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- Suppose $A \leq_M B$ holds:
  - $A \leq_T B$ holds.
  - If B is recognizable, then A is recognizable.
  - If A is not recognizable, then B is not recognizable.
- Suppose $A \leq_T B$ holds:
  - $A \leq_M B$ does not necessarily hold.
  - If B is decidable, then A is decidable.
  - If A is undecidable, then B is decidable.
- A_{TM} and A'_{TM} are Turing reducible but not mapping reducible.
  - $A'_{\text{TM}} \not\leq_M A_{\text{TM}}$ since A'_{TM} is not recognizable but A_{TM} is.
- HALT_0 and HALT'_0 are Turing reducible but not mapping reducible.
  - $\text{HALT}'_0 \not\leq_M \text{HALT}_0$ since HALT'_0 is not recognizable but HALT_0 is.
- **Theorem**: If $A \leq_M B$, then $A' \leq_M B'$.
  - If $A \leq_M B$, then there exists a computable function f such that $w \in A$ iff $f(w) \in B$.
  - $w \in A$ iff $w \notin A'$ and $f(w) \in B$ iff $f(w) \notin B'$.
  - $w \notin A'$ iff $f(w) \notin B'$
  - $w \in A'$ iff $f(w) \in B'$. Hence, $A' \leq_M B'$.
- **Theorem**: L_0 is undecidable. (Intuition: $A_{\text{TM}} \leq_T L_0$)
  - Let a decider for L_0 be H := On input $\lfloor M \rfloor$: Accept if $L(M) = \emptyset$. Reject if $L(M) \neq \emptyset$.
  - Define $M'(M, w)$:= On input x: Reject if $x \neq w$. Otherwise, run M on w. Accept if M accepts w.
  - Construct a decider D := On input (M, w): Construct $M'(M, w)$. Run H on $\lfloor M' \rfloor$. Accept if H rejects $\lfloor M' \rfloor$. Reject if H accepts $\lfloor M' \rfloor$.
  - If H decides L_0, then D decides A_{TM} . A_{TM} is undecidable, so L_0 is undecidable.
- **Theorem**: L_1 is undecidable. (Intuition: $A_{\text{TM}} \leq_T L_1$)
  - Let a decider for L_1 be H := On input $\lfloor M \rfloor$: Accept if $L(M) = \Sigma^*$. Reject if $L(M) \neq \Sigma^*$.
  - Define $M'(M, w)$:= On input x: Accept if $x \neq w$. Otherwise, run M on w. Reject if M accepts w.
  - Construct a decider D := On input (M, w): Construct $M'(M, w)$. Run H on $\lfloor M' \rfloor$. Accept if H rejects $\lfloor M' \rfloor$. Reject if H accepts $\lfloor M' \rfloor$.
  - If H decides L_1, then D decides A_{TM} . A_{TM} is undecidable, so L_1 is undecidable.
- **Theorem**: L_2 is undecidable. (Intuition: $A_{\text{TM}} \leq_T L_2$)
  - Let a decider for L_2 be H := On input $\lfloor M \rfloor$: Accept if M accepts ε. Reject if M rejects ε.
  - Define $M'(M, w)$:= On input x: Accept if $x \neq \varepsilon$. Otherwise, run M on w. Accept if M accepts w.
  - Construct a decider D := On input (M, w): Construct $M'(M, w)$. Run H on $\lfloor M' \rfloor$. Accept if H accepts $\lfloor M' \rfloor$. Reject if H rejects $\lfloor M' \rfloor$.

- If H decides L_2, then D decides A_{TM}. A_{TM} is undecidable, so L_2 is undecidable.

- **Theorem**: L_5 is undecidable. (Intuition: $A_{\text{TM}} \leq_T L_5$)
  - Let a decider for L_5 be H := On input $\lfloor M \rfloor$: Accept if $L(M)$ is regular. Reject if $L(M)$ is non-regular.
  - Define $M'(M, w)$:= On input x: Accept if x has the form $0^n 1^n$. Otherwise, run M on w. Accept if M accepts w.
  - Construct a decider D := On input (M, w): Construct $M'(M, w)$. Run H on $\lfloor M' \rfloor$. Accept if H accepts $\lfloor M' \rfloor$. Reject if H rejects $\lfloor M' \rfloor$.
  - If H decides L_5, then D decides A_{TM}. A_{TM} is undecidable, so L_5 is undecidable.