# Solutions to Exercise #12

(範圍: Graph Theory)

1. How many regions are there in any planar drawing of a connected planar graph with 15 vertices and 34 edges? (10%)

Sol: Since $|V| - |E| + r = 2$, there are 21 regions.

2. Is $K_{3,2}$ a planar graph ? Why? (10%)

Sol: $K_{3,2}$ a planar graph, because it has a planar drawing (omitted here).

3. Prove the second corollary on page 96 of lecture notes. (15%)

Sol: Denote the $k$ connected components by $G_i(V_i, E_i)$'s, where $1 \leq i \leq k$. Then, $|V_i| - |E_i| + r_i = 2$ for each component. Since $|V| = |V_1| + |V_2| + \ldots + |V_k|$, $|E| = |E_1| + |E_2| + \ldots + |E_k|$ and $r = (r_1 + r_2 + \ldots + r_k) - (k - 1)$, we have $|V| - |E| + r = k + 1$.

4. P. 666: 4 (only for (c) and (d)). (10%)

Sol: (c) $P(9, 5) = 9 \times 8 \times 7 \times 6 \times 5 = 15120$. (d) $P(n, m)$.

5. For the graph of Figure 11.72(b), give all maximal cliques of sizes greater than two. Which is the maximum clique? (10%)

Sol: maximal cliques: $\{t, v, w\}$, $\{t, u, w\}$, $\{u, w, x\}$, $\{w, x, y, z\}$.
maximum clique: $\{w, x, y, z\}$.

6. How to modify a maximum clique algorithm (i.e., an algorithm that can find a maximum clique of a graph) so that it can be used to find a minimum vertex cover of a graph? (10%)

Sol: Notice that $V - V'$ is a minimum vertex cover of $G$ if and only if $V'$ is a maximum clique of $\overline{G}$. Suppose that we are required to find a minimum vertex cover of a graph $G = (V, E)$. We feed the maximum clique algorithm with $\overline{G}$. If $V'$ is the output maximum clique, then $V - V'$ is a minimum vertex cover of $G$.

7. Prove the theorem on page 124 of lecture notes. (15%)

Sol: Notice that $F = \sum_{e \in E(S;\bar{S})} f(e) - \sum_{e \in E(\bar{S};S)} f(e)$, $c(S) = \sum_{e \in E(S;\bar{S})} c(e)$, and $f(e) \le c(e)$.

Hence, $F = c(S)$ if and only if $\sum_{e \in E(S;\bar{S})} f(e) = \sum_{e \in E(S;\bar{S})} c(e)$ and $\sum_{e \in E(\bar{S};S)} f(e) = 0$,

which hold if and only if (a) and (b) hold.

8. P. 658: 4 (only for Example 13.12). (20%)

Sol: Ford & Fulkerson's algorithm: the following augmenting paths can be found.

(1) $(a, c_1, b, h, m_2, z)$, $F = 0 + 15 = 15$.
(2) $(a, c_2, d, h, m_1, z)$, $F = 15 + 15 = 30$.
(3) $(a, c_3, d, b, g, m_1, z)$, $F = 30 + 10 = 40$.
(4) $(a, c_2, b, g, h, j, m_2, z)$, $F = 40 + 5 = 45$.

Edmonds & Karp's algorithm: the following augmenting paths can be found.

(1) $(a, c_1, b, g, m_1, z)$, $F = 0 + 10 = 10$.
(2) $(a, c_1, b, h, m_1, z)$, $F = 10 + 5 = 15$.
(3) $(a, c_2, b, h, m_1, z)$, $F = 15 + 5 = 20$.
(4) $(a, c_2, d, h, m_1, z)$, $F = 20 + 5 = 25$.
(5) $(a, c_2, d, h, m_2, z)$, $F = 25 + 10 = 35$.
(6) $(a, c_3, d, j, m_2, z)$, $F = 35 + 5 = 40$.
(7) $(a, c_3, c_2, b, h, m_2, z)$, $F = 40 + 5 = 45$.

The maximum total flow is 45 ($\{(g, m_1), (h, m_1), (h, m_2), (j, m_2)\}$ is a minimum cut).