Homework 3 B00401062 羅文斌

4.3

Let $f_0(x) = (1/2)x^{\mathrm{T}}Px + q^{\mathrm{T}}x + r$, which is quadratic and differentiable. The optimality condition for differentiable f_0 is $\nabla f_0(x^*)^{\mathrm{T}}(y - x^*) \geq 0$ for all feasible y.

$\nabla f_0(x) = (1/2)(P + P^{\mathrm{T}})x + q = Px + q$ (because P is symmetric).

$\nabla f_0(x^*)^{\mathrm{T}}(y - x^*) = (Px^* + q)^{\mathrm{T}}(y - x^*) = (\begin{bmatrix} 13 & 12 & -2 \\ 12 & 17 & 6 \\ -2 & 6 & 12 \end{bmatrix}\begin{bmatrix} 1 \\ 1/2 \\ -1 \end{bmatrix} + \begin{bmatrix} -22 \\ -14.5 \\ 13 \end{bmatrix})^{\mathrm{T}}(\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} - \begin{bmatrix} 1 \\ 1/2 \\ -1 \end{bmatrix})$

$= -1(y_1 - 1) + 2(y_3 + 1) \geq -1(1 - 1) + 2(-1 + 1) = 0$ (because $-1 \leq y_i \leq 1, i = 1, 2, 3$).

The optimality condition holds for all feasible y satisfying $-1 \leq y_i \leq 1, i = 1, 2, 3$.

Therefore, $x^* = (1, 1/2, -1)$ is optimal.

4.11

a. Minimize $\|Ax - b\|_\infty$

≡ Minimize $\max|Ax - b|$ (i.e. minimize $\max\{|a_1^{\mathrm{T}}x - b_1|, \ldots, |a_m^{\mathrm{T}}x - b_m|\}$)

≡ Minimize t subject to $|Ax - b| \preccurlyeq t \cdot 1$ (i.e. $|a_i^{\mathrm{T}}x - b_i| \leq t, i = 1, \ldots m$, so the max element of $|Ax - b|$ is also upper-bounded by t)

≡ Minimize t subject to $-t \cdot 1 \preccurlyeq Ax - b \preccurlyeq t \cdot 1$

b. Minimize $\|Ax - b\|_1$

≡ Minimize $1^{\mathrm{T}}|Ax - b|$ (i.e. minimize $\sum_{i=1}^{m}|a_i^{\mathrm{T}}x - b_i|$)

≡ Minimize $1^{\mathrm{T}}t$ subject to $|Ax - b| \preccurlyeq t$ (i.e. $|a_i^{\mathrm{T}}x - b_i| \leq t_i, i = 1, \ldots, m$, so $\sum_{i=1}^{m}|a_i^{\mathrm{T}}x - b_i|$ is upper-bounded by $\sum_{i=1}^{m} t_i = 1^{\mathrm{T}}t$)

≡ Minimize $1^{\mathrm{T}}t$ subject to $-t \preccurlyeq Ax - b \preccurlyeq t$

c. Minimize $\|Ax - b\|_1$ subject to $\|x\|_\infty \leq 1$

≡ Minimize $1^{\mathrm{T}}|Ax - b|$ subject to $\max|x| \leq 1$ (i.e. minimize $\sum_{i=1}^{m}|a_i^{\mathrm{T}}x - b_i|$ subject to $\max\{|x_1|, \ldots, |x_m|\} \leq 1$)

≡ Minimize $1^{\mathrm{T}}t$ subject to $|Ax - b| \preccurlyeq t$ and $|x| \preccurlyeq 1$ (i.e. $|a_i^{\mathrm{T}}x - b_i| \leq t_i, i = 1, \ldots, m$, so $\sum_{i=1}^{m}|a_i^{\mathrm{T}}x - b_i|$ is upper-bounded by $\sum_{i=1}^{m} t_i = 1^{\mathrm{T}}t$, and $|x_i| \leq 1, i = 1, \ldots, m$, so the max element of $|x|$ is also upper-bounded by 1)

≡ Minimize $1^{\mathrm{T}}t$ subject to $-t \preccurlyeq Ax - b \preccurlyeq t$ and $-1 \preccurlyeq x \preccurlyeq 1$

d. Minimize $\|x\|_1$ subject to $\|Ax - b\|_\infty \leq 1$

≡ Minimize $1^T|x|$ subject to $\max|Ax - b| \leq 1$ (i.e. minimize $\sum_{i=1}^{m}|x_i|$ subject to $\max\{|a_1^T x - b_1|, \ldots, |a_m^T x - b_m|\} \leq 1$)

≡ Minimize $1^T t$ subject to $|x| \preccurlyeq t$ and $|Ax - b| \preccurlyeq 1$ (i.e. $|x_i| \leq t_i, i = 1, \ldots, m$, so $\sum_{i=1}^{m}|x_i|$ is upper-bounded by $\sum_{i=1}^{m} t_i = 1^T t$, and $|a_i^T x - b_i| \leq 1, i = 1, \ldots, m$, so the max element of $|Ax - b|$ is also upper-bounded by 1)

≡ Minimize $1^T t$ subject to $-t \preccurlyeq x \preccurlyeq t$ and $-1 \preccurlyeq Ax - b \preccurlyeq 1$

e. Minimize $\|Ax - b\|_1 + \|x\|_\infty$

≡ Minimize $1^T|Ax - b| + \max|x|$ (i.e. minimize $\sum_{i=1}^{m}|a_i^T x - b_i| + \max\{|x_1|, \ldots, |x_m|\}$)

≡ Minimize $1^T t + s$ subject to $|Ax - b| \preccurlyeq t$ and $|x| \preccurlyeq s \cdot 1$ (i.e. $|a_i^T x - b_i| \leq t_i, i = 1, \ldots, m$, so $\sum_{i=1}^{m}|a_i^T x - b_i|$ is upper-bounded by $\sum_{i=1}^{m} t_i = 1^T t$, and $|x_i| \leq s, i = 1, \ldots m$, so the max element of $|x|$ is also upper-bounded by s)

≡ Minimize $1^T t + s$ subject to $-t \preccurlyeq Ax - b \preccurlyeq t$ and $-s \cdot 1 \preccurlyeq x \preccurlyeq s \cdot 1$

4.23

Minimize $\|Ax - b\|_4 = (\sum_{i=1}^{m}(a_i^T x - b_i)^4)^{1/4}$

≡ Minimize $\sum_{i=1}^{m}(a_i^T x - b_i)^4$ (i.e. if we minimize $\sum_{i=1}^{m}(a_i^T x - b_i)^4$, we also minimize the original question because $f(x) = x^{1/4}$ is strictly increasing)

≡ Minimize $\sum_{i=1}^{m} t_i^2$ subject to $(a_i^T x - b_i)^2 \leq t_i, i = 1, \ldots, m$ (because if $(a_i^T x - b_i)^2 \leq t_i$, $i = 1, \ldots, m$, then $\sum_{i=1}^{m}(a_i^T x - b_i)^4$ is upper-bounded by $\sum_{i=1}^{m} t_i^2$)

≡ Minimize $t^T t$ subject to $a_i^T x - b_i = s_i$ and $s_i^2 \leq t_i, i = 1, \ldots, m$ (s is introduced such that the optimization problem is a QCQP, i.e. quadratic objective + quadratic inequality constraints + affine equality constraints)