King Fahd University For Petroleum and Minerals

PHYS430 Project

The Apparent Time-Asymmetry in the Macroscopic World

Authors:
Abdulelah Hakami
Mohammed Alsadah

Submitted to:
Dr. Jihad Alsadah

Date:
May 19, 2024

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Abdulelah Hakami, Mohammed Alsadah

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Abstract

The concepts of entropy and time have always been linked together, while "the laws of physics" are symmetric in time, entropy introduces a law that breaks that symmetry. Elaborating on entropy, this paper will discuss three types of entropy: Boltzmann's, Gibbs's, and Shannon's entropy, highlighting differences between each type. Although laws like Newton's laws or electrodynamics laws, exhibit time symmetry, the "arrow of time" is an asymmetric concept that contradicts basic laws. Two approaches will be elaborated upon explaining this matter. Lastly, exploring Loschmidt's paradox that highlights a tension between the statistical behavior of large systems and the time-reversal symmetry of microscopic laws.

1 Introduction

In thermodynamics, entropy is a quantity used to measure the number of ways a system can be arranged. Entropy can also be viewed as a measure of randomness, hence a system with high entropy has many ways in which its particles could be arranged. Put another way, the system is in a state of high disorder or randomness. Entropy can be used to define a direction of time simply by considering low entropy macrostates to be the past while the inevitable high entropy macrostates to be the future. That is to say every system will evolve to a state of high entropy and thus we can say that this is where time is flowing.

In our world, most phenomena that we observe are irreversible, meaning they tend to increase entropy. For example, if we pour milk into coffee, then the two will mix until they become uniformly distributed in the cup and it is nearly impossible to see the two separate from one another. Another example is the process of magnetizing a ferromagnet, which exhibit the so called hysteresis phenomenon which is an irreversible process. From these examples we can see that many processes tend to increase in entropy and therefore irreversible, yet if we examine the laws of dynamics and Maxwell's electromagnetism and even quantum mechanics [1], we see that they are time reversal invariant. If we define time reversal transformation $T : t \mapsto -t$, the position $x(t)$ will be the same under T, that is $Tx(t) = x(-t)$ which implies $v(t) = -\dot{x}(-t)$ is the velocity and $a(t) = \ddot{x}(-t)$ is the acceleration and therefore satisfies $F(t) = ma(t)$. This means that Newton's second law is invariant under time reversal.

Now since the laws of motion are time reversal invariant, why is it the case that we only observe those processes which occurs from the past to the future and not the other way around if they are equally valid? In other words, how can we explain the discrepancy between irreversible macroscopic phenomena and the allowed or possible reversibility of microscopic processes giving rise to these macrostates?

There are many arrows of time, that is a specific direction in which event evolve and progress. Among them, the thermodynamic arrow of time which can be defined as the direction in time in which entropy increases, also we have the psychological arrow of time which is related to how we as humans perceive events. We can remember the past but not the future which we can only predict, and this is because of how we associate past events with causes and future events as effects generated from those causes. Another arrow of time is the one related to electrodynamics for which the field

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equations are time reversal invariant yet the only accepted and observed fields are those calculated from the retarded potentials. In addition to all these arrows, is the cosmological arrow of time which can be identified as the direction in which the universe expands [2]. It is speculated that if the cosmological arrow of time were to reverse direction (that is the universe started to contract) then all the other arrows will follow such reversal. There are also other arrows of time found in particle physics and quantum mechanics but we will restrict ourselves to talk about the thermodynamic arrow.

2 Types of entropy

Entropy has many interpretations and definitions, each for a specified audience. Therefore, we will pick the one we believe defines it the best: A quantity describes how the system will eventually arrange itself. As stated in the **2nd law of thermodynamics**, entropy tends to increase, which is an end destination of any system. Another interesting interpretation of entropy is the flow of time, which flows in only one direction, similar to a system of particles, flows from higher to lower energy. Multiplicity Ω is defined as how many arrangements we can make with the **microstates** for any given system, which is the main part of calculating the entropy. In the following sections, we shall expand on different treatments for Ω to find the entropy.

2.1 Boltzmann's entropy

The entropy of a system can be written as

$$S = k_B \ln(\Omega), \tag{1}$$

where k_B is the Boltzmann constant, and Ω is the microstates of the system. This equation is well-known as **Boltzmann's entropy** equation, the only thing you need to know is Ω to calculate it. Depending on the type of the system, Ω varies vastly on how it can be calculated, however, all can be found by modifying

$$\Omega(N, n) = \left(\begin{array}{c} N \\ n \end{array} \right) = \frac{N!}{n! \cdot (N - n)!}, \tag{2}$$

for **two-state system**. Hence, if we dealing with Einstein solid, for example, we modify N to be $q + N - 1$ and n to q, where q is the total number of energy units, N is the total number of harmonic oscillators. Eq.(1) is also used to find the entropy of an ideal gas, relating it to the multiplicity. Back between 1872 and 1875, Ludwig Boltzmann came up with such an equation in his kinetic theory of gases. By ignoring the interactions between molecules, the proper thermodynamics entropy for the special case of ideal gases.

2.2 Gibbs entropy

To our knowledge, finding Ω is not always easy, especially when we are talking about a large system of gas. Hence, we can manipulate Ω in order to calculate it, by introducing some approximations. Consider that you have a box filled with gas, and there is 3 total macrostates in such a system, as in Fig.(1). Ω for this case will be the sum of microstates in all 3 macrostates, which is 10. The first approximation is $\Omega \approx W_{max}$, where W_{max} is the macrostate with maximum number of microstate, which is 5. Yes, this seems absurd ($5 = 10$ that is), however, for a very large N, this approximation is relatively close to the actual value, not only that, we can find the state with the highest number of macrostates easier than calculating all microstates. Another name for W_{max} is the **most probable macrostate**. Now, we can approximate Eq.(1) as

$$S \approx k_B \ln(W_{max}), \tag{3}$$

which is a very good approximation for large system of particles. It is a good time now to discuss how to find W_{max}, which will invoke the second approximation there. Back to our example above, the **macrostate multiplicity** W can be calculated using

$$W = \frac{N!}{\prod_i^J N_i!}, \tag{4}$$

Figure 1: Box with 3 different macrostates
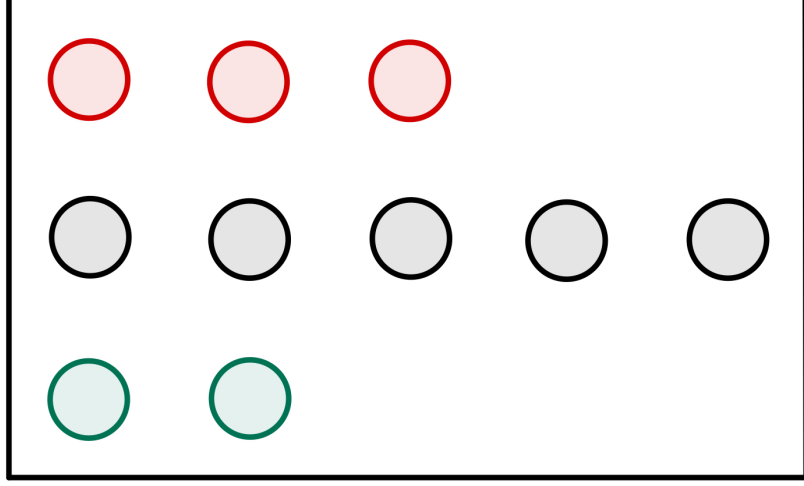
where the product runs from state i until state J, and N is the total number of particles. To make the substitution, we will take the natural log of W as

$$\ln(W) = \ln\left(\frac{N!}{\prod_i^J N_i!}\right) = \ln(N!) - \ln(\prod_i^J N_i!) = \ln(N!) - \sum_i^J \ln(N_i!). \tag{5}$$

Now, we are ready to invoke the second approximation, which is **Stirling's approximation**. Eq.(5) can then be written as

$$\ln(W) = N\ln(N) - N - \sum_i^J (N_i \ln(N_i) - N_i), \tag{6}$$

in the summation part of the equation, if we sum over all the energy states, we will end up with the total number of particles, making the equation simple as

$$\ln(W) = N\ln(N) - N - \sum_i^J (N_i \ln(N_i)) + N = N\ln(N) - \sum_i^J (N_i \ln(N_i)). \tag{7}$$

Now, what is N_i? It is the number of particles in state i, which is equal to the product of the total number of particles N and the probability state P_i in state i, $N_i = NP_i$. Substituting back into Eq.(7)

$$\ln(W) = N\ln(N) - \sum_i^J (NP_i \ln(NP_i)) = N\ln(N) - \sum_i^J (NP_i \ln(N)) - \sum_i^J (NP_i \ln(P_i)), \tag{8}$$

the first summation in the second term is only summing over all the probabilities, which is just 1, therefore canceling $N\ln(N)$ terms all together. Hence, ending up with

$$\ln(W) = -N\sum_i^J (P_i \ln(P_i)). \tag{9}$$

Plugging Eq.(9) into Eq.(3), we end up with the equation

$$S = -kN\sum_i^J (P_i \ln(P_i)). \tag{10}$$

This equation is called **the Gibbs entropy**, and it is a great approximation for the entropy using the probability states of our system [3]. We are not going to use it in the previous example, because it violated the condition of N should be **very** large. However, This result can help us in deriving many thermodynamic properties.

2.3 Shannon's entropy

The previous sections considered both Boltzmann's and Gibbs' definitions of entropy, one focused on the system as a whole, while the other approximated by considering the state holding the maximum microstates. Notice in these definitions, both considered systems of particles or similar analogies to it. Now, can we describe the outcome of such systems? We know plotting the entropy in terms of microstates, will result in a distribution, but this is not enough. We know any distribution has uncertainties, and how much information we can use to describe the outcome of a distribution. Let us say we have a group of 5 physics students, the certainty of one of them being the next physics club president is a certain outcome (he is in the last year and already worked with previous presidents of the club, thus very knowledgeable). On the other hand, 100 senior students of mechanical engineering who want to be the next mechanical club president is an uncertain outcome. How can we *quantify* this uncertainty? To answer this question, we must fix our point of view on one thing (this may vary greatly depending on how we treat our point), an information standpoint [4, 5]. How much information on average, we would need to encode an outcome from a distribution? This shows how we quantify uncertainties into information (the more uncertain you are, the more information you need to know to describe an outcome). To show this better, consider the example of a coin, two people separated by a partition, as in Fig.(2), one has a coin, and we want to send him a code

Figure 2: Two people separated by a partition, the left is sending a code to the right without specifically telling what outcome he got from a coin toss
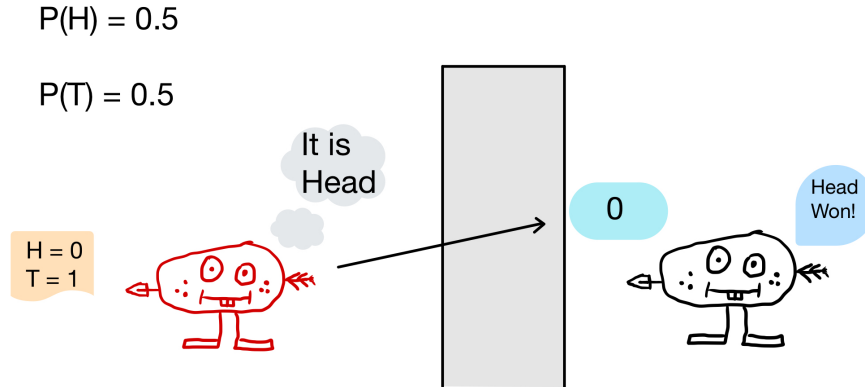

(known beforehand) that will let him know the outcome, without passing the wall. We can send him a 0 and he will know it was a head, 1 if it is a tale. Notice, we only sent him a one **bit** code, that described the outcome entirely. Thus, this distribution has an entropy of *one bit*, simple as that. Back to the club president analogy, we have now 8 equally probable students to become the next leader of the club, we can encode a *three bit* code to the guy behind the wall, knowing who has won, again, describing the outcome fully using three bits (entropy of three in other words). Now it is becoming apparent, that for N equally probable outcomes, the entropy will be

$$S_s = \log_2 N, \tag{11}$$

which is trivial as $\log_2 2^N = N$. Let us test if this also holds for outcomes other than the power of two (spoilers, it works). For 5 equally probable states (same as the physics club example), the encoding outcomes are in Fig.(3). Notice, we ended up with 3 *unused* combinations, which is not efficient in describing the outcomes (the number of bits is 3, while the number of outcomes is 1, thus, $3/1 = 3$). This ratio is called the **average bits per outcome**. To find the most efficient encoding

Figure 3: 5 outcomes encoding possibilities
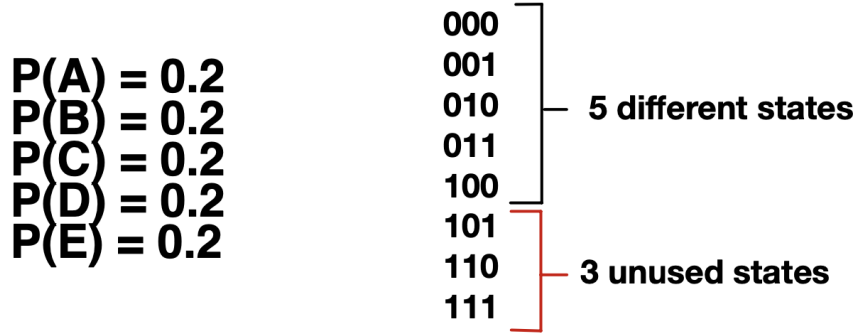
scheme for our example, we are looking for

$$2^B = 3^N, \tag{12}$$

taking the logarithm

$$\frac{B}{N} = \log_2 3 = 1.58, \tag{13}$$

this is the **entropy** expression, which shows the average bits per outcome is 1.58. We have shown, that for a uniform distribution, the entropy is given by $\log_2 N$, where N is the number of bits. However, most distributions are non-uniform distributions, we can still describe such distributions by assigning bits to outcomes. That is

$$P_N = \frac{1}{N}, \tag{14}$$

which tells us that for an arbitrary outcome having the probability P_N, we need $-\log_2 P_N$ bits to encode that outcome. Notice, that we get the original equation (Eq.(11)), as

$$-\log_2 P_N = -\log_2 \frac{1}{N} = \log_2 N. \tag{15}$$

Lastly, to get the expected number of bits for all the outcomes by multiplying each outcome by the corresponding probabilities

$$S_s = -\sum_i^N P_i \log_2 P_i, \tag{16}$$

This is called **The Shannon's entropy** formula. Which is the measurement of how much information on average, is needed to describe outcomes of a distribution. The more uncertain you are, the more information you need to describe the outcome, and vice versa (the lower entropy is, the more certain of what the outcome is).

3   Entropy, the second law, and time

The physicist Arthur Eddington put forth the concept of the "*arrow of time*" which means that time is running in only one direction or that it is unidirectional. In his book, *The Nature of the Physical World*, Sir Arthur describes a common property of the physical laws with regard to time by saying that the movement of states in a sequential manner from past to future is the doing of an event while running this sequence backwardly is the undoing of that even [6]. Then he goes on to say that if the laws are invariant under time reversal (of doing and undoing) which is what we showed in the introduction for the case of Newton's second law, then a consequence of that, is the inability to differentiate between past and future. However, the second law of thermodynamics which states that entropy must increase, contradicts this inability by introducing the element of randomness.

   Offered explanations to this puzzle try to explain why the second law of thermodynamics is time asymmetric. That is entropy increases with time, but if it was time-symmetric law, then it would

be equally probable to see entropy decrease or stay the same. This would mean that reversible processes would occur more often than we observe.

Explanations for this mystery come from two different approaches. The first is the two-asymmetry approach. This approach takes the stance that to explain the second law, two factors come into play, one related to the dynamics of the laws and the other to the boundary condition of the early universe. An example of a dynamical factor is the assumption of molecular chaos made by Boltzmann in his derivation of the second law through his H-theorem. As for the boundary condition, the universe needs to start at low entropy so that this increase of entropy dictated by the second law can take place. Therefore the two-asymmetry approach has two asymmetries, one in the dynamics and the other in the boundary conditions.

In the one-asymmetry approach, there is only one asymmetry involved and it is related to the boundary condition of low entropy. The problem of dynamical asymmetry is eliminated by using laws that respect time reversibility. An example of this approach is offered by Boltzmann to address the objections made by Loschmidt (which will be explained later) about his H-theorem. The new idea of Boltzmann is to consider deterministic laws like those of Newton which are time-symmetric. Boltzmann formulated a way to measure the number of possibilities for a system to be in a certain macrostate. such a measure is time-symmetric and only depends on the initial entropy of the system. This means that if we started with some gas sample confined to some place, then if we calculate the probability of dispersal, it will be high. Therefore, the only factor that governs the asymmetry in time is the boundary condition, of whether the system started at low or high temperature [7].

4 Loschmidt's paradox

4.1 H-Theorem

In his 1872 account "Further Studies on the Thermal Equilibrium of Gas Molecules" [8] Ludwig Boltzmann attempted to derive the second law of thermodynamics by showing that a certain quantity or function E (later named H) is always decreasing. This famous result is called the *H-theorem* and it is a consequence of Boltzmann transport equation which describes the evolution of the distribution function $f(\mathbf{r}, \mathbf{v}, t)$ that determines the number of molecules in different states at different times

$$\frac{\partial f}{\partial t} + \frac{\mathbf{p}}{m} \cdot \nabla f + \mathbf{F} \cdot \frac{\partial f}{\partial \mathbf{p}} = \left(\frac{\partial f}{\partial t} \right)_{coll}. \tag{17}$$

This is a partial differential equation of the distribution function $f(\mathbf{r}, \mathbf{v}, t)$. Where the first term $\frac{\partial f}{\partial t}$ is the change in the distribution due to all factors like external forces, diffusion, and collisions. $\mathbf{p}$ is the momentum of particles and m is the mass of each. $\mathbf{F}$ is the external force acting on the particles and $(\partial f/\partial t)_{coll}$ is the change in the density function over time due to collisions which is zero for collision-less fluid.

Boltzmann defined the function H to be

$$H(t) = \int f(\mathbf{r}, \mathbf{v}, t) \ln f(\mathbf{r}, \mathbf{v}, t) \, d^3\mathbf{r} \, d^3\mathbf{v} \tag{18}$$

Where the integral is carried over the phase space and $d^3\mathbf{r} = dx \, dy \, dz$ and $d^3\mathbf{v} = dv_x dv_y dv_z$. By using his equation, he was able to show (under certain assumption) that the function H monotonically decrease, that is

$$\frac{dH}{dt} \leq 0 \tag{19}$$

Where equality holds at equilibrium. Since entropy, S is proportional to H up to a negative sign, we can see that $dS/dt \geq 0$. Notice that we are assuming that the number of molecules or particles is large so that we can consider the distribution to be differentiable. Boltzmann assumed also that only two-body collisions take place because he considered dilute gas. He also assumed the density function to be position independent $f(\mathbf{r}, \mathbf{v}, t) = f(\mathbf{v}, t)$ and that the velocities of two colliding particles are uncorrelated making us able to write the density of the two particles $F(v_1, v_2, t)$ with velocities v_1 and v_2 colliding in the interval $[t, t + \Delta t]$ as the product of the individual densities [9]

$$F(v_1, v_2, t) = f(v_1, t)f(v_2, t). \tag{20}$$

This assumption is called *Stosszahlansatz* or the assumption of *molecular chaos*.

4.2   The paradox

After publishing his results, Boltzmann was faced with a lot of criticisms to his H-theorem. Among these objections is the one made by Josef Loschmidt in 1876. Loschmidt objected on how Boltzmann derived a non-symmetric law- namely the second law of thermodynamics- from time reversal invariant laws- those of Newtonian mechanics- and that this can be done by introducing a non-symmetric assumption while deriving his result. Loschmidt pointed out that this assumption is that of molecular chaos which states that the velocities of two colliding molecules do not correlate prior to their collision.

5   Conclusion

Entropy has many definitions as we saw in our article. It can be used to define a direction for time called the arrow of time in the direction where entropy increases. The main issue in this article was to discuss the apparent time asymmetry in the macroscopic processes while the laws describing them are time reversal invariant. In addition to how changing the perspective of the system, may lead to different definition of entropy. Boltzmann's, Gibbs's, and Shannon's entropy encounter the same overall principle, while being treated with respect to the system. Moreover, we discussed the H-theorem proved by Boltzmann in 1872 that tries to derive the second law of thermodynamics from the laws of dynamics. The loschmidt's paradox objects on Boltzmann derivation by saying that Boltzmann introduced an element of irreversibility by using the assumption of molecular chaos.

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