Mathematical logic

- Propositional calculus
  - Three components: variables, operators (e.g. ∧, ∨, ¬), formulas.
  - A formula is **satisfiable** if you can assign true/false to its variables s.t. the formula yields true.
    - $p \rightarrow \neg p$ is not satisfiable.
  - A set X of formulas is satisfiable if there is an assignment to all variables in X s.t. for every formula $x \in X$, x yields true under the assignment.
  - Is set X of infinitely many formulas is satisfiable?
- Compactness theorem
  - For a set X of formula, X is satisfiable iff every finite subset of X is satisfiable.
  - i.e. X is not satisfiable iff any finite subset of X is not satisfiable.
  - **Tautology**: a formula which is always true regardless of the value of its variables.
  - $X \vdash \alpha$: α is provable from a set X of formulas.
  - Rule 1: $X \vdash \alpha$ if $\alpha \in X$
  - Rule 2: $X \vdash \alpha$ and $Y \vdash \alpha$ whereas $Y \supseteq X$ or $X \supseteq Y$
  - Rule 3: $X \vdash \alpha \wedge \beta$ iff $X \vdash \alpha$ and $X \vdash \beta$
  - Rule 4: $X \cup \{\alpha\} \vdash \beta$ and $X \cup \{\neg\alpha\} \vdash \beta$, then $X \vdash \beta$
  - Rule 5: $X \vdash \alpha$ and $X \vdash \neg\alpha$, then $X \vdash \beta$ for every β
  - $\varnothing \vdash \alpha$, then α is tautology, e.g. $p \rightarrow \neg p$
  - Example: if $\{p\} \vdash p$, and $\{p, p \wedge \neg q\} \vdash p$, then $\{p, p \wedge \neg q\} \vdash p \wedge \neg q$
  - $X \vDash \alpha$: every satisfying assignment of X is also satisfiable assignment of α, i.e. $X \vdash \alpha$ iff $X \vDash \alpha$
- First order logic
  - First order quantification: "for all", 'there is", etc.
  - Define vocabulary (a set of symbols) (τ)
    - Constant symbols (c)
    - Relation symbols (R)
    - Function symbols
  - A first order formula set vocabulary τ is defined as follows:
    - If R is a relation symbol in τ and x1, x2, … xk are variable, then R(x1, x2, …xk) is a formula.
    - If f1 and f2 are formulas, then so are $f1 \wedge f2$, $f1 \vee f2$, $\neg f1$, $\neg f2$
    - If f is a formula and x is a variable, then "there exist x", "for all x" is in first order formulas.
    - If x and y are variables, then $x = y$ is a formula
    - "There exists x for all y" is not equal to "for all x there exists y"
  - $G \vDash f$ reads f holds in G
  - A formula f is satisfiable if there is G s.t. $G \vDash f$
  - A set X of formulas is satisfiable if there is G s.t. $G \vDash f$ for every f in X
  - A structure over vocabulary τ is $\langle A, R_1, R_2, \ldots R_k \rangle$
    - A is a set of elements called **universe** or **domain**.
    - For each R_i in τ, $R_i \subseteq A^*A^*\ldots{}^*A$
  - If F is a formula over τ, A is a structure over τ, then f is true/false in A
  - F is satisfiable if there is A s.t. $A \vDash f$

- X is satisfiable if there is A s.t. A $\models$ f for every f in X
- Cardinals of math
  - $|A| = |B|$ if there is a bijection from A to B
  - $|A| < |B|$ if there is injection from A to B but no bijection.
  - $|A| < |2^A|$ for every A regardless A is countable or uncountable.
  - $Z_i = 2^{Z}$i-1
- Proof system for first-order logic
- Godel's completeness theorem
- Godel's incompleteness theorem