2017-09-28

- A **nondeterministic finite state automaton (NFA)** $A = \langle \Sigma, Q, q_0, F, \delta \rangle$, where δ can be expressed as: (1) *Transition relation*: $\delta \subseteq Q \times \Sigma \times Q$; (2) *Transition function*: $\delta : Q \times \Sigma \to 2^Q$.
- **Theorem**: If A is a DFA, and A is also an NFA.
- Example of NFA: $A = \langle \Sigma, Q, q_0, F, \delta \rangle$, where $\Sigma = \{a, b\}, Q = \{q_0, q_1, q_2\}, q_0, F = \{q_2\}$,
$\delta = \{(q_0, a, q_0), (q_0, b, q_0), (q_1, a, q_1), (q_1, b, q_1), (q_2, a, q_2), (q_2, b, q_2), (q_0, a, q_1), (q_1, b, q_2)\}$
- On input word w, a run of A on w is $p_0, a_1, p_1, a_2, p_2, \ldots, a_n, p_n$ s.t. $(p_{i-1}, a_i, p_i) \in \delta$.
- A run is an accepting run if it ends with a final state, i.e. $p_n \in F$.
- An NFA A accepts w if there is an accepting run of A on w.
- **Theorem**: For every NFA A, there is an NFA A' s.t. A' will never get stuck, and $L(A) = L(A')$. How? Introduce an error state s.t. for every state, there is at least one transition for all alphabets.
- **Theorem**: NFA languages are closed under complement, $\cap$ (intersection) and $\cup$ (union).
- **Theorem**: For every NFA A, there is a DFA A' s.t. $L(A) = L(A')$.
  - Let an NFA $A = \langle \Sigma, Q, q_0, F, \delta \rangle$ and a DFA $A' = \langle \Sigma, Q', q_0', F', \delta' \rangle$ represents the same language L. Then, $Q' = 2^Q, q_0' = \{q_0\}, F' = \{X \in Q' | X \cap F \neq \emptyset\}, \delta' : Q' \times \Sigma \to Q'$ is defined for $X \in Q', a \in \Sigma$, let $\delta'(X, a) = \{q \mid \exists p \in X \text{ s.t. } (p, a, q) \in \delta\}$ or $\delta'(X, a) = \bigcup_{x \in X} \delta(x, a)$.
  - Both $L(A) \subseteq L(A')$ and $L(A') \subseteq L(A)$ should hold.
- **Lemma**: $L(A) \subseteq L(A')$ proved by induction
  - *Hypothesis*: For every accepting run of A on w: $p_0, a_1, p_1, a_2, p_2, \ldots, a_n, p_n$ s.t. $p_n \in F$, there is an accepting run of A' on w: $X_0, a_1, X_1, a_2, X_2, \ldots, a_n, X_n$ s.t. $X_i \ni p_i$ for $i = 1, 2, \ldots, n$, and $X_n \in F'$.
  - *Basis*: $|w| = 0$, i.e. $w = \epsilon$. The run of A on w is $q_0 \in F$, and the run of A' on w is $\{q_0\} \in F'$.
  - *Induction*: Consider $w = a_1 a_2 \ldots a_n a_{n+1}$. An accepting run of A on w is $p_0, a_1, p_1, a_2, p_2, \ldots, a_n, p_n, a_{n+1}, p_{n+1}$, s.t. $p_{n+1} \in F$ and $(p_n, a_{n+1}, p_{n+1}) \in \delta$. Then, the run of A' on w is $X_0, a_1, X_1, a_2, X_2, \ldots, a_n, X_n, a_{n+1}, X_{n+1}$. $X_n \ni p_n$ and $(p_n, a_{n+1}, p_{n+1}) \in \delta$ implies $X_{n+1} \ni p_{n+1}$. $p_{n+1} \in F$ and $X_{n+1} \ni p_{n+1}$ implies $X_{n+1} \cap F \neq \emptyset$, i.e. $X_{n+1} \in F'$.
- **Lemma**: $L(A') \subseteq L(A)$ proved by induction
  - *Hypothesis*: For the accepting run of A' on w: $X_0, a_1, X_1, a_2, X_2, \ldots, a_n, X_n$ s.t. $X_n \in F'$, there is an accepting run of A on w: $p_0, a_1, p_1, a_2, p_2, \ldots, a_n, p_n$ s.t. $p_i \in X_i$ for $i = 1, 2, \ldots, n$, and $p_n \in F$.
  - *Basis*.
  - *Induction*.
- Every NFA language is also regular.
- ϵ-NFA: $A = \langle \Sigma, Q, q_0, F, \delta \rangle$, where $\delta \subseteq Q \times (\Sigma \cup \{\epsilon\}) \times Q$.
- **Theorem**: For every ϵ-NFA, there is an NFA A' s.t. $L(A) = L(A')$