Homework 9

1. $1, x, x^2$ are three basis. Let $\{q_1, q_2, q_3\}$ be an orthonormal set.
$q_1' = 1 \Rightarrow q_1 = \frac{1}{\sqrt{\int_{-1}^{1} 1dx}} = \frac{\sqrt{2}}{2}$
$q_2' = x - (\int_{-1}^{1} \frac{\sqrt{2}}{2} xdx)\frac{\sqrt{2}}{2} = x \Rightarrow q_2 = \frac{x}{\sqrt{\int_{-1}^{1} x^2dx}} = \frac{\sqrt{6}}{2}x$
$q_3' = x^2 - (\int_{-1}^{1} \frac{\sqrt{2}}{2} x^2dx)\frac{\sqrt{2}}{2} - (\int_{-1}^{1} \frac{\sqrt{6}}{2} x^3dx)\frac{\sqrt{6}}{2}x = x^2 - \frac{1}{3} \Rightarrow q_3 = \frac{x^2-\frac{1}{3}}{\sqrt{\int_{-1}^{1}(x^2-\frac{1}{3})^2dx}} = \frac{3\sqrt{10}}{4}x^2 - \frac{\sqrt{10}}{4}$

2. Let $\{q_1, q_2, q_3\}$ be an orthonormal set than spans the column space of A.
$q_1' = (1,0,1) \Rightarrow q_1 = \frac{1}{\sqrt{2}}(1,0,1)$
$q_2' = (2,1,4) - (\frac{1}{\sqrt{2}}(1,0,1)\cdot(2,1,4))\frac{1}{\sqrt{2}}(1,0,1) = (-1,1,1) \Rightarrow q_2 = \frac{1}{\sqrt{3}}(-1,1,1)$
$q_3' = (3,1,6) - (\frac{1}{\sqrt{2}}(1,0,1)\cdot(3,1,6))\frac{1}{\sqrt{2}}(1,0,1) - (\frac{1}{\sqrt{3}}(-1,1,1)\cdot(3,1,6))\frac{1}{\sqrt{3}}(-1,1,1)$
$= (-\frac{1}{6}, -\frac{1}{3}, \frac{1}{6}) \Rightarrow q_3 = \sqrt{6}(-\frac{1}{6}, -\frac{1}{3}, \frac{1}{6}) = \frac{1}{\sqrt{6}}(-1,-2,1)$
Hence, $A = QR = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}\begin{bmatrix} \sqrt{2} & \frac{6}{\sqrt{2}} & \frac{9}{\sqrt{2}} \\ 0 & \sqrt{3} & \frac{4}{\sqrt{3}} \\ 0 & 0 & \frac{1}{\sqrt{6}} \end{bmatrix}$.

3. $\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{vmatrix} = -x_1 + x_3 + x_4$. $(1,1,1,1) - \frac{(-1,0,1,1)\cdot(1,1,1,1)}{(-1,0,1,1)\cdot(-1,0,1,1)}(-1,0,1,1) = (\frac{4}{3}, 1, \frac{2}{3}, \frac{2}{3})$.

4. Let $\{q_1, q_2\}$ be an orthonormal set than spans the column space of A.
$q_1' = (2,1,2) \Rightarrow q_1 = \frac{1}{3}(2,1,2)$
$q_2' = (1,1,1) - (\frac{1}{3}(2,1,2)\cdot(1,1,1))\frac{1}{3}(2,1,2) = \frac{1}{9}(-1,4,-1) \Rightarrow q_2 = \frac{1}{3\sqrt{2}}(-1,4,-1)$
Hence, $A = QR = \begin{bmatrix} \frac{2}{3} & \frac{-1}{3\sqrt{2}} \\ \frac{1}{3} & \frac{4}{3\sqrt{2}} \\ \frac{2}{3} & \frac{-1}{3\sqrt{2}} \end{bmatrix}\begin{bmatrix} 3 & \frac{5}{3} \\ 0 & \frac{\sqrt{2}}{3} \end{bmatrix}$.
$Ax = b \Rightarrow \tilde{x} = R^{-1}Q^{\top}b = \frac{1}{\sqrt{2}}\begin{bmatrix} \frac{\sqrt{2}}{3} & -\frac{5}{3} \\ 0 & 3 \end{bmatrix}\begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{-1}{3\sqrt{2}} & \frac{4}{3\sqrt{2}} & \frac{-1}{3\sqrt{2}} \end{bmatrix}\begin{bmatrix} 12 \\ 6 \\ 18 \end{bmatrix} = \begin{bmatrix} 9 \\ -3 \end{bmatrix}$

5. The parallelepiped is the region defined by $0 \le 2x - 2y + z \le 4, 0 \le 2y - z \le 2, 0 \le z \le 2$.
Let $u = 2x - 2y + z, v = 2y - z, w = z$, where $0 \le u \le 4, 0 \le v \le 2, 0 \le w \le 2$.
Then, $x = \frac{1}{2}(u+v), y = \frac{1}{2}(v+w), z = w$.
$\int (x-y)dV = \iiint (x-y)dxdydz = \int_0^2 \int_0^2 \int_0^4 (\frac{1}{2}(u+v) - \frac{1}{2}(v+w))\begin{vmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 \end{vmatrix} dudvdw =$
$\frac{1}{8}\int_0^2 \int_0^2 \int_0^4 (u-w)dudvdw = \frac{1}{2}\int_0^2 \int_0^2 (2-w)dvdw = \int_0^2 (2-w)dw = 2.$