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- Properties of transpose: $(A+B)^{\mathrm{T}} = A^{\mathrm{T}} + B^{\mathrm{T}}$; $(A^{\mathrm{T}})^{\mathrm{T}} = A$; $(AB)^{\mathrm{T}} = B^{\mathrm{T}} A^{\mathrm{T}}$; $(A^{-1})^{\mathrm{T}} = (A^{\mathrm{T}})^{-1}$.
- A **symmetric matrix** is a matrix that equals its own transpose, i.e. $A^{\mathrm{T}} = A$.
- A symmetric matrix need *not* be invertible.
- If A is symmetric and can be factored into $A = LDU$ without row exchanges, then U is the transpose of L. The symmetric factorization becomes $A = LDL^{\mathrm{T}}$.
- F is a **field** if the system $(F, +, \cdot)$ satisfies the following conditions:
  - Associativity of $+$ and $\cdot$
  - Commutativity of $+$ and $\cdot$
  - Distributivity of $\cdot$ over $+$
  - **Zero**: $\exists 0 \in F, \forall a \in F$, s.t. $a + 0 = 0 + a = a$
  - $\forall a \in F, \exists -a \in F$, s.t. $a + (-a) = (-a) + a = 0$
  - **Unity**: $\exists 1 \in F, \forall a \in F$, s.t. $a \cdot 1 = 1 \cdot a = a$
  - $\forall a \neq 0 \in F, \exists a^{-1} \in F$, s.t. $a \cdot a^{-1} = a^{-1} \cdot a = 1$
- $\mathbb{C}, \mathbb{R}, \mathbb{Q}$ are fields. $\mathbb{Z}, \mathbb{N}$ are not.
- V is a **vector space** over F (denoted as V/F) if addition and scalar multiplication are defined on V and satisfy the algebraic rules of vector algebra:
  - Addition is associative.
  - Addition is commutative.
  - $\exists 0 \in V, \forall v \in V$, s.t. $v + 0 = 0 + v = v$
  - $\forall v \in V, \exists -v \in V$, s.t. $v + (-v) = (-v) + v = 0$
  - $\exists 1 \in F, \forall v \in V$, s.t. $v \cdot 1 = 1 \cdot v = v$
  - $\forall v \in V, \forall \lambda, \mu \in F$, s.t. $(\lambda\mu)v = \lambda(\mu v)$
  - $\forall v \in V, \forall \lambda, \mu \in F$, s.t. $(\lambda + \mu)v = \lambda v + \mu v$
  - $\forall v_1, v_2 \in V, \forall \lambda \in F$, s.t. $\lambda(v_1 + v_2) = \lambda v_1 + \lambda v_2$
- Examples of vector spaces:
  - $\mathbb{R}^n/\mathbb{R}$
  - $\mathbb{R}^{m \times n}/\mathbb{R}$
  - Let V be the set of all symmetric matrices over $\mathbb{R}$. $V/\mathbb{R}$
  - Let V be the set of all real-valued functions defined on $[0, 1]$. $V/\mathbb{R}$
  - Let V be the set of all positive $\mathbb{R}$. Define $x + y = xy$ and $cx = x^c$. $V/\mathbb{R}$
- S is a **subspace** of V over F (denoted as $S \leq V$) iff
  - S is a nonempty subset of V s.t. S itself is also a vector space over F.
  - S is closed under addition and scalar multiplication, i.e. $\forall x, y \in S\ \forall \alpha \in F$ s.t. $\alpha x + y \in S$.
- Examples of subspaces of $\mathbb{R}^{m \times n}/\mathbb{R}$:
  - The set of all symmetric matrices.
  - The set of all upper/lower matrices.

    - The set of diagonal matrices.
- The zero vector belongs to every subspace.
- If w_1 and w_2 are subspaces of V over F, then $w_1 \cap w_2 \neq \emptyset$, because $w_1 \cap w_2$ contains at least the zero vector.
- Let $V = \mathbb{R}^2/\mathbb{R}$, the followings are subspaces of V: origin, all lines through origin, V itself.
- **Column space** of $A \in \mathbb{R}^{m\times n}$ (denoted as $C(A) \leq \mathbb{R}^m/\mathbb{R}$) is the set of all combinations of columns of A.
- The system $Ax = b$ is solvable iff $b \in C(A)$, i.e. $\exists x$ s.t. $Ax = b$ iff $b \in C(A)$.
- Examples of column spaces:
    - $A = 0_{m\times n} \implies C(A) = 0$
    - $A = I_m \implies C(A) = \mathbb{R}^m$
- **Nullspace** of $A \in \mathbb{R}^{m\times n}$ (denoted as $N(A) \leq \mathbb{R}^n/\mathbb{R}$) is $\{x \in \mathbb{R}^n \mid Ax = 0\}$
- The system $Ax = 0$ is called a **homogeneous equation**.
- The solution set of $Ax = b$ is *not* a subspace of $\mathbb{R}^n/\mathbb{R}$.