

INTRODUCTION

Principal component analysis (PCA) is a statistical procedure that is used to reduce the dimensionality. It uses an orthogonal transformation to convert a set of observations of possibly correlated variables into a set of values of linearly uncorrelated variables called principal components. It is often used as a dimensionality reduction technique.

Steps Involved in the PCA

Step 1: Standardize the dataset.

Step 2: Calculate the covariance matrix for the features in the dataset.

Step 3: Calculate the eigenvalues and eigenvectors for the covariance matrix.

Step 4: Sort Eigen values and their corresponding eigenvectors.

Step 5: Pick k eigenvalues and form a matrix of eigenvectors.

Step 6: Transform the original matrix.

Let's go to each step one by one.

1. Standardize the Dataset

Assume we have the below dataset which has 4 features and a total of 5 training examples.

f1	f2	f3	f4
1	2	3	4
5	5	6	7
1	4	2	3
5	3	2	1
8	1	2	2

Dataset matrix

First, we need to standardize the dataset and for that, we need to calculate the mean and standard deviation for each feature.

$$x_{new} = \frac{x - \mu}{\sigma}$$

Standardization formula

	f1	f2	f3	f4
μ =	4	3	3	3.4
σ =	3	1.58114	1.73205	2.30217

Mean and standard deviation before standardization

After applying the formula for each feature in the dataset is transformed as below:

f1	f2	f3	f4
-1	-0.63246	0	0.26062
0.33333	1.26491	1.73205	1.56374
-1	0.63246	-0.57735	-0.17375
0.33333	0	-0.57735	-1.04249
1.33333	-1.26491	-0.57735	-0.60812

Standardized Dataset

2. Calculate the covariance matrix for the whole dataset

The formula to calculate the covariance matrix:

For Population

$$\text{Cov}(x,y) = \frac{\sum (x_i - \bar{x}) * (y_i - \bar{y})}{N}$$

For Sample

$$\text{Cov}(x,y) = \frac{\sum (x_i - \bar{x}) * (y_i - \bar{y})}{(N - 1)}$$

Covariance Formula

The covariance matrix for the given dataset will be calculated as below

	f1	f2	f3	f4
f1	var(f1)	cov(f1,f2)	cov(f1,f3)	cov(f1,f4)
f2	cov(f2,f1)	var(f2)	cov(f2,f3)	cov(f2,f4)
f3	cov(f3,f1)	cov(f3,f2)	var(f3)	cov(f3,f4)
f4	cov(f4,f1)	cov(f4,f2)	cov(f4,f3)	var(f4)

Since we have standardized the dataset, so the mean for each feature is 0 and the standard deviation is 1.

$$\text{var}(f1) = ((-1.0-0)^2 + (0.33-0)^2 + (-1.0-0)^2 + (0.33-0)^2 + (1.33-0)^2)/5$$

$$\text{var}(f1) = 0.8$$

$$\text{cov}(f1,f2) =$$

$$((-1.0-0)*(-0.632456-0) +$$

$$(0.33-0)*(1.264911-0) +$$

$$(-1.0-0)*(0.632456-0)+$$

$$(0.33-0)*(0.000000 -0)+$$

$$(1.33-0)*(-1.264911-0))/5$$

$$\text{cov}(f1,f2) = -0.25298$$

In the similar way we can calculate the other covariances and which will result in the below covariance matrix

	f1	f2	f3	f4
f1	0.8	-0.25298	0.03849	-0.14479
f2	-0.25298	0.8	0.51121	0.4945
f3	0.03849	0.51121	0.8	0.75236
f4	-0.14479	0.4945	0.75236	0.8

3. Calculate eigenvalues and eigen vectors.

An eigenvector is a nonzero vector that changes at most by a scalar factor when that linear transformation is applied to it. The corresponding eigenvalue is the factor by which the eigenvector is scaled.

Let A be a square matrix (in our case the covariance matrix), v a vector and λ a scalar that satisfies $Av = \lambda v$, then λ is called eigenvalue associated with eigenvector v of A .

Rearranging the above equation,

$$Av - \lambda v = 0 ; (A - \lambda I)v = 0$$

Since we have already know v is a non- zero vector, only way this equation can be equal to zero, if

$$\det(A - \lambda I) = 0$$

	f1	f2	f3	f4
f1	$0.8 - \lambda$	-0.25298	0.03849	-0.14479
f2	-0.25298	$0.8 - \lambda$	0.51121	0.4945
f3	0.03849	0.51121	$0.8 - \lambda$	0.75236
f4	-0.14479	0.4945	0.75236	$0.8 - \lambda$

$$A - \lambda I = 0$$

Solving the above equation = 0

$$\lambda = 2.51579324, 1.0652885, 0.39388704, 0.02503121$$

Eigenvectors:

Solving the $(A - \lambda I)v = 0$ equation for v vector with different λ values:

$$\begin{pmatrix} 0.800000 - \lambda & -(0.252982) & 0.038490 & -(0.144791) \\ -(0.252982) & 0.800000 - \lambda & 0.511208 & 0.494498 \\ 0.038490 & 0.511208 & 0.800000 - \lambda & 0.752355 \\ -(0.144791) & 0.494498 & 0.752355 & 0.800000 - \lambda \end{pmatrix} \times \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = 0$$

For $\lambda = 2.51579324$, solving the above equation using Cramer's rule, the values for v vector are

$$v1 = 0.16195986$$

$$v2 = -0.52404813$$

$$v3 = -0.58589647$$

$$v4 = -0.59654663$$

Going by the same approach, we can calculate the eigen vectors for the other eigen values. We can form a matrix using the eigen vectors.

	e1	e2	e3	e4
	0.161960	-0.917059	-0.307071	0.196162
	-0.524048	0.206922	-0.817319	0.120610
	-0.585896	-0.320539	0.188250	-0.720099
	-0.596547	-0.115935	0.449733	0.654547

eigenvectors(4 * 4 matrix)

4. Sort eigenvalues and their corresponding eigenvectors.

Since eigen values are already sorted in this case so no need to sort them again.

5. Pick k eigenvalues and form a matrix of eigenvectors

If we choose the top 2 eigenvectors, the matrix will look like this:

	e1	e2
	0.161960	-0.917059
	-0.524048	0.206922
	-0.585896	-0.320539
	-0.596547	-0.115935

Top 2 eigenvectors(4*2 matrix)

6. Transform the original matrix.

Feature matrix * top k eigenvectors = Transformed Data

f1	f2	f3	f4		e1	e2		nf1	nf2
-1.000000	-0.632456	0.000000	0.260623		0.161960	-0.917059		0.014003	0.755975
0.333333	1.264911	1.732051	1.563740	*	-0.524048	0.206922	=	-2.556534	-0.780432
-1.000000	0.632456	-0.577350	-0.173749		-0.585896	-0.320539		-0.051480	1.253135
0.333333	0.000000	-0.577350	-1.042493		-0.596547	-0.115935		1.014150	0.000239
1.333333	-1.264911	-0.577350	-0.608121					1.579861	-1.228917
			(5,4)		(4,2)			(5,2)	