Homework 3

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1.

1 $\exists x \exists y (S(x,y) \lor S(y,x))$	premise
2 x_0	
3 $\exists y (S(x_0,y) \lor S(y,x_0))$	assumption
4 y_0	
5 $S(x_0,y_0) \lor S(y_0,x_0)$	assumption
6 $S(x_0,y_0)$	assumption
7 $\exists y S(x_0,y)$	$\exists$yi : 6
8 $\exists x \exists y S(x,y)$	$\exists$xi : 7
9 $S(y_0,x_0)$	assumption
10 $\exists y S(y_0,y)$	$\exists$yi : 9
11 $\exists x \exists y S(x,y)$	$\exists$xi : 10
12 $\exists x \exists y S(x,y)$	$\lor$ e : 5, 6 − 8, 9 − 11
13 $\exists x \exists y S(x,y)$	$\exists$ye : 3, 4 − 12
14 $\exists x \exists y S(x,y)$	$\exists$xe : 1, 2 − 13

2.

1 $\forall x \forall y \forall z (S(x,y) \land S(y,z) \implies S(x,z))$	premise
2 $\forall x \neg S(x,x)$	premise
3 x_0	
4 $\forall y \forall z (S(x_0,y) \land S(y,z) \implies S(x_0,z))$	$\forall$xe : 1
5 $\neg S(x_0,x_0)$	$\forall$xe : 2
6 y_0	
7 $\forall z (S(x_0,y_0) \land S(y_0,z) \implies S(x_0,z))$	$\forall$ye : 4
8 $S(x_0,y_0) \land S(y_0,x_0) \implies S(x_0,x_0)$	$\forall$ze : 7
9 $\neg(S(x_0,y_0) \land S(y_0,x_0))$	MT : 8, 5
10 $S(x_0,y_0)$	assumption
11 $S(y_0,x_0)$	assumption
12 $S(x_0,y_0) \land S(y_0,x_0)$	$\land$ i : 10, 11
13 $\perp$	$\perp$ i : 12, 9
14 $\neg S(y_0,x_0)$	$\neg$i : 11 − 13
15 $S(x_0,y_0) \implies \neg S(y_0,x_0)$	$\implies$ i : 10 − 14
16 $\forall y (S(x_0,y) \implies \neg S(y,x_0))$	$\forall$yi : 6 − 15
17 $\forall x \forall y (S(x,y) \implies \neg S(y,x))$	$\forall$xi : 3 − 16

3.

1	$\exists x \exists y (S(x,y) \vee S(y,x))$	premise
2	$\neg \exists x S(x,x)$	premise
3	x_0	
4	$\exists y (S(x_0,y) \vee S(y,x_0))$	assumption
5	y_0	
6	$S(x_0,y_0) \vee S(y_0,x_0)$	assumption
7	$x_0 = y_0$	assumption
8	$S(x_0,y_0)$	assumption
9	$S(y_0,y_0)$	$=\mathrm{e} : 7,8$
10	$\exists x S(x,x)$	$\exists x\mathrm{i} : 9$
11	$\bot$	$\bot\mathrm{i} : 10,2$
12	$S(y_0,x_0)$	assumption
13	$S(y_0,y_0)$	$=\mathrm{e} : 7,12$
14	$\exists x S(x,x)$	$\exists x\mathrm{i} : 13$
15	$\bot$	$\bot\mathrm{i} : 14,2$
16	$\bot$	$\vee\mathrm{e} : 6, 8-11, 12-15$
17	$\neg(x_0 = y_0)$	$\neg\mathrm{i} : 7$
18	$\exists y \neg(x_0 = y)$	$\exists y\mathrm{i} : 17$
19	$\exists x \exists y \neg(x = y)$	$\exists x\mathrm{i} : 18$
20	$\exists x \exists y \neg(x = y)$	$\exists y\mathrm{e} : 4, 5-19$
21	$\exists x \exists y \neg(x = y)$	$\exists x\mathrm{e} : 1, 3-20$

4. Consider $\forall x(P(x) \vee Q(x)) \vDash \forall x P(x) \vee \forall x Q(x)$. Let M be a model where $A = \{a,b\}$, $P^M = \{a\}$, and $Q^M = \{b\}$. Since $a \in P^M$ and $b \in Q^M$, $M \vDash \forall x(P(x) \vee Q(x))$. Since $b \notin P^M$ and $a \notin Q^M$, $M \nvDash \forall x(P(x) \vee Q(x))$. Hence, $\forall x(P(x) \vee Q(x)) \nvDash \forall x P(x) \vee \forall x Q(x)$. The soundness theorem says if $\forall x(P(x) \vee Q(x)) \vdash \forall x P(x) \vee \forall x Q(x)$ is valid, then $\forall x(P(x) \vee Q(x)) \vDash \forall x P(x) \vee \forall x Q(x)$ holds. Since $M \nvDash \forall x(P(x) \vee Q(x))$, $\forall x(P(x) \vee Q(x)) \nvdash \forall x P(x) \vee \forall x Q(x)$.

5. Let M be a model that $M \vDash \forall x \neg \varphi$. Assume $M \nvDash \neg \exists x \varphi$. There exists an element a in the universe of M s.t. φ computes to T. But since $M \vDash \forall x \neg \varphi$, $\neg \varphi$ computes to T, or φ computes to F, regardless of a. This is a contradiction. Hence, $M \vDash \neg \exists x \varphi$.