Homework 2

1. 1. True.
      $S \to aSbS \to aaSbSbS \to aaaSbSbSbS \to aaabSbSbS \to aaabbSbS \to aaabbbS \to aaabbb$. (Figure 1)
   2. False. $aabbb$, a prefix of $aabbb$, has more b's than a's.
   3. False. abb, a prefix of $abba$, has more b's than a's.
   4. True.
      $S \to aSbS \to aaSbSbS \to aabSbS \to aabbS \to aabbaSbS \to aabbaaSbSbS \to aabbaaaSbSbSbS$
      $\to aabbaaabSbSbS \to aabbaaabbSbS \to aabbaaabbbS \to aabbaaabbb$. (Figure 2)
   5. False. b, a prefix of $baababba$, has more b's than a's.
2. 1. $\Sigma = \{a, b\}$, $V = \{S\}$, S is the start variable, R:
      - $S \to a|aS|aSb$
   2. $\Sigma = \{a, b\}$, $V = \{S\}$, S is the start variable, R:
      - $S \to b|Sb|aSb$
   3. $\Sigma = \{a, b\}$, $V = \{S\}$, S is the start variable, R:
      - $S \to aaSb|\epsilon$
   4. $\Sigma = \{a, b, \$\}$, $V = \{S, X\}$, S is the start variable, R:
      - $S \to X\$$
      - $X \to aXa|bXb|\$$
   5. $L_5 = L_1 \cup L_2 \cup \{\Sigma^* ba \Sigma^*\}$. $\Sigma = \{a, b\}$, $V = \{S, S_1, S_2, S_3, X\}$, S is the start variable, R:
      - $S \to S_1|S_2|S_3$
      - $S_1 \to a|aS_1|aS_1b$
      - $S_2 \to b|S_2b|aS_2b$
      - $S_3 \to XbaX$
      - $X \to XX|a|b|\epsilon$
3. 1. Suppose $L_1 = \{a^k b^m c^n | k \le m \le n\}$ is context-free. $\exists p \in \mathbb{N}$, for every $w \in L_1$, if $|w| \ge p$, then w can be partitioned into five parts $w = sxyzt$, where $|xz| \ge 1$ and $|xyz| \le p$, s.t. $sx^i yz^i t \in L_1 \ \forall i \ge 0$. Consider $w = a^k b^k c^k \in L_1$, where $k \ge p$. There are five possible cases of xyz:
      - **Case 1**: $xyz = a^\alpha$ for some $1 \le \alpha \le p$. Consider $i = 2$.
        $sx^i yz^i t = a^{k'} b^k c^k$ has $k' \ge k + 1 > k$. So, $sx^i yz^i t \notin L_1$.
      - **Case 2**: $xyz = a^\alpha b^\beta$ for some $1 \le \alpha + \beta \le p$. Consider $i = 2$.
        $sx^i yz^i t = a^{k'} b^{k''} c^k$ has either $k' \ge k + 1 > k$ or $k'' \ge k + 1 > k$. So, $sx^i yz^i t \notin L_1$.
      - **Case 3**: $xyz = b^\beta$ for some $1 \le \beta \le p$. Consider $i = 2$.
        $sx^i yz^i t = a^k b^{k'} c^k$ has $k' \ge k + 1 > k$. So, $sx^i yz^i t \notin L_1$.

- **Case 4**: $xyz = b^\beta c^\gamma$ for some $1 \le \beta+\gamma \le p$. Consider $i=0$.
  $sx^i yz^i t = a^k b^{k'} c^{k''}$ has either $k' < k$ or $k'' < k$. So, $sx^i yz^i t \notin L_1$.
- **Case 5**: $xyz = c^\gamma$ for some $1 \le \gamma \le p$. Consider $i=0$.
  $sx^i yz^i t = a^k b^k c^{k'}$ has $k' < k$. So, $sx^i yz^i t \notin L_1$.

- **Conclusion**: This is a contradiction to the pumping lemma, so L_1 is not context-free.

2. Suppose $L_2 = \{a^n | n \text{ is a prime number}\}$ is context-free. $\exists p \in \mathbb{N}$, for every $w \in L_2$, if $|w| \ge p$, then w can be partitioned into five parts $w = sxyzt$, where $|xz| \ge 1$ and $|xyz| \le p$, s.t. $sx^i yz^i t \in L_2$ $\forall i \ge 0$. Consider $w = a^n \in L_2$, where $n \ge p+2$. Consider $i = |syt|$. Then, $|sx^i yz^i t|$ $= |syt| + |xz||syt| = (|xz|+1)|syt|$, which is not a prime number because $|xz|+1 \ge 2$ and $|syt| \ge 2$.

- **Conclusion**: This is a contradiction to the pumping lemma, so L_2 is not context-free.

4.
- $\Rightarrow$ (only if):
  - **IH**: Every prefix of $w \in L(G)$ where $|w| \le n$ has at least as many a's as b's.
  - **Basis**: $w \in L(G)$ where $|w| = 0$, i.e., ϵ, is always true for the hypothesis.
  - **IS**: **Assumption**: Given $w \in L(G)$ where $|w| = n+1$.
    w can be generated in either ways: (1) aS, where $S \to^* w' \in L(G)$ and $|w'| = n$, and (2) $aSbS$, where first $S \to^* w_1 \in L(G)$, second $S \to^* w_2 \in L(G)$, and $|w_1| + |w_2| = n-1$. Consider two cases.
    - **Case (1)**: Every prefix of w has at least as many a's as b's, so every prefix of aw' also has at least as many a's as b's.
    - **Case (2)**: Every prefix of w_1 and w_2 has at least as many a's as b's, so every prefix of aw_1bw_2, including aw_1, aw_1b and aw_1bw_2 itself, has at least as many a's as b's.
  - **Conclusion**: The IH is true for $n+1$, and therefore, for all $n \ge 0$.
- $\Leftarrow$ (if):
  - **IH**: If every prefix of w where $|w| \le n$ has at least as many a's as b's, then $w \in L(G)$.
  - **Basis**: w where $|w| = 0$, i.e., ϵ, has at least as many a's as b's and is in $L(G)$, because there is a derivation: $S \to \epsilon$.
  - **IS**: **Assumption**: Given w where $|w| = n+1$ s.t. every prefix of w has at least as many a's as b's.
    There exists a partition of w in either way: (1) aw', where $|w'| = n$ and every prefix of w' has at least as many a's as b's. (2) aw_1bw_2, where $|w_1| + |w_2| = n-1$ and every prefix of w_1 and w_2 has at least as many a's as b's. Otherwise, w cannot be partitioned in either way. Also, w can never start with b. So, $w = aw_1'bw_2'bw_3'$, s.t. $w_1', w_2' \in L(G)$ and both have as many a's as b's. However, w has a prefix, $aw_1'bw_2'b$, that has more b's than a's, which is a contradiction to the assumption. Therefore, $|w|$ can always be partitioned in either way. Consider two cases.
    - **Case (1)**: $w' \in L(G)$, so there is a derivation: $S \to aS \to^* aw' = w \in L(G)$.
    - **Case (2)**: $w_1, w_2 \in L(G)$, so there is a derivation: $S \to aSbS \to^* aw_1bw_2 = w \in L(G)$.
  - **Conclusion**: The IH is true for $n+1$, and therefore, for all $n \ge 0$.

5. Prove. Let $A_1 = (\Sigma_1, \Gamma_1, Q_1, q_{1,0}, F_1, \delta_1)$ be a PDA that recognizes L_1 and $A_2 = (\Sigma_2, Q_2, q_{2,0}, F_2, \delta_2)$ be a DFA that recognizes L_2. A PDA $A = (\Sigma, \Gamma, Q, q_0, F, \delta)$ that accepts $L_1 \cap L_2$ is constructed as follows: $\Sigma = \Sigma_1 \cap \Sigma_2$; $\Gamma = \Gamma_1$; $Q = Q_1 \times Q_2$; $q_0 = (q_{1,0}, q_{2,0})$; $F = F_1 \times F_2$;

$$\delta((s_1, s_2), x, a) = \{((s'_1, s'_2), b) | (s'_1, b) \in \delta_1(s_1, x, a) \wedge s'_2 = \delta_2(s_2, x)\}$$

$L(A) = L_1 \cap L_2$ is proved by showing that for any $w \in \Sigma^*$, $(q_0, \varepsilon) \xrightarrow{w\ *}_A ((s_1, s_2), \sigma)$ iff $(q_{1,0}, \varepsilon) \xrightarrow{w\ *}_{A_1} (s_1, \sigma)$ and $q_{2,0} \xrightarrow{w\ *}_{A_2} s_2$.

- $(q_0, \varepsilon) \xrightarrow{w\ *}_A ((s_1, s_2), \sigma) \Rightarrow (q_{1,0}, \varepsilon) \xrightarrow{w\ *}_{A_1} (s_1, \sigma)$ and $q_{2,0} \xrightarrow{w\ *}_{A_2} s_2$:
  - **IH**: Suppose the formula is true for any w of length n.
  - **Basis**: Given w where $|w| = 0$, i.e., ϵ. The run of A on ϵ is (q_0, ε). The run of A_1 on ϵ is $(q_{1,0}, \varepsilon)$. The run of A_2 on ϵ is $q_{2,0}$. Therefore, the IH is true for $n = 0$.
  - **IS**: **Assumption**: Given w where $|w| = n + 1$ s.t. the run of A on w is

    $$(q_0, \varepsilon) \to_A^* ((s_{1,n}, s_{2,n}), \sigma_n) \xrightarrow{w_{n+1}}_A ((s_{1,n+1}, s_{2,n+1}), \sigma_{n+1})$$

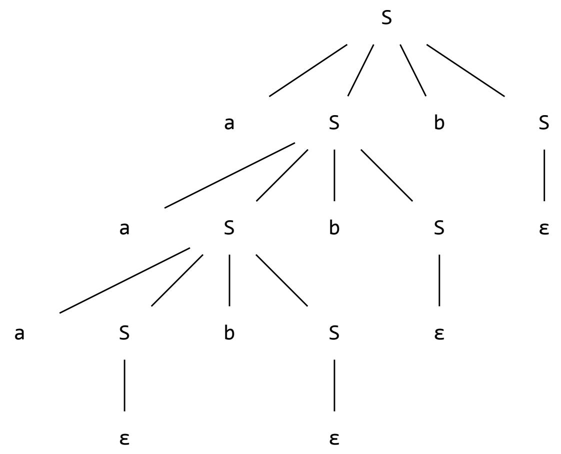
    From the assumption, we know there exist $a, b \in \Gamma^*$ s.t. stack σ_{n+1} is obtained from stack σ_n by popping a and pushing b and $((s_{1,n+1}, s_{2,n+1}), b) \in \delta((s_{1,n}, s_{2,n}), w_{n+1}, a)$.
    - Consider the run of A_1 on w: $(q_{1,0}, \varepsilon) \to_{A_1}^* (s_{1,n}, \sigma_n) \xrightarrow{w_{n+1}}_{A_1} (s_{1,n+1}, \sigma_{n+1})$ because there exist $a, b \in \Gamma^*$ s.t. $(s_{1,n+1}, b) \in \delta_1(s_{1,n}, w_{n+1}, a)$ and stack σ_{n+1} is obtained from stack σ_n by popping a and pushing b.
    - Consider the run of A_2 on w: $q_{2,0} \to_{A_2}^* s_{2,n} \xrightarrow{w_{n+1}}_{A_2} s_{2,n+1}$ because $s_{2,n+1} = \delta_2(s_{2,n}, w_{n+1})$.
  - **Conclusion**: The IH is true for $n + 1$, and therefore, for all $n \geq 0$.
- $(q_0, \varepsilon) \xrightarrow{w\ *}_A ((s_1, s_2), \sigma) \Leftarrow (q_{1,0}, \varepsilon) \xrightarrow{w\ *}_{A_1} (s_1, \sigma)$ and $q_{2,0} \xrightarrow{w\ *}_{A_2} s_2$:
  - **IH**: Suppose the formula is true for any w of length n.
  - **Basis**: Given w where $|w| = 0$, i.e., ϵ. The run of A_1 on Σ is $(q_{1,0}, \varepsilon)$. The run of A_2 on ϵ is $q_{2,0}$. The run of A on ϵ is (q_0, ε). Therefore, the IH is true for $n = 0$.
  - **IS**: **Assumption**: Given w where $|w| = n + 1$ s.t.
    the run of A_1 on w is $(q_{1,0}, \varepsilon) \to_{A_1}^* (s_{1,n}, \sigma_n) \xrightarrow{w_{n+1}}_{A_1} (s_{1,n+1}, \sigma_{n+1})$, and
    the run of A_2 on w is $q_{2,0} \to_{A_2}^* s_{2,n} \xrightarrow{w_{n+1}}_{A_2} s_{2,n+1}$.
    From the assumption, we know there exist $a, b \in \Gamma^*$ s.t. $(s_{1,n+1}, b) \in \delta_1(s_{1,n}, w_{n+1}, a)$ and stack σ_{n+1} is obtained from stack σ_n by popping a and pushing b, and that $s_{2,n+1} = \delta_2(s_{2,n}, w_{n+1})$. Consider the run of A on w:

    $$(q_0, \varepsilon) \to_A^* ((s_{1,n}, s_{2,n}), \sigma_n) \xrightarrow{w_{n+1}}_A ((s_{1,n+1}, s_{2,n+1}), \sigma_{n+1})$$

    because there exist $a, b \in \Gamma^*$ s.t. stack σ_{n+1} is obtained from stack σ_n by popping a and pushing b and $((s_{1,n+1}, s_{2,n+1}), b) \in \delta((s_{1,n}, s_{2,n}), w_{n+1}, a)$.
  - **Conclusion**: The IH is true for $n + 1$, and therefore, for all $n \geq 0$.
- **Conclusion**: $(s_1, s_2) \in F$ iff $s_1 \in F_1$ and $s_2 \in F_2$. For any $w \in \Sigma^*$, w is accepted by A iff w is

accepted by both A_1 and A_2. Therefore, $L_1 \cap L_2$ is context-free because it is recognized by some PDA.

- Figure 1.



- Figure 2.