Solutions to Exercise #11

(範圍: Graph Theory)

1. How many different Hamiltonian cycles are there in K_5? (10%)

Sol: $\frac{4!}{2} = 12$.

2. Prove the theorem on page 68 of lecture notes. (15%)

Sol: The proof is similar to the proof on pages 65 to 67 of lecture notes. The traversal starts at vertex u, traverses a new edge whenever it is possible, and finally stops at vertex v. If there are untraversed edges, continue the traversal as done at Steps 2 to 4 on pages 66 and 67 of lecture notes.

3. P. 564: 22. (40%)

Sol:

(a) If $x \neq v$ and $y \neq v$, then $\deg(x) + \deg(y) = (n-2) \times 2$.
Otherwise, $\deg(x) + \deg(y) = 2 + (n-2) = n$.

(b) Yes.
There is a Hamilton cycle in G_n, which can be assured by the theorem on page 79 of lecture notes.

(c) $|E| = \frac{(n-2)\times(n-1)}{2} - 1 + 2 = \frac{(n-2)\times(n-1)}{2} + 1$.

(d) No.

4. Given a graph G, how to determine 0/1 matrices B, B^2, B^3, …, $B^{|V|-1}$ so that for $1 \leq k \leq |V| - 1$ and $i \neq j$, $B^k(i, j) = 1$ if and only if there exists an i-to-j walk of length $\leq k$ in G? (15%)

Sol: Let $B = A + A^0$, where A is the adjacency matrix of G. The addition of A^0 to A is equivalent to adding a loop to each node of G. Thus, an i-to-j walk can increase its length by traversing loops.

5. Given a graph G, how to determine matrices C, C^2, C^3, …, $C^{|V|-1}$ so that for $1 \leq k \leq |V| - 1$ and $i \neq j$, $C^k(i, j)$ tells the number of different i-to-j walks of length k in G?

(20%)

Sol: Let $C = A$, and perform the matrix operation as ordinary mathematical addition and multiplication, e.g., $1+1=2$ and $2\times 3=6$. When $C^{k-l}(i, r) > 0$ and $C^{l}(r, j) > 0$ represent the number of different i-to-r walks of length $k-l$ and the number of different r-to-j walks of length l in G, respectively, there are $C^{k-l}(i, r)\times C^{l}(r, j)$ different i-to-j walks of length k in G, which go via r and have the i-to-r (r-to-j) walks of length $k-l$ (l).