Homework 2 B00401062 羅文斌

2.17(c)

The halfspace $C = \{(v,t) | f^\mathrm{T} v + gt \leq h\}$ under perspective function $P(C) = \{v/t | (v,t) \in C, t > 0\}$

Let $x = v/t$. Then, $P(C) = \{x | f^\mathrm{T} x \leq -g + h/t, t > 0\}$

The shape of $P(C)$ depends on the sign of h and the value of t. There are 3 situations to be considered:

(1) If $h > 0$, then $P(C) = \lim_{t\to 0}\{x | f^\mathrm{T} x \leq -g + h/t, t > 0\} = \{x | f^\mathrm{T} x \leq \infty\} = \mathbb{R}^\mathrm{n}$

(2) If $h = 0$, then $P(C) = \{x | f^\mathrm{T} x \leq -g + h/t, t > 0\}|_{h=0} = \{x | f^\mathrm{T} x \leq -g\}$, which is a closed halfspace.

(3) If $h < 0$, then $P(C) = \lim_{t\to\infty}\{x | f^\mathrm{T} x \leq -g + h/t, t > 0\} = \{x | f^\mathrm{T} x < -g\}$, which is an open halfspace.

2.24(a)

Let $f(x_1, x_2) = x_1 x_2$. The closed convex set $C = \{x \in \mathbb{R}^2_+ | x_1 x_2 \geq 1\}$ can be thought of as the intersection of an infinite number of supporting hyperplanes in $\mathbb{R}^2$ passing through $(t, \frac{1}{t})$ for all $t > 0$.

Also, $\nabla f = \frac{\partial f}{\partial x} = (x_2, x_1) = (\frac{1}{t}, t)$.

Therefore, the tangent line passing through $(t, \frac{1}{t})$ is $L : \frac{1}{t}(x_1 - t) + t(x_2 - \frac{1}{t}) = 0$.

The convex set C is the intersection of $\frac{1}{t}(x_1 - t) + t(x_2 - \frac{1}{t}) \geq 0$ for all $t > 0$.

$\frac{1}{t}(x_1 - t) + t(x_2 - \frac{1}{t}) \geq 0$

$\frac{1}{t}x_1 + t x_2 \geq 2$

$C = \{x \in \mathbb{R}^2_+ | \frac{1}{t}x_1 + t x_2 \geq 2, \forall t > 0\}$

2.24(b)

The convex set $C = \{x \in \mathbb{R}^n | ||x||_\infty \leq 1\}$ is a cube in $\mathbb{R}^n$. The boundary points of this convex set form the surface of the cube.

The supporting hyperplane at $\hat{x} = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_\mathrm{n})$ can be expressed as $a^\mathrm{T} x = a^\mathrm{T} \hat{x}$, such that $a^\mathrm{T} x \geq a^\mathrm{T} \hat{x}$ for all $x \in C$, where $a = (a_1, a_2, \ldots, a_\mathrm{n})$ is the normal vector of the supporting hyperplane in $\mathbb{R}^n$.

The value of a_i depends on the value of x_i. There are 3 situations to be considered:

(1) If $\hat{x}_i = 1$, then $a_i x_i \geq a_i \hat{x}_i = a_i \Rightarrow a_i(x_i - 1) \geq 0$. Because $x_i \leq 1$, $a_i < 0$.

(2) If $-1 < \hat{x}_i < 1$, then $a_i x_i \geq a_i \hat{x}_i$. In order for this inqeuality to hold for all x_i, a_i has to be 0.

(3) If $\hat{x}_i = -1$, then $a_i x_i \geq a_i \hat{x}_i = -a_i \Rightarrow a_i(x_i + 1) \geq 0$. Because $x_i \geq -1$, $a_i > 0$.

3.3

$f : \mathbb{R} \rightarrow \mathbb{R}$ is increasing and convex on its domain (a, b), meaning $\theta f(a) + (1-\theta)f(b) \geq f(\theta a + (1-\theta)b), 0 \leq \theta \leq 1$, according to the first-order condition.

Now the the function $g(f(x)) = x$ with domain $(f(a), f(b))$ is of our interest.

$\theta f(a) + (1-\theta)f(b) \geq f(\theta a + (1-\theta)b), 0 \leq \theta \leq 1$

$g(\theta f(a) + (1-\theta)f(b)) \geq g(f(\theta a + (1-\theta)b)) = \theta a + (1-\theta)b = \theta g(f(a)) + (1-\theta)g(f(b))$

Let $(a', b') = (f(a), f(b))$, the inequality above can be simplified as $g(\theta a' + (1-\theta)b') \geq \theta g(a') + (1-\theta)g(b')$, which is exactly the first-order condition of a concave function. Therefore, g is a concave function.