpoints, second is that we can have an infinitely many ways to choose the tags.

We want also to introduce the idea of **refinement of a partition**. let $\mathcal{P}$ and Q be two partitions of the interval $I; = [a, b]$. Then the partition Q is said to be a refinement of $\mathcal{P}$ if for all elements $A \in Q$ there exists an element $B \in \mathcal{P}$ such that $A \subseteq B$. Also we say that Q is finer than $\mathcal{P}$. we see by little work that $\mathcal{P}_1$ and $\mathcal{P}_2$ are not refinements of each other, but $\mathcal{P}_3 := \{[-1, -3/4], [-3/4, -2/3], [-2/3, -1/2], [-1/2, -1/5], [-1/5, 1]\}$ is a refinement of $\mathcal{P}_2$ since:

$$[-1, -3/4] \subseteq [-1, -2/3]$$
$$[-3/4, -2/3] \subseteq [-1, -2/3]$$
$$[-2/3, -1/2] \subseteq [-2/3, -1/5]$$
$$[-1/2, -1/5] \subseteq [-2/3, -1/5]$$
$$[-1/5, 1] \subseteq [-1/5, 1]$$

Lastly, we define the **common refinement**. Given two partitions $\mathcal{P}$ and $\mathcal{P}'$ and a closed bounded interval I. The common partition is defined as: [4]

$$\mathcal{P} \# \mathcal{P}' := \{A \cap B | A \in \mathcal{P}, B \in \mathcal{P}'\} \tag{1}$$

As a lemma, the set of common refinement is also a partition of I and it is a refinement of both $\mathcal{P}$ and $\mathcal{P}'$

3 Definition of Darboux Integral

Let $f : [a, b] \to \mathbb{R}$ be bounded on $I := [a, b]$ and let $\mathcal{P}$ be a partition of I that has an arbitrary subinterval $[x_{k-1}, x_k]$ where $k = 1, 2, 3, ..., n$ we define the following quantities:

$$m_k := inf\{f(x) \,|\, x \in [x_{k-1}, x_k]\} \tag{2}$$

and

$$M_k := sup\{f(x) \,|\, x \in [x_{k-1}, x_k]\} \tag{3}$$

We also have the following definitions:

$$L(f;\mathcal{P}) := \sum_{i=1}^{n} m_k(x_k - x_{k-1}) \tag{4}$$

and

$$U(f;\mathcal{P}) := \sum_{i=1}^{n} M_k(x_k - x_{k-1}) \tag{5}$$

We call these sums, Darboux sums and we call $L(f;\mathcal{P})$ the lower sum of f with respect to $\mathcal{P}$. This sum, multiply the minimum value of the function with the length of that interval for each interval and add them up. The other sum is called an upper sum of f with respect to $\mathcal{P}$ and is denoted by $U(f;\mathcal{P})$. This sum is the opposite of the lower sum because it take the maximum value of the function on an interval and multiply it by the length of the interval and add all terms.

It seems an intuitive idea that $L(f;\mathcal{P}) \geq U(f;\mathcal{P})$ since the lower sum has to do with mimimum of the function but we need to prove this.

Statement: let $f : I \to \mathbb{R}$, for a given $\mathcal{P}$ of I we have $L(f;\mathcal{P}) \leq U(f;\mathcal{P})$.

Proof: first we can see that $m_k \leq M_k$ for $k = 1, 2, ..., n$ and that $x_k - x_{k-1} > 0$ then for an arbitrary interval $m_k(x_k - x_{k-1}) \leq M_k(x_k - x_{k-1})$ so when summing all terms

$$\sum_{i=1}^{n} m_k(x_k - x_{k-1}) \leq \sum_{i=1}^{n} M_k(x_k - x_{k-1}) \Longleftrightarrow L(f;\mathcal{P}) \leq U(f;\mathcal{P})$$

■

We can picture the lower sum as an underestimate of the area under the curve of a function $f : [a, b] \to \mathbb{R}$ and the upper sum as an overestimate. we next proof that a finer partition yields a greater lower sum and lesser upper sum.[1]

Statement: let $f : [a, b] \to \mathbb{R}$. If $\mathcal{P}$ is a partition of $[a, b]$ and if Q is a refinement of $\mathcal{P}$. then $L(f;\mathcal{P}) \leq L(f;Q)$ and $U(f;Q) \leq U(f;\mathcal{P})$.

Proof: let

$$\mathcal{P} = \{[x_0, x_1], [x_1, x_2], ..., [x_{k-1}, x_k], ..., [x_{n-1}, x_n]\}$$

and let Q be a refinement of $\mathcal{P}$ such that

$$Q = \{[x_0, x_1], [x_1, x_2], ..., [x_{k-1}, z], [z, x_k], ..., [x_{n-1}, x_n]\}$$

we define $m'_k = inf\{f(x) \mid x \in [x_{k-1}, z]\} = inf(A)$ and $m''_k = inf\{f(x) \mid x \in [z, x_k]\} = inf(B)$ We also have $m_k = inf\{f(x) \mid x \in [x_{k-1}, x_k]\} = inf(C)$ but we see that $C = A \cup B$ which means: $inf(C) = inf(A \cup B)$ and using the following property of infimum

$$inf(A \cup B) = inf\{inf(A), inf(B)\}$$

we conclude that $m_k \leq m'_k$, $m_k \leq m''_k$. Now we have:

$$m_k(x_k - x_{k-1}) = m_k(z - x_{k-1}) + m_k(x_k - z) \leq m'_k(z - x_{k-1}) + m''_k(x_k - z)$$

by adding the sum $\sum_{i=1}^{n} m_k(x_j - x_{j-1})$ with $j \neq k$ we get equation 3 on the left side and $L(f;Q)$ on the right side, thus $L(f;\mathcal{P}) \leq L(f;Q)$. if we refine Q then the same inequality will hold and hence the inequality hold for any arbitrary refinement of $\mathcal{P}$. In a similar fashion we can prove $U(f;Q) \leq U(f;\mathcal{P})$ by observing that the infimum property would be replaced by supremum and we will have instead $M_k \geq M'_k$, $M_k \geq M''_k$. ■

Before we move to the concept of upper and lower limit we prove the following statement.

Statment: Given a function $f : I \to \mathbb{R}$, such that f is bounded on I. If $\mathcal{P}$ and $\mathcal{P}'$ are two partitions of I then $L(f;\mathcal{P}) \leq U(f;\mathcal{P}')$

Proof: first we take the common refinement of the two partitions $Q = \mathcal{P} \# \mathcal{P}'$. Since Q is a refinement of $\mathcal{P}$ and $\mathcal{P}'$ then from the previous statements we have $L(f;\mathcal{P}) \leq L(f;Q)$ and $L(f;Q) \leq U(f;Q)$ and also $U(f;Q) \leq U(f;\mathcal{P}')$ putting all that together we see that:

$$L(f;\mathcal{P}) \leq L(f;Q) \leq U(f;Q) \leq U(f;\mathcal{P}') \Rightarrow L(f;\mathcal{P}) \leq U(f;\mathcal{P}')$$

■

3.1 Upper and lower integrals

We start this concept by introducing the set of all partitions. We denote the set which contains all the partitions of an interval $I = [a, b]$ by $\mathscr{P}(I)$. This set give rise to two quantities which we call the lower and upper integrals respectively as follows:

$$L(f) := sup\{L(f;\mathcal{P}) \mid \mathcal{P} \in \mathscr{P}(I)\} = \underline{\int_a^b} f \tag{6}$$

Also for upper integral:

$$U(f) := inf\{U(f;\mathcal{P}) \mid \mathcal{P} \in \mathscr{P}(I)\} = \overline{\int_a^b} f \tag{7}$$

for which $f : I \to \mathbb{R}$ is bounded. Since f is bounded then by completeness property, a supremum and an infimum exist and we have:

$$m_I := inf\{f(x) \mid x \in I\} \quad , \quad M_I := sup\{f(x) \mid x \in I\} \tag{8}$$

Since $m_I \leq m_k$ for all $k \in \mathbb{N}$ such that $1 \leq k \leq n$, then by comparison we have: [3]

$$m_I(b-a) = \sum_{k=1}^{n} m_I(x_k - x_{k-1}) \leq \sum_{k=1}^{n} m_k(x_k - x_{k-1}) = L(f;\mathcal{P})$$

also

$$U(f;\mathcal{P}) = \sum_{k=1}^{n} M_k(x_k - x_{k-1}) \leq \sum_{k=1}^{n} M_I(x_k - x_{k-1}) = M_I(b-a)$$

so that

$$m_I(b-a) \leq L(f;\mathcal{P}) \leq U(f;\mathcal{P}) \leq M_I(b-a)$$

which implies $m_I(b-a) \leq L(f)$ and $U(f) \leq M_I(b-a)$ next we want to prove the following inequality $L(f) \leq U(f)$.

Statement: suppose we have a function $f : I \to \mathbb{R}$ that is bounded on I, then $L(f)$ and $U(f)$ exist and $L(f) \leq U(f)$.

Proof: given two partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ we know from the previous theorem that $L(f;\mathcal{P}_1) \leq U(f;\mathcal{P}_2)$. We see that $U(f;\mathcal{P}_2)$ is an upper bound for $\{L(f;\mathcal{P}) \mid \mathcal{P} \in \mathscr{P}(I)\}$ and since $L(f)$ is the supremum or the least upper bound then $L(f) \leq U(f;\mathcal{P}_2)$, Also since $L(f)$ is a lower bound for $\{U(f;\mathcal{P}) \mid \mathcal{P} \in \mathscr{P}(I)\}$ then $L(f) \leq U(f)$ because $U(f)$ is the infimum or the greatest lower bound. ■

We are now in a position where we defined everything needed to state the definition of the Darboux integral.

3.2 The Definition

Suppose f is a function such that $f : I \to \mathbb{R}$ is a bounded on I. If $L(f) = U(f)$ that is the lower integral is equal to the upper integral then we say that f is Darboux integrable on I and the Darboux integral have the value $L(f) = U(f)$. Equivalently, we can say that if f is not Darboux integrable on I then $L(f) \neq U(f)$ in that case $L(f) < U(f)$.
We should notice that the value is unique without proving this since the Darboux integral is the value of $L(f)$ or $U(f)$ both of which are supremum and infimum respectively of the set of Darboux sums and we know that supremum and infimum whenever they exist they are unique.

4 The Riemann Integral

We now focus our attention on the Riemann integral. Let $f : I \to \mathbb{R}$ and let $\dot{\mathcal{P}}$ be the tagged partition set defined by $\dot{\mathcal{P}} = \{[x_{i-1}, x_i], t_i\}_{i=1}^{n}$ we define the *Riemann sum* as follows:

$$S(f; \dot{\mathcal{P}}) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) \tag{9}$$

you maybe noticed that if we chose the sample point or the tag t_i to be the value such that $f(t_i) \leq f(x)$ for all $x \in [x_{i-1}, x_i]$ then $f(t_i)$ is the infimum of the interval and we have a lower Darboux sum. likewise if $f(t_i)$ is chosen to yield the supremum we will have an upper Darboux sum. So in some sense the definition of the Riemann integral will be more general than the Darboux integral.

Definition:[5] Suppose f is a function defined on $[a, b]$ then we say that f is Riemann integrable if we can find a number L obeying the following condition: for all $\epsilon > 0$ there is a number $\delta > 0$ such that if $||\dot{\mathcal{P}}|| < \delta$, then $|S(f; \dot{\mathcal{P}}) - L| < \epsilon$. L represent the value of the Riemann integral and we write this as

$$\int_a^b f(x)dx = L \tag{10}$$

Unlike the Darboux integral, the value of the Riemann integral must be proven that it is unique.

Statement: if f is Riemann integrable function, then the value of the integral is unique.

Proof: assume that we have two values for the integral L_1 and L_2 for which both satisfy the definition of the Riemann integral. Now for all $\epsilon > 0$ there exist $\delta_1 > 0$ such that any tagged partition with $||\dot{\mathcal{P}}_1|| < \delta_1$ then $|S(f; \dot{\mathcal{P}}_1) - L_1| < \epsilon/2$ and also any tagged partition with $||\dot{\mathcal{P}}_2|| < \delta_2$ it follows that $|S(f; \dot{\mathcal{P}}_2) - L_2| < \epsilon/2$. Now we take $\delta = min\{\delta_1, \delta_2\}$ (we do this so that both epsilon inequalities hold) and we say for $\dot{\mathcal{P}}$ with $||\dot{\mathcal{P}}|| < \delta$ it follows that $|S(f; \dot{\mathcal{P}}) - L_1| < \epsilon/2$ and $|S(f; \dot{\mathcal{P}}) - L_2| < \epsilon/2$ (as we said, we took the minimum so that both inequalities are true for $\mathcal{P}$). Now we manipulate the expression $|L_1 - L_2|$ and we use the triangle inequality:

$$|L_1 - L_2| = |L_1 + S(f; \dot{\mathcal{P}}) - S(f; \dot{\mathcal{P}}) - L_2| \leq |S(f; \dot{\mathcal{P}}) - L_1| + |S(f; \dot{\mathcal{P}}) - L_2| \leq \epsilon/2 + \epsilon/2 = \epsilon$$

$$\Rightarrow 0 \leq |L_1 - L_2| < \epsilon$$

for this last inequality to hold for all ϵ the only possible choice is $|L_1 - L_2| = 0 \Rightarrow L_1 = L_2$ ■

In the next section we establish the equivalence of the two approaches or definitions.

5 Equivalence of Riemann and Darboux integral

Riemann integral definition is equivalent to the Darboux integral definition. This means that whenever a function f is Riemann integrable it follows that it is Darboux integrable and if f is Darboux integrable then it is Riemann integrable, moreover the value of the value of the two integrals are equal.

before we show the proof of equivalence we need to state four theorems that will be used in the proof of equivalence. For brevity of discussion we omit their proofs and only state them.

5.1 Four Theorems

Theorem 1:[2] if f is a Riemann integrable function on the interval $[a, b]$ then f is bounded on $[a, b]$
Theorem 2:[2] A real-valued function f defined on $[a, b]$ is Riemann integrable if and only if for all $\epsilon > 0$ we can find at least two functions $\alpha(x)$ and $\omega(x)$ that are Riemann integrable and satisfy the two conditions: $\alpha(x) \leq f(x) \leq \omega(x)$ for all $x \in [a, b]$ and

$$\int_a^b (\omega - \alpha) < \epsilon$$

This is the so called *Squeeze Theorem*.
Theorem 3:[2] let ψ be a real valued *step function* defined on $[a, b]$, then it follows that ψ is Riemann integrable function.
Note: *step functions* are defined to be functions with domain $[a, b]$ such that this domain is divided into subintervals that does not overlap with the function having a constant value on each subinterval.

Theorem 4:[2] let f be real valued bounded function defined on $I = [a, b]$, then f is Darboux integrable if and only if for all $\epsilon > 0$ there exist a partition $\mathcal{P}$ to the interval $[a, b]$ such that

$$U(f; \mathcal{P}) - L(f; \mathcal{P}) < \epsilon$$

we call this the *integrability criterion*

5.2 The Theorem of Equivalence

Statement: A function $f : I \to \mathbb{R}$ is Riemann integrable if and only if it is Darboux integrable with the two integrals being equal.

Proof: ($\Rightarrow$) let f be a Riemann integrable function on I with the integral equal L. from definition we have that for all $\epsilon > 0$ there exist a partition $\dot{\mathcal{P}}$ with $||\dot{\mathcal{P}}|| < \delta$ such that $|S(f, \dot{\mathcal{P}}) - L| < \epsilon$ or:

$$L - \epsilon < S(f; \dot{\mathcal{P}}) < L + \epsilon \tag{11}$$

Now we tag $\dot{\mathcal{P}} = \{[x_{k-1}, x_k], t_k\}_{k=1}^n$ with tags $t_k \in [x_{k-1}, x_k]$ in such a way that the following inequality hold:

$$f(t_k) < m_k + \frac{\epsilon}{b - a}$$

where m_k is defined as for the left equation in (8). multiplying both sides by $b-a$ and observing that $\sum_{k=1}^n (x_k - x_{k-1}) = b - a$

$$f(t_k)(b - a) < m_k(b - a) + \epsilon$$

$$f(t_k)[(x_n - x_{n-1}) + ... + (x_1 - x_0)] < m_k[(x_n - x_{n-1}) + ... + (x_1 - x_0)] + \epsilon$$

$$\Rightarrow \sum_{k=1}^n f(t_k)(x_k - x_{k-1}) = S(f; \dot{\mathcal{P}}) < \sum_{k=1}^n m_k(x_k - x_{k-1}) + \epsilon = L(f; \mathcal{P}) + \epsilon \leq L(f) + \epsilon$$

using the last inequality together with (11) we have $L - 2\epsilon < L(f)$ (do not confuse the Riemann integral value L with the lower integral $L(f)$). Since $L - 2\epsilon < L(f)$ holds for all $\epsilon > 0$ then we can make it so small that $L \leq L(f)$. applying small changes to the previous argument we can see by similar procedure that $U(f) \leq L$. Therefore we have $L \leq L(f) \leq U(f) \leq L$ so the only possible choice for this inequality to be satisfied is to have $L(f) = U(f) = L$ which means that the lower integral is equal to the upper integral and that their values is equal to the Riemann integral. So by definition, f is Darboux integrable.

($\Leftarrow$) assume that f is Darboux integrable. Then we say by theorem 4 (the integrability criterion) that for all $\epsilon > 0$ there exist a partition $\mathcal{P}$ that satisfy $U(f; \mathcal{P}) - L(f; \mathcal{P}) < \epsilon$.

Now, we let $\alpha(x) = m_k$ and $\omega(x) = M_k$ where m_k and M_k are defined in (8) and $x \in [x_{k-1}, x_k)$ with $k = 1, 2, ..., n-1$ (note that the interval is open from the right since we want to construct step functions with non-overlapping subintervals). For $x \in [x_{n-1}, x_n]$ we let $\alpha(x) = m_n$ and $\omega(x) = M_n$. Since $\alpha(x)$ is always the minimum of all subintervals and $\omega(x)$ is the maximum we conclude that

$$\alpha(x) \leq f(x) \leq \omega(x) \qquad \forall x \in I = [a, b] \tag{12}$$

Since $\alpha(x)$ and $\omega(x)$ are step functions then by theorem 3 we can assure that they are Riemann integrable. Taking the integral of the two functions we get:

$$\int_a^b \alpha = \sum_{k=1}^{n} m_k(x_k - x_{k-1}) = L(f; \mathcal{P})$$

$$\int_a^b \omega = \sum_{k=1}^{n} M_k(x_k - x_{k-1})) = U(f; \mathcal{P})$$

$$\Rightarrow \int_a^b (\omega - \alpha) = U(f; \mathcal{P}) - L(f; \mathcal{P}) < \epsilon \tag{13}$$

Thus from (12) and (13) it follows that f is Riemann integrable by squeeze theorem (theorem 2). Now, since $\mathcal{P}$ is arbitrary throughout the proof we see that the value of the Riemann integral is bounded by:

$$L(f; \mathcal{P}) \leq \int_a^b f(x)dx \leq U(f; \mathcal{P})$$

for any $\mathcal{P}$, the last inequality always holds. Therefore if we keep refining $\mathcal{P}$ we get:

$$L(f) = \int_a^b f(x)dx = U(f)$$

Thus establishing the equivalence ■

6 Conclusion

In this article we constructed or formulated two integrals, namely the Darboux integral and the Riemann integral. We saw that the Darboux integral is approximated by two sums called the upper and lower Darboux sums and by refining the partition over which we are summing we get a better approximation until the two sums coincide to give the exact value of the integral. As for the Riemann integral, we saw that it was defined by what we called Riemann sum over a tagged partition. The idea was that the sum gets better as the norm of the partition shrinks in size until it give the value of the Riemann integral. We established the equivalence of the two definitions, meaning that whenever one definition can be applied to a function then we can apply the other definition and they must give the same result.

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