2017-10-17

- The solution to $Ax = b$, where A is a square matrix:
  - A is not singular $\Rightarrow x = A^{-1}b$.
  - A is singular and $b = 0 \Rightarrow x$ is in the nullspace of A.
  - A is singular and $b \neq 0 \Rightarrow$ no solution or infinitely many solution.
- A matrix $R_{m \times n}$ is called a **row echelon matrix** if
  - The nonzero rows come first and the pivots are the first nonzero entries in those rows.
  - Below each pivot is a column of zeros.
  - Each pivot lies to the right of the pivot in the row above.
- A matrix $R_{m \times n}$ is called a **row-reduced echelon matrix** if
  - The nonzero rows come first and the pivots are the first nonzero entries in those rows and *normalized to be 1*.
  - *Above and below* each pivot is a column of zeros.
  - Each pivot lies to the right of the pivot in the row above.
- To any matrix $A_{m \times n}$, there exist a permuatation matrix P, a lower triangular matrix L w/ unit diagnoal, and a row echelon matrix $U_{m \times n}$ such that $PA = LU$. Every matrix $A_{m \times n}$ is **row equivalent** to a row echelon matrix.
- Homogeneous cases ($b = 0$):
  - The components of x which correspond to columns w/ pivots are called **pivot variables**, and those corresponding to columns w/o pivots are called **free variables**.
  - The pivot variables are expressed in terms of free variables by back substitution.
  - Set one free variable to be 1 and the others to be zero and solve $Ux = 0$ for pivot variables.
  - The combination of the solutions from each free variable is the solution.
  - The solution set of $Ax = 0$ is the nullspace of A, i.e. $x \in N(A)$.
- If a homogeneous system $A_{m \times n}x = 0$ has more unknowns than equations ($m < n$), it has a nontrivial solution.
- The nullspace is a subspace of the same dimension (degree of freedom) as the number of free variables.
- Inhomogeneous cases ($b \neq 0$):
  - The pivot variables are expressed in terms of free variables by back substitution.
  - *Particular solution*: Set all free variables to 0 and solve $Ux = L^{-1}b$ for pivot variables.
  - *Homogeneous solution*: Set one free variable to be 1 and the others to be zero and solve $Ux = 0$ for pivot variables.
  - $x = x_p + x_h$, where x is the general solution, x_p is the particular solution, and x_h is the homogeneous solution.
- The set of general solutions is not a subspace since it does not contain the zero vector (origin). It is parallel to the nullspace of A.
- Given a matrix $A_{m \times n}$, if there are r pivots, there are r pivot variables and $n - r$ free variables. The

number of pivots, r, is called the **rank** of A.

- Suppose elimination reduces $A_{m\times n}x = b$ to $Ux = L^{-1}b$ and there are r pivots. Then, the last $m - r$ rows of U are zeros, and there are $n - r$ free variables.
  - If $r = m$, threre are $n - r$ free variables and the column space $C(A) = \mathbb{R}^m$. There's always a solution, which is the sum of particular solution and a homogeneous solution.
  - If $r = n$, there are no free variables and the nullspace contains $x = 0$ only, i.e. $N(A) = \{0\}$. There is a solution only if the last $m - r$ elements of $L^{-1}b$ are zeros as well.
- Let V be a vector space over F. A nonempty subset S of V is said to be **linearly dependent** if there exist distinct vectors $v_1, v_2, \ldots, v_n$ in S and at least one nonzero scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ in F s.t. $\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n = 0$; Otherwise, S are **linearly independent**.
- To show that $v_1, v_2, \ldots, v_n$ are linearly independent, we should verify that if $\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n = 0$ for some $\alpha_i \in F$, then α_i must be zero for all i.
- In $\mathbb{R}^2$, if v_1 and v_2 are not colinear iff they are linearly independent. Any three vectors in $\mathbb{R}^2$ are linearly dependent.
- If $v_1 = v_2$, the the set $\{v_1, v_2, \ldots, v_n\}$ are linearly dependent.
- Any set which contains zero vector is linearly dependent.