# 2017-12-14

- **Theorem**: PCP is undecidable.
  - From any TM $M$ and input $w$, we can construct an instance $P$ where a match is an accepting of $M$ on $w$.
  - Dominos $P'$: (1) $[\frac{\#}{\#q_0w_1\ldots w_n\#}]$; (2) $[\frac{qa}{br}]$ if $\delta(q,a)=(r,b,\mathrm{R})$; (3) $[\frac{cqa}{rcb}]$ if $\delta(q,a)=(r,b,\mathrm{L})$; (4) $[\frac{a}{a}]$ for every $a\in\Gamma$; (5) $[\frac{\#}{\#}]$ and $[\frac{\#}{\sqcup\#}]$; (6) $[\frac{aq_{\text{accept}}}{q_{\text{accept}}}]$ and $[\frac{q_{\text{accept}}a}{q_{\text{accept}}}]$ for every $a\in\Gamma$; (7) $[\frac{q_{\text{accept}}\#\#}{\#}]$.
  - Define $*u$, $u*$, and $*u*$ to be the three strings: (1) $*u = *u1*\ldots*u_n$; (2) $u* = u1*\ldots*u_n*$; (3) $*u* = *u1*\ldots*u_n*$.
  - Dominos $P'$: $[\frac{t_1}{b_1}]=[\frac{\#}{\#q_0w_1\ldots w_n\#}]$, …, $[\frac{t_k}{b_k}]$.
  - Dominos $P$: $[\frac{*t_1}{*b_1*}]$, $[\frac{*t_1}{b_1*}]$, …, $[\frac{*t_k}{b_k*}]$, $[\frac{*@}{@}]$.
- **Theorem**: CFL-Intersection is undecidable. (Intuition: PCP $\leq_T$ CFL-Intersection)
  - Let a decider for CFL-Intersection be $H$ := On input $(G_1,G_2)$: Accept if $L(G_1)\cap L(G_2)\neq\emptyset$. Reject if $L(G_1)\cap L(G_2)=\emptyset$.
  - Define $G_1'((u_1,v_1),\ldots,(u_n,v_n))$ := Let $\Sigma$ be the set of alphabets used in $((u_1,v_1),\ldots,(u_n,v_n))$. $L(G_1')=\{w\#w^R \mid w\in\Sigma^*\}$.
  - Define $G_2'((u_1,v_1),\ldots,(u_n,v_n))$ := $S\to u_iSv_i^R \mid \#$ for $i=1,\ldots,n$.
  - Construct a decider $D$ := On input $((u_1,v_1),\ldots,(u_n,v_n))$: Construct $G_1'((u_1,v_1),\ldots,(u_n,v_n))$ and $G_2'((u_1,v_1),\ldots,(u_n,v_n))$. Run $H$ on $(G_1',G_2')$. Accept if $H$ accepts $(G_1',G_2')$. Reject if $H$ rejects $(G_1',G_2')$.
  - If $H$ decides CFL-Intersection, then $D$ decides PCP. PCP is undecidable, so CFL-Intersection is undecidable.
- **Theorem**: CFL-Universality ($\mathrm{ALL}_{\mathrm{CFG}}$) is undecidable. (Intuition: $A_{\mathrm{TM}}\leq_T\mathrm{ALL}_{\mathrm{CFG}}$)
  - Let a decider for $\mathrm{ALL}_{\mathrm{CFG}}$ be $H$ := On input $\lfloor G\rfloor$: Accept if $L(G)=\Sigma^*$. Reject if $L(G)\neq\Sigma^*$.
  - Define $G'(M,w)$ := $L(G')=\{C_0\#C_1\#\ldots\#C_n\}$ which satisfies any one of the following:
    - $C_0$ is not the initial configuration.
    - $C_n$ is not an accepting configuration.
    - There exists $0\leq i<n$ such that $C_i\vdash C_{i+1}$ does not obey the transition functions for $M$.
  - Construct a decider $D$ := On input $(M,w)$: Construct $G'(M,w)$. Run $H$ on $\lfloor G'\rfloor$. Accept if $H$ rejects $\lfloor G'\rfloor$. Reject if $H$ accepts $\lfloor G'\rfloor$.
  - If $H$ decides $\mathrm{ALL}_{\mathrm{CFG}}$, then $D$ decides $A_{\mathrm{TM}}$. $A_{\mathrm{TM}}$ is undecidable, so $\mathrm{ALL}_{\mathrm{CFG}}$ is undecidable.
- **Theorem**: CFL-Subset is undecidable.
  - Let a decider for CFL-Subset be $H$ := On input $(G_1,G_2)$: Accept if $L(G_1)\subseteq L(G_2)$. Reject if $L(G_1)\nsubseteq L(G_2)$.
  - Construct a decider $D$ := On input $\lfloor G\rfloor$: Run $H$ on $(G_1',G)$ where $L(G_1')=\Sigma^*$. Accept if $H$ accepts $(G_1',G)$. Reject if $H$ rejects $(G_1',G)$.
  - If $H$ decides CFL-Subset, then $D$ decides $\mathrm{ALL}_{\mathrm{CFG}}$. $\mathrm{ALL}_{\mathrm{CFG}}$ is undecidable, so CFL-Subset is undecidable.