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- Let A be an $n \times n$ matrix. If there exists a nonzero vector v such that $Av = \lambda v$, then λ is called an **eigenvalue** of A, and the vector v is called the **eigenvector** of A associated with λ.
- $Av = \lambda v \Rightarrow (A - \lambda I)v = 0$. Hence, v is in the nullspace of $A - \lambda I$.
- *Theorem*: λ is an eigenvalue of $A \Leftrightarrow \det(A - \lambda I) = 0$. And for every eigenvalue λ, there exists at least one nonzero eigenvector v associated with it.
- Eigenvectors are nonzero vectors and are never unique.
- A $n \times n$ matrix has n (not necessarily) real or complex eigenvalues.
- $\det(A - \lambda I) = 0$ is called the **characteristic polynomial** of A. For each eigenvalue λ, the nullspace of $A - \lambda I$ is called the **eigenspace** of A.
- The eigenvalue is also called the **characteristic value** or **latent value (root)**. The eigenvector is also called the **characteristic vector** or **latent vector**.
- A zero eigenvalue means dependent columns and rows and a zero determinant. All invertible matrices have $\lambda \neq 0$.
- For a triangular matrix, its eigenvalues are its diagonal elements.
- *Theorem*: Let A be an $n \times n$ matrix w/ n eigenvalues $\lambda_1, \ldots, \lambda_n$. Then, $\mathrm{trace} A = \sum_i \lambda_i$ and $\det A = \prod_i \lambda_i$.
- The multiplicity with which an eigenvalue appears is called the **algebraic multiplicity** of λ. The dimension of eigenspace is called the **geometric multiplicity** of λ.
- *Theorem*: algebraic multiplicity $\geq$ geometric multiplicity.
- A $n \times n$ matrix A is said to be diagonalizable if there exists a nonsingular matrix S such that $S^{-1}AS = \Lambda$.
- *Theorem*: Suppose $A_{n \times n}$ has n linearly independent eigenvectors $x_1, \ldots, x_n$. Let S be an $n \times n$ matrix w/ $x_1, \ldots, x_n$ as columns. Then $S^{-1}AS = \Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$, where λ_i satisfies $Ax_i = \lambda_i x_i$.
- If all eigenvalues are different, then all eigenvectors are linearly independent and all geometric and algebraic multiplicities are 1. In other words, a matrix with distinct eigenvalues can be diagonalized.
- Te diagonalizing matrix S is not unique. Repeated eigenvalues leave more freedom. For example, $S^{-1}AS = I$ for any nonsingular S.
- $AS = S\Lambda$ holds if and only if the columns of S are eigenvectors of A.
- Not all matrices possess n linearly independent eigenvectors, and therefore, not all matrices are diagonalizable.
- Diagonalizability is connected with the eigenvectors. Invertibility is connected with eigenvalues.
- The only connection between diagonalizability and invertibility probably is that "diagonalization can fail only if there are repeated eigenvalues".
- *Theorem*: The eigenvectors corresponding to distinct eigenvalues are linearly independent.
- $Q = I - 2uu^\top$ is called a **Householder transformation**. The Householder transformation is a reflection which reflects about the axis perpendicular to u.