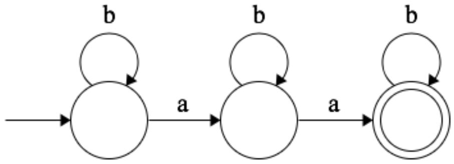
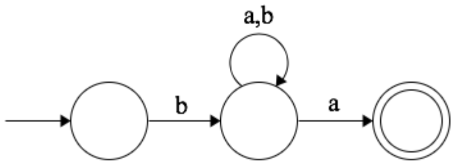
Homework 1

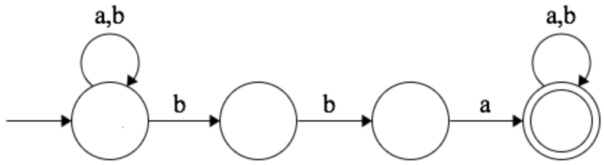
1. a.



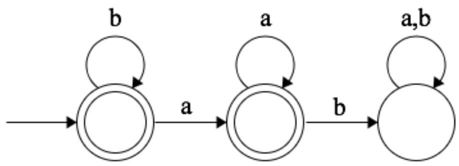
   b.



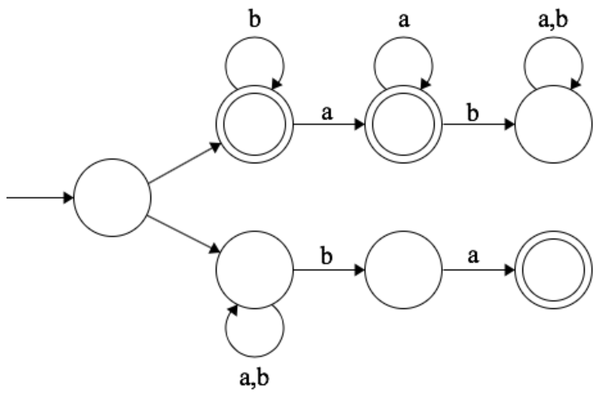
   c.



   d.



   e.



2. a. $b^*ab^*ab^*$

   b. $b\Sigma^*a$

   c. $\Sigma^*bba\Sigma^*$

   d. b^*a^*

   e. $b^*a^* \cup \Sigma^*ba$

3. a. a^*ba^*

   b. $(a \cup ba^*b)^*$

   c. $(\Sigma^2)^*$

   d. Suppose L is regular and $A = \langle \Sigma, Q, q_0, F, \delta \rangle$ is its DFA. Consider a word $a^m ba^n \in L$, where $m, n \in \mathbb{N}$, $m \le n$, and $m > |Q|$. Then, a^m can be divided into three parts $a^m = a^u a^v a^w$, and $m = u + v + w$ s.t. $a^u (a^v)^k a^w ba^n \in L$ for every nonnegative integer $k \ge 0$. However, there is a contradiction because $a^u (a^v)^k a^w ba^n \notin L$ when $u + kv + w > n$, or $k > (n - u - w)/v$. Therefore, L is not regular.

   e. Suppose L is regular and $A = \langle \Sigma, Q, q_0, F, \delta \rangle$ is its DFA. Consider a word a^k, where k is a prime number, and $k > |Q|$. Then, a^k can be divided into three parts $a^k = a^u a^v a^w$, and $k = u + v + w$ s.t. $a^u (a^v)^n a^w \in L$ for every nonnegative integer $n \ge 0$. However, there is a contradiction because $a^u (a^v)^n a^w \notin L$ when $n = k + 1$, $u + vn + w = (v + 1)k$, which is not a prime number. Therefore, L is not regular.

4. a.

   (1) *Reflexivity*. For every $w \in \Sigma^*$, $uw \in L$ if and only if $uw \in L$ always holds. Therefore, $u \sim_L u$ holds.
   (2) *Symmetry*. Suppose $u \sim_L v$ holds. Then, for every $w \in \Sigma^*$, $uw \in L$ if and only if $vw \in L$, and vice versa, $vw \in L$ if and only if $uw \in L$. Therefore, $v \sim_L u$ holds.
   (3) *Transitivity*. Suppose $t \sim_L u$ and $u \sim_L v$. Then, for every $w \in \Sigma^*$, $tw \in L$ if and only if $vw \in L$ and $uw \in L$ if and only if $vw \in L$ together imply $tw \in L$ if and only if $vw \in L$. Therefore, $t \sim_L v$ holds.
   To sum up, $\sim_L$ is an equivalence relation.

   b. L is a regular language if and only if $\#(\sim_L)$ is finite.
   $\Rightarrow$. Suppose L is regular. Let $A = \langle \Sigma, Q, q_0, F, \delta \rangle$ be its DFA, where $|Q| = k$. If $\#(\sim_L) > k$, and there must exist two words u and v in two different equivalence classes, s.t. the run of A on u and the run of A on v end in the same state. Then, $\forall w \in \Sigma^*$, $uw \in L$ if and only if $vw \in L$, which is a contradiction to that u and v are from two different equivalence classes. Therefore, $\#(\sim_L) \le k$, which is finite.
   $\Leftarrow$. Suppose $\#(\sim_L) = k$, which is finite. And the equivalence classes are $L_1, L_2, \ldots, L_k$, s.t. $\bigcup_i L_i = \Sigma^*$ and $L_i \cap L_j = \emptyset$ for $1 \le i \ne j \le k$. Then, a DFA $A = \langle \Sigma, Q, q_0, F, \delta \rangle$ can be constructed accordingly s.t. L is recognized by A.

   - $Q = \{q_1, q_2, \ldots, q_k\}$ s.t. q_i corresponds to L_i, for $1 \le i \le k$.
   - $q_0 = q_i$ s.t. $\epsilon \in L_i$.
   - $F = \{q_i | L_i \subseteq L\}$.
   - $\delta(q_i, a) = q_j$ s.t. $\forall w \in L_i$, $wa \in L_j$.

   Given a word $w = a_1 a_2 \ldots a_n \in L$, and the run of A on w is $p_0 a_1 p_1 \ldots a_n p_n$. Since $w \in L_i$ if and only if the run of A on w ends at q_i, therefore, $w \in L_{p_n}$ (which is one of the equivalence classes). Altogether, $w \in L_{p_n}$ and $w \in L$ implies $L_{p_n} \subseteq L$ and $p_n \in F$. Any word $w \in L$ is accepted by A, and L is recognized by A.