Ordinary Differential Equations in Simulating Aeronautical Systems

Project Report

Course: MATH 202 – Ordinary Differential Equations
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Abstract

This report consolidates two key applications of ordinary differential equations (ODEs) in aeronautical engineering: the modeling of aircraft longitudinal dynamics and the simulation of Doppler shift in aeronautical communications. Both topics rely on linear and nonlinear ODEs to describe physical phenomena critical to flight stability and communication reliability.

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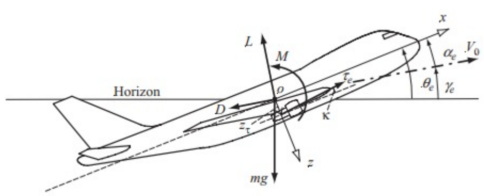
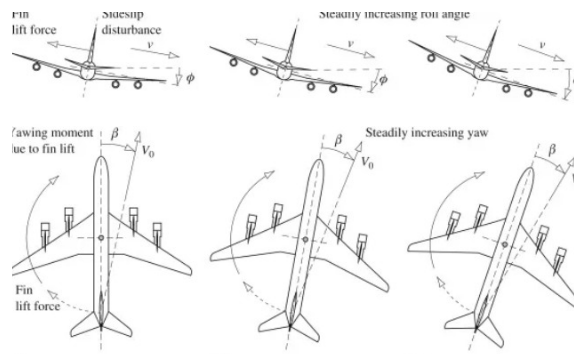


Figure 3.20 *Symmetric forces and moments acting on a trimmed aircraft.*



Aeroplanes are designed to fly at trimmed rectilinear flight with zero roll, sideslip and yaw angles. During flight the aeroplane may encounter a small perturbation(short period long period) about trim, and for small perturbations, the aeroplane is a classical example of a linear dynamic system and frequently the solution of its equations of motion is a prelude to flight control system design and analysis. The motion, or state, of any linear dynamic system may be described by a minimum set of variables called the state variables. The number of state variables required to completely describe the motion of the system is dependent on the number of degrees of freedom the system has.

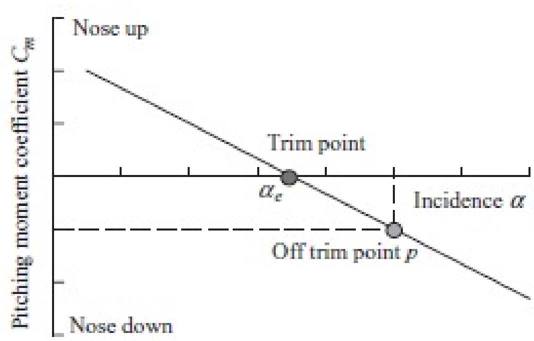


Figure 3.2 *Pitching moment variation with incidence for a stable aircraft.*

1.2 Equations of motion in state-space form:

The longitudinal state equation may be written in matrix form as

$$\begin{bmatrix} \dot{u} \\ \dot{w} \\ \dot{q} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} x_u & x_w & x_q & x_\theta \\ z_u & z_w & z_q & z_\theta \\ m_u & m_w & m_q & m_\theta \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ w \\ q \\ \theta \end{bmatrix} + \begin{bmatrix} x_\eta & x_\tau \\ z_\eta & z_\tau \\ m_\eta & m_\tau \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \eta \\ \tau \end{bmatrix}.$$

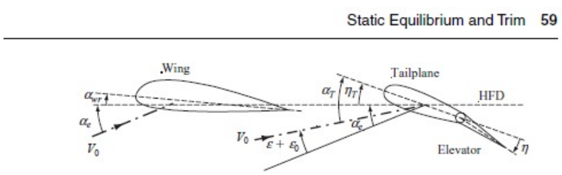
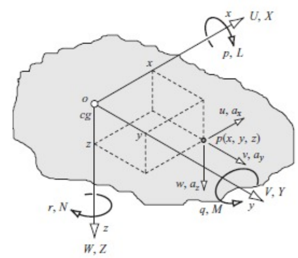


Figure 3.21 *Practical wing–tailplane aerodynamic geometry.*



q, w, u the components of the angular disturbance velocities are (p, q, r) with respect to the undisturbed aeroplane axes (oxyz).

θ pitch angle

η where aileron angle, elevator angle and rudder angle are denoted ξ, η and ζ respectively.

τ quantifies the thrust perturbation relative to the trim setting τ_e.

$x_u, x_w, x_q, x_\theta,\ z_u, \ldots,\ m_\eta, m_\tau$ The aerodynamic stability derivatives, referred to aeroplane body axes, in concise form.

After expansion the four first-order longitudinal equations yeild:

$$\dot{u} = x_u\, u + x_w\, w + x_q\, q + x_\theta\, \theta + x_\eta\, \eta + x_\tau\, \tau, \tag{1}$$

$$\dot{w} = z_u\, u + z_w\, w + z_q\, q + z_\theta\, \theta + z_\eta\, \eta + z_\tau\, \tau, \tag{2}$$

$$\dot{q} = m_u\, u + m_w\, w + m_q\, q + m_\theta\, \theta + m_\eta\, \eta + m_\tau\, \tau, \tag{3}$$

$$\dot{\theta} = q. \tag{4}$$

1.3 Solving ODEs

The ODEs can be solved by first-order techniques with assumption:

$$w(0) = 0, \quad q(0) = 0, \quad u(0) = 0.$$

1.3.1 Pitch attitude $\dot{\theta}$ – separation

Start from

$$\frac{d\theta}{dt} = q(t) \quad \Longleftrightarrow \quad d\theta = q(t)\,dt.$$

Integrate both sides from 0 to t:

$$\int_{\theta(0)}^{\theta(t)} d\theta' = \int_0^t q(t)\,dt,$$

so

$$\theta(t) = \theta(0) + \int_0^t q()\,\mathbf{dt}.$$

If $q(t) = q_0$ constant then $\theta(t) = q_0 t$.

1.3.2 Forward velocity $\dot{u}$ – linear ODE

Equation:

$$\dot{u} = x_u\,u + x_w\,w + x_q\,q + x_\theta\,\theta + x_\eta\,\eta + x_\tau\,\tau.$$

Treat w, q, θ, η, τ as fixed constants and use $u(0) = 0$. Define

$$C_1 = x_w\,w, \qquad C_2 = x_q\,q, \qquad C_3 = x_\theta\,\theta + x_\eta\,\eta + x_\tau\,\tau,$$

and let $C_\mathrm{s} = C_1 + C_2 + C_3$. The ODE becomes

$$\frac{du}{dt} - x_u\, u = C_{\rm s}.$$

Integrating-factor method :

- Integrating factor: $\mu(t) = e^{-x_u t}$.

- Multiply both sides by $\mu(t)$:

$$e^{-x_u t}\frac{du}{dt} - x_u e^{-x_u t} u = C_{\rm s} e^{-x_u t}.$$

$$\frac{d}{dt}\big(e^{-x_u t} u(t)\big) = C_{\rm s} e^{-x_u t}.$$

Integrate from 0 to t :

$$e^{-x_u t} u(t) - e^{-x_u\, 0} u(0) = C_{\rm s} \int_0^t e^{-x_u t}\, dt.$$

With $u(0) = 0$ this simplifies to

$$e^{-x_u t} u(t) = C_{\rm s} \int_0^t e^{-x_u t}\, dt.$$

Evaluating the integral gives: - If $x_u \neq 0$,

$$\int_0^t e^{-x_u t}\, dt = \frac{1 - e^{-x_u t}}{x_u},$$

so

$$e^{-x_u t} u(t) = C_{\rm s} \frac{1 - e^{-x_u t}}{x_u},$$

hence

$$u(t) = \frac{C_{\rm s}}{x_u}\big(e^{x_u t} - 1\big).$$

1.3.3 Vertical velocity $\dot{w}$ – Laplace transform

Equation:

$$\dot{w} = z_u\, u + z_w\, w + z_q\, q + z_\theta\, \theta + z_\eta\, \eta + z_\tau\, \tau.$$

Combine the terms except w as constants and set $w(0) = 0$.

$$C = z_u\, u + z_q\, q + z_\theta\, \theta + z_\eta\, \eta + z_\tau\, \tau.$$

Take Laplace transforms (with $W(s) = \mathcal{L}\{w(t)\}$ and $w(0) = 0$):

$$\mathcal{L}\{\dot{w}\} = sW(s) - w(0) = sW(s) = z_w W(s) + \frac{C}{s}.$$

Rearrange for $W(s)$:

$$(s - z_w)W(s) = \frac{C}{s} \quad \Longrightarrow \quad W(s) = \frac{C}{s(s - z_w)}.$$

Partial fractions:

$$\frac{C}{s(s - z_w)} = \frac{A}{s} + \frac{B}{s - z_w}.$$
$$C = A(s - z_w) + Bs.$$

Evaluate at convenient s: - $s = 0$ gives $C = -Az_w$ so $A = -\frac{C}{z_w}$.

- $s = z_w$ gives $C = Bz_w$ so $B = \frac{C}{z_w}$.

Thus

$$\frac{C}{s(s - z_w)} = -\frac{C}{z_w}\frac{1}{s} + \frac{C}{z_w}\frac{1}{s - z_w}.$$

Substitute into $W(s)$:

$$W(s) = -\frac{C}{z_w}\frac{1}{s} + \frac{C}{z_w}\frac{1}{s - z_w} \quad \Longrightarrow \quad w(t) = \frac{C}{z_w}\big(e^{z_w t} - 1\big).$$

2 Longitudinal Dynamics State-Space Model

2.1 Introduction:

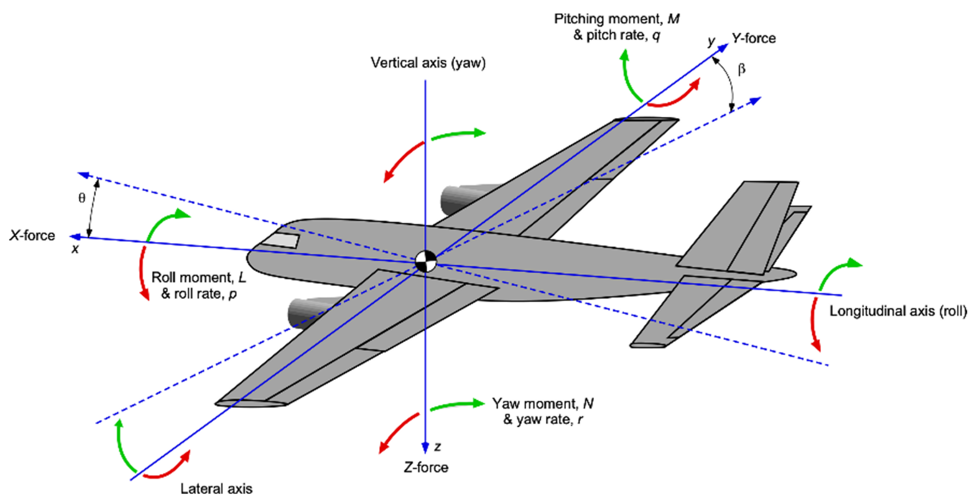


Figure 1: Aircraft Longitudinal Dynamics Variables

The longitudinal dynamics of an aircraft describe its motion in the vertical plane, including variations in forward speed, vertical motion, pitch rate, and pitch angle. These dynamics play a crucial role in determining flight stability and passenger comfort and are commonly analyzed using mathematical models based on ordinary differential equations. For small disturbances around a steady, level flight condition, the nonlinear equations of aircraft motion can be linearized, resulting in a coupled system of first-order linear ordinary differential equations. This linearized model captures the essential physical behavior of the aircraft while remaining mathematically tractable, making it well suited for analysis using ODE theory. In this report, the full longitudinal dynamics are studied as a four-state linear ODE system, with emphasis on the mathematical structure of the equations, the interpretation of system parameters, and the resulting dynamic modes of motion that govern aircraft behavior.

2.2 Modeling Assumptions:

To obtain a linear ODE model that is suitable for analytical and numerical study, the following assumptions are made:

- The aircraft operates near a steady, level flight condition (trim), where all forces and moments are initially balanced. - All motions are considered small perturbations about the trim condition, allowing higher-order nonlinear terms to be neglected. - Pitch angles and angle-of-attack variations are assumed to be small, so trigonometric functions can be approximated linearly. - Aerodynamic forces and moments are represented using linear stability derivatives that remain constant around the trim condition. - The aircraft is treated as a rigid body; structural flexibility and aeroelastic effects are neglected.

These assumptions reduce the complex nonlinear dynamics of an aircraft to a system of coupled linear ordinary differential equations with constant coefficients, allowing the focus of the project to remain on the mathematical analysis and behavior of the ODE system.

2.3 Definition of State Variables:

To model the longitudinal dynamics of the aircraft as a system of ordinary differential equations, a set of state variables is defined. These variables describe small deviations of the aircraft motion from the steady, level flight condition (trim). Each state variable has a clear physical meaning and represents an essential component of the longitudinal motion.

The state vector is defined as:

$$\mathbf{x}(t) = \begin{bmatrix} u(t) \\ w(t) \\ q(t) \\ \theta(t) \end{bmatrix}$$

where each component is described below.

- **Forward velocity deviation,** $u(t)$

This variable represents the small deviation of the aircraft's forward (longitudinal) velocity from its trim value. It reflects changes in air speed caused by disturbances or dynamic interactions with pitch motion. Variations in taffect aerodynamic forces such as drag and lift.

- **Vertical velocity deviation,** $w(t)$

This variable represents the small deviation of the aircraft's vertical velocity relative to the body-fixed reference frame. It is closely related to changes in angle of attack and plays an important role in the generation of lift and vertical acceleration.

- **Pitch rate,** $q(t)$

The pitch rate is the angular velocity of the aircraft about its lateral axis. It describes how fast the aircraft is rotating nose-up or nose-down and directly influences the evolution of the pitch angle. Pitch rate is a key dynamic variable in aircraft stability analysis.

- **Pitch angle,** $\theta(t)$

The pitch angle represents the orientation of the aircraft relative to the horizontal plane. It describes the aircraft's attitude and determines how gravity and aerodynamic forces are distributed along the body axes. The pitch angle evolves according to the pitch rate and provides an important link between rotational and translational motion.

These four state variables are sufficient to capture the essential longitudinal behavior of the aircraft under small perturbations. Together, they form a coupled system of first-order linear ordinary differential equations that describe how translational and rotational motions interact. This state selection allows the longitudinal dynamics to be expressed in a compact state-space form, making the system suitable for analytical study and numerical simulation using ODE techniques.

2.4 Derivation of the Linear ODE System:

2.4.1 Derivation of the First Longitudinal Equation ($\dot{u}$)

The general longitudinal equation for forward (body-axis) velocity U is obtained from Newton's second law along the x-body axis:

$$m\frac{dU}{dt} = X - mg\sin\theta$$

Where: m = mass X = total force along the x-axis g = gravity θ = pitch angle

At steady level flight (trim):

$$U = U_0, \theta = \theta_0, X = X_0, \frac{dU_0}{dt} = 0$$

Define small deviations from trim:

$$u = U - U_0, \theta' = \theta - \theta_0, X' = X - X_0$$

$$m\frac{d(U_0 + u)}{dt} = X_0 + X' - mg\sin(\theta_0 + \theta')$$

Since $\frac{dU_0}{dt} = 0$ and $X_0 = mg\sin\theta_0$ (trim condition), we get:

$$m\frac{du}{dt} = X' - mg(\sin(\theta_0 + \theta') - \sin\theta_0)$$

For small perturbations, $\sin(\theta_0 + \theta') \approx \sin\theta_0 + \theta' \cos\theta_0$.

Substitute into the equation:

$$m\frac{du}{dt} = X' - mg(\sin\theta_0 + \theta' \cos\theta_0 - \sin\theta_0) = X' - mg\cos\theta_0\theta'$$

For small trim angles $\theta_0 \approx 0$, $\cos\theta_0 \approx 1$. So:

$$m\frac{du}{dt} = X' - mg\theta'$$

The change in longitudinal force X' is expressed using stability derivatives:

$$X' = X_u u + X_w w + X_q q$$

Where: u = forward velocity deviation w = vertical velocity deviation q = pitch rate

Substitute X' into the equation:

$$\frac{du}{dt} = X_u u + X_w w + X_q q - mg\theta$$

$$\left| \frac{X_u}{m} \quad \frac{X_w}{m} \quad \frac{X_q}{m} \right|$$

2.4.2 Derivation of the Second Longitudinal Equation ($\dot{w}$)

The general longitudinal equation for vertical (body-axis) velocity W comes from Newton's second law along the z-body axis (positive downward in body frame):

$$m\frac{dW}{dt} = Z + mg\cos\theta\sin\phi$$

Where: m = mass Z = total force along z-axis g = gravity θ = pitch angle

ϕ = roll angle (here we assume no roll, so $\phi = 0$)

At trim condition:

$$W = W_0 = 0, Z = Z_0, \theta = \theta_0$$

So, the vertical acceleration is zero:

$$\frac{dW_0}{dt} = 0.$$

Define small deviations:

$$w = W - W_0, \theta' = \theta - \theta_0, Z' = Z - Z_0$$

Substitute into the nonlinear equation:

$$m\frac{d(W_0 + w)}{dt} = Z_0 + Z' + mg\cos(\theta_0)\sin\phi$$

Since $W_0 = 0, Z_0 = 0$ (trim), and $\phi = 0$, we get:

$$m\frac{dw}{dt} = Z'$$

The vertical aerodynamic force Z' is expressed in terms of stability derivatives:

$$Z' = Z_u u + Z_w w + Z_q q$$

Where: u = forward velocity deviation w = vertical velocity deviation q = pitch rate

When the aircraft pitches at speed U_0, a pitch rate q produces a vertical velocity component $w \approx U_0 q$. This is included in the linearized system as a term:

$$Z_q\, q + U_0\, q$$

So the total vertical velocity equation becomes:

$$m\frac{dw}{dt} = Z_u u + Z_w w + (Z_q + mU_0)q$$

Divide by m to get the linearized second equation:

$$\dot{w} = \tfrac{Z_u}{m}u + \tfrac{Z_w}{m}w + \tfrac{Z_q + mU_0}{m}q$$

2.4.3 Derivation of the Third Longitudinal Equation ($\dot{q}$)

The general rotational equation about the lateral axis (y-axis) comes from Euler's equation for rotation:

$$I_y \frac{dq}{dt} = M$$

Where: I_y = moment of inertia about the lateral (pitch) axis q pitch rate (rad/s)

$$M = \text{total pitching moment (about the center of gravity)}$$

At trim conditions, the pitching moment is zero: $M_0 = 0$.

Define a small perturbation in the pitching moment:

$$q = q', M = M' + M_0$$

Since $M_0 = 0$:

$$I_y \frac{dq}{dt} = M'$$

The pitching moment change M' is expressed using stability derivatives:

$$M' = M_u u + M_w w + M_q q$$

Where: u = forward velocity deviation w = vertical velocity deviation q = pitch rate

Each term represents how the pitch rate changes due to small deviations in velocity and angle of attack:

- $M_u u \rightarrow$ change in pitch moment due to forward speed change - $M_w w \rightarrow$ change in pitch moment due to vertical speed (related to angle of attack) - $M_q q \rightarrow$ damping moment proportional to pitch rate

Divide both sides by I_y to isolate $\dot{q}$:

$$\dot{q} = \frac{M_u}{I_y} u + \frac{M_w}{I_y} w + \frac{M_q}{I_y} q$$

2.4.4 Derivation of the Fourth Longitudinal Equation ($\dot{\theta}$)

The pitch rate q is the rate of change of the pitch angle θ:

$$q = \frac{d\theta}{dt}$$

This equation is exact and does not depend on mass, inertia, or aerodynamics.

Let $\theta = \theta_0 + \theta'$, where θ_0 is the trim pitch angle and θ' is a small deviation.

The derivative is:

$$\dot{\theta}' = \frac{d\theta'}{dt} = q$$

Since at trim, $\dot{\theta}_0 = 0$, the equation for the small deviation becomes:

$$\dot{\theta} = q$$

2.5 MATLAB Simulation of Longitudinal Dynamics

The simulation solves the linearized longitudinal ODE system numerically using MATLAB's.

The state vector is defined as:

$$x = \begin{bmatrix} u \\ w \\ q \\ \theta \end{bmatrix}$$

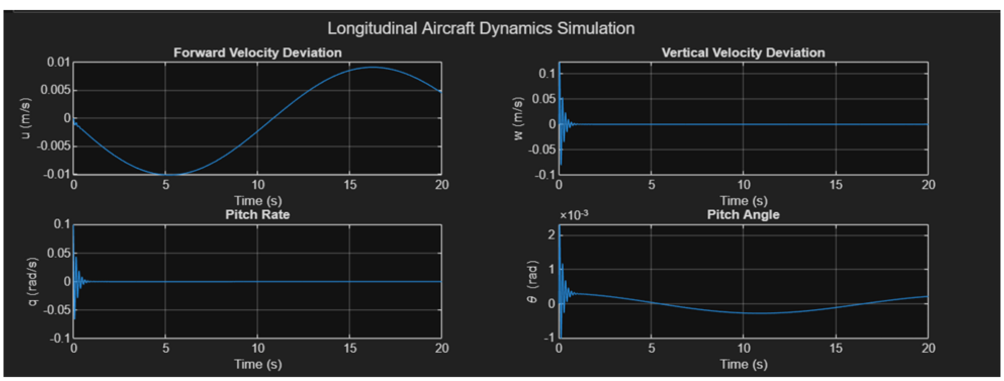


Figure 2: Longitudinal Aircraft Dynamics Simulation

The MATLAB simulation of the longitudinal dynamics produces time histories for the four state variables: forward velocity deviation (u), vertical velocity deviation (w), pitch rate (q), and pitch angle (θ). By examining these plots, the characteristic dynamic modes of the aircraft can be observed: the short-period mode and the phugoid mode.

2.5.1 Short-Period Mode

The short-period mode is a rapid oscillation primarily involving the pitch rate (q) and vertical velocity (w). In the simulation, the initial pitch rate disturbance causes an immediate oscillation in q and w with a relatively high frequency and moderate damping. The forward velocity u and pitch angle θ are only slightly affected during this mode. Physically, this mode corresponds to the aircraft nose pitching up and down quickly, with the angle of attack changing but the air speed and altitude remaining nearly constant. The period of this oscillation is typically short (around 1-2 seconds for small aircraft), and the mode damps out quickly.

2.5.2 Phugoid Mode

The phugoid mode is a slow, long-period oscillation involving primarily the forward velocity (u) and pitch angle (θ). In the simulation, after the short-period oscillations decay, a slower oscillation in u and θ can be observed. The vertical velocity w and pitch rate q show smaller variations. Physically, this mode represents the aircraft trading kinetic energy for potential energy: as the aircraft climbs, its speed decreases; as it descends, the speed increases. The phugoid has a low frequency and very light damping, resulting in oscillations that last tens of seconds.

2.6 Conclusion

In this project, the longitudinal dynamics of an aircraft were analyzed using a linearized system of ordinary differential equations. By defining four state variables—forward velocity deviation, vertical velocity deviation, pitch rate, and pitch angle—the complex nonlinear aircraft motion was reduced to a tractable set of linear ODEs. The derivation of these equations demonstrated how aerodynamic forces, pitching moments, and gravity interact to determine aircraft motion.

The MATLAB simulation of the system revealed the characteristic short-period and plugoid modes. The short-period mode showed a rapid, moderately damped oscillation in pitch rate and vertical velocity, while the plugoid mode exhibited a slow, lightly damped oscillation in forward speed and pitch angle. These results confirmed that linear ODE models, even with simplifying assumptions, can accurately capture essential features of aircraft longitudinal behavior.

Overall, this study highlights the usefulness of linear ODE analysis in understanding flight dynamics, predicting aircraft responses to disturbances, and providing a foundation for control system design. The approach can be extended to include control inputs, nonlinear effects, or more complex aircraft models for further study.

3 Doppler Shift in Aeronautical Communications

3.1 Introduction

when a subscriber is in motion in a flying vehicle the channel exhibits fading due to doppler shift and the frequency changes in spreading and creates fading in the time domain which is the envelope of the receive carrier exhibits deep fade as a function of time which creates a relationship between the envelope of fading as the frequency of the carrier and the vehicle speed which is very detrimental to the communication systems and has to be taken into consideration especially in aeronautical structures.

3.2 Observation and explanation

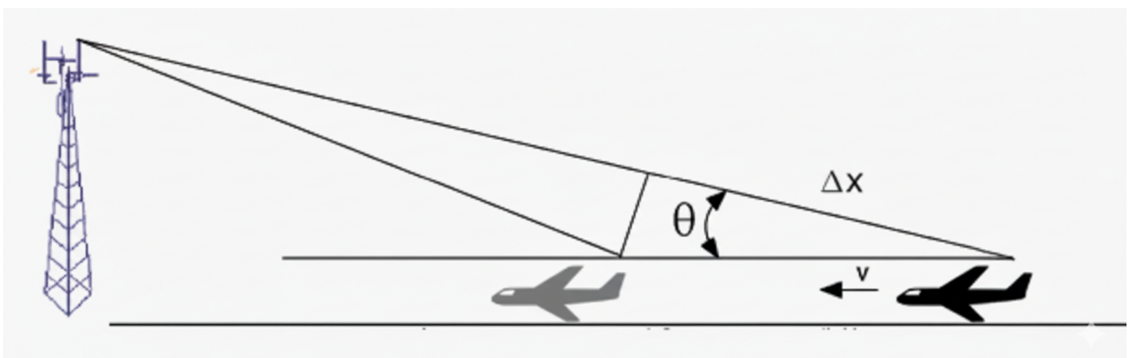


Figure 3: Doppler effect geometry diagram showing vehicle motion relative to base station

When a base station is transmitting it's signal encounters a time delay of $\tau = \frac{d}{c}$ where x is the distance from the receiver and c is the speed of light which will be expressed as $\tau = \frac{d(t)}{c}$ to a moving receiver that receives carrier signal f_c and as the spaceship moves the received signal at the antenna from the transmitter exhibits a phase shift. since we can express the phase as $\frac{\phi}{2\pi} = \frac{x}{\lambda}$ If moving distance λ corresponds to a phase change of 2π which gives $\Delta\phi = \frac{2\pi}{\lambda}\Delta x$ the changing in phase over time gained from the position variation $\Delta x = v\ \Delta t\ cos\theta$ as in figure 1 which makes $\Delta\phi = \frac{2\pi}{\lambda} v\ \Delta t\ cos\theta\ \rightarrow\ \frac{\Delta\phi}{\Delta t} = \frac{2\pi}{\lambda} v\ cos\theta$ this also results in a frequency shift and since for a constant frequency signal the derivative of phase equals the angular frequency so we get the instantaneous doppler frequency change as $f_d = \frac{1}{2\pi}\frac{\Delta\phi}{\Delta t}\ = \frac{v}{\lambda}\ cos\theta$ and since the wave speed equals the frequency multiplied by the wavelength $c\ =\ \lambda\ \times\ f$ we can substitute for λ to get the expression $f_d = \frac{f_c}{c}\ v\ cos\theta$ so the received frequency will be expression by $f_r = f_c(1 + \frac{v}{c}\ cos\theta)$ where if the transmitter and receiver are moving apart $(v > 0)$, $f_r < f_c$ and vice versa $(v < 0)$, $f_r > f_c$.

3.3 Theoretical bases

deriving the first-order differential equation that describes how the phase of the received signal evolves we can define the phase error due to Doppler as $\theta_e(t)$. For a receiver expecting frequency f_c, the phase of the received signal relative to its local oscillator is:

$$\phi_r(t) = 2\pi f_c(t - \frac{d(t)}{c})$$

Where the time delay is added

3.3.1 Step 1: Phase Error Definition

The phase error is the difference between received phase and the expected phase for an instantaneous stationary receiver at reference distance:

$$\theta_e(t) = \phi_r(t) - 2\pi f_c t = -2\pi f_c \frac{d(t)}{c}$$

3.3.2 Step 2: Differentiate to Get the Differential Equation

$$\frac{d\theta_e(t)}{dt} = -2\pi f_c \frac{1}{c} \frac{d}{dt} d(t)$$

Substitute $\frac{d}{dt} d(t) = -v \cos\theta(t)$:

$$\frac{d\theta_e(t)}{dt} = -2\pi f_c \frac{1}{c}(-v \cos\theta(t))$$
$$\frac{d\theta_e(t)}{dt} = 2\pi f_c \frac{v}{c} \cos\theta(t)$$

3.3.3 Step 3: Differentiate once for First-Order Model that depends on velocity

The Doppler frequency shift is:

$$f_d(t) = f_c \frac{v}{c} \cos\theta(t)$$

Therefore:

$$\frac{d\theta_e(t)}{dt} = 2\pi f_d(t)$$

$$\frac{d\theta_e(t)}{dt} = 2\pi f_c \frac{v}{c} \cos\theta(t)$$

3.3.4 Step 4: Differentiate Again for Second-Order Model that depends on acceleration

$$\frac{d^2\theta_e(t)}{dt^2} = 2\pi f_c \frac{1}{c} \frac{d^2x(t)}{dt^2} \cos\theta = 2\pi f_c \frac{a(t)}{c} \cos\theta$$

$$\frac{d^2\theta_e(t)}{dt^2} = 2\pi f_c \frac{a(t)}{c} \cos\theta$$

3.3.5 More Realistic General Case

For the general case with time-varying velocity and angle Differentiate using the product rule ($v(t)$ and $\cos\theta(t)$ both vary):

$$\frac{d^2\theta_e(t)}{dt^2} = 2\pi \frac{f_c}{c} [\frac{dv(t)}{dt} \cos\theta(t) - v(t)\sin\theta(t) \frac{d\theta(t)}{dt}]$$

3.4 Simulation

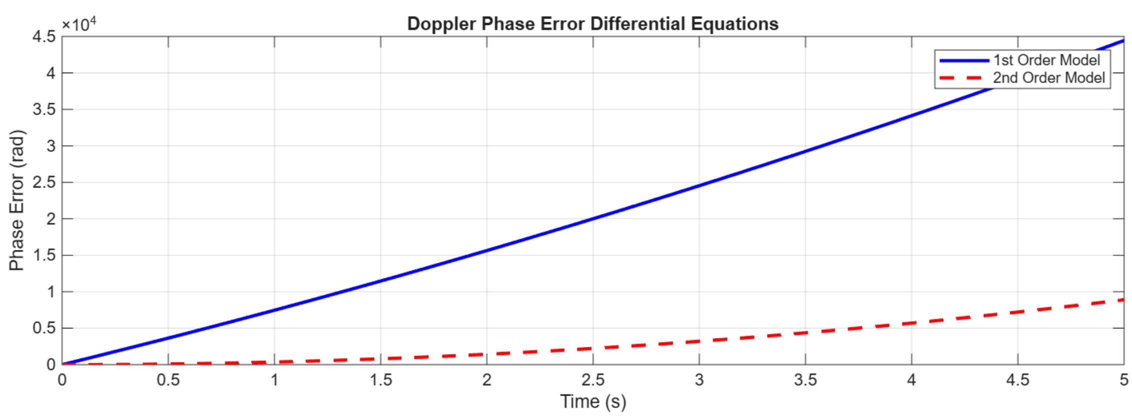


Figure 4: Simulation results showing phase error and Doppler frequency shift over time

Both models show accumulating phase error that is growing quadratically over time this is the carrier phase drift due to Doppler effect.

ODE	Describes	Impact	Receiver Challenge
First-Order Model	How velocity causes instantaneous frequency shift	Causes received signal frequency to differ from transmitted frequency	Must track this changing frequency
Second-Order Model	How acceleration causes the frequency shift to change over time	Makes frequency tracking more difficult and receiver must predict rate of change	Requires more sophisticated phase-locked loops (PLLs)

3.5 Conclusion

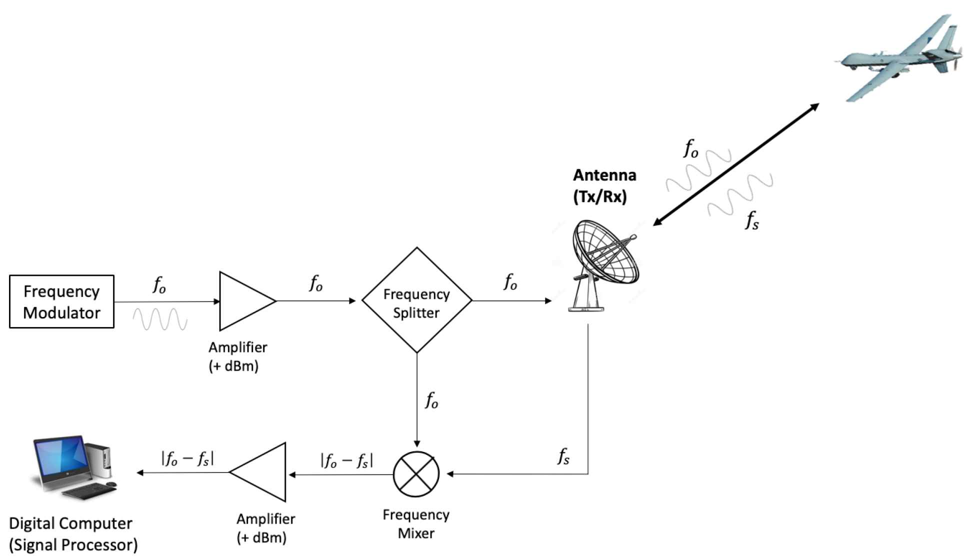


Figure 5: Satellite communication system architecture with Doppler compensation

Simulating Doppler shift enables the design of robust aeronautical communication systems that maintain reliable links despite high-velocity effects. It allows engineers to develop advanced tracking algorithms and compensation techniques that prevent signal degradation during flight. This proactive approach ensures seamless connectivity and safety for aircraft communications across varying operational scenarios.

4 Propeller Spin-Up Dynamics

4.1 Introduction

Propeller-driven systems are widely used in aerospace and marine engineering. When a propeller is powered by a motor, it does not instantaneously reach its steady rotational speed due to inertia and aerodynamic drag forces. Instead, the angular velocity increases gradually with time. This transient process is known as propeller spin-up.

Understanding propeller spin-up dynamics is important for performance prediction, actuator design, and control system stability. In this project, the spin-up behavior is modeled using an ordinary differential equation (ODE) and solved using numerical techniques.

4.2 Mathematical Model

The rotational motion of the propeller is governed by Newton's second law for rotation:

$$J\frac{d\Omega}{dt} = \tau(u) - k\Omega^2 \tag{1}$$

where:

- J is the propeller moment of inertia
- Ω is the angular speed
- $\tau(u)$ is the motor torque
- $k\Omega^2$ represents aerodynamic drag

For a step input in throttle, the motor torque is assumed constant:

$$\tau(u) = K_t u_0 \tag{2}$$

Substituting into Equation 1:

$$\frac{d\Omega}{dt} = \frac{K_t u_0 - k\Omega^2}{J} \tag{3}$$

This equation is a nonlinear first-order ordinary differential equation due to the quadratic drag term.

4.3 Exact Analytical Solution

For constant input torque, the exact solution of the propeller spin-up equation is:

$$\Omega(t) = \Omega_{ss} \tanh\left(\frac{\sqrt{K_t u_0 k}}{J} t\right) \tag{4}$$

where the steady-state angular speed is:

$$\Omega_{ss} = \sqrt{\frac{K_t u_0}{k}} \tag{5}$$

The exact solution serves as a reference to evaluate numerical accuracy.

4.4 Numerical Methods

4.4.1 Euler Method

Euler's method approximates the solution using the slope at the beginning of each time step:

$$\Omega_{n+1} = \Omega_n + hf(\Omega_n) \tag{6}$$

This approach assumes the slope remains constant throughout the step, which introduces significant error in nonlinear systems.

4.4.2 Runge–Kutta Fourth Order (RK4)

The RK4 method improves accuracy by calculating four intermediate slopes within each time step:

$$\Omega_{n+1} = \Omega_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{7}$$

where:

$$
\begin{aligned}
k_1 &= f(\Omega_n) \\
k_2 &= f\left(\Omega_n + \frac{h}{2}k_1\right) \\
k_3 &= f\left(\Omega_n + \frac{h}{2}k_2\right) \\
k_4 &= f\left(\Omega_n + hk_3\right)
\end{aligned}
$$

This method provides excellent accuracy and stability.

4.5 Results and Discussion

Simulation results show that Euler's method deviates from the exact solution, especially during the transient phase. RK4 closely matches the analytical solution, accurately capturing the nonlinear behavior of the propeller.

5 Actuator Lag Dynamics

5.1 Introduction

Actuators are essential components in control systems, but they cannot respond instantaneously to input commands. Physical limitations such as inertia, friction, and internal dynamics cause a delay in the response. This delay is referred to as actuator lag.

Actuator lag is commonly modeled as a first-order system, making it an important application of first-order differential equations.

5.2 Mathematical Model

The actuator lag model is given by:

$$\tau\frac{d\delta}{dt} + \delta = Ku(t) \tag{8}$$

where:

- $\delta(t)$ is the actuator output

- τ is the time constant
- K is the actuator gain
- $u(t)$ is the input command

Rewriting Equation 8:

$$\frac{d\delta}{dt} = \frac{Ku - \delta}{\tau} \tag{9}$$

This is a linear first-order ODE.

5.3 Exact Analytical Solution

For a unit step input and zero initial condition, the exact solution is:

$$\delta(t) = K\left(1 - e^{-t/\tau}\right) \tag{10}$$

The time constant τ determines how fast the actuator reaches steady state.

5.4 Numerical Methods

5.4.1 Euler Method

Euler's method applies a single slope per time step, which can lead to noticeable numerical error during transient response.

5.4.2 Runge–Kutta Fourth Order (RK4)

RK4 evaluates the slope four times within each step, significantly improving accuracy and closely matching the exact solution.

5.5 Results and Discussion

The Euler solution shows deviation from the analytical response, particularly at early times. RK4 provides excellent agreement with the exact solution across the entire simulation period.

5.6 Overall Conclusion

This project demonstrated the application of numerical methods to both nonlinear and linear dynamic systems. Euler's method is useful for conceptual understanding but suffers from accuracy limitations. The RK4 method consistently provides superior accuracy and is recommended for engineering simulations involving dynamic systems.

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