

Dynamic Programming

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January 12, 2015

Outline

Dynamic Programming

1-dimensional DP

2-dimensional DP

Interval DP

Tree DP

Subset DP

What is DP?

- ▶ Wikipedia definition: “method for solving complex problems by breaking them down into simpler subproblems”
- ▶ This definition will make sense once we see some examples
 - Actually, we'll only see problem solving examples today

Steps for Solving DP Problems

1. Define subproblems
2. Write down the recurrence that relates subproblems
3. Recognize and solve the base cases

► Each step is very important!

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1-dimensional DP Example

- ▶ Problem: given n , find the number of different ways to write n as the sum of 1, 3, 4
- ▶ Example: for $n = 5$, the answer is 6

$$\begin{aligned} 5 &= 1 + 1 + 1 + 1 + 1 \\ &= 1 + 1 + 3 \\ &= 1 + 3 + 1 \\ &= 3 + 1 + 1 \\ &= 1 + 4 \\ &= 4 + 1 \end{aligned}$$

1-dimensional DP Example

- ▶ Define subproblems
 - Let D_n be the number of ways to write n as the sum of 1, 3, 4
- ▶ Find the recurrence
 - Consider one possible solution $n = x_1 + x_2 + \cdots + x_m$
 - If $x_m = 1$, the rest of the terms must sum to $n - 1$
 - Thus, the number of sums that end with $x_m = 1$ is equal to D_{n-1}
 - Take other cases into account ($x_m = 3, x_m = 4$)

1-dimensional DP Example

- Recurrence is then

$$D_n = D_{n-1} + D_{n-3} + D_{n-4}$$

- Solve the base cases

- $D_0 = 1$
- $D_n = 0$ for all negative n
- Alternatively, can set: $D_0 = D_1 = D_2 = 1$, and $D_3 = 2$

- We're basically done!

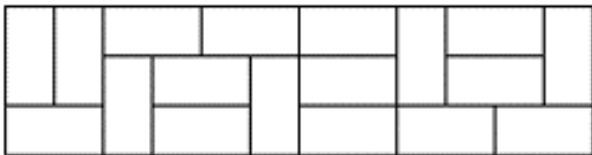
Implementation

```
D[0] = D[1] = D[2] = 1; D[3] = 2;  
for(i = 4; i <= n; i++)  
    D[i] = D[i-1] + D[i-3] + D[i-4];
```

- ▶ Very short!
- ▶ Extension: solving this for huge n , say $n \approx 10^{12}$
 - Recall the matrix form of Fibonacci numbers

POJ 2663: Tri Tiling

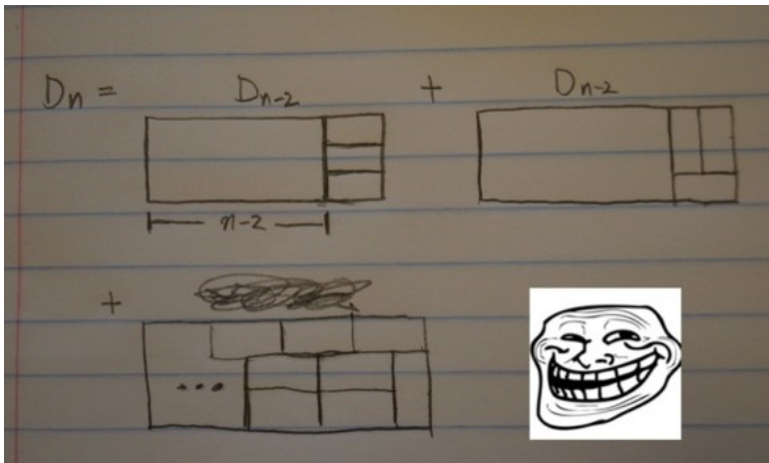
- ▶ Given n , find the number of ways to fill a $3 \times n$ board with dominoes
- ▶ Here is one possible solution for $n = 12$



POJ 2663: Tri Tiling

- ▶ Define subproblems
 - Define D_n as the number of ways to tile a $3 \times n$ board
- ▶ Find recurrence
 - Uuuhhhh...

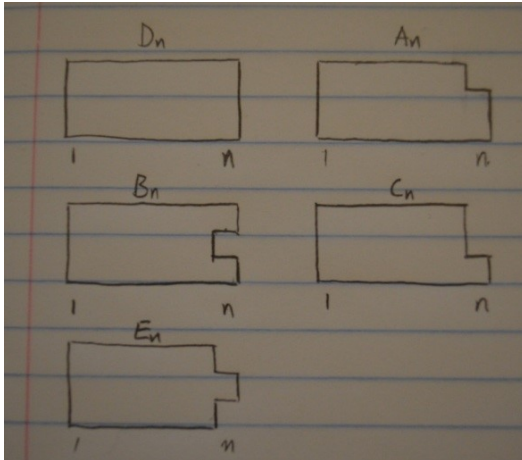
Troll Tiling



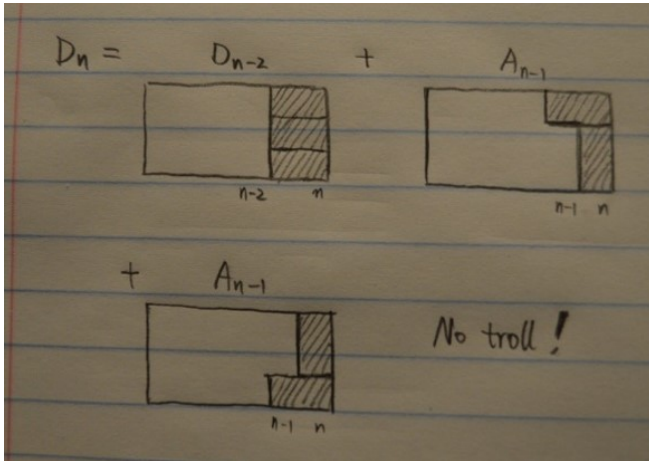
Defining Subproblems

- ▶ Obviously, the previous definition didn't work very well
- ▶ D_n 's don't relate in simple terms
- ▶ What if we introduce more subproblems?

Defining Subproblems



Finding Recurrences



Finding Recurrences

- ▶ Consider different ways to fill the n th column
 - And see what the remaining shape is
- ▶ Exercise:
 - Finding recurrences for A_n , B_n , C_n
 - Just for fun, why is B_n and E_n always zero?
- ▶ Extension: solving the problem for $n \times m$ grids, where n is small, say $n \leq 10$
 - How many subproblems should we consider?

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2-dimensional DP Example

- ▶ Problem: given two strings x and y , find the longest common subsequence (LCS) and print its length
- ▶ Example:
 - x : **A**BCBD**A**B
 - y : BDC**A**BC
 - “BCAB” is the longest subsequence found in both sequences, so the answer is 4

Solving the LCS Problem

- ▶ Define subproblems
 - Let D_{ij} be the length of the LCS of $x_{1\dots i}$ and $y_{1\dots j}$
- ▶ Find the recurrence
 - If $x_i = y_j$, they both contribute to the LCS
 - ▶ $D_{ij} = D_{i-1,j-1} + 1$
 - Otherwise, either x_i or y_j does not contribute to the LCS, so one can be dropped
 - ▶ $D_{ij} = \max\{D_{i-1,j}, D_{i,j-1}\}$
 - Find and solve the base cases: $D_{i0} = D_{0j} = 0$

Implementation

```
for(i = 0; i <= n; i++) D[i][0] = 0;
for(j = 0; j <= m; j++) D[0][j] = 0;
for(i = 1; i <= n; i++) {
    for(j = 1; j <= m; j++) {
        if(x[i] == y[j])
            D[i][j] = D[i-1][j-1] + 1;
        else
            D[i][j] = max(D[i-1][j], D[i][j-1]);
    }
}
```

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Interval DP Example

- ▶ Problem: given a string $x = x_{1\dots n}$, find the minimum number of characters that need to be inserted to make it a palindrome
- ▶ Example:
 - x : Ab3bd
 - Can get “dAb3bAd” or “Adb3bdA” by inserting 2 characters (one ‘d’, one ‘A’)

Interval DP Example

- ▶ Define subproblems
 - Let D_{ij} be the minimum number of characters that need to be inserted to make $x_{i...j}$ into a palindrome
- ▶ Find the recurrence
 - Consider a shortest palindrome $y_{1...k}$ containing $x_{i...j}$
 - Either $y_1 = x_i$ or $y_k = x_j$ (why?)
 - $y_{2...k-1}$ is then an optimal solution for $x_{i+1...j}$ or $x_{i...j-1}$ or $x_{i+1...j-1}$
 - ▶ Last case possible only if $y_1 = y_k = x_i = x_j$

Interval DP Example

- Find the recurrence

$$D_{ij} = \begin{cases} 1 + \min\{D_{i+1,j}, D_{i,j-1}\} & x_i \neq x_j \\ D_{i+1,j-1} & x_i = x_j \end{cases}$$

- Find and solve the base cases: $D_{ii} = D_{i,i-1} = 0$ for all i
- The entries of D must be filled in increasing order of $j - i$

Interval DP Example

```
// fill in base cases here
for(t = 2; t <= n; t++)
    for(i = 1, j = t; j <= n; i++, j++)
        // fill in D[i][j] here
```

- ▶ Note how we use an additional variable t to fill the table in correct order
- ▶ And yes, for loops can work with multiple variables

An Alternate Solution

- ▶ Reverse x to get x^R
- ▶ The answer is $n - L$, where L is the length of the LCS of x and x^R
- ▶ Exercise: Think about why this works

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Tree DP Example

- ▶ Problem: given a tree, color nodes black as many as possible without coloring two adjacent nodes
- ▶ Subproblems:
 - First, we arbitrarily decide the root node r
 - B_v : the optimal solution for a subtree having v as the root, where we color v black
 - W_v : the optimal solution for a subtree having v as the root, where we don't color v
 - Answer is $\max\{B_r, W_r\}$

Tree DP Example

- Find the recurrence
 - Crucial observation: once v 's color is determined, subtrees can be solved independently
 - If v is colored, its children must not be colored

$$B_v = 1 + \sum_{u \in \text{children}(v)} W_u$$

- If v is not colored, its children can have any color

$$W_v = 1 + \sum_{u \in \text{children}(v)} \max\{B_u, W_u\}$$

- Base cases: leaf nodes

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Subset DP Example

- ▶ Problem: given a weighted graph with n nodes, find the shortest path that visits every node exactly once (Traveling Salesman Problem)
- ▶ Wait, isn't this an NP-hard problem?
 - Yes, but we can solve it in $O(n^2 2^n)$ time
 - Note: brute force algorithm takes $O(n!)$ time

Subset DP Example

- ▶ Define subproblems
 - $D_{S,v}$: the length of the optimal path that visits every node in the set S exactly once and ends at v
 - There are approximately $n2^n$ subproblems
 - Answer is $\min_{v \in V} D_{V,v}$, where V is the given set of nodes
- ▶ Let's solve the base cases first
 - For each node v , $D_{\{v\},v} = 0$

Subset DP Example

- Find the recurrence
 - Consider a path that visits all nodes in S exactly once and ends at v
 - Right before arriving v , the path comes from some u in $S - \{v\}$
 - And that subpath has to be the optimal one that covers $S - \{v\}$, ending at u
 - We just try all possible candidates for u

$$D_{S,v} = \min_{u \in S - \{v\}} \left(D_{S - \{v\},u} + \text{cost}(u, v) \right)$$

Working with Subsets

- ▶ When working with subsets, it's good to have a nice representation of sets
- ▶ Idea: Use an integer to represent a set
 - Concise representation of subsets of small integers $\{0, 1, \dots\}$
 - If the i th (least significant) digit is 1, i is in the set
 - If the i th digit is 0, i is not in the set
 - e.g., $19 = 010011_{(2)}$ in binary represent a set $\{0, 1, 4\}$

Using Bitmasks

- ▶ Union of two sets x and y : $x \mid y$
- ▶ Intersection: $x \& y$
- ▶ Symmetric difference: $x \wedge y$
- ▶ Singleton set $\{i\}$: $1 \ll i$
- ▶ Membership test: $x \& (1 \ll i) \neq 0$

Conclusion

- ▶ Wikipedia definition: “a method for solving complex problems by breaking them down into simpler subproblems”
 - Does this make sense now?
- ▶ Remember the three steps!
 1. Defining subproblems
 2. Finding recurrences
 3. Solving the base cases