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- Background:
  - Geometry: to study geometry w/ linearity, e.g. lines in 2D, planes in 3D, and hyperplanes in n-dimensional space.
  - Abstract algebra:
    - Algebra is the study of basic mathematical structures, e.g. groups, rings, fields, etc.
    - Linear algebra studies one of the structures called *vector space*.
    - Followed by logical deduction from basic definitions, we can derive some important theorems.
  - Applied mathematics:
    - Linear algebra is widely used in applied science, like mechanics and differential equations.
    - Linear programming was developed during World War II.
    - Recently, it is widely applied to image processing and computer graphics, etc.
- Introduction:
  - The central problem of linear algebra is the solution of linear equations. The most important and simplest case is when the number of unknown equals the number of equations.
  - There are two ways to solve linear equations: (1) the method of elimination (*Gaussian elimination*), and (2) determinants (*Cramer's rule*).
  - Four aspects that we should look into:
    - The geometry of linear equations.
    - The interpretation of elimination as a factorization of the coefficient matrix, e.g. $Ax = b$, $A = LU$, A^{T}, A^{-1}, etc.
    - Irregular or singular cases, i.e. no solution or infinitely many solutions.
    - The number of operations to solve the system by elimination.
- A vector in an $n \times 1$ array w/ n real numbers. In the text, we usually write it as $(c_1, c_2, \ldots, c_n) \in \mathbb{R}^n$.
- Geometry of linear equations of two equations in two dimensions:
  - Approach 1: row picture → two lines in the plane.
  - Approach 2: column picture → linear combination of two vectors in the plane.
- Geometry of linear equations of three equations in three dimensions:
  - Approach 1: row picture → three planes in 3 dimensions.
  - Approach 2: column picture → linear combination of three vectors in 3 dimensions.
- Question: How to extend into n dimensions?

    - Each equation represents an $n-1$-dimensional hyperplane in n dimensions.
    - They intersect at a smaller set in lower dimensions or do not intersect at all.
- The singular cases of three (different) equations in three dimensions from the perspective of row picture:
    - Case 1: all three planes are perpendicular to a common plane → no solution.
    - Case 2: any two planes are parallel → no solution.
    - Case 3: three planes intersect in a line → infinitely many solutions.
    - Case 4: three planes are parallel → no solution.
- The singular cases of three (different) equations in three dimensions from the perspective of column picture: three vectors are coplanar (in the same plane).
    - The RHS vector is in the plane → infinitely many solutions.
    - The RHS vector is not in the plane → no solution.
- If n planes have no point in common, then the n columns lie in the same plane.