Homework 1

1. a. $\begin{bmatrix}1\\3\\2\end{bmatrix} = -\begin{bmatrix}1\\1\\0\end{bmatrix} + 2\begin{bmatrix}1\\2\\1\end{bmatrix}$

   b. (u,v,w) is parallel to $(1,1,1)\times(0,1,2) = (1,-2,1)$. Therefore, $(u,v,w) \in \{(1,-2,1)t \mid t \in \mathbb{R}\}$.

2. $\begin{bmatrix}2 & 3 & 1 & | & 8\\4 & 7 & 5 & | & 20\\0 & -2 & 2 & | & 0\end{bmatrix} \to \begin{bmatrix}2 & 3 & 1 & | & 8\\0 & 1 & 3 & | & 4\\0 & -2 & 2 & | & 0\end{bmatrix} \to \begin{bmatrix}2 & 3 & 1 & | & 8\\0 & 1 & 3 & | & 4\\0 & 0 & 8 & | & 8\end{bmatrix}$.

   Therefore, $z = 1$, $y = 4 - 3z = 1$, $x = (8 - z - 3y)/2 = 2$. Therefore, $(x,y,z) = (2,1,1)$.

3. $\begin{vmatrix}a & 2 & 3\\a & a & 4\\a & a & a\end{vmatrix} = a(a-4)(a-2) := 0$. Therefore, $a = 0,2,4$.

4. $\begin{bmatrix}k & 1 & 1 & | & 1\\1 & k & 1 & | & 1\\1 & 1 & k & | & 1\end{bmatrix} \to \begin{bmatrix}1 & 1 & k & | & 1\\1 & k & 1 & | & 1\\k & 1 & 1 & | & 1\end{bmatrix} \to \begin{bmatrix}1 & 1 & k & | & 1\\0 & k-1 & 1-k & | & 0\\0 & 1-k & 1-k^2 & | & 1-k\end{bmatrix} \to$
   $\begin{bmatrix}1 & 1 & k & | & 1\\0 & k-1 & 1-k & | & 0\\0 & 0 & 2-k-k^2 & | & 1-k\end{bmatrix}$

   a. No solution if $k-1 \neq 0$ and $2-k-k^2 = 0$, i.e. $k = -2$

   b. One solution if $k-1 \neq 0$ and $2-k-k^2 \neq 0$, i.e. $k \in \mathbb{R} - \{1,-2\}$

   c. Infinitely many solutions if $k-1 = 0$ and $2-k-k^2 = 0$, i.e. $k = 1$

5. Four equations $x_1 + x_4 = 600 + 400$, $800 + x_1 = 800 + x_2$, $x_2 + 1200 = x_3 + 1000$, $x_3 + x_4 = 1000 + 600$ can be simplified to $x_1 + x_4 = 1000$, $x_1 - x_2 = 0$, $x_2 - x_3 = -200$, $x_3 + x_4 = 1600$, which can be expressed in matrix form: $\begin{bmatrix}1 & 0 & 0 & 1\\1 & -1 & 0 & 0\\0 & 1 & -1 & 0\\0 & 0 & 1 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}1000\\0\\-200\\1600\end{bmatrix}$.

   By Gaussian elimination, $\begin{bmatrix}1 & 0 & 0 & 1 & | & 1000\\1 & -1 & 0 & 0 & | & 0\\0 & 1 & -1 & 0 & | & -200\\0 & 0 & 1 & 1 & | & 1600\end{bmatrix} \to \begin{bmatrix}1 & 0 & 0 & 1 & | & 1000\\0 & -1 & 0 & -1 & | & -1000\\0 & 1 & -1 & 0 & | & -200\\0 & 0 & 1 & 1 & | & 1600\end{bmatrix} \to$
   $\begin{bmatrix}1 & 0 & 0 & 1 & | & 1000\\0 & -1 & 0 & -1 & | & -1000\\0 & 0 & -1 & -1 & | & -1200\\0 & 0 & 1 & 1 & | & 1600\end{bmatrix} \to \begin{bmatrix}1 & 0 & 0 & 1 & | & 1000\\0 & -1 & 0 & -1 & | & -1000\\0 & 0 & -1 & -1 & | & -1200\\0 & 0 & 0 & 0 & | & 400\end{bmatrix}$. Therefore, there is no solution.

6. a. I_3 b. I_3 c. I_3 d. I_3

7. $(A+B)^2 = A(A+B) + B(A+B) = (A+B)(B+A) = A^2 + AB + BA + B^2$, i.e. (b), (c), (d) are guaranteed.

8. $E_{21} = \begin{bmatrix}1 & 0 & 0\\-4 & 1 & 0\\0 & 0 & 1\end{bmatrix}$, $E_{31} = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\2 & 0 & 1\end{bmatrix}$, $E_{32} = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & -2 & 1\end{bmatrix}$

9. $A\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix} = \begin{bmatrix}a+b & a+b\\c+d & c+d\end{bmatrix} := \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}A = \begin{bmatrix}a+c & b+d\\a+c & b+d\end{bmatrix}$. Therefore, $A \in \{\begin{bmatrix}a & b\\c & d\end{bmatrix} \mid b = c, a = d\}$