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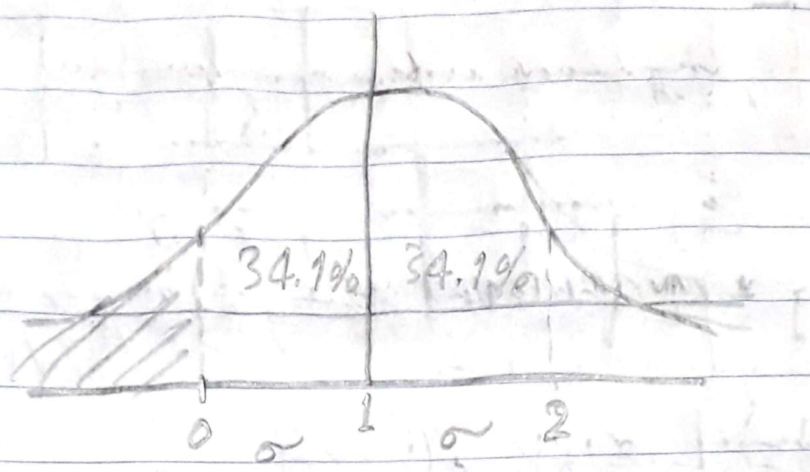
Task 1



1.B) $1 - CDF(0) = \underbrace{.5}_{\text{right}} + .341 \simeq .841$

$\therefore CDF(0) \simeq .159$ (Analytically)

It has the same percentages as standard normal because they both share the same value of <u>standard deviation</u> $\sigma = 1$

1.C) $P(x|z) = \dfrac{P(z|x)\,P(x)}{P(z)}$ ← Prior; ← Proportionality factor doesn't depend on x

$P(x|z) \propto P(z|x)\,P(x)$

Likelihood   Prior

*

1.D) Joint Probability $P(x, z)$

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Task 2:

2.A) $y = Ax + b$

$\Sigma_y = L \cdot L^T$

get L from Cholesky decomposition

$y(K) = (L \cdot x) * K + b \rightarrow \mu$

2.B) Sample mean: $\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$

for a single random variable X with n observations

$$\text{Sample Covariance} = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2$$

$\bar{x}$ is scalar

$\bar{x} = \bar{x}^T$

$\sum x_i = n\bar{x}$

$$= \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})(x_i - \bar{x})^T$$

$$= \frac{1}{n-1} \sum_{i=1}^{n} x_i x_i^T - \bar{x} x_i^T - x_i \bar{x}^T + \bar{x}\bar{x}^T$$

$$= \frac{1}{n-1} \left[\sum_{i=1}^{n} x_i^2 - \overbrace{2\bar{x} \sum_{i=1}^{n} x_i}^{2n\bar{x}^2} + n\bar{x}^2 \right]$$

$$= \frac{1}{n-1} \left[\sum_{i=1}^{n} x_i^2 - n\bar{x}^2 \right]$$

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Generalizing for K random variables where each x_i is a vector

X_i is a column vector of length K denoting the i^{th} observation of all K variables

$\bar{x}_j = \frac{1}{n} \sum_{i=1}^{n} x_{ij}$, $j = 1, 2, \dots K$, denotes j^{th} entry in $\bar{X}$

$\bar{X}$ is the sample mean vector

$\therefore \bar{X} = \frac{1}{n} \sum_{i=1}^{n} x_i = \begin{bmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_j \\ \vdots \\ \bar{x}_K \end{bmatrix}$

* Covariance Matrix: $\Sigma_{K \times K} = [\sigma_{rc}]$

$$\sigma_{rc} = \frac{1}{n-1} \sum_{i=1}^{n} (x_{ir} - \bar{x}_r)(x_{ic} - \bar{x}_c)$$

σ_{rc} is the covariance between r^{th} & c^{th} variables

x_r & x_c vectors of points
& $\bar{x}_r$ & $\bar{x}_c$ from $\bar{X}$ sample mean vector

Task 3:

3.A) $\begin{bmatrix} x \\ y \end{bmatrix}_t = I \begin{bmatrix} x \\ y \end{bmatrix}_{t-1} + \Delta t . I . \begin{bmatrix} v_x \\ v_y \end{bmatrix}_t + \begin{bmatrix} \eta_x \\ \eta_y \end{bmatrix}_t$

$\begin{bmatrix} \eta_x \\ \eta_y \end{bmatrix}_t \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix} \right)$ $\Delta t = .5$

$E\left\{ \begin{bmatrix} x \\ y \end{bmatrix}_t \right\} = E\left\{ I \begin{bmatrix} x \\ y \end{bmatrix}_{t-1} + \Delta t\, I \times \begin{bmatrix} v_x \\ v_y \end{bmatrix}_t + \begin{bmatrix} \eta_x \\ \eta_y \end{bmatrix}_t \right\}$

$= E\left\{ I \cdot \begin{bmatrix} x \\ y \end{bmatrix}_{t-1} \right\} + E\left\{ \Delta t\, I \begin{bmatrix} v_x \\ v_y \end{bmatrix}_t \right\} + \underbrace{E\left\{ \begin{bmatrix} \eta_x \\ \eta_y \end{bmatrix}_t \right\}}_{0}$

let $A = I_{2\times 2}$, $B = \Delta t\, I_{2 \times 2}$

$\therefore \begin{bmatrix} x \\ y \end{bmatrix}_t = A \begin{bmatrix} x \\ y \end{bmatrix}_{t-1} + B \overset{\text{fixed}}{u_t} + \begin{bmatrix} \eta_x \\ \eta_y \end{bmatrix}$

$\therefore E\left\{ \begin{bmatrix} x \\ y \end{bmatrix}_t \right\} = A E\left\{ \begin{bmatrix} x \\ y \end{bmatrix}_{t-1} \right\} + B E\{u_t\}$, $E\{u_t\} = u_t$ const.

$\therefore \boxed{\mu_t = A \mu_{t-1} + B u_t}$ (1)

$$\Sigma_t = E\left\{\left(\begin{bmatrix} x \\ y \end{bmatrix}_t - \mu_t\right)\left(\begin{bmatrix} x \\ y \end{bmatrix}_t - \mu_t\right)^T\right\}$$

$$= E\left\{\left[A\begin{bmatrix} x \\ y \end{bmatrix}_{t-1} + \cancel{Bu_t} + \begin{bmatrix} \eta_x \\ \eta_y \end{bmatrix}_t - A\mu_{t-1} - \cancel{Bu_t}\right] * \left[A\begin{bmatrix} x \\ y \end{bmatrix}_{t-1} + \cancel{Bu_t} + \begin{bmatrix} \eta_x \\ \eta_y \end{bmatrix}_t - A\mu_{t-1} - \cancel{Bu_t}\right]^T\right\}$$

$$= E\left\{\left[A\left(\begin{bmatrix} x \\ y \end{bmatrix}_{t-1} - \mu_{t-1}\right) + \begin{bmatrix} \eta_x \\ \eta_y \end{bmatrix}_t\right] \cdot \left[A\left(\begin{bmatrix} x \\ y \end{bmatrix}_{t-1} - \mu_{t-1}\right) + \begin{bmatrix} \eta_x \\ \eta_y \end{bmatrix}_t\right]^T\right\}$$

$$= E\left\{A\left(\begin{bmatrix} x \\ y \end{bmatrix}_{t-1} - \mu_{t-1}\right)\left(\begin{bmatrix} x \\ y \end{bmatrix}_{t-1} - \mu_{t-1}\right)^T A^T\right\} + E\left\{\begin{bmatrix} \eta_x \\ \eta_y \end{bmatrix}\begin{bmatrix} \eta_x \\ \eta_y \end{bmatrix}^T\right\}$$

$\begin{bmatrix} x \\ y \end{bmatrix}$ & η uncorrelated, $E\left\{\begin{bmatrix} \eta_x \\ \eta_y \end{bmatrix}\right\} = 0$, so other two terms $E = 0$

$$= A\,E\left\{\left(\begin{bmatrix} x \\ y \end{bmatrix}_{t-1} - \mu_{t-1}\right)\left(\begin{bmatrix} x \\ y \end{bmatrix}_{t-1} - \mu_{t-1}\right)^T\right\} A^T + E\{\eta \cdot \eta^T\}$$

$$\boxed{\Sigma_t = A\,\Sigma_{t-1}\,A^T + R} \quad (2) \qquad , R = \Sigma_{\eta_t} = \begin{bmatrix} .1 & 0 \\ 0 & .1 \end{bmatrix} = .1 * I_{2\times 2}$$

$$\therefore \boxed{\begin{aligned} \mu_t &= A\mu_{t-1} + Bu_t \\ \Sigma_t &= A\Sigma_{t-1}A^T + R \end{aligned}}$$

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3.D)

$$\begin{bmatrix}x\\y\\\theta\end{bmatrix}_t = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}x\\y\\\theta\end{bmatrix}_{t-1} + \begin{bmatrix}\cos(\theta)\Delta t & 0\\ \sin(\theta)\Delta t & 0\\ 0 & \Delta t\end{bmatrix}_t \begin{bmatrix}v\\ \omega\end{bmatrix}_t + \begin{bmatrix}\eta_x\\\eta_y\\\eta_\theta\end{bmatrix}$$

$$, \begin{bmatrix}\eta_x\\\eta_y\\\eta_\theta\end{bmatrix} \sim N\left(\begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}0.2&0&0\\0&0.2&0\\0&0&0.5\end{bmatrix}\right), \quad \begin{bmatrix}x\\y\\\theta\end{bmatrix}_0 \sim N\left(\begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}0.1&0&0\\0&0.1&0\\0&0&0.05\end{bmatrix}\right)$$

$$\therefore A = I, \quad B = \begin{bmatrix}\cos(\theta)\Delta t & 0\\ \sin(\theta)\Delta t & 0\\ 0 & \Delta t\end{bmatrix}_t \quad \text{non-linear}$$

$$R = \begin{bmatrix}.2&0&0\\0&.2&0\\0&0&.5\end{bmatrix}$$

following the lecture

* Linearization: $x_t = g(x_{t-1}, \mu_t) \simeq g(\mu_{t-1}, \mu_t) + G_t\overbrace{(x_{t-1} - \mu_{t-1})}^{\Delta x}$

$$G_t = \left.\frac{\partial g(x_{t-1}, \mu_t)}{\partial x_{t-1}}\right|_{\mu_{t-1}} = \begin{array}{c}\begin{bmatrix}1 & 0 & -\sin(\theta_{t-1})\Delta t\, v_t\\ 0 & 1 & \cos(\theta_{t-1})\Delta t\, v_t\\ 0 & 0 & 1\end{bmatrix}\\ \begin{array}{ccc}\partial/\partial x & \partial/\partial y & \partial/\partial\theta\end{array}\end{array}$$

* Covariance Propagation:

$$\therefore x_t \sim N(g(\mu_{t-1}, \mu_t), G_t \Sigma_t G_t^T + R)$$

$$E\left\{\begin{bmatrix}x\\y\\\theta\end{bmatrix}_t\right\} = \underbrace{E\left\{\begin{bmatrix}x\\y\\\theta\end{bmatrix}_{t-1}\right\}}_{\mu_{t-1}} + E\left\{B_t\cdot\begin{bmatrix}v\\\omega\end{bmatrix}\right\} + \underbrace{E\left\{\begin{bmatrix}\eta_x\\\eta_y\\\eta_\theta\end{bmatrix}\right\}}_{0}$$

$$\mu_t = \mu_{t-1} + E\left\{B_t\cdot\begin{bmatrix}v\\\omega\end{bmatrix}\right\}$$

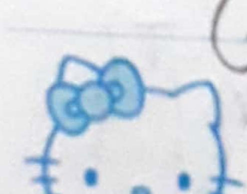
$$\boxed{\mu_t = \mu_{t-1} + B(\mu_{t-1})\begin{bmatrix}v\\\omega\end{bmatrix}} \qquad A = I_{3\times3}$$

$$\Sigma_t = G_{t-1}\Sigma_{t-1}G_{t-1}^T + R$$

3.E) $$\begin{bmatrix}x\\y\\\theta\end{bmatrix}_t = \begin{bmatrix}x\\y\\\theta\end{bmatrix}_{t-1} + \begin{bmatrix}\cos(\theta)\Delta t & 0\\ \sin(\theta)\Delta t & 0\\ 0 & \Delta t\end{bmatrix}_t \begin{bmatrix}v+\eta_v\\ \omega+\eta_\omega\end{bmatrix}_t, \quad \begin{bmatrix}\eta_v\\ \eta_\omega\end{bmatrix} \sim N\left(\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}2 & 0\\ 0 & 0.1\end{bmatrix}\right)$$

$$= \begin{bmatrix}x\\y\\\theta\end{bmatrix} + \begin{bmatrix}\cos(\theta)\,\Delta t\,(v+\eta_v)\\ \sin(\theta)\,\Delta t\,(v+\eta_v)\\ \Delta t\,(\omega+\eta_\omega)\end{bmatrix}_t$$

$$G_t = \begin{array}{c} \begin{bmatrix}1 & 0 & -\sin(\theta)\Delta t(v+\eta_v)\\ 0 & 1 & \cos(\theta)\Delta t(v+\eta_v)\\ 0 & 0 & 1\end{bmatrix}_t \\ \begin{array}{ccc}\partial/\partial x & \partial/\partial y & \partial/\partial\theta\end{array}\end{array}$$



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$$= Cov(x_{t-1}) + B(\mu_{t-1}) E\{\eta\eta^T\} B^T(\mu_{t-1})$$

$$\boxed{\Sigma_t = G \Sigma_{t-1} G^T + B(\mu_{t-1}) R B^T(\mu_{t-1})}$$