Homework 10

1. a. F. Suppose $A = I_{2\times2}$. Adding 0 times row 1 to row 2 gives $B = A$. $\det B = 1 \neq 0 \cdot \det A = 0$.
   b. T.
   c. T.
   d. T.
2. $\det A = -11$.
   a. $\det \frac{1}{2}A = (\frac{1}{2})^4 \det A = -\frac{11}{16}$.
   b. $\det -A = (-1)^4 \det A = -11$.
   c. $\det A^2 = (\det A)^2 = 121$.
   d. $\det A^{-1} = -\frac{1}{11}$.
3. Let $A = \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & \frac{c-a}{b-a} & 1 \end{bmatrix} \begin{bmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & 0 & c^2-a^2-\frac{(b^2-a^2)(c-a)}{b-a} \end{bmatrix}$.
   $\det A = (b-a)(c^2-a^2-\frac{(b^2-a^2)(c-a)}{b-a}) = bc^2 - ac^2 - ba^2 - b^2c + a^2c + ab^2 = (b-a)(c-a)(c-b)$.
4. a. $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} n_1 & n_1 & n_1 & n_1 \\ 0 & n_2-n_1 & n_2-n_1 & n_2-n_1 \\ 0 & 0 & n_3-n_2 & n_3-n_2 \\ 0 & 0 & 0 & n_4-n_3 \end{bmatrix}$
   b. $\det A = n_1(n_2-n_1)(n_3-n_2)(n_4-n_3)$.
5. $S_n = 3 \cdot S_{n-1} - S_{n-2}$. $S_1 = 3$. $S_2 = 8$. Hence, $S_n = \frac{5+3\sqrt{5}}{10}(\frac{3+\sqrt{5}}{2})^n + \frac{5-3\sqrt{5}}{10}(\frac{3-\sqrt{5}}{2})^n$.
6. a. When $t = 0, A = \begin{bmatrix} 2 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 5 & 5 \end{bmatrix}$.
   $$A^{-1} = \frac{1}{\det A} \begin{bmatrix} \begin{vmatrix} 1 & 0 \\ 5 & 5 \end{vmatrix} & -\begin{vmatrix} 0 & 5 \\ 5 & 5 \end{vmatrix} & \begin{vmatrix} 0 & 5 \\ 1 & 0 \end{vmatrix} \\ -\begin{vmatrix} 0 & 0 \\ 0 & 5 \end{vmatrix} & \begin{vmatrix} 2 & 5 \\ 0 & 5 \end{vmatrix} & -\begin{vmatrix} 2 & 5 \\ 0 & 0 \end{vmatrix} \\ \begin{vmatrix} 0 & 1 \\ 0 & 5 \end{vmatrix} & -\begin{vmatrix} 2 & 0 \\ 0 & 5 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 5 & 25 & -5 \\ 0 & 10 & 0 \\ 0 & -10 & 2 \end{bmatrix}.$$
   b. $\det A = 2t^2 + 4t + 10 \geq 8 > 0$. Hence, A^{-1} always exists.
7. $\det \begin{vmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{vmatrix} = a^3 - 3a + 2$. There is a solution when $a \neq 1, -2$.
   $$x = \frac{1}{a^3-3a+2} \begin{vmatrix} 1 & 1 & 1 \\ 0 & a & 1 \\ 0 & 1 & a \end{vmatrix} = \frac{a^2-1}{a^3-3a+2}.$$
   $$y = \frac{1}{a^3-3a+2} \begin{vmatrix} a & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & a \end{vmatrix} = \frac{1-a}{a^3-3a+2}.$$
   $$z = \frac{1}{a^3-3a+2} \begin{vmatrix} a & 1 & 1 \\ 1 & a & 0 \\ 1 & 1 & 0 \end{vmatrix} = \frac{1-a}{a^3-3a+2}.$$