Homework 11

1. $T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

   a. Eigenvalues $\lambda = -1, -1, 0, 0$. Eigenvectors $v = (0, 0, 1, 0), (1, 0, 0, 0), (0, 0, 0, 1), (0, 1, 0, 0)$.

   b. $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \mathrm{T} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

2. $\lambda_1 + \lambda_2 = 7$ and $\lambda_1\lambda_2 = 12$. Eigenvalues $\lambda = 4, 3$.

3. a. $\det(A - I - \lambda I) = \lambda^4 - 6\lambda^2 - 8\lambda - 3 = (\lambda - 3)(\lambda + 1)^3 := 0$. $\lambda = 3, -1, -1, -1$.

   b. $\det(A - I) = -3$.

4. a. Nullspace: $\{u\}$. Column space: $\{v, w\}$.

   b. $Av = 3v$ and $Aw = 5w$. Hence, $A(\frac{1}{3}v + \frac{1}{5}w) = v + w$.
   $x_p = \frac{1}{3}v + \frac{1}{5}w$. $x = \alpha u + \frac{1}{3}v + \frac{1}{5}w$ where $\alpha \in \mathbb{R}$.

   c. u, v, w are independent. Hence, u is not in $\mathrm{span}\{v, w\}$, and thus, not in the column space of A.
   Hence, $Ax = u$ has no solution.

5. $A = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$. $u^\top A^{-1} u = \frac{3}{16}$.

6. $\det(Q - \lambda I) = (\cos\theta - \lambda)^2 + \sin^2\theta = \lambda^2 - 2\lambda\cos\theta + 1 := 0$. $\lambda = \cos\theta \pm i\sin\theta$.
   When $\lambda = \cos\theta + i\sin\theta$, $v = (i, 1)$.
   When $\lambda = \cos\theta - i\sin\theta$, $v = (-i, 1)$.