Homework 4

1. REACH $\in$ NLog-complete, i.e., REACH $\in$ NLog and for every $L \in$ NLog, $L \leq_{\log}$ REACH. Also, NLog = coNLog.
   - REACH $\in$ NLog = coNLog. Hence, REACH $\in$ coNLog.
   - For every $L \in$ NLog = coNLog, $L \leq_{\log}$ REACH. Hence, for every $L \in$ coNLog, $L \leq_{\log}$ REACH.
   - **Conclusion**: REACH $\in$ coNLog-complete.
2. 1. Let $L_1, L_2 \in$ NP. Then, for $i = 1, 2$, there exists an NTM M_i that decides L_i in polynomial time. An NTM M that decides $L_1 \cup L_2$ is constructed as follows: "On input w: Run M_1 and M_2 on w. Accept if M_1 accepts or M_2 accepts. Reject, otherwise." Since, M_1 and M_2 decides in polynomial time, M also decides in polynomial time. **Conclusion**: $L_1 \cup L_2 \in$ NP.
   2. Let $L_1, L_2 \in$ NP. Then, for $i = 1, 2$, there exists an NTM M_i that decides L_i in polynomial time. An NTM M that decides $L_1 \cap L_2$ is constructed as follows: "On input w: Run M_1 and M_2 on w. Accept if M_1 accepts and M_2 accepts. Reject, otherwise." Since, M_1 and M_2 decides in polynomial time, M also decides in polynomial time. **Conclusion**: $L_1 \cap L_2 \in$ NP.
3. 1. Let $L_1, L_2 \in$ coNP. Then, for $i = 1, 2$, there exists an NTM M_i that decides $\Sigma^* - L_i$ in polynomial time. An NTM M that decides $\Sigma^* - L_1 \cup L_2 = (\Sigma^* - L_1) \cap (\Sigma^* - L_2)$ is constructed as follows: "On input w: Run M_1 and M_2 on w. Accept if M_1 accepts and M_2 accepts. Reject, otherwise." Since, M_1 and M_2 decides in polynomial time, M also decides in polynomial time. Hence, $\Sigma^* - L_1 \cup L_2 \in$ NP. **Conclusion**: $L_1 \cup L_2 \in$ coNP.
   2. Let $L_1, L_2 \in$ coNP. Then, for $i = 1, 2$, there exists an NTM M_i that decides $\Sigma^* - L_i$ in polynomial time. An NTM M that decides $\Sigma^* - L_1 \cap L_2 = (\Sigma^* - L_1) \cup (\Sigma^* - L_2)$ is constructed as follows: "On input w: Run M_1 and M_2 on w. Accept if M_1 accepts or M_2 accepts. Reject, otherwise." Since, M_1 and M_2 decides in polynomial time, M also decides in polynomial time. Hence, $\Sigma^* - L_1 \cap L_2 \in$ NP. **Conclusion**: $L_1 \cap L_2 \in$ coNP.
4. - Suppose SAT $\in$ coNP. Then, Σ^* – SAT $\in$ NP. Since SAT is NP-hard, Σ^* – SAT $\leq_p$ SAT.
   - **Claim 1**: If $A \leq_p B$, then $\Sigma^* - A \leq_p \Sigma^* - B$. **Proof**: If $A \leq_p B$, then there exists a function f such that $w \in A$ iff $f(w) \in B$. Hence, $w \notin A$ iff $f(w) \notin B$, i.e., $w \in \Sigma^* - A$ iff $f(w) \in \Sigma^* - B$. Hence, $\Sigma^* - A \leq_p \Sigma^* - B$.
   - By **Claim 1** and Σ^* – SAT $\leq_p$ SAT, we have SAT $\leq_p$ Σ^* – SAT. Hence, Σ^* – SAT is NP-hard, i.e., for every $L \in$ NP, $L \leq_p$ Σ^* – SAT. By **Claim 1** again, we have $\Sigma^* - L \leq_p$ SAT. Since SAT $\in$ NP, $\Sigma^* - L \in$ NP, i.e., $L \in$ coNP. Hence, NP $\subseteq$ coNP. **Conclusion**: If SAT $\in$ coNP, then NP $\subseteq$ coNP.
   - Suppose NP $\subseteq$ coNP. Let $L \in$ coNP. By definition, $\Sigma^* - L \in$ NP $\subseteq$ coNP. Hence, $\Sigma^* - L \in$ coNP and $L \in$ NP. Hence, coNP $\subseteq$ NP. Hence, NP = coNP. **Conclusion**: If NP $\subseteq$ coNP, then NP = coNP.