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SENSITIVITY ANALYSIS AND NUMERICAL MODELING OF INFLUENZA PROPAGATION AND INTERVENTION STRATEGIES UNDER THE FRACTAL-FRACTIONAL OPERATOR

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Abstract

This paper introduces a novel non-linear fractal fractional model to comprehensively analyze influenza epidemics. By integrating fractional calculus and considering non-linear interactions among individuals, the model utilizes the Fractal-Fractional (FF) operator. This operator, combining fractal and fractional calculus, establishes a unique framework for investigating influenza virus propagation dynamics and potential vaccination strategies. Amid growing concerns over influenza outbreaks, fractional derivatives are employed to address intricate challenges. The proposed model sheds light on virus spread dynamics and countermeasures. The integration of the Fractal-Fractional operator enriches analysis, while the application of the fixed point theory of Schauder and Banach demonstrates solution existence and uniqueness. Model validation employs numerical simulations with MATLAB[12] and the Adams-Bashforth method, confirming its ability to capture influenza propagation dynamics accurately. Ulam-Hyers stability techniques ensure the model's reliability. Beyond its scientific contributions, the model underscores the significance of studying influenza epidemics via mathematical modeling in understanding disease dynamics and guiding effective intervention strategies. Through a synergy of mathematical innovation and epidemiological insights, this study establishes a robust foundation for more effective strategies against influenza epidemics.

Keywords: Influenza Epidemic; Fractal-Fractional Operator; Vaccination Effect, Numerical Algorithms, Hyers-Ulam Stability.

1. INTRODUCTION

Influenza, a highly contagious respiratory illness caused by influenza viruses, stands as a critical global health concern. Each year, this viral affliction exacts a significant toll, resulting in numerous human suffering and fatalities, while incurring healthcare costs that soar into the billions. The impact of influenza is particularly pronounced among both the elderly and pediatric populations. The influenza virus is categorized into three distinct types: A, B, and C, with influenza A viruses carrying a notably greater impact on human health. Throughout history, the origins of influenza outbreaks, epidemics, and the defining pandemics of the last century can be traced back to specific HA and NA subtypes of influenza A viruses. These subtypes consistently lead to heightened rates of morbidity and mortality, underscoring the necessity for continued vigilance. Given the elevated vulnerability to illness and the substantial mortality associated with influenza, substantial efforts have been dedicated to unraveling the intricate disease dynamics. In this context, mathematical models have emerged as invaluable tools, providing nuanced insights into disease progression and the multifaceted implications of diverse preventive and control strategies. The effectiveness of such strategies has been explored by Deguchi and Tagasugi,[1] who investigated the impact of influenza vaccination on the elderly during an epidemic, uncovering reduced mortality and morbidity risks. Earn, Dushoff, and Levin[2] delved into the intricate interplay between ecology and evolution in flu dynamics, shedding light on the complex patterns inherent in the disease's behavior. Meanwhile, Lin, Andreasen, and Levin[3] introduced a linear three-strain model to elucidate the drift dynamics of influenza A, enhancing our understanding of its underlying mechanisms. Amid discussions of pandemic readiness, Webby and Webster[4] underscored the global state of preparedness and highlighted potential challenges in managing pandemic influenza outbreaks, emphasizing the critical need for comprehensive strategies and preparedness measures.

Vaccination has held a pivotal role in the arsenal against infectious diseases, representing a crucial strategy for disease control. Diseases like pertussis, measles, rabies, influenza, poliovirus, hepatitis B, and encephalitis B have all been the focus of vaccination campaigns, demonstrating the broad scope of this approach. In the context of influenza, the success of public health largely hinges on effective vaccination programs that mitigate disease transmission. Beyond the virological aspects, the influence of human behavior emerges as a pivotal factor in determining vaccination coverage. In most vaccination efforts, voluntarism prevails, granting individuals the autonomy to choose whether to be vaccinated. This individual-level decision significantly shapes population immunity and consequently plays a vital role in the potential scale of epidemics. Researchers like Vardavas et al.[5] and Galvani et al.[6] have probed into the intricate interplay of human cognition and behavior within this context, presenting two distinct perspectives. Their work emphasizes the necessity for precisely designed and incentive-driven vaccination programs as a cornerstone in achieving herd immunity. Numerous studies[7–9] have underscored the effectiveness of mass vaccination strategies in thwarting epidemics. Yet, a comprehensive evaluation of their practicality is still lacking. Amid the complex landscape of disease control, the importance of conventional and sustained vaccination strategies cannot be overstated, as evidenced by the triumphant global eradication of smallpox.[10,11] Vardavas et al.[5] delved into the potential of preventing influenza epidemics through voluntary vaccination, while Galvani et al.[6] examined the alignment of long-standing influenza vaccination policies with individual self-interest. Longini et al.[7] explored pandemic containment using antiviral agents, while Viboud et al.[8] employed the method of analogues to predict the spread of influenza epidemics. Viboud et al.[9] conducted cross-country analyses of influenza epidemics, and Anderson and May[10] provided a comprehensive resource on infectious diseases. Finally, Coburn et al.[11] modeled influenza epidemics and pandemics, offering insights into the future of H1N1 swine flu.

The literature relevant to the application of fractional calculus in epidemiological models includes several pioneering works. Atangana and Baleanu's contributions in 2014 and 2018 laid foundational principles by introducing new fractional derivatives with non-local and non-singular kernels, applying these to heat transfer models.[13,14] Their work emphasizes the benefits of fractional derivatives in capturing complex dynamic behaviors. In 2022, Zarin et al. extended these concepts to epidemic modeling, specifically the fractional-order dynamics of the Chagas-HIV epidemic model, showcasing the utility of different fractional operators.[15] This approach was also applied by Jitsinchayakul et al.[16] and Khan et al.[17] to model the COVID-19 epidemic, incorporating harmonic mean type incidence rates and analyzing stability under fractional dynamics. Further, Zarin et al.[18] explored the dynamics of a five-grade Leishmania epidemic model using a fractional operator with a Mittag–Leffler kernel, highlighting the flexibility and depth fractional models provide. Additionally, Zarin et al.[19] applied fractional modeling to real COVID-19 data from Pakistan under the ABC operator, demonstrating practical applicability and effectiveness in real-world scenarios. These studies collectively illustrate the robustness and versatility of fractional calculus in capturing the intricate dynamics of various epidemic models. The literature on fractional calculus and its applications spans a diverse range of fields. Baleanu and Atangana (2015) explored fractional calculus operators and subordination chains, providing a theoretical foundation for further applications in nonlinear sciences.[20] Zhang et al. (2020) extended these concepts to the realm of electromagnetic fields, utilizing fractal fractional derivatives to model complex wave behaviors.[21] Wei et al. (2019) introduced a novel fractal fractional calculus operator for viscoelasticity modeling, demonstrating its effectiveness in capturing material properties.[22] In 2022, Zarin et al. applied fractional modeling to the propagation of computer viruses, highlighting the versatility of these mathematical tools in diverse contexts.[23] Zarin (2022) further delved into the numerical analysis of a fractional-order hepatitis B virus model, incorporating harmonic mean type incidence rates.[24] Additionally, Zarin et al. (2021) investigated the rabies virus through fractional modeling and optimal control analysis, emphasizing the utility of convex incidence rates.[25] The application of fractional calculus to epidemic modeling is further illustrated by Zarin et al. (2021), who analyzed a fractional COVID-19 epidemic model using the Caputo operator, providing insights into the disease's dynamics.[26] Liu et al. (2022) explored the numerical dynamics and fractional modeling of a hepatitis B virus model with non-singular and nonlocal kernels, showcasing the depth and flexibility of fractional approaches in capturing complex biological processes.[27] These studies collectively underscore the broad applicability and effectiveness of fractional calculus in modeling and understanding diverse physical and biological systems.

Atangana and colleagues introduced a novel non-local operator for fractional differential equations known as the fractal-fractional (FF) operator, which integrates both fractal and fractional elements.[28] They demonstrated the operator's existence and uniqueness, establishing it as a crucial tool in various scientific fields, such as epidemiology. In one study, Khan and his team applied different fractional-fractal operators to numerically address the competitive dynamics between rural and commercial banks, identifying the optimal combination of fractional and fractal orders for improved data fitting.[29] Another research effort utilized the FF operator to examine the dynamics of two avian influenza models.[30] Further applications of the FF operator are documented in additional studies,[31,32] including a comprehensive mathematical model for the coronavirus outbreak in China, which was developed using a fractal order derivative.[33]

This study presents an innovative mathematical model that employs fractional calculus and the Fractal-Fractional (FF) operator to thoroughly investigate the dynamics of Influenza epidemics. To the best of the author's knowledge, this is the first application of the FF operator to the complex dynamics of Influenza transmission. The model addresses non-linear interactions among individuals, offering valuable insights into Influenza propagation. It also explores the potential of fractional derivatives in overcoming challenges related to Influenza viruses, illustrating how the virus spreads within a susceptible population and can be mitigated through vaccination strategies. Numerical simulations and stability analysis convincingly demonstrate the model's effectiveness, aiding in the design of more effective strategies to contain Influenza outbreaks. This study introduces two significant extensions to the existing $\mathbb{SVEIR}$ model.[34] The first extension involves transforming the initial model into a non-integer-order model using the FF derivative, which captures complex and nuanced dynamics, particularly in the context of Influenza epidemics. The second extension incorporates a fractal time dimension, providing deeper insights into the dynamics of Influenza transmission. The generalized fractional order Ψ_1 offers flexibility in capturing complex behaviors, while the FF operator ${}_0^{FFM}D_t^{\Psi_1,\Phi_1}$ provides an innovative method for modeling Influenza transmission dynamics. These extensions are both mathematically rigorous and physically meaningful, bridging the gap between the initial model and more intricate real-world scenarios, thus enhancing the model's practical relevance and applicability in public health.

The structure of the manuscript is organized as follows: Section 1 offers a brief overview of influenza epidemics and highlights contributions from prominent scholars. Section 2 presents the essential background data required for formulating the proposed mathematical model. In Section 3, epidemiological models incorporating classical and fractional derivatives are described, along with an examination of the existence and uniqueness of the solution. Section 4 discusses the Hyers-Ulam stability of the proposed fractal-fractional model. Sensitivity analysis is conducted in Section 5. Section 6 outlines the numerical scheme and provides graphical results for the proposed model. Finally, Section 7 concludes with a summary of the findings and suggestions for future research directions.

2. BASIC DEFINITIONS

This section presents essential concepts and established results related to fractional order derivatives with fractional fractals, crucial for developing the fractional-fractal model under study.

Definition 1.[35] The Fractal-Fractional (FF) differentiation of a function $\mathcal{G}$ within the Riemann-Liouville (RL) framework is defined as:

$$ {}^{FFP}D_{0,t}^{\Psi_1,\Phi_1}[\mathcal{G}(t)] = \frac{1}{\Gamma(n-\Psi_1)}\frac{d}{d\mathfrak{M}^{\Phi_1}} \int_0^t (t-x)^{n-\Psi_1-1}\mathcal{G}(\mathfrak{M})\,d\mathfrak{M}, \tag{1}$$

where $n-1 < \Psi_1, \Phi_1 \leq n \in \mathbb{N}$ and $\frac{d\mathcal{G}(\mathfrak{M})}{d\mathfrak{M}^{\Phi_1}} = \lim_{t\to\mathfrak{M}} \frac{\mathcal{G}(t)-\mathcal{G}(\mathfrak{M})}{t^{\Phi_1}-\mathfrak{M}^{\Phi_1}}$.

Definition 2.[35] The function $\mathcal{G}$ can be FF differentiated in the sense of RL with an exponential kernel, as follows:

$$ {}^{FFM}D_{0,t}^{\Phi_1}[\mathcal{G}(t)] = \frac{G^*(\Psi_1)}{1-\Psi_1}\frac{d}{dt^{\Phi_1}} \int_0^t \exp\left(-\frac{\Psi_1}{1-\Psi_1}(t-\mathfrak{M})\right)\mathcal{G}(\mathfrak{M})\,d\mathfrak{M}, \tag{2}$$

where $n-1 < \Psi_1, \Phi_1 \leq n \in \mathbb{N}$ and $G^*(0) = G(1) = 1$.

Definition 3.[35] The RL case of the Fractal-Fractional differentiation of $\mathcal{G}$ using the Mittag-Leffler mapping as a non-singular kernel is defined as:

$$ {}^{FFM}D_{0,t}^{\Psi_1,\Phi_1}[\mathcal{G}(t)] = \frac{G^*(\Psi_1)}{1-\Psi_1}\frac{d}{dt^{\Phi_1}} \int_0^t E_{\Psi_1} \left(-\frac{\Psi_1}{1-\Psi_1}(t-\mathfrak{M})^{\Psi_1}\right)\mathcal{G}(\mathfrak{M})\,d\mathfrak{M}, \tag{3}$$

with $\Psi_1 > 0, \Phi_1 \leq 1 \in \mathbb{N}$ and $G^*(\Psi_1) = 1-\Psi_1+\frac{\Psi_1}{\Gamma(\Psi_1)}$.

Definition 4.[35] The FF integration is:

$$ {}^{FFP}I_{0,t}^{\varphi_1,\Phi_1}[\mathcal{G}(t)] = \frac{\Phi_1}{\Gamma(\Psi_1)}\int_0^t (t-\mathfrak{M}^*)^{\Psi_1-1} \mathfrak{M}^{*\Phi_1-1}\mathcal{G}(\mathfrak{M}^*)\,d\mathfrak{M}^*. \tag{4}$$

Definition 5.

$$ {}^{FFE}I_{0,t}^{\Psi_1,\Phi_1}[\mathcal{G}(t)] = \frac{\Psi_1\Phi_1}{G^*(\Psi_1)}\int_0^t \mathfrak{M}^{*\Psi_1-1}\mathcal{G}(\mathfrak{M}^*)\,d\mathfrak{M}^* + \frac{\Phi_1(1-\Psi_1)\,t^{\Phi_1-1}\mathcal{G}(t)}{G^*(\Psi_1)}. \tag{5}$$

Definition 6.

$$ {}^{FFM}I_{0,t}^{\Psi_1,\Phi_1}[\mathcal{G}(t)] = \frac{\Psi_1\Phi_1}{G^*(\Psi_1)\,\Gamma(\Psi_1)}\int_0^t \mathfrak{M}^{*\Phi_1-1} \mathcal{G}(\mathfrak{M}^*)(t-\mathfrak{M}^*)^{\Psi_1-1}\,d\mathfrak{M}^* + \frac{\Phi_1(1-\Psi_1)\,t^{\Phi_1-1}\mathcal{G}(t)}{G^*(\Psi_1)}. \tag{6}$$

3. MODEL FORMULATION

Recent studies have introduced the $\mathbb{SVEIR}$ model as a powerful tool for analyzing the transmission dynamics of influenza within a population.[34] This model is built on non-linear equations that describe the interactions among various state variables, offering a comprehensive understanding of disease spread. The system of equations is formulated as

Table B: Table showing the values of the model parameters

Parameter	Description	Value	Source
β	Contact rate	0.514000000	[34]
β_E	Ability to cause infection by exposed individuals ($0 \leqslant \beta_E \leqslant 1$)	1.3×10^{-8}	Fitted
β_I	Ability to cause infection by infectious individuals ($0 \leqslant \beta_I \leqslant 1$)	1.2644×10^{-8}	Fitted
$1-\beta_V$	Factor by which the vaccine reduces infection ($0 \leqslant \beta_V \leqslant 1$)	0.56	[34]
σ^{-1}	Mean duration of latency (days)	2.000000000	[34]
γ^{-1}	Mean recovery time for clinically ill (days)	5.000000000	[34]
δ^{-1}	Duration of immunity loss (days)	365.0000000	[34]
μ	Natural mortality rate	5.500×10^{-8}	[34]
r	Birth rate	7.140×10^{-5}	[34]
κ	Recovery rate of latents	1.857×10^{-4}	[34]
α	Flu induced mortality rate	9.300×10^{-6}	[34]
θ^{-1}	Duration of vaccine-induced immunity loss (days)	365.00000000	[34]
ϕ	Rate of vaccination	0.56	[34]

follows:

$$\begin{cases} \dfrac{d\mathbb{S}(t)}{dt} = -\beta\beta_E\dfrac{\mathbb{E}\mathbb{S}}{\mathbb{N}} - \beta\beta_I\dfrac{\mathbb{I}\mathbb{S}}{\mathbb{N}} - \phi\mathbb{S} - \mu\mathbb{S} + \delta\mathbb{R} + \theta\mathbb{V} \\ \qquad + r\mathbb{N} \\ \dfrac{d\mathbb{V}(t)}{dt} = -\beta\beta_E\beta_V\dfrac{\mathbb{E}\mathbb{V}}{\mathbb{N}} - \beta\beta_I\beta_V\dfrac{\mathbb{I}\mathbb{V}}{\mathbb{N}} - \mu\mathbb{V} - \theta\mathbb{V} + \phi\mathbb{S} \\ \dfrac{d\mathbb{E}(t)}{dt} = \beta\beta_E\dfrac{\mathbb{E}\mathbb{S}}{\mathbb{N}} + \beta\beta_I\dfrac{\mathbb{I}\mathbb{S}}{\mathbb{N}} + \beta\beta_E\beta_V\dfrac{\mathbb{E}\mathbb{V}}{\mathbb{N}} \\ \qquad + \beta\beta_I\beta_V\dfrac{\mathbb{I}\mathbb{V}}{\mathbb{N}} - (\mu+\kappa+\sigma)\mathbb{E} \\ \dfrac{d\mathbb{I}(t)}{dt} = \sigma\mathbb{E} - (\mu+\alpha+\gamma)\mathbb{I} \\ \dfrac{d\mathbb{R}(t)}{dt} = \kappa\mathbb{E} + \gamma\mathbb{I} - \mu\mathbb{R} - \delta\mathbb{R}. \\ \mathbb{S}(t), \quad \mathbb{V}(t), \quad \mathbb{E}(t), \quad \mathbb{I}(t), \quad \mathbb{R}(t) \geq 0. \end{cases} \tag{7}$$

Here, $\mathbb{S}(t)$, $\mathbb{V}(t)$, $\mathbb{E}(t)$, $\mathbb{I}(t)$, and $\mathbb{R}(t)$ denote the numbers of susceptible, vaccinated, exposed, infectious, and recovered individuals at time t, respectively. The model parameters and their values, detailed in Table B, enable the design of strategies to mitigate the spread of the influenza virus. This research enhances our understanding of the $\mathbb{SVEIR}$ model's capacity to simulate the transmission dynamics of influenza and informs the development of more effective intervention measures. The numerical values and descriptions of the parameters are provided in Table B. The basic reproduction number R_0 and the equilibrium points of the model is given by:[34]

$$\mathfrak{R}_{vac} = \frac{\beta\left(r\beta_E + \alpha\beta_E + \gamma\beta_E + \sigma\beta_I\right)\left(r + \theta + \beta_V\phi\right)}{(r+\alpha+\gamma)(r+\kappa+\sigma)(r+\theta+\phi)}. \tag{8}$$

3.1. Fractal-Fractional Model

In this study, we extend the influenza epidemic model (7) by incorporating non-integer-order derivatives of order Ψ_1, transforming it into a fractional-fractal model. This approach leverages the versatility of fractional derivatives, which encompass both globalization and heredity characteristics—attributes that traditional integer-order derivatives in epidemiology often overlook. The generalized fractional order is denoted by Ψ_1, and the fractal dimension by Φ_1, reflecting the fractal nature of many real-world phenomena. The inclusion of a fractal time dimension provides a deeper understanding of the epidemic dynamics. The fractional-order system for the influenza epidemic model is formulated as:

$$\begin{cases} {}_0^{FFM}D_t^{\Psi_1,\Phi_1}\left([\mathbb{S}(t)\right] = -\beta\beta_E\dfrac{\mathbb{E}\mathbb{S}}{\mathbb{N}} - \beta\beta_I\dfrac{\mathbb{I}\mathbb{S}}{\mathbb{N}} - \phi\mathbb{S} - \mu\mathbb{S} \\ \qquad + \delta\mathbb{R} + \theta\mathbb{V} + r\mathbb{N} \\ {}_0^{FFM}D_t^{\Psi_1,\Phi_1}\left[\mathbb{V}(t)\right] = -\beta\beta_E\beta_V\dfrac{\mathbb{E}\mathbb{V}}{\mathbb{N}} - \beta\beta_I\beta_V\dfrac{\mathbb{I}\mathbb{V}}{\mathbb{N}} \\ \qquad - \mu\mathbb{V} - \theta\mathbb{V} + \phi\mathbb{S} \\ {}_0^{FFM}D_t^{\Psi_1,\Phi_1}\left[\mathbb{E}(t)\right] = \beta\beta_E\dfrac{\mathbb{E}\mathbb{S}}{\mathbb{N}} + \beta\beta_I\dfrac{\mathbb{I}\mathbb{S}}{\mathbb{N}} + \beta\beta_E\beta_V\dfrac{\mathbb{E}\mathbb{V}}{\mathbb{N}} \\ \qquad + \beta\beta_I\beta_V\dfrac{\mathbb{I}\mathbb{V}}{\mathbb{N}} - (\mu+\kappa+\sigma)\mathbb{E} \\ {}_0^{FFM}D_t^{\Psi_1,\Phi_1}\left[\mathbb{I}(t)\right] = \sigma\mathbb{E} - (\mu+\alpha+\gamma)\mathbb{I} \\ {}_0^{FFM}D_t^{\Psi_1,\Phi_1}\left[\mathbb{R}(t)\right] = \kappa\mathbb{E} + \gamma\mathbb{I} - \mu\mathbb{R} - \delta\mathbb{R}. \\ \mathbb{S}(0) = \mathbb{S}_0, \mathbb{V}(0) = \mathbb{V}_0, \mathbb{E}(0) = \mathbb{E}_0, \mathbb{I}(0) = \mathbb{I}_0, \mathbb{R}(0) = \mathbb{R}_0. \end{cases} \tag{9}$$

For $t > 0$, the initial conditions are given by $\mathbb{S}(0) = \mathbb{S}_0$, $\mathbb{V}(0) = \mathbb{V}_0$, $\mathbb{E}(0) = \mathbb{E}_0$, $\mathbb{I}(0) = \mathbb{I}_0$, and $\mathbb{R}(0) =$

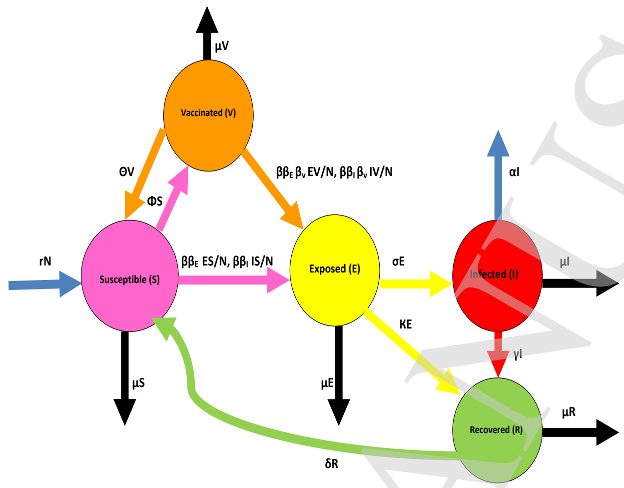


Fig. 1: The flow diagram of SVEIR model.

$\mathbb{R}_0$, ensuring that $\min(\mathbb{S}_0, \mathbb{V}_0, \mathbb{E}_0, \mathbb{I}(0), \mathbb{R}_0) \geq 0$. The roles of the state variables and parameters remain consistent with the natural order model (7), maintaining dimensional integrity and physical relevance. The parameter $0 < \Psi_1 \leq 1$ specifies the order of the fractional derivative, while the operator ${}_0^{FFM}D_t^{\Psi_1,\Phi_1}$ represents the Fractal-Fractional (FF) derivative, such as the AB-fractional or ABFF operator. When $\Psi_1 = 1$, the system (9) reverts to the natural order model (7). Similarly, setting $\Phi_1 = 1$ converts the ABFF model (9) into a standard fractional model. The incorporation of the fractal time dimension, Φ_1, is crucial for capturing the complex dynamics of real-world systems, many of which exhibit fractal characteristics. This enhanced model provides a more profound understanding of the influenza epidemic.

3.2. Existence of Solution

The existence of solutions for the proposed FF system described by equation (9) can be established using fixed-point theory.[35] This approach leads to the following results:

$$\begin{aligned}
\mathbb{S}(t)-\mathbb{S}(0) &= \frac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\int_0^t \mathfrak{M}\Phi_1-1\,(t-\mathfrak{M})^{\Psi_1-1}\left(-\beta\beta_E\frac{\mathbb{E}\mathbb{S}}{\mathbb{N}}-\beta\beta_I\frac{\mathbb{I}\mathbb{S}}{\mathbb{N}}-\phi\mathbb{S}-\mu\mathbb{S}+\delta\mathbb{R}+\theta\mathbb{V}+r\mathbb{N}\right)d\mathfrak{M}\\
&\quad+\frac{\Phi_1(1-\Psi_1)t^2-1}{G(\Psi_1)}\left(-\beta\beta_E\frac{\mathbb{E}\mathbb{S}}{\mathbb{N}}-\beta\beta_I\frac{\mathbb{I}\mathbb{S}}{\mathbb{N}}-\phi\mathbb{S}-\mu\mathbb{S}+\delta\mathbb{R}+\theta\mathbb{V}+r\mathbb{N}\right),\\
\mathbb{V}(t)-\mathbb{V}(0) &= \frac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\int_0^t \mathfrak{M}\Phi_1-1\,(t-\mathfrak{M})^{\Psi_1-1}\left(-\beta\beta_E\beta_V\frac{\mathbb{E}\mathbb{V}}{\mathbb{N}}-\beta\beta_I\beta_V\frac{\mathbb{I}\mathbb{V}}{\mathbb{N}}-\mu\mathbb{V}-\theta\mathbb{V}+\phi\mathbb{S}\right)d\mathfrak{M}\\
&\quad+\frac{\Phi_1(1-\Psi_1)t^2-1}{G(\Psi_1)}\left(-\beta\beta_E\beta_V\frac{\mathbb{E}\mathbb{V}}{\mathbb{N}}-\beta\beta_I\beta_V\frac{\mathbb{I}\mathbb{V}}{\mathbb{N}}-\mu\mathbb{V}-\theta\mathbb{V}+\phi\mathbb{S}\right),\\
\mathbb{E}(t)-\mathbb{E}(0) &= \frac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\int_0^t \mathfrak{M}\Phi_1-1\,(t-\mathfrak{M})^{\Psi_1-1}\left(\beta\beta_E\frac{\mathbb{E}\mathbb{S}}{\mathbb{N}}+\beta\beta_I\frac{\mathbb{I}\mathbb{S}}{\mathbb{N}}+\beta\beta_E\beta_V\frac{\mathbb{E}\mathbb{V}}{\mathbb{N}}+\beta\beta_I\beta_V\frac{\mathbb{I}\mathbb{V}}{\mathbb{N}}-(\mu+\kappa+\sigma)\mathbb{E}\right)d\mathfrak{M}\\
&\quad+\frac{\Phi_1(1-\Psi_1)t^2-1}{G(\Psi_1)}\left(\beta\beta_E\frac{\mathbb{E}\mathbb{S}}{\mathbb{N}}+\beta\beta_I\frac{\mathbb{I}\mathbb{S}}{\mathbb{N}}+\beta\beta_E\beta_V\frac{\mathbb{E}\mathbb{V}}{\mathbb{N}}+\beta\beta_I\beta_V\frac{\mathbb{I}\mathbb{V}}{\mathbb{N}}-(\mu+\kappa+\sigma)\mathbb{E}\right),\\
\mathbb{I}(t)-\mathbb{I}(0) &= \frac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\int_0^t \mathfrak{M}\Phi_1-1\,(t-\mathfrak{M})^{\Psi_1-1}\left(\sigma\mathbb{E}-(r+\alpha+\gamma)\mathbb{I}\right)d\mathfrak{M}\\
&\quad+\frac{\Phi_1(1-\Psi_1)t^2-1}{G(\Psi_1)}\left(\sigma\mathbb{E}-(r+\alpha+\gamma)\mathbb{I}\right),\\
\mathbb{R}(t)-\mathbb{R}(0) &= \frac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\int_0^t \mathfrak{M}\Phi_1-1\,(t-\mathfrak{M})^{\Psi_1-1}\left(\kappa\mathbb{E}+\gamma\mathbb{I}-\mu\mathbb{R}-\delta\mathbb{R}\right)d\mathfrak{M}\\
&\quad+\frac{\Phi_1(1-\Psi_1)t^2-1}{G(\Psi_1)}\left(\kappa\mathbb{E}+\gamma\mathbb{I}-\mu\mathbb{R}-\delta\mathbb{R}\right).
\end{aligned}\tag{10}$$

Next, we define the mapping $\mathbb{D}i^*$ with fixed parameters γi, where i is an element of the set of natural numbers $\mathbb{N}_1^5$. Consequently, we obtain:

$$\begin{cases}
\mathbb{D}_1(t,\mathbb{S}) = -\beta\beta_E\frac{\mathbb{E}\mathbb{S}}{\mathbb{N}}-\beta\beta_I\frac{\mathbb{I}\mathbb{S}}{\mathbb{N}}-\phi\mathbb{S}-\mu\mathbb{S}+\delta\mathbb{R}+\theta\mathbb{V}+r\mathbb{N}\\
\mathbb{D}_2(t,\mathbb{V}) = -\beta\beta_E\beta_V\frac{\mathbb{E}\mathbb{V}}{\mathbb{N}}-\beta\beta_I\beta_V\frac{\mathbb{I}\mathbb{V}}{\mathbb{N}}-\mu\mathbb{V}-\theta\mathbb{V}+\phi\mathbb{S}\\
\mathbb{D}_3(t,\mathbb{E}) = \beta\beta_E\frac{\mathbb{E}\mathbb{S}}{\mathbb{N}}+\beta\beta_I\frac{\mathbb{I}\mathbb{S}}{\mathbb{N}}+\beta\beta_E\beta_V\frac{\mathbb{E}\mathbb{V}}{\mathbb{N}}+\beta\beta_I\beta_V\frac{\mathbb{I}\mathbb{V}}{\mathbb{N}}-(\mu+\kappa+\sigma)\mathbb{E}\\
\mathbb{D}_4(t,\mathbb{I}) = \sigma\mathbb{E}-(\mu+\alpha+\gamma)\mathbb{I}\\
\mathbb{D}_5(t,\mathbb{R}) = \kappa\mathbb{E}+\gamma\mathbb{I}-\mu\mathbb{R}-\delta\mathbb{R}.
\end{cases}\tag{11}$$

Assuming the continuity of the operators $\mathbb{S}(t)$, $\mathbb{V}(t)$, $\mathbb{E}(t)$, $\mathbb{I}(t)$, and $\mathbb{R}(t)$, along with their corresponding adjoint operators $\mathbb{S}^*(t)$, $\mathbb{V}^*(t)$, $\mathbb{E}^*(t)$, $\mathbb{I}^*(t)$, and $\mathbb{R}^*(t)$, we make the following considerations for deriving the results ($\mathbf{H}^*$): The functions are assumed to be contained within $\mathbf{K}[0,1]$, and their norms satisfy the conditions $\|\mathbb{S}\|\le\Psi_{11}$, $\|\mathbb{V}\|\le\Psi_{12}$, $\|\mathbb{E}\|\le\Psi_{13}$, $\|\mathbb{I}\|\le\Psi_{14}$, and $\|\mathbb{R}\|\le\Psi_{15}$, where $\Psi_{11},\Psi_{12},\Psi_{13},\Psi_{14},\Psi_{15}>0$ are constants.

Theorem 7. *Given the assumptions in* H^*, *it can be demonstrated that the kernels* $\mathbb{D}i$, *for* $i=1,2,3,4,5$, *satisfy the Lipschitz condition with* $\Psi_{1i}<1$ *for* $i\in\mathbb{N}_1^5$.

Proof. Now, we prove that $\mathbb{D}_1(t,\mathbb{S})$ fulfils the condition of Lipschitz. Utilizing $\mathbb{S}(t), \mathbb{S}^*(t)$, thus gives

$$\begin{aligned}\|\mathbb{D}_1(t,\mathbb{S})-\mathbb{D}_1(t,\mathbb{S}^*)\| = &\|(-\beta\beta_E\mathbb{E}\mathbb{S}-\beta\beta_I\mathbb{I}\mathbb{S}+\alpha\mathbb{I}\mathbb{S}-\phi\mathbb{S}-r\mathbb{S}+\delta\mathbb{R}+\theta\mathbb{V}+r)-\\ &(-\beta\beta_E\mathbb{E}\mathbb{S}^*-\beta\beta_I\mathbb{I}\mathbb{S}^*+\alpha\mathbb{I}\mathbb{S}^*-\phi\mathbb{S}^*-r\mathbb{S}^*+\delta\mathbb{R}+\theta\mathbb{V}+r)\\ =&\|(-\beta\beta_E\mathbb{E}-\beta\beta_I\mathbb{I}+\alpha\mathbb{I}-\phi-r)(\mathbb{S}^*-\mathbb{S})\|\\ \le&\|-(\beta\beta_E\mathbb{E}+\beta\beta_I\mathbb{I}-\alpha\mathbb{I}+\phi+r)\|\,\|(\mathbb{S}^*-\mathbb{S})\|\\ \le&\Psi_{11}\|(\mathbb{S}-\mathbb{S}^*)\|.\end{aligned} \tag{12}$$

Taking into account

$$\Psi_{11}=(\beta\beta_E M_1+\beta\beta_I M_2-\alpha M_2+\phi+r), \quad M_1=\max_{t\in J}\|\mathbb{E}(t)\|, \quad M_2=\max_{t\in J}\|\mathbb{I}(t)\|,$$

one reaches

$$\|\mathbb{D}_1(t,\mathbb{S})-\mathbb{D}_1(t,\mathbb{S}^*)\|\le\Psi_{11}\|\mathbb{S}-\mathbb{S}^*\|. \tag{13}$$

Therefore, it can be seen that $\mathbb{D}_1$ adheres to the Lipschitz condition with $\Psi_{11}<1$. By applying the same process, it follows that $\mathbb{D}_2(t,\mathbb{V})$ also meets the Lipschitz condition. Using this method, we can derive the following results:

$$\begin{aligned}&\|\mathbb{D}_2(t,\mathbb{V})-\mathbb{D}_2(t,\mathbb{V}^*)\|\le\Psi_{12}\|\mathbb{V}-\mathbb{V}^*\|,\\ &\|\mathbb{D}_3(t,\mathbb{E})-\mathbb{D}_3(t,\mathbb{E}_2^*)\|\le\Psi_{13}\|\mathbb{E}-\mathbb{E}^*\|,\\ &\|\mathbb{D}_4(t,\mathbb{I})-\mathbb{D}_4(t,\mathbb{I}^*)\|\le\Psi_{14}\|\mathbb{I}-\mathbb{I}^*\|,\\ &\|\mathbb{D}_5(t,\mathbb{R})-\mathbb{D}_5(t,\mathbb{R}^*)\|\le\Psi_{16}\|\mathbb{R}-\mathbb{R}^*\|.\end{aligned} \tag{14}$$

By expressing equation (11) with the kernels $\mathbb{D}_i$ and assuming the initial conditions $\mathbb{S}(0)=\mathbb{V}(0)=\mathbb{E}(0)=\mathbb{I}(0)=\mathbb{Z}(0)=\mathbb{R}(0)=0$, the equation can be reformulated as follows:

$$\begin{cases}\mathbb{S}(t)=\dfrac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\displaystyle\int_0^t\mathfrak{M}^{\Phi_1-1}(t-\mathfrak{M})^{\Psi_1-1}\mathbb{D}_1(\mathfrak{M},\mathbb{S}(\mathfrak{M}))\,d\mathfrak{M}+\dfrac{\Phi_1(1-\Psi_1)t^{\Phi_1}-1}{G(\Psi_1)}\mathbb{D}_1(t,\mathbb{S}(t)),\\ \mathbb{V}(t)=\dfrac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\displaystyle\int_0^t\mathfrak{M}^{\Phi_1-1}(t-\mathfrak{M})^{\Psi_1-1}\mathbb{D}_2(\mathfrak{M},\mathbb{V}(\mathfrak{M}))\,d\mathfrak{M}+\dfrac{\Phi_1(1-\Psi_1)t^{\Phi_1}-1}{G(\Psi_1)}\mathbb{D}_2(t,\mathbb{V}(t)),\\ \mathbb{E}(t)=\dfrac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\displaystyle\int_0^t\mathfrak{M}^{\Phi_1-1}(t-\mathfrak{M})^{\Psi_1-1}\mathbb{D}_3(\mathfrak{M},\mathbb{E}(\mathfrak{M}))\,d\mathfrak{M}+\dfrac{\Phi_1(1-\Psi_1)t^{\Phi_1}-1}{G(\Psi_1)}\mathbb{D}_3(t,\mathbb{E}(t)),\\ \mathbb{I}(t)=\dfrac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\displaystyle\int_0^t\mathfrak{M}^{\Phi_1-1}(t-\mathfrak{M})^{\Psi_1-1}\mathbb{D}_4(\mathfrak{M},\mathbb{I}(\mathfrak{M}))\,d\mathfrak{M}+\dfrac{\Phi_1(1-\Psi_1)t^{\Phi_1}-1}{G(\Psi_1)}\mathbb{D}_4(t,\mathbb{I}(t)),\\ \mathbb{R}(t)=\dfrac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\displaystyle\int_0^t\mathfrak{M}^{\Phi_1-1}(t-\mathfrak{M})^{\Psi_1-1}\mathbb{D}_5(\mathfrak{M},\mathbb{R}(\mathfrak{M}))\,d\mathfrak{M}+\dfrac{\Phi_1(1-\Psi_1)t^{\Phi_1}-1}{G(\Psi_1)}\mathbb{D}_5(t,\mathbb{R}(t)).\end{cases} \tag{15}$$

Now, we state the following formulas recursively.

$$\begin{cases} \mathbb{S}_n(t) = \dfrac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\displaystyle\int_0^t \mathfrak{M}^{\Phi_1-1}(t-\mathfrak{M})^{\Psi_1-1}\mathbb{D}_1(\mathfrak{M},\mathbb{S}_{n-1}(\mathfrak{M}))\,d\mathfrak{M} + \dfrac{\Phi_1(1-\Psi_1)t^{\Phi_1}-1}{G(\Psi_1)}\mathbb{D}_1(t,\mathbb{S}_{n-1}(t)), \\ \mathbb{V}_n(t) = \dfrac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\displaystyle\int_0^t \mathfrak{M}^{\Phi_1-1}(t-\mathfrak{M})^{\Psi_1-1}\mathbb{D}_2(\mathfrak{M},\mathbb{V}_{n-1}(\mathfrak{M}))\,d\mathfrak{M} + \dfrac{\Phi_1(1-\Psi_1)t^{\Phi_1}-1}{G(\Psi_1)}\mathbb{D}_2(t,\mathbb{V}_{n-1}(t)), \\ \mathbb{E}_n(t) = \dfrac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\displaystyle\int_0^t \mathfrak{M}^{\Phi_1-1}(t-\mathfrak{M})^{\Psi_1-1}\mathbb{D}_3(\mathfrak{M},\mathbb{E}_{n-1}(\mathfrak{M}))\,d\mathfrak{M} + \dfrac{\Phi_1(1-\Psi_1)t^{\Phi_1}-1}{G(\Psi_1)}\mathbb{D}_3(t,\mathbb{E}_{n-1}(t)), \\ \mathbb{I}_n(t) = \dfrac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\displaystyle\int_0^t \mathfrak{M}^{\Phi_1-1}(t-\mathfrak{M})^{\Psi_1-1}\mathbb{D}_4(\mathfrak{M},\mathbb{E}_{cn-1}(\mathfrak{M}))\,d\mathfrak{M} + \dfrac{\Phi_1(1-\Psi_1)t^{\Phi_1}-1}{G(\Psi_1)}\mathbb{D}_4(t,\mathbb{E}_{n-1}(t)), \\ \mathbb{R}_n(t) = \dfrac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\displaystyle\int_0^t \mathfrak{M}^{\Phi_1-1}(t-\mathfrak{M})^{\Psi_1-1}\mathbb{D}_5(\mathfrak{M},\mathbb{R}_{n-1}(\mathfrak{M}))\,d\mathfrak{M} + \dfrac{\Phi_1(1-\Psi_1)t^{\Phi_1}-1}{G(\Psi_1)}\mathbb{D}_5(t,\mathbb{R}_{n-1}(t)). \end{cases} \tag{16}$$

$$\begin{cases} \mathbb{S}_{n+1}(t) = \dfrac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\displaystyle\int_0^t \mathfrak{M}^{\Phi_1-1}(t-\mathfrak{M})^{\Psi_1-1}\mathbb{D}_1(\mathfrak{M},\mathbb{S}_n(\mathfrak{M}))\,d\mathfrak{M} + \dfrac{\Phi_1(1-\Psi_1)t^{\Phi_1}-1}{G(\Psi_1)}\mathbb{D}_1(t,\mathbb{S}_n(t)), \\ \mathbb{V}_{n+1}(t) = \dfrac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\displaystyle\int_0^t \mathfrak{M}^{\Phi_1-1}(t-\mathfrak{M})^{\Psi_1-1}\mathbb{D}_2(\mathfrak{M},\mathbb{V}_n(\mathfrak{M}))\,d\mathfrak{M} + \dfrac{\Phi_1(1-\Psi_1)t^{\Phi_1}-1}{G(\Psi_1)}\mathbb{D}_2(t,\mathbb{V}_n(t)), \\ \mathbb{E}_{n+1}(t) = \dfrac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\displaystyle\int_0^t \mathfrak{M}^{\Phi_1-1}(t-\mathfrak{M})^{\Psi_1-1}\mathbb{D}_3(\mathfrak{M},\mathbb{E}_n(\mathfrak{M}))\,d\mathfrak{M} + \dfrac{\Phi_1(1-\Psi_1)t^{\Phi_1}-1}{G(\Psi_1)}\mathbb{D}_3(t,\mathbb{E}_n(t)), \\ \mathbb{I}_{n+1}(t) = \dfrac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\displaystyle\int_0^t \mathfrak{M}^{\Phi_1-1}(t-\mathfrak{M})^{\Psi_1-1}\mathbb{D}_4(\mathfrak{M},\mathbb{E}_n(\mathfrak{M}))\,d\mathfrak{M} + \dfrac{\Phi_1(1-\Psi_1)t^{\Phi_1}-1}{G(\Psi_1)}\mathbb{D}_4(t,\mathbb{I}_n(t)), \\ \mathbb{R}_{n+1}(t) = \dfrac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\displaystyle\int_0^t \mathfrak{M}^{\Phi_1-1}(t-\mathfrak{M})^{\Psi_1-1}\mathbb{D}_5(\mathfrak{M},\mathbb{R}_n(\mathfrak{M}))\,d\mathfrak{M} + \dfrac{\Phi_1(1-\Psi_1)t^{\Phi_1}-1}{G(\Psi_1)}\mathbb{D}_5(t,\mathbb{R}_n(t)). \end{cases} \tag{17}$$

Furthermore, let us examine the following differences, denoted as D^*:

$$\begin{aligned}
D^{*}\left(\mathbb{S}_{n+1}(t)\right) &= \mathbb{S}_{n+1}-\mathbb{S}_{n} \\
&= \frac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\int_0^t \mathfrak{M}^{\Phi_1-1}(t-\mathfrak{M})^{\Psi_1-1}\mathbb{D}_1(\mathfrak{M},\mathbb{S}_n(\mathfrak{M}))\,d\mathfrak{M}+\frac{\Phi_1(1-\Psi_1)t^{\Phi_1}-1}{G(\Psi_1)}\mathbb{D}_1(t,\mathbb{S}_n(t)) \\
&- \left(\frac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\int_0^t \mathfrak{M}^{\Phi_1-1}(t-\mathfrak{M})^{\Psi_1-1}\mathbb{D}_1(\mathfrak{M},\mathbb{S}_{n-1}(\mathfrak{M}))\,d\mathfrak{M}+\frac{\Phi_1(1-\Psi_1)t^{\Phi_1}-1}{G(\Psi_1)}\mathbb{D}_1(t,\mathbb{S}_{n-1}(t))\right), \\
D^{*}\left(\mathbb{V}_{n+1}(t)\right) &= \mathbb{V}_{n+1}-\mathbb{V}_{n} \\
&= \frac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\int_0^t \mathfrak{M}^{\Phi_1-1}(t-\mathfrak{M})^{\Psi_1-1}\mathbb{D}_2(\mathfrak{M},\mathbb{V}_n(\mathfrak{M}))\,d\mathfrak{M}+\frac{\Phi_1(1-\Psi_1)t^{\Phi_1}-1}{G(\Psi_1)}\mathbb{D}_2(t,\mathbb{V}_n(t)) \\
&- \left(\frac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\int_0^t \mathfrak{M}^{\Phi_1-1}(t-\mathfrak{M})^{\Psi_1-1}\mathbb{D}_2(\mathfrak{M},\mathbb{V}_{n-1}(\mathfrak{M}))\,d\mathfrak{M}+\frac{\Phi_1(1-\Psi_1)t^{\Phi_1}-1}{G(\Psi_1)}\mathbb{D}_2(t,\mathbb{V}_{n-1}(t))\right), \\
D^{*}\left(\mathbb{E}_{n+1}(t)\right) &= \mathbb{E}_{n+1}-\mathbb{E}_{n} \\
&= \frac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\int_0^t \mathfrak{M}^{\Phi_1-1}(t-\mathfrak{M})^{\Psi_1-1}\mathbb{D}_3(\mathfrak{M},\mathbb{E}_n(\mathfrak{M}))\,d\mathfrak{M}+\frac{\Phi_1(1-\Psi_1)t^{\Phi_1}-1}{G(\Psi_1)}\mathbb{D}_3(t,\mathbb{E}_n(t)) \\
&- \left(\frac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\int_0^t \mathfrak{M}^{\Phi_1-1}(t-\mathfrak{M})^{\Psi_1-1}\mathbb{D}_3(\mathfrak{M},\mathbb{E}_{n-1}(\mathfrak{M}))\,d\mathfrak{M}+\frac{\Phi_1(1-\Psi_1)t^{\Phi_1}-1}{G(\Psi_1)}\mathbb{D}_3(t,\mathbb{E}_{n-1}(t))\right), \\
D^{*}\left(\mathbb{I}_{n+1}(t)\right) &= \mathbb{E}_{n+1}-\mathbb{E}_{n} \\
&= \frac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\int_0^t \mathfrak{M}^{\Phi_1-1}(t-\mathfrak{M})^{\Psi_1-1}\mathbb{D}_3(\mathfrak{M},\mathbb{E}_n(\mathfrak{M}))\,d\mathfrak{M}+\frac{\Phi_1(1-\Psi_1)t^{\Phi_1}-1}{G(\Psi_1)}\mathbb{D}_3(t,\mathbb{E}_n(t)) \\
&- \left(\frac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\int_0^t \mathfrak{M}^{\Phi_1-1}(t-\mathfrak{M})^{\Psi_1-1}\mathbb{D}_3(\mathfrak{M},\mathbb{E}_{n-1}(\mathfrak{M}))\,d\mathfrak{M}+\frac{\Phi_1(1-\Psi_1)t^{\Phi_1}-1}{G(\Psi_1)}\mathbb{D}_3(t,\mathbb{E}_{n-1}(t))\right), \\
D^{*}\left(\mathbb{R}_{n+1}(t)\right) &= \mathbb{R}_{n+1}-\mathbb{R}_{n} \\
&= \frac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\int_0^t \mathfrak{M}^{\Phi_1-1}(t-\mathfrak{M})^{\Psi_1-1}\mathbb{D}_5(\mathfrak{M},\mathbb{Z}_n(\mathfrak{M}))\,d\mathfrak{M}+\frac{\Phi_1(1-\Psi_1)t^{\Phi_1}-1}{G(\Psi_1)}\mathbb{D}_5(t,\mathbb{R}_n(t)) \\
&- \left(\frac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\int_0^t \mathfrak{M}^{\Phi_1-1}(t-\mathfrak{M})^{\Psi_1-1}\mathbb{D}_5(\mathfrak{M},\mathbb{R}_{n-1}(\mathfrak{M}))\,d\mathfrak{M}+\frac{\Phi_1(1-\Psi_1)t^{\Phi_1}-1}{G(\Psi_1)}\mathbb{D}_4(t,\mathbb{R}_{n-1}(t))\right).
\end{aligned} \tag{18}$$

Imposing norm on D^*, we get:

$$\begin{aligned}
\|D^{*}\left(\mathbb{S}_{n+1}(t)\right)\| &= \|\mathbb{S}_{n+1}-\mathbb{S}_{n}\| \\
&= \|\frac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\int_0^t \mathfrak{M}^{\Phi_1-1}(t-\mathfrak{M})^{\Psi_1-1}\mathbb{D}_1(\mathfrak{M},\mathbb{S}_n(\mathfrak{M}))\,d\mathfrak{M}+\frac{\Phi_1(1-\Psi_1)t^{\Phi_1}-1}{G(\Psi_1)}\mathbb{D}_1(t,\mathbb{S}_n(t)) \\
&- \left(\frac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\int_0^t \mathfrak{M}^{\Phi_1-1}(t-\mathfrak{M})^{\Psi_1-1}\mathbb{D}_1(\mathfrak{M},\mathbb{S}_{n-1}(\mathfrak{M}))\,d\mathfrak{M}+\frac{\Phi_1(1-\Psi_1)t^{\Phi_1}-1}{G(\Psi_1)}\mathbb{D}_1(t,\mathbb{S}_{n-1}(t))\right)\|, \\
&= \frac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\int_0^t \mathfrak{M}^{\Phi_1-1}(t-\mathfrak{M})^{\Psi_1-1}\|\mathbb{D}_1(\mathfrak{M},\mathbb{S}_n(\mathfrak{M}))-\mathbb{D}_1(\mathfrak{M},\mathbb{S}_{n-1}(\mathfrak{M}))\|\,d\mathfrak{M} \\
&+ \frac{\Phi_1(1-\Psi_1)t^{\Phi_1}-1}{G(\Psi_1)}\|\mathbb{D}_1(t,\mathbb{S}_n(t))-\mathbb{D}_1(t,\mathbb{S}_{n-1}(t))\|,
\end{aligned} \tag{19}$$

By proceeding in a similar fashion, we derive the final equation:

$$\begin{matrix} \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \end{matrix}$$

$$
\begin{aligned}
\|D^*(\mathbb{R}_{n+1}(t))\| &= \|\mathbb{R}_{n+1}-\mathbb{R}_n\| \\
&= \|\frac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\int_0^t \mathfrak{M}^{\Phi_1-1}(t-\mathfrak{M})^{\Psi_1-1}\mathbb{D}_5(\mathfrak{M},\mathbb{R}_n(\mathfrak{M}))\,d\mathfrak{M}+\frac{\Phi_1(1-\Psi_1)t^{\Phi_1}-1}{G(\Psi_1)}\mathbb{D}_5(t,\mathbb{R}_n(t)) \\
&-\left(\frac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\int_0^t \mathfrak{M}^{\Phi_1-1}(t-\mathfrak{M})^{\Psi_1-1}\mathbb{D}_5(\mathfrak{M},\mathbb{R}_{n-1}(\mathfrak{M}))\,d\mathfrak{M}+\frac{\Phi_1(1-\Psi_1)t^{\Phi_1}-1}{G(\Psi_1)}\mathbb{D}_5(t,\mathbb{R}_{n-1}(t))\right)\|, \\
&= \frac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\int_0^t \mathfrak{M}^{\Phi_1-1}(t-\mathfrak{M})^{\Psi_1-1}\|\mathbb{D}_5(\mathfrak{M},\mathbb{R}_n(\mathfrak{M}))-\mathbb{D}_5(\mathfrak{M},\mathbb{R}_{n-1}(\mathfrak{M}))\|\,d\mathfrak{M} \\
&+\frac{\Phi_1(1-\Psi_1)t^{\Phi_1}-1}{G(\Psi_1)}\|\mathbb{D}_5(t,\mathbb{R}_n(t))-\mathbb{D}_5(t,\mathbb{R}_{n-1}(t))\|.
\end{aligned}
\tag{20}
$$

Theorem 8. *The proposed Fractal-Fractional model admits a solution if the following condition is satisfied:*

$$
\sigma=\max\{\phi_1,\phi_2,\phi_3,\phi_4,\phi_5\}<1. \tag{21}
$$

Proof: To establish this theorem, we begin by defining the following mappings:

$$
\begin{cases}
\mathbf{H}_1 n(t)=\mathbb{S}_{n+1}(t)-\mathbb{S}(t)\\
\mathbf{H}_2 n(t)=\mathbb{V}_{n+1}(t)-\mathbb{V}(t)\\
\mathbf{H}_3 n(t)=\mathbb{E}_{n+1}(t)-\mathbb{E}(t)\\
\mathbf{H}_4 n(t)=\mathbb{I}_{n+1}(t)-\mathbb{I}(t)\\
\mathbf{H}_5 n(t)=\mathbb{R}_{n+1}(t)-\mathbb{R}(t).
\end{cases}
\tag{22}
$$

Imposing Norm,

$$
\begin{aligned}
\|\mathbf{H}_1 n(t)\| &= \|\mathbb{S}_{n+1}-\mathbb{S}_n\| \\
&= \|\frac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\int_0^t \mathfrak{M}^{\Phi_1-1}(t-\mathfrak{M})^{\Psi_1-1}\mathbb{D}_1(\mathfrak{M},\mathbb{S}_n(\mathfrak{M}))\,d\mathfrak{M}+\frac{\Phi_1(1-\Psi_1)t^{\Phi_1}-1}{G(\Psi_1)}\mathbb{D}_1(t,\mathbb{S}_n(t)) \\
&-\left(\frac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\int_0^t \mathfrak{M}^{\Phi_1-1}(t-\mathfrak{M})^{\Psi_1-1}\mathbb{D}_1(\mathfrak{M},\mathbb{S}_{n-1}(\mathfrak{M}))\,d\mathfrak{M}+\frac{\Phi_1(1-\Psi_1)t^{\Phi_1}-1}{G(\Psi_1)}\mathbb{D}_1(t,\mathbb{S}_{n-1}(t))\right)\|, \\
&= \frac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\int_0^t \mathfrak{M}^{\Phi_1-1}(t-\mathfrak{M})^{\Psi_1-1}\|\mathbb{D}_1(\mathfrak{M},\mathbb{S}_n(\mathfrak{M}))-\mathbb{D}_1(\mathfrak{M},\mathbb{S}_{n-1}(\mathfrak{M}))\|\,d\mathfrak{M} \\
&+\frac{\Phi_1(1-\Psi_1)t^{\Phi_1}-1}{G(\Psi_1)}\|\mathbb{D}_1(t,\mathbb{S}_n(t))-\mathbb{D}_1(t,\mathbb{S}_{n-1}(t))\|, \\
&\le \left(\frac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\int_0^t \mathfrak{M}^{\Phi_1-1}(t-\mathfrak{M})^{\Psi_1-1}+\frac{\Phi_1(1-\Psi_1)t^{\Phi_1}-1}{G(\Psi_1)}\right)\phi_1\|\mathbb{S}_n-\mathbb{S}\|, \\
&\le \left(\frac{\Psi_1\Phi_1\Gamma\Phi_1}{G(\Psi_1)\Gamma(\Psi_1+\Phi_1)}+\frac{\Phi_1(1-\Psi_1)}{G(\Psi_1)}\right)\phi_1\|\mathbb{S}_n-\mathbb{S}\|, \\
&\le \left(\frac{\Psi_1\Phi_1\Gamma\Phi_1}{G(\Psi_1\Gamma(\Psi_1+\Phi_1))}+\frac{\Phi_1(1-\Psi_1)}{G(\Psi_1)}\right)^n\sigma^n\|\mathbb{S}_1-\mathbb{S}\|,
\end{aligned}
\tag{23}
$$

where $\sigma<1$ and as $n\to\infty$ so $\mathbb{S}_n\to\mathbb{S}$, and using the formula,

$B\left(u,v\right) = \left(b-a\right)^{-u+v+1}\int_a^b \left(x-a\right)^{u-1}\left(b-x\right)^{v-1} d\mathfrak{M}$ and as $t \in [0,1]$ so $t^{-1-\Psi_1+\Phi_1} \leq 1$ and $t^{\Phi_1} \leq 1$.

$$
\begin{aligned}
\|\mathbf{H}_2 n(t)\| &= \|\mathbb{V}_{n+1} - \mathbb{V}_n\| \\
&= \|\frac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\int_0^t \mathfrak{M}^{\Phi_1-1}\left(t-\mathfrak{M}\right)^{\Psi_1-1}\mathbb{D}_2\left(\mathfrak{M},\mathbb{V}_n\left(\mathfrak{M}\right)\right)d\mathfrak{M} + \frac{\Phi_1\left(1-\Psi_1\right)t^{\Phi_1}-1}{G(\Psi_1)}\mathbb{D}_2\left(t,\mathbb{V}_n(t)\right) \\
&- \left(\frac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\int_0^t \mathfrak{M}^{\Phi_1-1}\left(t-\mathfrak{M}\right)^{\Psi_1-1}\mathbb{D}_2\left(\mathfrak{M},\mathbb{V}_{n-1}\left(\mathfrak{M}\right)\right)d\mathfrak{M} + \frac{\Phi_1\left(1-\Psi_1\right)t^{\Phi_1}-1}{G(\Psi_1)}\mathbb{D}_2\left(t,\mathbb{V}_{n-1}(t)\right)\right)\|, \\
&= \frac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\int_0^t \mathfrak{M}^{\Phi_1-1}\left(t-\mathfrak{M}\right)^{\Psi_1-1}\|\mathbb{D}_2\left(\mathfrak{M},\mathbb{V}_n\left(\mathfrak{M}\right)\right) - \mathbb{D}_2\left(\mathfrak{M},\mathbb{V}_{n-1}\left(\mathfrak{M}\right)\right)\| d\mathfrak{M} \\
&+ \frac{\Phi_1\left(1-\Psi_1\right)t^{\Phi_1}-1}{G(\Psi_1)}\|\mathbb{D}_2\left(t,\mathbb{V}_n(t)\right) - \mathbb{D}_2\left(t,\mathbb{V}_{n-1}(t)\right)\|, \\
&\leq \left(\frac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\int_0^t \mathfrak{M}^{\Phi_1-1}\left(t-\mathfrak{M}\right)^{\Psi_1-1} + \frac{\Phi_1\left(1-\Psi_1\right)t^{\Phi_1}-1}{G(\Psi_1)}\right)\phi_2\|\mathbb{V}_n - \mathbb{V}\|, \\
&\leq \left(\frac{\Psi_1\Phi_1\Gamma\Phi_1}{G(\Psi_1)\Gamma(\Psi_1+\Phi_1)} + \frac{\Phi_1\left(1-\Psi_1\right)}{G(\Psi_1)}\right)\phi_2\|\mathbb{V}_n - \mathbb{V}\|, \\
&\leq \left(\frac{\Psi_1\Phi_1\Gamma\Phi_1}{G\left(\Psi_1\Gamma\left(\Psi_1+\Phi_1\right)\right)} + \frac{\Phi_1\left(1-\Psi_1\right)}{G(\Psi_1)}\right)^n\sigma^n\|\mathbb{V}_1 - \mathbb{V}\|,
\end{aligned}
\tag{24}
$$

where $\sigma < 1$ and as $n \to \infty$, $\mathbb{V}_n \to \mathbb{V}$.
Continuing in this manner, we arrive at the final equation as follows:

$$
\begin{matrix} \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \end{matrix}
$$

$$
\begin{aligned}
\|\mathbf{H}_5 n(t)\| &= \|\mathbb{R}_{n+1} - \mathbb{R}_n\| \\
&= \|\frac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\int_0^t \mathfrak{M}^{\Phi_1-1}\left(t-\mathfrak{M}\right)^{\Psi_1-1}\mathbb{D}_5\left(\mathfrak{M},\mathbb{R}_n\left(\mathfrak{M}\right)\right)d\mathfrak{M} + \frac{\Phi_1\left(1-\Psi_1\right)t^{\Phi_1}-1}{G(\Psi_1)}\mathbb{D}_5\left(t,\mathbb{R}_n(t)\right) \\
&- \left(\frac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\int_0^t \mathfrak{M}^{\Phi_1-1}\left(t-\mathfrak{M}\right)^{\Psi_1-1}\mathbb{D}_5\left(\mathfrak{M},\mathbb{R}_{n-1}\left(\mathfrak{M}\right)\right)d\mathfrak{M} + \frac{\Phi_1\left(1-\Psi_1\right)t^{\Phi_1}-1}{G(\Psi_1)}\mathbb{D}_5\left(t,\mathbb{R}_{n-1}(t)\right)\right)\|, \\
&= \frac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\int_0^t \mathfrak{M}^{\Phi_1-1}\left(t-\mathfrak{M}\right)^{\Psi_1-1}\|\mathbb{D}_5\left(\mathfrak{M},\mathbb{R}_n\left(\mathfrak{M}\right)\right) - \mathbb{D}_5\left(\mathfrak{M},\mathbb{R}_{n-1}\left(\mathfrak{M}\right)\right)\| d\mathfrak{M} \\
&+ \frac{\Phi_1\left(1-\Psi_1\right)t^{\Phi_1}-1}{G(\Psi_1)}\|\mathbb{D}_5\left(t,\mathbb{R}_n(t)\right) - \mathbb{D}_5\left(t,\mathbb{R}_{n-1}(t)\right)\|, \\
&\leq \left(\frac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\int_0^t \mathfrak{M}^{\Phi_1-1}\left(t-\mathfrak{M}\right)^{\Psi_1-1} + \frac{\Phi_1\left(1-\Psi_1\right)t^{\Phi_1}-1}{G(\Psi_1)}\right)\phi_5\|\mathbb{R}_n - \mathbb{R}\|, \\
&\leq \left(\frac{\Psi_1\Phi_1\Gamma\Phi_1}{G(\Psi_1)\Gamma(\Psi_1+\Phi_1)} + \frac{\Phi_1\left(1-\Psi_1\right)}{G(\Psi_1)}\right)\phi_5\|\mathbb{R}_n - \mathbb{R}\|, \\
&\leq \left(\frac{\Psi_1\Phi_1\Gamma\Phi_1}{G\left(\Psi_1\Gamma\left(\Psi_1+\Phi_1\right)\right)} + \frac{\Phi_1\left(1-\Psi_1\right)}{G(\Psi_1)}\right)^n\sigma^n\|\mathbb{R}_1 - \mathbb{R}\|,
\end{aligned}
\tag{25}
$$

where $\sigma < 1$ and as $n \to \infty$, we have $\mathbb{R}_n \to \mathbb{R}$. Therefore, as n tends to infinity, $\mathbf{H}_i n$ approaches zero for all $i \in \mathbb{N}^5$ when $\sigma < 1$.

3.3. Unique Solution

Theorem 9. *The Fractal-Fractional system describing the propagation of the influenza virus in a population has a unique solution if the following condition is satisfied:*

$$\left(\frac{\Psi_1\Phi_1\Gamma\Phi_1}{G(\Psi_1\Gamma(\Psi_1+\Phi_1))}+\frac{\Phi_1(1-\Psi_1)}{G(\Psi_1)}\right)\Phi_{1i}\leq 1,\quad i\in\mathbb{N}_1^5. \tag{26}$$

Proof: Assume, for the sake of contradiction, that there exist multiple solutions to the FF system, denoted as $\mathbb{S}^*,\mathbb{V}^*,\mathbb{E}^*,\mathbb{I}^*$, and $\mathbb{R}^*$, that satisfy the model given in equation (9). We have:

$$\begin{aligned}
\mathbb{S}^*(t) &= \tfrac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\int_0^t \mathfrak{M}^{\Phi_1-1}(t-\mathfrak{M})^{\Psi_1-1}\mathbb{D}_1(\mathfrak{M},\mathbb{S}^*(\mathfrak{M}))\,d\mathfrak{M}+\tfrac{\Phi_1(1-\Psi_1)t^{\Phi_1-1}}{G(\Psi_1)}\mathbb{D}_1(t,\mathbb{S}^*(t)),\\
\mathbb{V}^*(t) &= \tfrac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\int_0^t \mathfrak{M}^{\Phi_1-1}(t-\mathfrak{M})^{\Psi_1-1}\mathbb{D}_2(\mathfrak{M},\mathbb{V}^*(\mathfrak{M}))\,d\mathfrak{M}+\tfrac{\Phi_1(1-\Psi_1)t^{\Phi_1-1}}{G(\Psi_1)}\mathbb{D}_2(t,\mathbb{V}^*(t)),\\
\mathbb{E}^*(t) &= \tfrac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\int_0^t \mathfrak{M}^{\Phi_1-1}(t-\mathfrak{M})^{\Psi_1-1}\mathbb{D}_3(\mathfrak{M},\mathbb{E}^*(\mathfrak{M}))\,d\mathfrak{M}+\tfrac{\Phi_1(1-\Psi_1)t^{\Phi_1-1}}{G(\Psi_1)}\mathbb{D}_3(t,\mathbb{E}^*(t)),\\
\mathbb{I}^*(t) &= \tfrac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\int_0^t \mathfrak{M}^{\Phi_1-1}(t-\mathfrak{M})^{\Psi_1-1}\mathbb{D}_4(\mathfrak{M},\mathbb{I}^*(\mathfrak{M}))\,d\mathfrak{M}+\tfrac{\Phi_1(1-\Psi_1)t^{\Phi_1-1}}{G(\Psi_1)}\mathbb{D}_4(t,\mathbb{I}^*(t)),\\
\mathbb{R}^*(t) &= \tfrac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\int_0^t \mathfrak{M}^{\Phi_1-1}(t-\mathfrak{M})^{\Psi_1-1}\mathbb{D}_5(\mathfrak{M},\mathbb{R}^*(\mathfrak{M}))\,d\mathfrak{M}+\tfrac{\Phi_1(1-\Psi_1)t^{\Phi_1-1}}{G(\Psi_1)}\mathbb{D}_5(t,\mathbb{R}^*(t)).
\end{aligned} \tag{27}$$

$$\begin{aligned}
\|\mathbb{S}-\mathbb{S}^*\| &= \|\frac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\int_0^t \mathfrak{M}^{\Phi_1-1}(t-\mathfrak{M})^{\Psi_1-1}\mathbb{D}_1(\mathfrak{M},L(\mathfrak{M}))\,d\mathfrak{M}+\frac{\Phi_1(1-\Psi_1)t^{\Phi_1}-1}{G(\Psi_1)}\mathbb{D}_1(t,St))\\
&-\left(\frac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\int_0^t \mathfrak{M}^{\Phi_1-1}(t-\mathfrak{M})^{\Psi_1-1}\mathbb{D}_1(\mathfrak{M},\mathbb{S}^*(\mathfrak{M}))\,d\mathfrak{M}+\frac{\Phi_1(1-\Psi_1)t^{\Phi_1}-1}{G(\Psi_1)}\mathbb{D}_1(t,\mathbb{S}^*(t))\right)\|,\\
&=\frac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\int_0^t \mathfrak{M}^{\Phi_1-1}(t-\mathfrak{M})^{\Psi_1-1}\|\mathbb{D}_1(\mathfrak{M},\mathbb{S}(\mathfrak{M}))-\mathbb{D}_1(\mathfrak{M},\mathbb{S}^*(\mathfrak{M}))\|\,d\mathfrak{M}\\
&+\frac{\Phi_1(1-\Psi_1)t^{\Phi_1}-1}{G(\Psi_1)}\|\mathbb{D}_1(t,\mathbb{S}(t))-\mathbb{D}_1(t,\mathbb{S}^*(t))\|,\\
&\leq\left(\frac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\int_0^t \mathfrak{M}^{\Phi_1-1}(t-\mathfrak{M})^{\Psi_1-1}+\frac{\Phi_1(1-\Psi_1)t^{\Phi_1}-1}{G(\Psi_1)}\right)\Phi_{11}\|\mathbb{S}-\mathbb{S}^*\|\\
&\times\left[1-\left(\frac{\Psi_1\Phi_1\Gamma\Phi_1}{G(\Psi_1\Gamma(\Psi_1+\Phi_1))}+\frac{\Phi_1(1-\Psi_1)}{G(\Psi_1)}\right)\Phi_{11}\right]\|\mathbb{S}-\mathbb{S}^*\|\leq 0.
\end{aligned} \tag{28}$$

The given equation is valid if and only if $|\mathbb{S}-\mathbb{S}^*|=0$, which indicates that $\mathbb{S}$ must be equal to $\mathbb{S}^*$. Likewise, examining the differences between the norms of $\mathbb{V}(t)$ and $\mathbb{V}^*(t)$ leads to:

$$\begin{aligned}
\|\mathbb{V}-\mathbb{V}^*\| &= \|\frac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\int_0^t \mathfrak{M}^{\Phi_1-1}(t-\mathfrak{M})^{\Psi_1-1}\mathbb{D}_2(\mathfrak{M},\mathbb{V}(\mathfrak{M}))\,d\mathfrak{M}+\frac{\Phi_1(1-\Psi_1)t^{\Phi_1}-1}{G(\Psi_1)}\mathbb{D}_2(t,\mathbb{V}(t))\\
&-\left(\frac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\int_0^t \mathfrak{M}^{\Phi_1-1}(t-\mathfrak{M})^{\Psi_1-1}\mathbb{D}_2(\mathfrak{M},\mathbb{V}^*(\mathfrak{M}))\,d\mathfrak{M}+\frac{\Phi_1(1-\Psi_1)t^{\Phi_1}-1}{G(\Psi_1)}\mathbb{D}_2(t,\mathbb{V}^*(t))\right)\|,\\
&=\frac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\int_0^t \mathfrak{M}^{\Phi_1-1}(t-\mathfrak{M})^{\Psi_1-1}\|\mathbb{D}_2(\mathfrak{M},\mathbb{V}(\mathfrak{M}))-\mathbb{D}_2(\mathfrak{M},\mathbb{V}^*(\mathfrak{M}))\|\,d\mathfrak{M}\\
&+\frac{\Phi_1(1-\Psi_1)t^{\Phi_1}-1}{G(\Psi_1)}\|\mathbb{D}_2(t,\mathbb{V}(t))-\mathbb{D}_2(t,\mathbb{V}^*(t))\|,\\
&\leq\left(\frac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\int_0^t \mathfrak{M}^{\Phi_1-1}(t-\mathfrak{M})^{\Psi_1-1}+\frac{\Phi_1(1-\Psi_1)t^{\Phi_1}-1}{G(\Psi_1)}\right)\Phi_{12}\|\mathbb{V}-\mathbb{V}^*\|\\
&\times\left[1-\left(\frac{\Psi_1\Phi_1\Gamma\Phi_1}{G(\Psi_1\Gamma(\Psi_1+\Phi_1))}+\frac{\Phi_1(1-\Psi_1)}{G(\Psi_1)}\right)\Phi_{12}\right]\|\mathbb{V}-\mathbb{V}^*\|\leq 0.
\end{aligned} \tag{29}$$

The inequality is satisfied if $\|\mathbb{V}-\mathbb{V}^*\|=0$, thus $\mathbb{V}=\mathbb{V}^*$. By continuing this process in the same manner, we eventually reach the final equation as follows:

$$\begin{aligned}
&\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots\\
\|\mathbb{R}-\mathbb{R}^*\| &= \| \frac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\int_0^t \mathfrak{M}^{\Phi_1-1}(t-\mathfrak{M})^{\Psi_1-1}\mathbb{D}_5(\mathfrak{M},\mathbb{R}(\mathfrak{M}))\,d\mathfrak{M} + \frac{\Phi_1(1-\Psi_1)t^{\Phi_1}-1}{G(\Psi_1)}\mathbb{D}_5(t,\mathbb{R}(t))\\
&\quad -\left(\frac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\int_0^t \mathfrak{M}^{\Phi_1-1}(t-\mathfrak{M})^{\Psi_1-1}\mathbb{D}_5(\mathfrak{M},\mathbb{R}^*(\mathfrak{M}))\,d\mathfrak{M} + \frac{\Phi_1(1-\Psi_1)t^{\Phi_1}-1}{G(\Psi_1)}\mathbb{D}_5(t,\mathbb{R}^*(t))\right)\|,\\
&= \frac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\int_0^t \mathfrak{M}^{\Phi_1-1}(t-\mathfrak{M})^{\Psi_1-1}\|\mathbb{D}_5(\mathfrak{M},\mathbb{R}(\mathfrak{M}))-\mathbb{D}_5(\mathfrak{M},\mathbb{R}^*(\mathfrak{M}))\|\,d\mathfrak{M}\\
&\quad + \frac{\Phi_1(1-\Psi_1)t^{\Phi_1}-1}{G(\Psi_1)}\|\mathbb{D}_5(t,\mathbb{R}(t))-\mathbb{D}_5(t,\mathbb{R}^*(t))\|,\\
&\le \left(\frac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\int_0^t \mathfrak{M}^{\Phi_1-1}(t-\mathfrak{M})^{\Psi_1-1} + \frac{\Phi_1(1-\Psi_1)t^{\Phi_1}-1}{G(\Psi_1)}\right)\Phi_{16}\|\mathbb{R}-\mathbb{R}^*\|\\
&\quad \times\left[1-\left(\frac{\Psi_1\Phi_1\Gamma\Phi_1}{G(\Psi_1\Gamma(\Psi_1+\Phi_1))}+\frac{\Phi_1(1-\Psi_1)}{G(\Psi_1)}\right)\Phi_{16}\right]\|\mathbb{R}-\mathbb{R}^*\|\le 0.
\end{aligned} \tag{30}$$

If the norm of the difference between $\mathbb{R}$ and $\mathbb{R}^*$ is zero, then the inequality is satisfied, indicating that $\mathbb{R}$ is equal to $\mathbb{R}^*$.

4. Fixed Point Stability

Hyers-Ulam stability, introduced by Hyers and later generalized by Ulam, examines the behavior of solutions to mathematical equations under small perturbations.[36,37] This concept is crucial for Fractal Fractional epidemic models, which study the spread of infectious diseases using fractional calculus and fractal geometry. These models account for the complex dynamics of epidemics, including heterogeneity, nonlinearity, and memory effects. Hyers-Ulam stability ensures that the solutions of Fractal Fractional epidemic models remain stable under small changes in initial conditions. This stability is vital for verifying the robustness of the model's predictions, such as the rapid spread of an infectious disease, and for guiding public health policies. Additionally, Hyers-Ulam stability helps assess the impact of control measures like vaccination or quarantine, ensuring the reliability of these models in informing public health strategies. In summary, Hyers-Ulam stability is essential for validating the robustness and reliability of Fractal Fractional epidemic models in predicting the spread of infectious diseases.

Definition 10.[35] The FF integration described in (10) is considered HU stable if there exists a constant $k_i^* > 0$ for all $i \in \mathbb{N}^5$ such that for every $\delta_i > 0$ with $i \in \mathbb{N}_1^4$, the following criteria are met:

$$\begin{aligned}
&\left|\mathbb{S}(t)-\frac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\int_0^t \mathfrak{M}^{\varphi_3-1}(t-\mathfrak{M})^{\Psi_1-1}\mathbb{D}_1(x,\mathbb{S}(x))d\mathfrak{M}+\frac{\Phi_1(1-\Psi_1)t^{\Phi_1-1}}{G(\Psi_1)}\mathbb{D}_1(t,\mathbb{S}(t))\right|\le\delta_1,\\
&\left|\mathbb{V}(t)-\frac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\int_0^t \mathfrak{M}^{\Phi_1-1}(t-x)^{\Psi_1-1}\mathbb{D}_2(x,\mathbb{V}(\mathfrak{M}^*))d\mathfrak{M}+\frac{\Phi_1(1-\Psi_1)t^{\Phi_1-1}}{G(\Psi_1)}\mathbb{D}_2(t,\mathbb{V}(t))\right|\le\delta_2,\\
&\left|\mathbb{E}(t)-\frac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\int_0^t \mathfrak{M}^{\Phi_1-1}(t-\mathfrak{M})\Psi_1-1\mathbb{D}_3(\mathfrak{M},\mathbb{E}(\mathfrak{M}))\,d\mathfrak{M}+\frac{\Phi_1(1-\Psi_1)t_1^{\Phi_1-1}}{G(\Psi_1)}\mathbb{D}_3(t,\mathbb{E}(t))\right|\le\delta_3,\\
&\left|\mathbb{I}(t)-\frac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\int_0^t \mathfrak{M}^{\Phi_1-1}(t-\mathfrak{M})\Psi_1-1\mathbb{D}_4(\mathfrak{M},N(\mathfrak{M}))\,d\mathfrak{M}+\frac{\Phi_1(1-\Psi_1)t_1^{\Phi_1-1}}{G(\Psi_1)}\mathbb{D}_4(t,\mathbb{I}(t))\right|\le\delta_4,\\
&\left|\mathbb{R}(t)-\frac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\int_0^t \mathfrak{M}^{\Phi_1-1}(t-\mathfrak{M})\Psi_1-1\mathbb{D}_5(\mathfrak{M},Q(\mathfrak{M}))\,d\mathfrak{M}+\frac{\Phi_1(1-\Psi_1)t_1^{\Phi_1-1}}{G(\Psi_1)}\mathbb{D}_5(t,\mathbb{R}(t))\right|\le\delta_5.
\end{aligned} \tag{31}$$

A numerical solution exists for the model $\mathbb{S}^*(t), \mathbb{V}^*(t), \mathbb{E}^*(t), \mathbb{I}^*(t), \mathbb{R}^*(t)$ that satisfies the given system, such that

$$\begin{aligned}
|\mathbb{S}-\mathbb{S}^*| = &|\frac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\int_0^t \mathfrak{M}^{\Phi_1-1}(t-\mathfrak{M})^{\Psi_1-1}\mathbb{D}_1(\mathfrak{M},\mathbb{S}(\mathfrak{M}))\,d\mathfrak{M} + \frac{\Phi_1(1-\Psi_1)t^{\Phi_1}-1}{G(\Psi_1)}\mathbb{D}_1(t,\mathbb{S}(t)) \\
&-\left(\frac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\int_0^t \mathfrak{M}^{\Phi_1-1}(t-\mathfrak{M})^{\Psi_1-1}\mathbb{D}_1(\mathfrak{M},\mathbb{S}^*(\mathfrak{M}))\,d\mathfrak{M} + \frac{\Phi_1(1-\Psi_1)t^{\Phi_1}-1}{G(\Psi_1)}\mathbb{D}_1(t,\mathbb{S}^*(t))\right)|, \\
&= \frac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\int_0^t \mathfrak{M}^{\Phi_1-1}(t-\mathfrak{M})^{\Psi_1-1}\left|\mathbb{D}_1(\mathfrak{M},\mathbb{S}(\mathfrak{M}))-\mathbb{D}_1(\mathfrak{M},\mathbb{S}^*(\mathfrak{M}))\right|d\mathfrak{M} \\
&+\frac{\Phi_1(1-\Psi_1)t^{\Phi_1}-1}{G(\Psi_1)}\left|\mathbb{D}_1(t,\mathbb{S}(t))-\mathbb{D}_1(t,\mathbb{S}^*(t))\right|, \\
&\le \left(\frac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\int_0^t \mathfrak{M}^{\Phi_1-1}(t-\mathfrak{M})^{\Psi_1-1} + \frac{\Phi_1(1-\Psi_1)t^{\Phi_1}-1}{G(\Psi_1)}\right)\Phi_{11}|\mathbb{S}-\mathbb{S}^*| \\
&\times\left[1-\left(\frac{\Psi_1\Phi_1\Gamma\Phi_1}{G(\Psi_1\Gamma(\Psi_1+\Phi_1))}+\frac{\Phi_1(1-\Psi_1)}{G(\Psi_1)}\right)\Phi_{11}\right]|\mathbb{S}-\mathbb{S}^*| \le 0.
\end{aligned} \tag{32}$$

Considering

$$\left\{\zeta_1 = \left[1-\left(\frac{\Psi_1\Phi_1\Gamma\varphi_3}{G(\Psi_1\Gamma(\Psi_1+\Phi_1))}+\frac{\Phi_1(1-\Psi_1)}{G(\Psi_1)}\right)\Phi_{11}\right]|\mathbb{S}-\mathbb{S}^*|, \eta_1 = \Phi_{11}, \right. \tag{33}$$

the inequality can be reformulated as $|\mathbb{S}-\mathbb{S}^*| \le \zeta_1\eta_1$.

$$\begin{aligned}
|\mathbb{V}-\mathbb{V}^*| = &|\frac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\int_0^t \mathfrak{M}^{\Phi_1-1}(t-\mathfrak{M})^{\Psi_1-1}\mathbb{D}_2(\mathfrak{M},\mathbb{V}(\mathfrak{M}))\,d\mathfrak{M} + \frac{\Phi_1(1-\Psi_1)t^{\Phi_1}-1}{G(\Psi_1)}\mathbb{D}_2(t,\mathbb{V}(t)) \\
&-\left(\frac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\int_0^t \mathfrak{M}^{\Phi_1-1}(t-\mathfrak{M})^{\Psi_1-1}\mathbb{D}_2(\mathfrak{M},\mathbb{V}^*(\mathfrak{M}))\,d\mathfrak{M} + \frac{\Phi_1(1-\Psi_1)t^{\Phi_1}-1}{G(\Psi_1)}\mathbb{D}_2(t,\mathbb{V}^*(t))\right)|, \\
&= \frac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\int_0^t \mathfrak{M}^{\Phi_1-1}(t-\mathfrak{M})^{\Psi_1-1}\left|\mathbb{D}_2(\mathfrak{M},\mathbb{V}(\mathfrak{M}))-\mathbb{D}_2(\mathfrak{M},\mathbb{V}^*(\mathfrak{M}))\right|d\mathfrak{M} \\
&+\frac{\Phi_1(1-\Psi_1)t^{\Phi_1}-1}{G(\Psi_1)}\left|\mathbb{D}_2(t,\mathbb{V}(t))-\mathbb{D}_2(t,\mathbb{V}^*(t))\right|, \\
&\le \left(\frac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\int_0^t \mathfrak{M}^{\Phi_1-1}(t-\mathfrak{M})^{\Psi_1-1} + \frac{\Phi_1(1-\Psi_1)t^{\Phi_1}-1}{G(\Psi_1)}\right)\Phi_{12}|\mathbb{V}-\mathbb{V}^*| \\
&\times\left[1-\left(\frac{\Psi_1\Phi_1\Gamma\Phi_1}{G(\Psi_1\Gamma(\Psi_1+\Phi_1))}+\frac{\Phi_1(1-\Psi_1)}{G(\Psi_1)}\right)\Phi_{12}\right]|\mathbb{V}-\mathbb{V}^*| \le 0.
\end{aligned} \tag{34}$$

Letting

$$\left\{\zeta_2 = \left[1-\left(\frac{\Psi_1\Phi_1\Gamma\varphi_3}{G(\Psi_1\Gamma(\Psi_1+\Phi_1))}+\frac{\Phi_1(1-\Psi_1)}{G(\Psi_1)}\right)\Phi_{12}\right]|\mathbb{V}-\mathbb{V}^*|, \eta_2 = \Phi_{12}, \right. \tag{35}$$

Thus, the inequality transforms into $|\mathbb{V}-\mathbb{V}^*| \le \zeta_2\eta_2$.
Following this pattern, we arrive at the final equation:

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$
\begin{aligned}
|\mathbb{R}-\mathbb{R}^*| = {} & |\frac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\int_0^t \mathfrak{M}^{\Phi_1-1}(t-\mathfrak{M})^{\Psi_1-1}\mathbb{D}_5(\mathfrak{M},\mathbb{R}(\mathfrak{M}))\,d\mathfrak{M} + \frac{\Phi_1(1-\Psi_1)t^{\Phi_1}-1}{G(\Psi_1)}\mathbb{D}_5(t,\mathbb{R}(t)) \\
& - \left(\frac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\int_0^t \mathfrak{M}^{\Phi_1-1}(t-\mathfrak{M})^{\Psi_1-1}\mathbb{D}_5(\mathfrak{M},\mathbb{R}^*(\mathfrak{M}))\,d\mathfrak{M} + \frac{\Phi_1(1-\Psi_1)t^{\Phi_1}-1}{G(\Psi_1)}\mathbb{D}_5(t,\mathbb{R}^*(t))\right)|, \\
= {} & \frac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\int_0^t \mathfrak{M}^{\Phi_1-1}(t-\mathfrak{M})^{\Psi_1-1}|\mathbb{D}_5(\mathfrak{M},\mathbb{R}(\mathfrak{M}))-\mathbb{D}_5(\mathfrak{M},\mathbb{R}^*(\mathfrak{M}))|\,d\mathfrak{M} \\
& + \frac{\Phi_1(1-\Psi_1)t^{\Phi_1}-1}{G(\Psi_1)}|\mathbb{D}_5(t,\mathbb{R}(t))-\mathbb{D}_5(t,\mathbb{R}^*(t))|, \\
\le {} & \left(\frac{\Psi_1\Phi_1}{G(\Psi_1)\Gamma(\Psi_1)}\int_0^t \mathfrak{M}^{\Phi_1-1}(t-\mathfrak{M})^{\Psi_1-1} + \frac{\Phi_1(1-\Psi_1)t^{\Phi_1}-1}{G(\Psi_1)}\right)\Phi_{16}|\mathbb{R}-\mathbb{R}^*| \\
& \times\left[1-\left(\frac{\Psi_1\Phi_1\Gamma\Phi_1}{G(\Psi_1\Gamma(\Psi_1+\Phi_1))}+\frac{\Phi_1(1-\Psi_1)}{G(\Psi_1)}\right)\Phi_{16}\right]|\mathbb{R}-\mathbb{R}^*| \le 0.
\end{aligned}
\tag{36}
$$

Letting

$$
\left\{\zeta_5 = \left[1-\left(\frac{\Psi_1\Phi_1\Gamma\varphi_5}{G(\Psi_1\Gamma(\Psi_1+\Phi_1))}+\frac{\Phi_1(1-\Psi_1)}{G(\Psi_1)}\right)\Phi_{14}\right]|\mathbb{R}-\mathbb{R}^*|, \eta_5 = \Phi_{16}, \right. \tag{37}
$$

so the inequality above becomes $|\mathbb{R}-\mathbb{R}^*| \le \zeta_5\eta_5$. Therefore, based on the definition of HU stability,[35] the proposed FF influenza virus model is considered stable.

5. Sensitivity Analysis

To perform sensitivity analysis on the given expression for $\mathfrak{R}_{vac}$, we'll calculate the partial derivatives of $\mathfrak{R}_{vac}$ with respect to its parameters. This will help us understand how changes in each parameter affect the value of $\mathfrak{R}_{vac}$. The sensitivity of $\mathfrak{R}_{vac}$ with respect to each parameter is calculated by taking the partial derivative of $\mathfrak{R}_{vac}$ with respect to that parameter.

1. Sensitivity with respect to β_E:

$$\frac{\partial\mathfrak{R}_{vac}}{\partial\beta_E} = \frac{(r\beta_E+\alpha\beta_E+\gamma\beta_E+\sigma\beta_I)(r+\theta+\beta_V\phi)}{(r+\alpha+\gamma)(r+\kappa+\sigma)(r+\theta+\phi)} \tag{38}$$

2. Sensitivity with respect to β_I:

$$\frac{\partial\mathfrak{R}_{vac}}{\partial\beta_I} = \frac{\beta\,(r\beta_E+\alpha\beta_E+\gamma\beta_E+\sigma\beta_I)\,(r+\theta+\beta_V\phi)}{(r+\alpha+\gamma)(r+\kappa+\sigma)(r+\theta+\phi)} \tag{39}$$

3. Sensitivity with respect to β_V:

$$\frac{\partial\mathfrak{R}_{vac}}{\partial\beta_V} = \frac{\beta(r\beta_E+\alpha\beta_E+\gamma\beta_E+\sigma\beta_I)(r+\theta+\phi)}{(r+\alpha+\gamma)(r+\kappa+\sigma)(r+\theta+\phi)} \tag{40}$$

4. Sensitivity with respect to α:

$$\frac{\partial\mathfrak{R}_{vac}}{\partial\alpha} = \frac{-\beta\beta_E(r\beta_E+\alpha\beta_E+\gamma\beta_E+\sigma\beta_I)(r+\theta+\beta_V\phi)}{(r+\alpha+\gamma)^2(r+\kappa+\sigma)(r+\theta+\phi)} \tag{41}$$

5. Sensitivity with respect to γ:

$$\frac{\partial\mathfrak{R}_{vac}}{\partial\gamma} = \frac{-\beta\beta_E(r\beta_E+\alpha\beta_E+\gamma\beta_E+\sigma\beta_I)(r+\theta+\beta_V\phi)}{(r+\alpha+\gamma)^2(r+\kappa+\sigma)(r+\theta+\phi)} \tag{42}$$

6. Sensitivity with respect to σ:

$$\frac{\partial\mathfrak{R}_{vac}}{\partial\sigma} = \frac{-\beta\beta_I(r\beta_E+\alpha\beta_E+\gamma\beta_E+\sigma\beta_I)(r+\theta+\beta_V\phi)}{(r+\alpha+\gamma)(r+\kappa+\sigma)^2(r+\theta+\phi)} \tag{43}$$

7. Sensitivity with respect to κ:

$$\frac{\partial \mathfrak{R}_{vac}}{\partial \kappa} = \frac{-\beta\beta_V(r\beta_E + \alpha\beta_E + \gamma\beta_E + \sigma\beta_I)(r + \theta + \beta_V\phi)}{(r + \alpha + \gamma)(r + \kappa + \sigma)^2(r + \theta + \phi)} \tag{44}$$

8. Sensitivity with respect to θ:

$$\frac{\partial \mathfrak{R}_{vac}}{\partial \theta} = \frac{\beta(r\beta_E + \alpha\beta_E + \gamma\beta_E + \sigma\beta_I)(r + \beta_V\phi)}{(r + \alpha + \gamma)(r + \kappa + \sigma)(r + \theta + \phi)^2} \tag{45}$$

9. Sensitivity with respect to ϕ:

$$\frac{\partial \mathfrak{R}_{vac}}{\partial \phi} = \frac{\beta(r\beta_E + \alpha\beta_E + \gamma\beta_E + \sigma\beta_I)(r + \theta + \beta_V\phi)}{(r + \alpha + \gamma)(r + \kappa + \sigma)(r + \theta + \phi)^2} \tag{46}$$

These sensitivity derivatives will allow you to understand how changes in each parameter affect the value of $\mathfrak{R}_{vac}$.

5.1. Sensitivity indices

Now we can proceed to calculate the sensitivity indices for each parameter:
First, calculate $\mathfrak{R}_{vac}$:

$$\begin{aligned}\mathfrak{R}_{vac} &= \frac{\beta\left(r\beta_E + \alpha\beta_E + \gamma\beta_E + \sigma\beta_I\right)\left(r + \theta + \beta_V\phi\right)}{(r + \alpha + \gamma)(r + \kappa + \sigma)(r + \theta + \phi)} \\ &= \frac{0.514 \cdot (7.14 \times 10^{-5} \cdot 0.25 + 9.3 \times 10^{-6} \cdot 0.25 + 5 \cdot 10^{-8} \cdot 0.25 + 2 \cdot 1.0)}{(7.14 \times 10^{-5} + 9.3 \times 10^{-6} + 5 \cdot 10^{-8})(7.14 \times 10^{-5} + 1.857 \times 10^{-4} + 2.0)(7.14 \times 10^{-5} + 365.0 \cdot 0.56)} \\ &= 12.70996516\end{aligned}$$

Now, we'll calculate each sensitivity index using the formulas derived earlier:
1. Sensitivity index with respect to β_E:

$$\begin{aligned}\text{Sensitivity index}_{\beta_E} &= \frac{\partial \mathfrak{R}_{vac}/\partial \beta_E \times \beta_E}{\mathfrak{R}_{vac}} = \frac{\beta(r + \theta + \beta_V\phi) \cdot \beta_E}{\mathfrak{R}_{vac}} \\ &= \frac{0.514 \cdot (7.14 \times 10^{-5} + 365.0 \cdot 0.56) \cdot 0.25}{12.70996516} = 0.019089184.\end{aligned} \tag{47}$$

2. Sensitivity index with respect to β_I:

$$\begin{aligned}\text{Sensitivity index}_{\beta_I} &= \frac{\partial \mathfrak{R}_{vac}/\partial \beta_I \times \beta_I}{\mathfrak{R}_{vac}} = \frac{\beta\sigma\left(r + \theta + \beta_V\phi\right) \cdot \beta_I}{\mathfrak{R}_{vac}} \\ &= \frac{0.514 \cdot 2.0 \cdot (7.14 \times 10^{-5} + 365.0 \cdot 0.56) \cdot 1.0}{12.70996516} = 0.062505656.\end{aligned} \tag{48}$$

3. Sensitivity index with respect to β_V:

$$\begin{aligned}\text{Sensitivity index}_{\beta_V} &= \frac{\partial \mathfrak{R}_{vac}/\partial \beta_V \times (1 - \beta_V)}{\mathfrak{R}_{vac}} = \frac{\beta\phi\left(r\beta_E + \alpha\beta_E + \gamma\beta_E + \sigma\beta_I\right) \cdot (1 - \beta_V)}{\mathfrak{R}_{vac}} \\ &= \frac{0.514 \cdot 0.56 \cdot (7.14 \times 10^{-5} \cdot 0.25 + 9.3 \times 10^{-6} \cdot 0.25 + 5 \cdot 10^{-8} \cdot 0.25 + 2 \cdot 1.0) \cdot (1 - 0.56)}{12.70996516} \\ &= -0.12774771.\end{aligned} \tag{49}$$

4. Sensitivity index with respect to α:

$$\begin{aligned}\text{Sensitivity index}_{\alpha} &= \frac{\partial \mathfrak{R}_{vac}/\partial \alpha \times \alpha}{\mathfrak{R}_{vac}} = \frac{-\beta\beta_E(r + \theta + \beta_V\phi) \cdot \alpha}{\mathfrak{R}_{vac}} \\ &= \frac{-0.514 \cdot 0.25 \cdot (7.14 \times 10^{-5} + 365.0 \cdot 0.56) \cdot 9.3 \times 10^{-6}}{12.70996516} = -0.000257001.\end{aligned} \tag{50}$$

5. Sensitivity index with respect to γ:

$$\begin{aligned}\text{Sensitivity index}_\gamma &= \frac{\partial \mathfrak{R}_{vac}/\partial \gamma \times \gamma}{\mathfrak{R}_{vac}} = \frac{-\beta\beta_E(r+\theta+\beta_V\phi)\cdot\gamma}{\mathfrak{R}_{vac}} \\ &= \frac{-0.514\cdot 0.25\cdot(7.14\times 10^{-5}+365.0\cdot 0.56)\cdot 5\times 10^{-8}}{12.70996516} = -1.69598\times 10^{-8}.\end{aligned} \tag{51}$$

6. Sensitivity index with respect to σ:

$$\begin{aligned}\text{Sensitivity index}_\sigma &= \frac{\partial \mathfrak{R}_{vac}/\partial \sigma \times (1/\sigma)}{\mathfrak{R}_{vac}} = \frac{-\beta\beta_I(r+\theta+\beta_V\phi)\cdot(1/\sigma)}{\mathfrak{R}_{vac}} \\ &= \frac{-0.514\cdot 1.0\cdot(7.14\times 10^{-5}+365.0\cdot 0.56)\cdot(1/2.0)}{12.70996516} = -0.10646021.\end{aligned} \tag{52}$$

7. Sensitivity index with respect to κ:

$$\begin{aligned}\text{Sensitivity index}_\kappa &= \frac{\partial \mathfrak{R}_{vac}/\partial \kappa \times (1/\kappa)}{\mathfrak{R}_{vac}} = \frac{-\beta\beta_I(r+\theta+\beta_V\phi)\cdot(1/\kappa)}{\mathfrak{R}_{vac}} \\ &= \frac{-0.514\cdot 1.0\cdot(7.14\times 10^{-5}+365.0\cdot 0.56)\cdot(1/1.857\times 10^{-4})}{12.70996516} = -0.16102451.\end{aligned} \tag{53}$$

8. Sensitivity index with respect to θ:

$$\begin{aligned}\text{Sensitivity index}_\theta &= \frac{\partial \mathfrak{R}_{vac}/\partial \theta \times (\theta^{-1})}{\mathfrak{R}_{vac}} = \frac{\beta\,(r\beta_E+\alpha\beta_E+\gamma\beta_E+\sigma\beta_I)\,(r+\beta_V\phi)\cdot(\theta^{-1})}{\mathfrak{R}_{vac}} \\ &= \frac{0.514\cdot(7.14\times 10^{-5}\cdot 0.25+9.3\times 10^{-6}\cdot 0.25+5\cdot 10^{-8}\cdot 0.25+2\cdot 1.0)(7.14\times 10^{-5}+365.0\cdot 0.56)\cdot(365.0^{-1})}{12.70996516} \\ &= 0.003431662.\end{aligned} \tag{54}$$

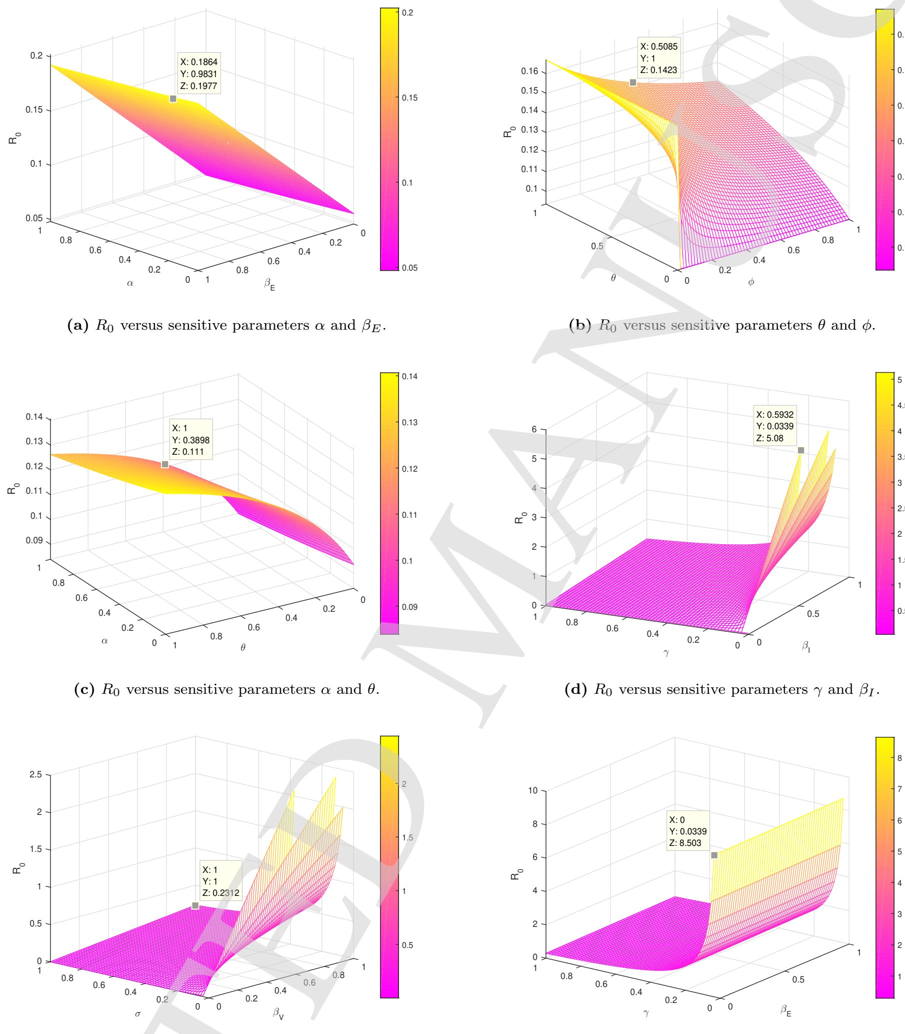
9. Sensitivity index with respect to ϕ:

$$\begin{aligned}\text{Sensitivity index}_\phi &= \frac{\partial \mathfrak{R}_{vac}/\partial \phi \times \phi}{\mathfrak{R}_{vac}} = \frac{\beta\beta_V\,(r\beta_E+\alpha\beta_E+\gamma\beta_E+\sigma\beta_I)\,(r+\theta+\beta_V\phi)\cdot\phi}{\mathfrak{R}_{vac}} \\ &= \frac{0.514\cdot 0.56\cdot(7.14\times 10^{-5}\cdot 0.25+9.3\times 10^{-6}\cdot 0.25+5\cdot 10^{-8}\cdot 0.25+2\cdot 1.0)\cdot(7.14\times 10^{-5}+365.0\cdot 0.56)\cdot 0.56}{12.70996516} \\ &= 0.000229577.\end{aligned} \tag{55}$$

These are the sensitivity indices calculated using the provided parameter values and formulas for each parameter.

6. Adams-Bashforth Numerical Method

The Adams-Bashforth method is a popular technique for numerically solving ordinary differential equations (ODEs). This method can be adapted to fractional differential equations (FDEs) by incorporating fractional derivatives, such as the Caputo or Riemann-Liouville derivatives. The flexibility and precision of the Adams-Bashforth method in handling higher-order terms make it a valuable tool for solving complex problems in various fields, including engineering, physics, and finance. By extending this method to fractional orders, it provides a robust framework for addressing the unique challenges posed by FDEs, ensuring both efficiency and accuracy in the solutions.



(a) R_0 versus sensitive parameters α and β_E.

(b) R_0 versus sensitive parameters θ and ϕ.

(c) R_0 versus sensitive parameters α and θ.

(d) R_0 versus sensitive parameters γ and β_I.

(e) R_0 versus sensitive parameters σ and β_V.

(f) R_0 versus sensitive parameters γ and β_E.

Fig. 2: The plots examine how changes in various parameters relate to R_0 through sensitivity analysis.

6.1. Numerical scheme

The influenza virus system (9) can be formulated using the Fractal-Fractional Atangana-Baleanu-Caputo derivative method:

$$
\begin{aligned}
{}^{FFM}_{\ 0}D_t^{\Psi_1,\Phi_1}[\mathbb{S}(t)] &= -\beta\beta_E\frac{\mathbb{E}\mathbb{S}}{\mathbb{N}} - \beta\beta_I\frac{\mathbb{I}\mathbb{S}}{\mathbb{N}} - \phi\mathbb{S} - \mu\mathbb{S} + \delta\mathbb{R} + \theta\mathbb{V} + r\mathbb{N}\\
{}^{FFM}_{\ 0}D_t^{\Psi_1,\Phi_1}[\mathbb{V}(t)] &= -\beta\beta_E\beta_V\frac{\mathbb{E}\mathbb{V}}{\mathbb{N}} - \beta\beta_I\beta_V\frac{\mathbb{I}\mathbb{V}}{\mathbb{N}} - \mu\mathbb{V} - \theta\mathbb{V} + \phi\mathbb{S}\\
{}^{FFM}_{\ 0}D_t^{\Psi_1,\Phi_1}[\mathbb{E}(t)] &= \beta\beta_E\frac{\mathbb{E}\mathbb{S}}{\mathbb{N}} + \beta\beta_I\frac{\mathbb{I}\mathbb{S}}{\mathbb{N}} + \beta\beta_E\beta_V\frac{\mathbb{E}\mathbb{V}}{\mathbb{N}} + \beta\beta_I\beta_V\frac{\mathbb{I}\mathbb{V}}{\mathbb{N}} - (\mu+\kappa+\sigma)\mathbb{E}\\
{}^{FFM}_{\ 0}D_t^{\Psi_1,\Phi_1}[\mathbb{I}(t)] &= \sigma\mathbb{E} - (r+\alpha+\gamma)\mathbb{I}\\
{}^{FFM}_{\ 0}D_t^{\Psi_1,\Phi_1}[\mathbb{Z}(t)] &= \eta_1\mathbb{E}(t) + \eta_2\mathbb{I}(t) - (d_1+\varphi)\,\mathbb{Z}(t)\\
{}^{FFM}_{\ 0}D_t^{\Psi_1,\Phi_1}[\mathbb{R}(t)] &= \kappa\mathbb{E} + \gamma\mathbb{I} - \eta\mathbb{R} - \delta\mathbb{R}
\end{aligned}
\tag{56}
$$

Subsequently, the problem (56) can be rewritten as

$$\begin{cases} {}^{FFM}\mathbb{D}_{0\ 1}^{\Psi}\mathbb{S} = \Phi_1 t^{\Phi_1 - 1}\mathcal{G}_1(s, \mathbb{S}, \mathbb{V}, \mathbb{E}, \mathbb{I}, \mathbb{R}), \\ {}^{FFM}\mathbb{D}_{0\ 1}^{\Psi}\mathbb{V} = \Phi_1 t^{\Phi_1 - 1}\mathcal{G}_2(s, \mathbb{S}, \mathbb{V}, \mathbb{E}, \mathbb{I}, \mathbb{R}), \\ {}^{FFM}\mathbb{D}_{0\ 1}^{\Psi}\mathbb{E} = \Phi_1 t^{\Phi_1 - 1}\mathcal{G}_3(s, \mathbb{S}, \mathbb{V}, \mathbb{E}, \mathbb{I}, \mathbb{R}), \\ {}^{FFM}\mathbb{D}_{0\ 1}^{\Psi}\mathbb{I} = \Phi_1 t^{\Phi_1 - 1}\mathcal{G}_4(s, \mathbb{S}, \mathbb{V}, \mathbb{E}, \mathbb{I}, \mathbb{R}), \\ {}^{FFM}\mathbb{D}_{0\ 1}^{\Psi}\mathbb{R} = \Phi_1 t^{\Phi_1 - 1}\mathcal{G}_6(s, \mathbb{S}, \mathbb{V}, \mathbb{E}, \mathbb{I}, \mathbb{R}). \end{cases} \tag{57}$$

Applying the Fractal-Fractional integration to equation (56) yields the following result:

$$\begin{cases} \mathbb{S}(t) - \mathbb{S}(0) = \dfrac{\Phi_1(1-\Psi_1)t^{\Phi_1-1}}{W(\Psi_1)}\mathcal{G}_1(s, \mathbb{S}, \mathbb{V}, \mathbb{E}, \mathbb{I}, \mathbb{R}) + \dfrac{\Phi_1\Psi_1}{W(\Psi_1)}\displaystyle\int_0^t s^{\Phi_1-1}(t-s)^{\Psi_1-1}\mathcal{G}_1(s, \mathbb{S}, \mathbb{V}, \mathbb{E}, \mathbb{I}, \mathbb{R})ds, \\ \mathbb{V}(t) - \mathbb{V}(0) = \dfrac{\Phi_1(1-\Psi_1)t^{\Phi_1-1}}{W(\Psi_1)}\mathcal{G}_2(s, \mathbb{S}, \mathbb{V}, \mathbb{E}, \mathbb{I}, \mathbb{R}) + \dfrac{\Phi_1\Psi_1}{W(\Psi_1)}\displaystyle\int_0^t s^{\Phi_1-1}(t-s)^{\Psi_1-1}\mathcal{G}_2(s, \mathbb{S}, \mathbb{V}, \mathbb{E}, \mathbb{I}, \mathbb{R})ds, \\ \mathbb{E}(t) - \mathbb{E}(0) = \dfrac{\Phi_1(1-\Psi_1)t^{\Phi_1-1}}{W(\Psi_1)}\mathcal{G}_3(s, \mathbb{S}, \mathbb{V}, \mathbb{E}, \mathbb{I}, \mathbb{R}) + \dfrac{\Phi_1\Psi_1}{W(\Psi_1)}\displaystyle\int_0^t s^{\Phi_1-1}(t-s)^{\Psi_1-1}\mathcal{G}_3(s, \mathbb{S}, \mathbb{V}, \mathbb{E}, \mathbb{I}, \mathbb{R})ds, \\ \mathbb{I}(t) - \mathbb{I}(0) = \dfrac{\Phi_1(1-\Psi_1)t^{\Phi_1-1}}{W(\Psi_1)}\mathcal{G}_4(s, \mathbb{S}, \mathbb{V}, \mathbb{E}, \mathbb{I}, \mathbb{R}) + \dfrac{\Phi_1\Psi_1}{W(\Psi_1)}\displaystyle\int_0^t s^{\Phi_1-1}(t-s)^{\Psi_1-1}\mathcal{G}_3(s, \mathbb{S}, \mathbb{V}, \mathbb{E}, \mathbb{I}, \mathbb{R})ds, \\ \mathbb{R}(t) - \mathbb{R}(0) = \dfrac{\Phi_1(1-\Psi_1)t^{\Phi_1-1}}{W(\Psi_1)}\mathcal{G}_5(s, \mathbb{S}, \mathbb{V}, \mathbb{E}, \mathbb{I}, \mathbb{R}) + \dfrac{\Phi_1\Psi_1}{W(\Psi_1)}\displaystyle\int_0^t s^{\Phi_1-1}(t-s)^{\Psi_1-1}\mathcal{G}_4(s, \mathbb{S}, \mathbb{V}, \mathbb{E}, \mathbb{I}, \mathbb{R})ds. \end{cases} \tag{58}$$

Set $t = t_{m+1}$ for $m = 0, 1, 2, ...$, it follows that

$$\begin{cases} \mathbb{S}(t_{m+1}) - \mathbb{S}(0) = \dfrac{\Phi_1(1-\Psi_1)t_m^{\Phi_1-1}}{W(\Psi_1)}\mathcal{G}_1(t_m, \mathbb{S}, \mathbb{V}, \mathbb{E}, \mathbb{I}, \mathbb{R}) \\ \qquad + \dfrac{\Phi_1\Psi_1}{W(\Psi_1)}\displaystyle\sum_{n=1}^{m}\int_{t_n}^{t_{n+1}} s^{\Phi_1-1}(t_{m+1}-s)^{\Psi_1-1}\mathcal{G}_1(s, \mathbb{S}, \mathbb{V}, \mathbb{E}, \mathbb{I}, \mathbb{R})ds, \\ \mathbb{V}(t_{m+1}) - \mathbb{V}(0) = \dfrac{\Phi_1(1-\Psi_1)t_m^{\Phi_1-1}}{W(\Psi_1)}\mathcal{G}_2(t_m, \mathbb{S}, \mathbb{V}, \mathbb{E}, \mathbb{I}, \mathbb{R}) \\ \qquad + \dfrac{\Phi_1\Psi_1}{W(\Psi_1)}\displaystyle\sum_{n=1}^{m}\int_{t_n}^{t_{n+1}} s^{\Phi_1-1}(t_{m+1}-s)^{\Psi_1-1}\mathcal{G}_2(s, \mathbb{S}, \mathbb{V}, \mathbb{E}, \mathbb{I}, \mathbb{R})ds, \\ \mathbb{E}(t_{m+1}) - \mathbb{E}(0) = \dfrac{\Phi_1(1-\Psi_1)t_m^{\Phi_1-1}}{W(\Psi_1)}\mathcal{G}_3(t_m, \mathbb{S}, \mathbb{V}, \mathbb{E}, \mathbb{I}, \mathbb{R}) \\ \qquad + \dfrac{\Phi_1\Psi_1}{W(\Psi_1)}\displaystyle\sum_{n=1}^{m}\int_{t_n}^{t_{n+1}} s^{\Phi_1-1}(t_{m+1}-s)^{\Psi_1-1}\mathcal{G}_3(s, \mathbb{S}, \mathbb{V}, \mathbb{E}, \mathbb{I}, \mathbb{R})ds, \\ \mathbb{I}(t_{m+1}) - \mathbb{I}(0) = \dfrac{\Phi_1(1-\Psi_1)t_m^{\Phi_1-1}}{W(\Psi_1)}\mathcal{G}_4(t_m, \mathbb{S}, \mathbb{V}, \mathbb{E}, \mathbb{I}, \mathbb{R}) \\ \qquad + \dfrac{\Phi_1\Psi_1}{W(\Psi_1)}\displaystyle\sum_{n=1}^{m}\int_{t_n}^{t_{n+1}} s^{\Phi_1-1}(t_{m+1}-s)^{\Psi_1-1}\mathcal{G}_5(s, \mathbb{S}, \mathbb{V}, \mathbb{E}, \mathbb{I}, \mathbb{R})ds, \\ \mathbb{R}(t_{m+1}) - \mathbb{R}(0) = \dfrac{\Phi_1(1-\Psi_1)t_m^{\Phi_1-1}}{W(\Psi_1)}\mathcal{G}_5(t_m, \mathbb{S}, \mathbb{V}, \mathbb{E}, \mathbb{I}, \mathbb{R}) \\ \qquad + \dfrac{\Phi_1\Psi_1}{W(\Psi_1)}\displaystyle\sum_{n=1}^{m}\int_{t_n}^{t_{n+1}} s^{\Phi_1-1}(t_{m+1}-s)^{\Psi_1-1}\mathcal{G}_5(s, \mathbb{S}, \mathbb{V}, \mathbb{E}, \mathbb{I}, \mathbb{R})ds. \end{cases} \tag{59}$$

We will utilize an interpolation polynomial to estimate the functions $s^{\Phi_1-1}\mathbb{A}(t_m,\mathbb{S},\mathbb{V},\mathbb{E},\mathbb{I},\mathbb{R})$, with h representing the difference $t_{n+1}-t_n$. The interpolation polynomial is formulated as follows:

$$x_i(s) := s^{\Phi_1-1}\mathbb{A}_i(s,\mathbb{S},\mathbb{V},\mathbb{E},\mathbb{I},\mathbb{R})$$
$$\begin{cases} \left[\dfrac{(t-t_{n-1})}{h} t_n^{\Phi_1-1}\mathbb{A}_i\left(t_n,\mathbb{S}(t_n),\mathbb{V}(t_n),\mathbb{E}(t_n),\mathbb{I}(t_n),\mathbb{R}(t_n)\right)\right. \\ \left. -\dfrac{(t-t_n)}{h} t_{n-1}^{\Phi_1-1}\mathbb{A}_i\left(t_{n-1},\mathbb{S}(t_{n-1}),\mathbb{V}(t_{n-1}),\mathbb{E}(t_{n-1}),\mathbb{I}(t_{n-1}),\mathbb{R}(t_{n-1})\right)\right]. \end{cases}$$

Substituting the final equation into (59), we can express (59) as follows:

$$\begin{cases} \mathbb{S}(t_{m+1})-\mathbb{S}(0)=\frac{\Phi_1(1-\Psi_1)t_m^{\Phi_1-1}}{W(\Psi_1)}\mathcal{G}_1\left(t_m,\mathbb{S},\mathbb{V},\mathbb{E},\mathbb{I},\mathbb{R}\right)+\frac{\Phi_1\Psi_1}{W(\Psi_1)}\sum_{n=1}^{m}\int_{t_n}^{t_{n+1}}\left(t_{m+1}-s\right)^{\Psi_1-1}x_1(s)ds, \\ \mathbb{V}(t_{m+1})-\mathbb{V}(0)=\frac{\Phi_1(1-\Psi_1)t_m^{\Phi_1-1}}{W(\Psi_1)}\mathcal{G}_2\left(t_m,\mathbb{S},\mathbb{V},\mathbb{E},\mathbb{I},\mathbb{R}\right)+\frac{\Phi_1\Psi_1}{W(\Psi_1)}\sum_{n=1}^{m}\int_{t_n}^{t_{n+1}}\left(t_{m+1}-s\right)^{\Psi_1-1}x_2(s)ds, \\ \mathbb{E}(t_{m+1})-\mathbb{E}(0)=\frac{\Phi_1(1-\Psi_1)t_m^{\Phi_1-1}}{W(\Psi_1)}\mathcal{G}_3\left(t_m,\mathbb{S},\mathbb{V},\mathbb{E},\mathbb{I},\mathbb{R}\right)+\frac{\Phi_1\Psi_1}{W(\Psi_1)}\sum_{n=1}^{m}\int_{t_n}^{t_{n+1}}\left(t_{m+1}-s\right)^{\Psi_1-1}x_3(s)ds, \\ \mathbb{I}(t_{m+1})-\mathbb{I}(0)=\frac{\Phi_1(1-\Psi_1)t_m^{\Phi_1-1}}{W(\Psi_1)}\mathcal{G}_4\left(t_m,\mathbb{S},\mathbb{V},\mathbb{E},\mathbb{I},\mathbb{R}\right)+\frac{\Phi_1\Psi_1}{W(\Psi_1)}\sum_{n=1}^{m}\int_{t_n}^{t_{n+1}}\left(t_{m+1}-s\right)^{\Psi_1-1}x_4(s)ds, \\ \mathbb{R}(t_{m+1})-\mathbb{R}(0)=\frac{\Phi_1(1-\Psi_1)t_m^{\Phi_1-1}}{W(\Psi_1)}\mathcal{G}_5\left(t_m,\mathbb{S},\mathbb{V},\mathbb{E},\mathbb{I},\mathbb{R}\right)+\frac{\Phi_1\Psi_1}{W(\Psi_1)}\sum_{n=1}^{m}\int_{t_n}^{t_{n+1}}\left(t_{m+1}-s\right)^{\Psi_1-1}x_5(s)ds. \end{cases} \tag{60}$$

By the functions $x_i(s)$, (60) becomes

$$\mathbb{S}\left(t_{m+1}\right)=\begin{cases} \mathbb{S}(0)+\frac{\Phi_1(1-\Psi_1)t_m^{\Phi_1-1}}{W(\Psi_1)}\mathcal{G}_1\left(t_m,\mathbb{S},\mathbb{V},\mathbb{E},\mathbb{I},\mathbb{R}\right) \\ +\frac{\Phi_1\Psi_1}{W(\Psi_1)}\sum_{n=1}^{m}\left[\frac{t_n^{\Phi_1-1}\mathcal{G}_1(t_n,\mathbb{S}(t_n),\mathbb{V}(t_n),\mathbb{E}(t_n),\mathbb{I}(t_n),\mathbb{R}(t_n))}{h}\mathbb{N}_1\right. \\ \left.-\frac{t_{n-1}^{\Phi_1-1}\mathbb{A}_1(t_{n-1},\mathbb{S}(t_{n-1}),\mathbb{V}(t_{n-1}),\mathbb{E}(t_{n-1}),\mathbb{I}(t_{n-1}),\mathbb{R}(t_{n-1}))}{h}\mathbb{N}_2\right]. \end{cases} \tag{61}$$

$$\mathbb{V}\left(t_{m+1}\right)=\begin{cases} \mathbb{V}(0)+\frac{\Phi_1(1-\Psi_1)t_m^{\Phi_1-1}}{W(\Psi_1)}\mathcal{G}_2\left(t_m,\mathbb{S},\mathbb{V},\mathbb{E},\mathbb{I},\mathbb{R}\right) \\ +\frac{\Phi_1\Psi_1}{W(\Psi_1)}\sum_{n=1}^{m}\left[\frac{t_n^{\Phi_1-1}\mathcal{G}_2(t_n,\mathbb{S}(t_n),\mathbb{V}(t_n),\mathbb{E}(t_n),\mathbb{I}(t_n),\mathbb{R}(t_n))}{h}\mathbb{N}_1\right. \\ \left.-\frac{t_{n-1}^{\Phi_1-1}\mathbb{A}_2(t_{n-1},\mathbb{S}(t_{n-1}),\mathbb{V}(t_{n-1}),\mathbb{E}(t_{n-1}),\mathbb{I}(t_{n-1}),\mathbb{R}(t_{n-1}))}{h}\mathbb{N}_2\right]. \end{cases} \tag{62}$$

$$\mathbb{E}\left(t_{m+1}\right)=\begin{cases} \mathbb{E}(0)+\frac{\Phi_1(1-\Psi_1)t_m^{\Phi_1-1}}{W(\Psi_1)}\mathcal{G}_3\left(t_m,\mathbb{S},\mathbb{V},\mathbb{E},\mathbb{I},\mathbb{R}\right) \\ +\frac{\Phi_1\Psi_1}{W(\Psi_1)}\sum_{n=1}^{m}\left[\frac{t_n^{\Phi_1-1}\mathbb{A}_3(t_n,\mathbb{S}(t_n),\mathbb{V}(t_n),\mathbb{E}(t_n),\mathbb{I}(t_n),\mathbb{R}(t_n))}{h}\mathbb{N}_1\right. \\ \left.-\frac{t_{n-1}^{\Phi_1-1}\mathbb{A}_3(t_{n-1},\mathbb{S}(t_{n-1}),\mathbb{V}(t_{n-1}),\mathbb{E}(t_{n-1}),\mathbb{I}(t_{n-1}),\mathbb{R}(t_{n-1}))}{h}\mathbb{N}_2\right]. \end{cases} \tag{63}$$

$$\mathbb{I}\left(t_{m+1}\right)=\begin{cases} \mathbb{I}(0)+\frac{\Phi_1(1-\Psi_1)t_m^{\Phi_1-1}}{W(\Psi_1)}\mathcal{G}_5\left(t_m,\mathbb{S},\mathbb{V},\mathbb{E},\mathbb{I},\mathbb{R}\right) \\ +\frac{\Phi_1\Psi_1}{W(\Psi_1)}\sum_{n=1}^{m}\left[\frac{t_n^{\Phi_1-1}\mathcal{G}_5(t_n,\mathbb{S}(t_n),\mathbb{V}(t_n),\mathbb{E}(t_n),\mathbb{I}(t_n),\mathbb{R}(t_n))}{h}\mathbb{N}_1\right. \\ \left.-\frac{t_{n-1}^{\Phi_1-1}\mathbb{A}_4(t_{n-1},\mathbb{S}(t_{n-1}),\mathbb{V}(t_{n-1}),\mathbb{E}(t_{n-1}),\mathbb{I}(t_{n-1}),\mathbb{R}(t_{n-1}))}{h}\mathbb{N}_2\right]. \end{cases} \tag{64}$$

$$\mathbb{R}\left(t_{m+1}\right)=\begin{cases}\mathbb{R}(0)+\frac{\Phi_1(1-\Psi_1)t_m^{\Phi_1-1}}{W(\Psi_1)}\mathcal{G}_5\left(t_m,\mathbb{S},\mathbb{V},\mathbb{E},\mathbb{I},\mathbb{R}\right)\\ +\frac{\Phi_1\Psi_1}{W(\Psi_1)}\sum_{n=1}^{m}\left[\frac{t_n^{\Phi_1-1}\mathcal{G}_5(t_n,\mathbb{S}(t_n),\mathbb{V}(t_n),\mathbb{E}(t_n),\mathbb{I}(t_n),\mathbb{R}(t_n))}{h}\mathbb{N}_1\right.\\ \left.-\frac{t_{n-1}^{\Phi_1-1}\mathbb{A}_4(t_{n-1},\mathbb{S}(t_{n-1}),\mathbb{V}(t_{n-1}),\mathbb{E}(t_{n-1}),\mathbb{I}(t_{n-1}),\mathbb{R}(t_{n-1}))}{h}\mathbb{N}_2\right].\end{cases} \tag{65}$$

Where

$$\mathbb{N}_1=\int_{t_n}^{t_{n+1}}(t_{m+1}-s)^{\Psi_1-1}(t-t_{n-1})ds,$$

and

$$\mathbb{N}_2=\int_{t_n}^{t_{n+1}}(t_{m+1}-s)^{\Psi_1-1}(t-t_n)ds.$$

Easy calculation yields the following result:

$$\begin{aligned}\mathbb{N}_1=&-\frac{1}{p}\left[(t_{n+1}-t_{n-1})(t_{m+1}-t_{n+1})_1^{\Psi}-(t_n-t_{n-1})(t_{m+1}-t_n)_1^{\Psi}\right]\\ &-\frac{1}{\Psi_1(\Psi_1+1)}\left[(t_{m+1}-t_{n+1})^{\Psi_1+1}-(t_{m+1}-t_n)^{\Psi_1+1}\right],\end{aligned}$$

and

$$\begin{aligned}\mathbb{N}_2=&-\frac{1}{p}\left[(t_{n+1}-t_n)(t_{m+1}-t_{n+1})_1^{\Psi}\right]\\ &-\frac{1}{\Psi_1(\Psi_1+1)}\left[(t_{m+1}-t_{n+1})^{\Psi_1+1}-(t_{m+1}-t_n)^{\Psi_1+1}\right].\end{aligned}$$

Put $t_n=nh$, we get

$$\mathbb{N}_1=\frac{h^{\Phi_1+1}}{\Phi_1(\Phi_1+1)}\left[(m+1-n)(m-n+2+\Phi_1)-(m-n)_1^{\Phi}(m-n+2+2\Phi_1)\right], \tag{66}$$

and

$$\mathbb{N}_2=\frac{h^{\Psi_1+1}}{\Psi_1(\Psi_1+1)}\left[(m+1-n)^{\Psi_1+1}-(m-n)_1^{\Psi}(m-n+1+\Psi_1)\right]. \tag{67}$$

Substituting (66) and (67) into equations (61-64), we get

$$\mathbb{S}(t_{m+1})=\begin{cases}\mathbb{S}(0)+\frac{\Phi_1(1-\Psi_1)t_m}{W(\Psi_1)}\mathcal{G}_1\left(t_m,\mathbb{S},\mathbb{V},\mathbb{E},\mathbb{I},\mathbb{R}\right)\\ +\frac{\Phi_1\Psi_1}{W(\Psi_1)}\sum_{n=1}^{m}\left[t_n^{\Phi_1-1}\mathcal{G}_1\left(t_n,\mathbb{S}(t_n),\mathbb{V}(t_n),\mathbb{E}(t_n),\mathbb{I}(t_n),\mathbb{R}(t_n)\right)\right.\\ \frac{h_1^{\Psi}}{\Psi_1(\Psi_1+1)}\left[(m+1-n)_1^{\Psi}(m-n+2+\Psi_1)-(m-n)_1^{\Psi}(m-n+2+2\Psi_1)\right]\\ -t_{n-1}^{\Phi_1-1}\mathbb{A}_1\left(t_{n-1},\mathbb{S}(t_{n-1}),\mathbb{V}(t_{n-1}),\mathbb{E}(t_{n-1}),\mathbb{I}(t_{n-1}),\mathbb{R}(t_{n-1})\right)\\ \left.\frac{h_1^{\Psi}}{\Psi_1(\Psi_1+1)}\left[(m+1-n)^{\Psi_1+1}-(m-n)^{\Psi_1}(m-n+1+\Psi_1)\right]\right].\end{cases}$$

$$\mathbb{V}(t_{m+1}) = \begin{cases} \mathbb{V}(0) + \frac{\Phi_1(1-\Psi_1)t_m}{W(\Psi_1)}\mathcal{G}_2\left(t_m, \mathbb{S}, \mathbb{V}, \mathbb{E}, \mathbb{I}, \mathbb{R}\right) \\ +\frac{\Phi_1\Psi_1}{W(\Psi_1)}\sum_{n=1}^{m}\left[t_n^{\Phi_1-1}\mathcal{G}_2\left(t_n, \mathbb{S}(t_n), \mathbb{V}(t_n), \mathbb{E}(t_n), \mathbb{I}(t_n), \mathbb{R}(t_n)\right)\right. \\ \frac{h_1^{\Psi}}{\Psi_1(\Psi_1+1)}\left[(m+1-n)_1^{\Psi}(m-n+2+\Psi_1)-(m-n)_1^{\Psi}(m-n+2+2\Psi_1)\right] \\ -t_{n-1}^{\Phi_1-1}\mathbb{A}_2\left(t_{n-1}, \mathbb{S}(t_{n-1}), \mathbb{V}(t_{n-1}), \mathbb{E}(t_{n-1}), \mathbb{I}(t_{n-1}), \mathbb{R}(t_{n-1})\right) \\ \left.\frac{h_1^{\Psi}}{\Psi_1(\Psi_1+1)}\left[(m+1-n)^{\Psi_1+1}-(m-n)^{\Psi_1}(m-n+1+\Psi_1)\right]\right]. \end{cases}$$

$$\mathbb{E}(t_{m+1}) = \begin{cases} \mathbb{E}(0) + \frac{\Phi_1(1-\Psi_1)t_m}{W(\Psi_1)}\mathcal{G}_3\left(t_m, \mathbb{S}, \mathbb{V}, \mathbb{E}, \mathbb{I}, \mathbb{R}\right) \\ +\frac{\Phi_1\Psi_1}{W(\Psi_1)}\sum_{n=1}^{m}\left[t_n^{\Phi_1-1}\mathcal{G}_3\left(t_n, \mathbb{S}(t_n), \mathbb{V}(t_n), \mathbb{E}(t_n), \mathbb{I}(t_n), \mathbb{R}(t_n)\right)\right. \\ \frac{h_1^{\Psi}}{\Psi_1(\Psi_1+1)}\left[(m+1-n)_1^{\Psi}(m-n+2+\Psi_1)-(m-n)_1^{\Psi}(m-n+2+2\Psi_1)\right] \\ -t_{n-1}^{\Phi_1-1}\mathbb{A}_3\left(t_{n-1}, \mathbb{S}(t_{n-1}), \mathbb{V}(t_{n-1}), \mathbb{E}(t_{n-1}), \mathbb{I}(t_{n-1}), \mathbb{R}(t_{n-1})\right) \\ \left.\frac{h_1^{\Psi}}{\Psi_1(\Psi_1+1)}\left[(m+1-n)^{\Psi_1+1}-(m-n)^{\Psi_1}(m-n+1+\Psi_1)\right]\right]. \end{cases}$$

$$\mathbb{I}(t_{m+1}) = \begin{cases} \mathbb{I}(0) + \frac{\Phi_1(1-\Psi_1)t_m}{W(\Psi_1)}\mathcal{G}_3\left(t_m, \mathbb{S}, \mathbb{V}, \mathbb{E}, \mathbb{I}, \mathbb{R}\right) \\ +\frac{\Phi_1\Psi_1}{W(\Psi_1)}\sum_{n=1}^{m}\left[t_n^{\Phi_1-1}\mathcal{G}_3\left(t_n, \mathbb{S}(t_n), \mathbb{V}(t_n), \mathbb{E}(t_n), \mathbb{I}(t_n), \mathbb{R}(t_n)\right)\right. \\ \frac{h_1^{\Psi}}{\Psi_1(\Psi_1+1)}\left[(m+1-n)_1^{\Psi}(m-n+2+\Psi_1)-(m-n)_1^{\Psi}(m-n+2+2\Psi_1)\right] \\ -t_{n-1}^{\Phi_1-1}\mathbb{A}_4\left(t_{n-1}, \mathbb{S}(t_{n-1}), \mathbb{V}(t_{n-1}), \mathbb{E}(t_{n-1}), \mathbb{I}(t_{n-1}), \mathbb{R}(t_{n-1})\right) \\ \left.\frac{h_1^{\Psi}}{\Psi_1(\Psi_1+1)}\left[(m+1-n)^{\Psi_1+1}-(m-n)^{\Psi_1}(m-n+1+\Psi_1)\right]\right]. \end{cases}$$

$$\mathbb{R}(t_{m+1}) = \begin{cases} \mathbb{R}(0) + \frac{\Phi_1(1-\Psi_1)t_m}{W(\Psi_1)}\mathcal{G}_3\left(t_m, \mathbb{S}, \mathbb{V}, \mathbb{E}, \mathbb{I}, \mathbb{R}\right) \\ +\frac{\Phi_1\Psi_1}{W(\Psi_1)}\sum_{n=1}^{m}\left[t_n^{\Phi_1-1}\mathcal{G}_3\left(t_n, \mathbb{S}(t_n), \mathbb{V}(t_n), \mathbb{E}(t_n), \mathbb{I}(t_n), \mathbb{R}(t_n)\right)\right. \\ \frac{h_1^{\Psi}}{\Psi_1(\Psi_1+1)}\left[(m+1-n)_1^{\Psi}(m-n+2+\Psi_1)-(m-n)_1^{\Psi}(m-n+2+2\Psi_1)\right] \\ -t_{n-1}^{\Phi_1-1}\mathbb{A}_5\left(t_{n-1}, \mathbb{S}(t_{n-1}), \mathbb{V}(t_{n-1}), \mathbb{E}(t_{n-1}), \mathbb{I}(t_{n-1}), \mathbb{R}(t_{n-1})\right) \\ \left.\frac{h_1^{\Psi}}{\Psi_1(\Psi_1+1)}\left[(m+1-n)^{\Psi_1+1}-(m-n)^{\Psi_1}(m-n+1+\Psi_1)\right]\right]. \end{cases}$$

6.2. Graphical Results and Discussion

Understanding non-integer orders is crucial for analyzing the dynamic behavior of real-world systems. Recently, fractional-order modeling of non-linear systems has gained attention, typically using orders between 0.5 and 0.99. This range is chosen due to the inherent characteristics of many physical systems, where the fractional order significantly impacts system behavior. Combining fractal and fractional orders results in a fractal dimension that correlates with the fractional order, depending on the specific system being modeled. In this study, simulations were conducted using the Adams-Moulton method over 300 days with a step size of 10^{-2}. This new mathematical model differs from a prior study[34] that used an FF order derivative in an ABC sense. Comparing the classical model with the fractional derivative revealed the latter's superiority. Additionally, the model employing the fractal-fractional ABC operator exhibited improved crossover characteristics, setting it apart from the other two operators examined.

This section presents graphical results showing the dynamics of a fractal-fractional-order model (9), solved numerically as described in Section 6. Figures 3, 4, 5, 6, and 7 illustrate the evolution of different populations: susceptible ($\mathbb{S}$), vaccinated ($\mathbb{V}$), exposed ($\mathbb{E}$), infected ($\mathbb{I}$), and recovered ($\mathbb{R}$). Figure 3**(a)** shows the behavior of the susceptible population, influenced by fractional-order derivatives and a fractal dimension ranging from 0.60 to 0.95. Over time, the number of susceptible individuals decreases, consistent with other epidemiological models, due to no exposure to the virus. Figure 3**(b)** depicts the vaccinated population, showing a steady decline over 300 days as the fractional-order derivative approaches the classical value. This decline results from vaccinated individuals moving to the recovered class, with some becoming infected, indicating a reduced transmission risk early in the epidemic. Figure 3**(c)** shows the number of exposed individuals, which decreases as the fractional order and fractal dimension approach unity. The fractional order's sensitivity near unity causes this decline, as exposed and infectious individuals transition to the infected class within weeks. Figure 3**(d)** illustrates the infected population's dynamics, showing a similar decreasing trend as the fractional-order derivative and fractal dimension approach integer values. Figure 3**(e)** highlights the effect of the fractional order on the number of recovered individuals, increasing rapidly as the fractional-order derivative and fractal dimension approach classical values. This rise is due to vaccinated individuals joining the recovered class, with a higher fractional order accelerating this increase. The impact of parameters β_E and β_I on each state variable is also depicted in Figures (6) and (7).

7. CONCLUSION

This study presents a novel non-linear fractal fractional model to analyze the dynamics of influenza epidemics, marking a significant advancement in the field. Our findings emphasize the critical role of non-linear dynamics in understanding and controlling influenza spread. Utilizing the Fractal-Fractional order operator based on the Atangana-Baleanu approach, our model accurately represents the propagation of influenza strains. The model's memory response, validated through Fractal-Fractional derivatives, highlights its effectiveness in predictive analysis. We also provide an approximate solution scheme for the Fractional-Fractal model, along with a qualitative analysis of its solutions. The proposed fractal-fractional derivative operator demonstrates great potential for modeling complex dynamics in various applications. Theoretical aspects such as uniqueness and existence theorems, along with Ulam-Hyers stability, are thoroughly explored to analyze the Fractal-Fractional system. Graphical results illustrate the impact of non-integer and integer-order dimensions on model behavior, with some dimensions converging to 1 as time progresses. This comprehensive approach deepens our understanding of influenza dynamics and suggests improved control strategies. Overall, this research significantly enhances our knowledge of influenza epidemic behavior and opens new avenues for further study in this important area. Future studies can expand the current model to include additional factors influencing influenza spread, such as environmental variables, seasonal effects, and social behavior patterns. Incorporating these elements can provide a more comprehensive understanding of the dynamics of influenza transmission.

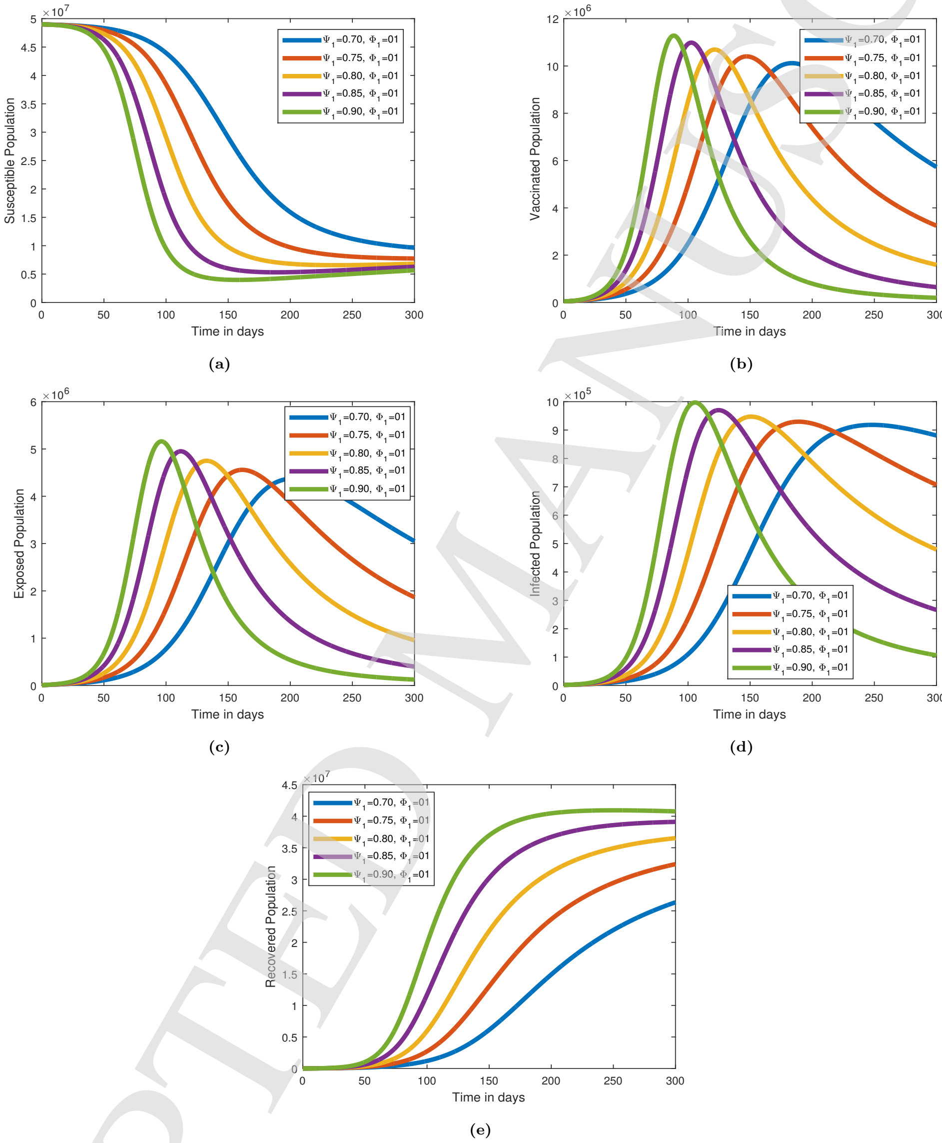


Fig. 3: Application of the Adams-Bashforth method to problem (9) generates numerical results for each state variable, illustrated here for the case where the fractal dimension $\Phi_1 = 1$.

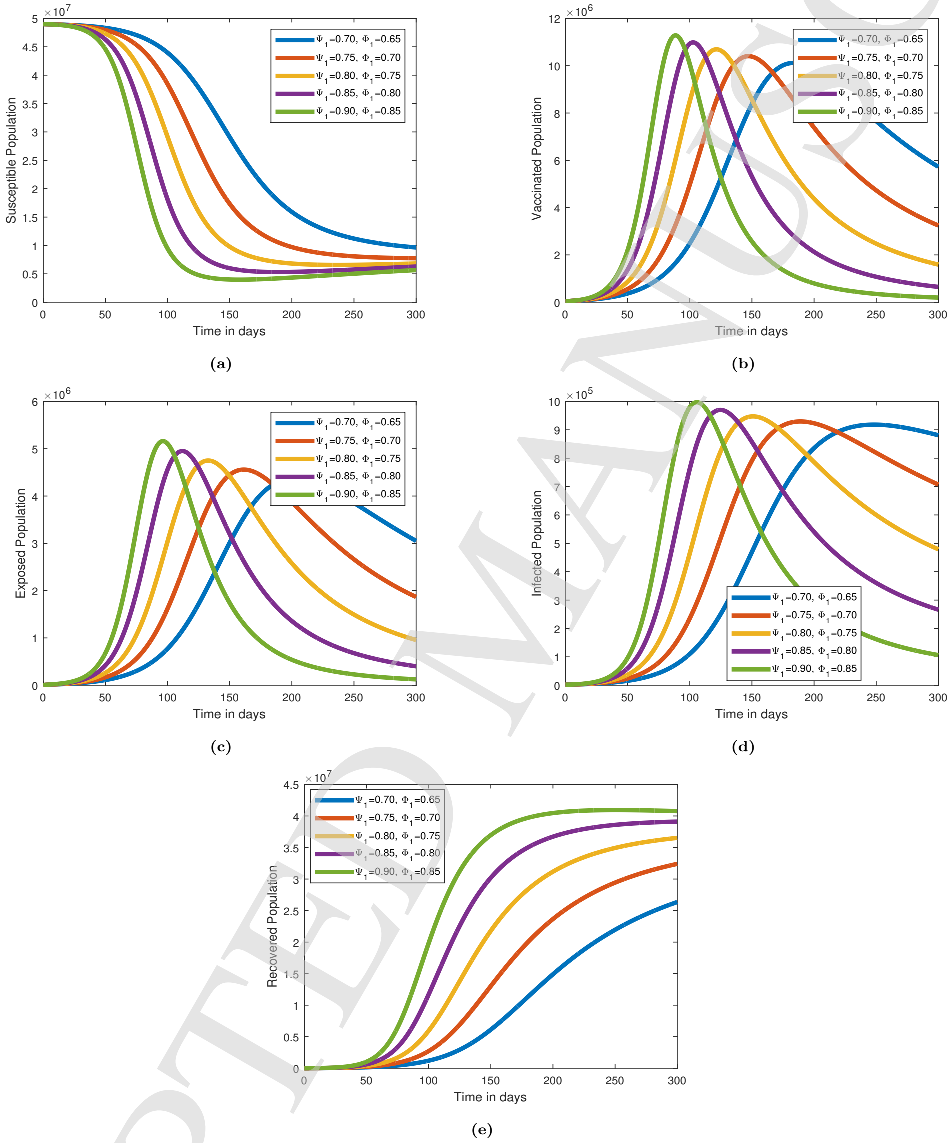


Fig. 4: Graphical representation of problem (9) utilizing the Adams-Bashforth method, highlighting the effects of different values of Φ_1 and Ψ_1 on the resulting curves.

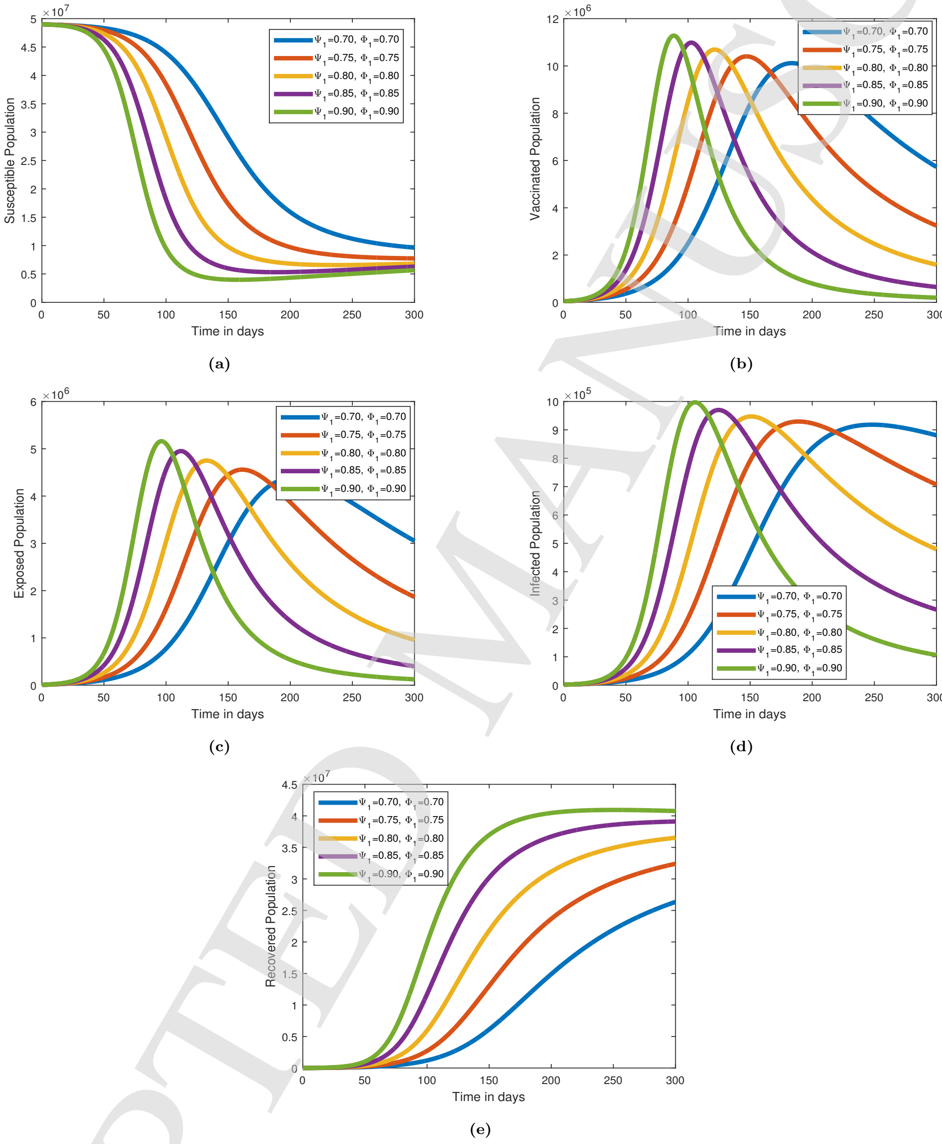


Fig. 5: Numerical solutions for each state variable in problem (9) were obtained using the Adams-Bashforth method with $\Phi_1 = \Psi_1$.

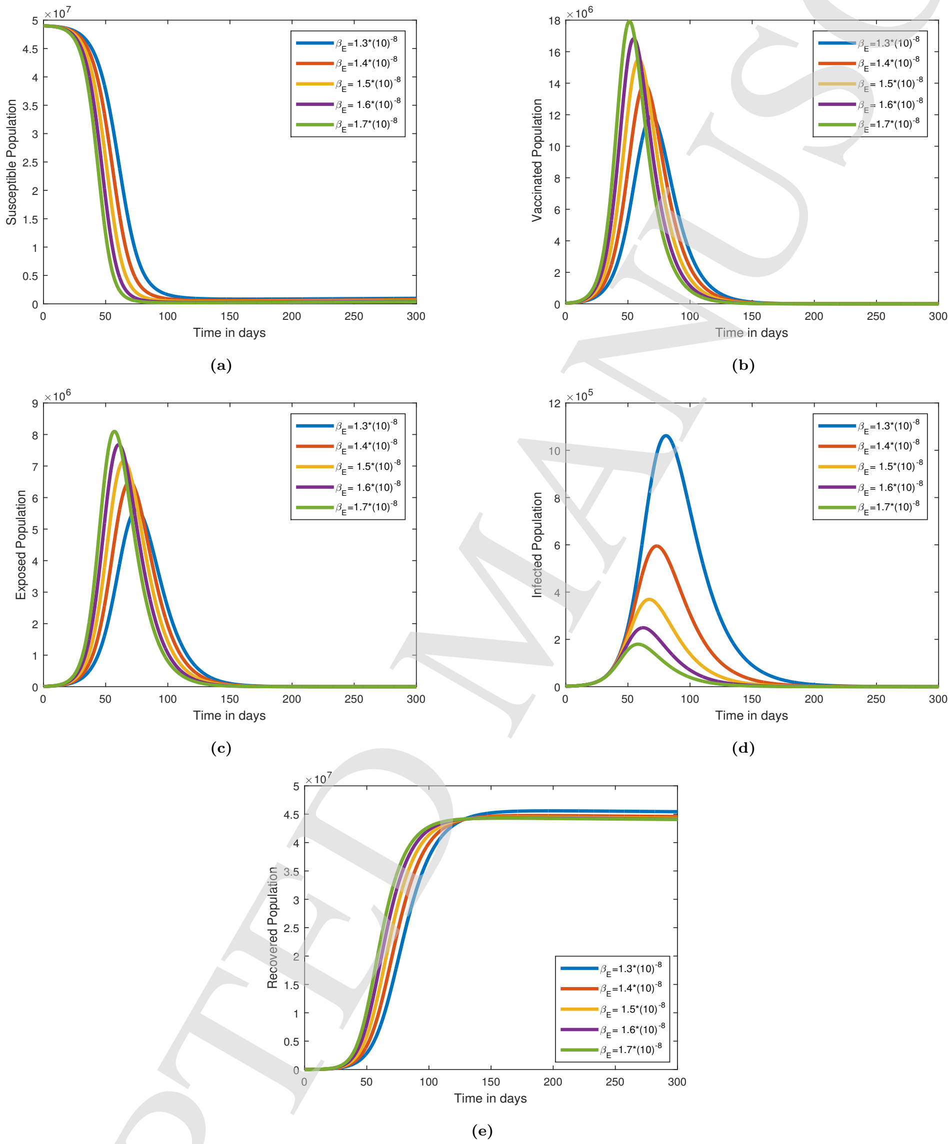


Fig. 6: Impact of changing the parameter β_E on individual state variables for Ψ_1 equal to Φ_1. The results are presented for each state variable: **(a)** - **(e)**.

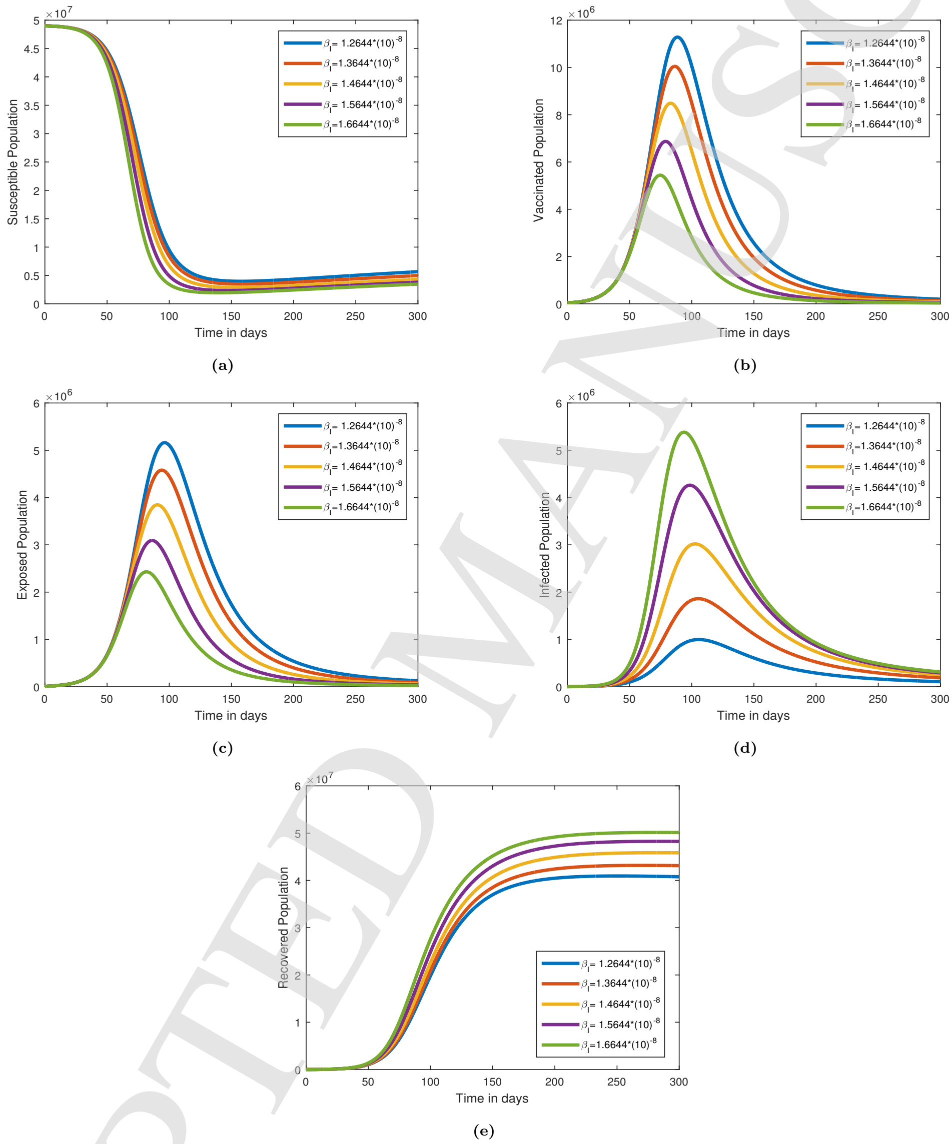


Fig. 7: Effect of parameter β_I on individual state variables within the considered problem (9), with a fixed fractional order $\Psi_1 = 1$ and fractal dimension $\Phi_1 = 1$.

AUTHOR CONTRIBUTIONS

All the authors have contributed equally to this paper. All the authors have read and agreed to the published version of the manuscript.

CONFLICTS OF INTEREST

The authors declare no conflict of interest.

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DATA AVAILABILITY STATEMENT

Data are contained within the article.

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