Discrete Mathematics Final

2017-05-04

- **Graphs**:
  - **Undirected graph** and **edges**
  - **Directed graph (digraph)** and **arcs**
  - **Multigraph** : E is a multiset of edges.
  - **Simple graph**: no *loops* and no more than one edge connecting the same pair of vertices.
  - **Complete graph**: each pair of distinct vertices is connected by an edge. Denoted as $K_{|V|}$
  - **Bipartite graph**: there exist X and Y such that $V = X \cup Y(X \cap Y = \emptyset)$ and $E = \{(i, j) | i \in X, j \in Y\}$.
  - **Complete bipartite graph**: each vertex of X is connected to each vertex of Y. Denoted as $K_{|X|,|Y|}$.
  - **Regular graph**: every vertex has the same degree.
  - **Subgraph**: if $G' = (V', E')$ is a subgraph of $G = (V, E)$, then $V' \subseteq V$ and $E' \subseteq E$.
  - **Spanning subgraph**: a subgraph that contains all the vertices of the original graph.
  - **Induced subgraph**: a subset of the vertices of the graph together with any edges connecting pairs of vertices in that subset.
  - **Underlying graph**: the undirected graph that replaces all arcs in a digraph with edges.

2017-05-11

- **Isomorphism**: $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are **isomorphic** iff there exists a one-to-one and onto mapping $f : V1 \to V2$ such that $(i, j) \in E_1 \leftrightarrow (f(i), f(j)) \in E_2$.
- **Walk**: an arbitrary sequence of vertices and edges
  - **Trail**: all edges are distinct.
  - **Path**: all vertices are distinct.
  - **Circuit**: a trail which starts and ends at the same vertex.
  - **Cycle**: a path which starts and ends at the same vertex.
- **Connected components**:
  - **Connected**: each pair of vertices forms the endpoints of a path.
  - **Component**: a maximal connected subgraph.

- **Connected graph**: a graph consists of one single connected component.
- **Strongly connected digraph**: each pair of vertices has a directed path to each other.
- **Weakly connected digraph**: whose underlying graph is connected.

- Theorems:
  - If $G = (V, E)$ is a connected graph with $|V| > 1$, then G contains either a vertex of degree 1 or a cycle (or both).
  - If $G = (V, E)$ is a connected graph, and $(i, j) \in E$ be an edge that is contained in one cycle of G. Then, $G - (i, j)$ remains connected.
  - Every n-vertex connected undirected graph contains at least $n - 1$ edges.
  - Every n-vertex strongly connected digraph contains at least n arcs.
- **Tree**:
  - **Tree**: a tree of n vertices is an n-vertex connected undirected graph that contains exactly $n - 1$ edges.
  - **Spanning tree**: a tree which is also a spanning subgraph of an undirected graph.
  - **Minimum spanning tree (MST)**: **Kruskal's** algorithm, **Prim's** algorithm, **Sollin's** algorithm.
- The number of spanning tress: $S(G) = S(G - e) + S(G \cdot e)$
  - $S(G - e)$: the number of spanning trees that do not contain edge e.
  - $S(G \cdot e)$: the number of spanning trees that contains edge e.
- Algorithms for searching MSTs: **Kruskal's algorithm**, **Prim's algorithm**, and **Sollin's algorithm**.
- G has a unique MST if its edge costs are all distinct.

2017-05-18

- **Connectivity**:
  - Let $G = (V, E)$ be a connected undirected graph. A subset S of V is called a **vertex cut** of G iff $G - S = (V - S, E - \{(i, j) | i \in S \vee j \in S, (i, j) \in E\})$ is disconnected.
  - A k-vertex cut is a vertex cut of k vertices.
  - The **connectivity** of G is the minimum k such that G has a k-vertex cut.
  - The connectivity of K_n, which has no vertex cut, is defined to be $n - 1$.
  - G is k-connected, if its connectivity ≥ k.
  - v is an **articulation point** of G iff $\{v\}$ is a vertex cut of G.
  - A connected graph G is **biconnected** (or 2-connected) iff G has no articulation point.
- Finding articulation points:
  - $\mathbf{DFN}(i)$: the visiting sequence of vertices by DFS.

- $\mathrm{L}(i)$: the least DFN reachable from i through a path consisting of zero or more (downward) tree edges followed by zero or one back edge.
- $i \in V$ is an articulation point of G iff either
  - i is the root and has at least two children.
  - i is not the root and has a child j with $\mathrm{L}(j) \geq \mathrm{DFN}(i)$.

- Theorem: Suppose G is a connected graph and T is a depth-first spanning tree of G. Then, G contains no cross edge with respect to T.
- **Edge connectivity**:
  - Given $G = (V, E)$, $S \subset E$ is an **edge cut** of G iff $G - S = (V, E - S)$ is disconnected.
  - A k-edge cut is an edge cut of k edges.
  - The **edge connectivity** of G is the minimum k such that G has a k-edge cut.
  - G is k-edge-connected, if its edge connectivity ≥ k.
  - (i, j) is a **bridge** of G iff $\{(i, j)\}$ is an edge cut of G.
- Finding bridges:
  - $\mathrm{DFN}(i)$ and $\mathrm{L}(i)$ are defined in the same way as in articulation points.
  - $(i, j) \in E$ is a bridge of G iff $\mathrm{L}(j) = \mathrm{DFN}(j)$ given $\mathrm{DFN}(i) < \mathrm{DFN}(j)$

2017-05-25

- **Euler trails** & **Euler circuits**: A trail (circuit) is called an Euler trail (Euler circuit) of G iff it traverses each edge of G exactly once.
- Theorem: Let $G = (V, E)$ be a connected undirected graph, where $|V| \geq 1$. Then,
  - G has an Euler trail, but not an Euler circuit, iff it has exactly two vertices of odd degrees.
  - G has an Euler circuit iff all vertices have even degrees.
- Theorem: Suppose that $G = (V, E)$ is a directed graph, where $|V| > 1$. Let d^{in} and d^{out} denote the indegree and outdegree of vertex i, respectively. Then, G has a u-to-v Euler trail iff the underlying graph of G is connected and either
  - $u = v$ and $d^{\mathrm{in}} = d^{\mathrm{out}}$ for every $i \in V$.
  - $u \neq v$, $d^{\mathrm{in}} = d^{\mathrm{out}}$ for every $i \in V - \{u, v\}$, $d_u^{\mathrm{in}} = d_u^{\mathrm{out}} - 1$, and $d_u^{\mathrm{in}} = d_u^{\mathrm{out}} + 1$.
- **Hamiltonian paths** & **Hamiltonian cycles**: A path (cycle) is called a Hamiltonian path (cycle) of G iff it goes through each vertex (exclusive of the starting vertex and ending vertex) of G exactly once.
- Theorem: Suppose that $G = (V, E)$ is a directed graph and between every two vertices u, v of G, there is one arc (either $\langle u, v\rangle$ or $\langle v, u\rangle$). Then, there exists a directed *Hamiltonian path* in G.
- Theorem: Suppose that $G = (V, E)$ is an undirected graph where $|V| = n$. Let d_i be the degree

of vertex v_i.

  - If $d_i + d_j \geq n - 1$ for every $(v_i, v_j) \notin E$ and $v_i \neq v_j$, then G has a *Hamiltonian path*.
  - If $d_i + d_j \geq n$ for every $(v_i, v_j) \notin E$ and $v_i \neq v_j$, then G has a *Hamiltonian cycle*.
- Theorem: Suppose that $G = (V, E)$ is an undirected graph where $|V| = n$. If for every $1 \leq i \leq \lfloor (n-1)/2 \rfloor$, G has fewer than i vertices with degrees at most i, then G has a Hamiltonian cycle. When n is odd, G has a Hamiltonian cycle, even if G has $(n-1)/2$ vertices of degrees $(n-1)/2$.
- Shortest paths: **Dijkstra's algorithm**.

2017-06-01

- Closure:
  - **Transitive closure**: $A^+ = \sum_{i=1}^{\infty} A^i$
  - **Reflexive transitive closure**: $A^+ = \sum_{i=0}^{\infty} A^i$, where A^0 is the identity.
  - A transitive relation implies *reachability*.
- **Planar graph**: A graph is **planar** iff it can be drawn so that no two edges cross. Such a drawing is called a **planar drawing**.
- Theorems: Let $G = (V, E)$ be a connected planar graph, and r be the number of regions.
  - $|V| - |E| + r = 2$
  - $|E| \leq 3|V| - 6$ if $|E| \geq 2$
  - Every planar drawing of a connected planar graph G has the same number $r = 2 - |V| + |E|$ of regions.
  - Suppose G has k connected components. Then, $|V| - |E| + r = k + 1$.
- **Contractible**: H is **contractible** to G iff G can be obtained from H by a series of elementary contractions.
- **Homeomorphic**: two graphs are said to be **homeomorphic** if they can be obtained from the same graph by adding vertices onto some of its edges, or one can be obtained from the other by the same way.
- Theorems:
  - A graph G is planar iff no subgraph of G is *contractible* to $K_{3,3}$ or K_5.
  - A graph G is planar iff no subgraph of G is *homeomorphic* to $K_{3,3}$ or K_5.
- **Matching**:
  - $M \subseteq E$ is a **matching** in $G = (V, E)$ if no two edges in M are incident on the same vertex.
  - **Maximal matching**: M is a **maximal matching** if there exists no matching M' with

$|M'| > |M|$ in G.
  - **Perfect matching**: M is a **perfect matching** if $|V| = 2 \times |M|$.
  - **Complete matching**: M is a **complete matching** iff $|M| = \min\{|S|, |R|\}$ for a bipartite graph.
- Theorem: Suppose that $G = (R \bigcup S, E)$ is a bipartite graph, where $|R| \leq |S|$ is assumed. For any $W \subseteq R$, let $\mathrm{ADJ}(W)$ be the set of vertices adjacent to any vertex in W. Then, G has a complete matching iff $|W| \leq |\mathrm{ADJ}(W)|$ for every $W \subseteq R$.

2017-06-08

- Cliques, independent sets, vertex covers:
  - **Clique**: a set of vertices every two of which are adjacent.
  - **Independent set**: a set of vertices no two of which are adjacent.
  - **Vertex cover**: a set of vertices such that each edge in the graph is incident with at least one vertex in the set.
- Theorem: suppose that $G = (V, E)$ is an undirected graph and $V' \subseteq V$. The following statements are equivalent:
  - V' is a *clique* of G.
  - V' is an *independent set* of $\bar{G}$.
  - $V - V'$ is a *vertex cover* of $\bar{G}$.
- Maximum flow and minimum cut:
  - Transport network $N = (V, E)$ has a pair of **source** node a and **sink** node z.
  - A **flow** is a function f from E to the set of nonnegative integers, satisfying
    - Capacity constraint: $0 \leq f(e) \leq c(e)$ for each $e \in E$
    - Conservation constraint: $\psi^+(v) = \psi^-(v)$ for $v \notin \{a, z\}$
  - The **total flow** (or **net flow**) of f is defined to be $F = \psi^-(a) = \psi^+(z)$.
  - The **maximum flow problem** is to determine f such that F is maximum.
  - **Cut**: $E(S; S') \cup E(S'; S)$ is called a cut (or a-z cut) of N, where $S \subset V, S = V - S, a \in S, z \in S'$.
  - *Capacity* of the cut induced by S: $c(S) = \sum_{e \in E(S;S')} c(e)$
  - **Minimum cut**: $E(S; S') \cup E(S'; S)$ is a minimum cut if $c(S)$ is minimum.
- Lemmas and theorems:
  - **Conservation of flow**: $F = f(S, S') - f(S', S)$ for any $S \subset V$ and $a \in S$.
  - If $F = c(S)$ for some $S \subset V$, then F is maximum and $c(S)$ is minimum.
  - $F = c(S)$ iff (a) $f(e) = c(e)$ for each $e \in E(S; S')$; (b) $f(e) = 0$ for each $e \in E(S'; S)$.

2017-06-15

- **Ford & Fulkerson's algorithm**
  - **Augmenting path**: a path in N is an *augmenting path* if its each forward edge is unsaturated and its each backward edge e has $f(e) > 0$.
  - The maximal increment of flow by an augmenting a-to-z path P is equal to $\Delta_P = \min\{\min\{c(e) - f(e) | e \text{ is a forward edge}\}, \min\{f(e) | e \text{ is a backward edge}\}\}$.
  - The updated flow $f^+(e) = \begin{cases} f(e) + \Delta_P, & \text{if e is a forward edge} \\ f(e) - \Delta_P, & \text{if e is a backward edge} \\ f(e), & \text{if e is not an edge of P} \end{cases}$.
  - F is maximum iff there is no augmenting a-to-z path in N.
  - The capacities must be rational numbers; otherwise, Ford & Fulkerson's algorithm may cause an infinite sequence of flow augmentations, and the flow finally converges to a value that is 1/4 of the maximum total flow.
  - Ford & Fulkerson's algorithm takes exponential time in the worst case.
- **Edmonds & Karp's algorithm**:
  - Overcomes the two flaws of Ford & Fulkerson's algorithm.
  - Uses BFS to find shortest augmenting paths iteratively.
- Coloring:
  - A **proper coloring** of a graph G is an assignment of colors to the vertices of G so that no two adjacent vertices are assigned with the same color.
  - The **chromatic number** of G, denoted by $\chi(G)$, is the smallest number of colors needed to properly color G.
  - A graph is **k-colorable** iff it can be properly colored with k colors.
- Properties and theorems:
  - $\chi(K_n) = n$
  - An undirected graph G is 2-colorable (i.e., bipartite) iff G has no cycle of odd length.