Probability Homework 2 B00401062 羅文斌

2.1-16

a. $P(2,1,6,10)$ means that no number smaller than 2 is chosen, and 5 out of 8 numbers greater than 2 are chosen. Therefore, $P(2,1,6,10)=\binom{8}{5}/\binom{10}{6}=4/15$.
b. $P(i,r,k,n)$ means that $r-1$ out of $i-1$ numbers smaller than i are chosen, and $k-r$ out of $n-i$ numbers greater than i are chosen. Therefore, $P(i,r,k,n)=\binom{i-1}{r-1}\binom{n-i}{k-r}/\binom{n}{k}$.

2.2-10

For a pair of fair six-sided dice, the probability of the sum is as follows:
$P(X=2)=\frac{1}{36}, P(X=3)=\frac{2}{36}, P(X=4)=\frac{3}{36}, P(X=5)=\frac{4}{36}, P(X=6)=\frac{5}{36}$
$P(X=7)=\frac{6}{36}$
$P(X=8)=\frac{5}{36}, P(X=9)=\frac{4}{36}, P(X=10)=\frac{3}{36}, P(X=11)=\frac{2}{36}, P(X=12)=\frac{1}{36}$
Let $f(x)$ denote the money you gain:

- **If you bet low**: $f(x)=\begin{cases}1, \text{ if } x\in\{2,3,4,5,6\}\\ -1, \text{ if } x\in\{7,8,9,10,11,12\}\end{cases}$. Therefore,
  $P(f(x)=1)=\frac{15}{36}, P(f(x)=-1)=\frac{21}{36}$. The expected value $E[X]=1\cdot\frac{15}{36}-1\cdot\frac{21}{36}=-\frac{1}{6}$
- **If you bet high**: $f(x)=\begin{cases}1, \text{ if } x\in\{8,9,10,11,12\}\\ -1, \text{ if } x\in\{2,3,4,5,6,7\}\end{cases}$. Therefore,
  $P(f(x)=1)=\frac{15}{36}, P(f(x)=-1)=\frac{21}{36}$. The expected value $E[X]=1\cdot\frac{15}{36}-1\cdot\frac{21}{36}=-\frac{1}{6}$
- **If you bet on {7}**: $f(x)=\begin{cases}4, \text{ if } x=7\\ -1, \text{ if } x\in\{2,3,4,5,6,8,9,10,11,12\}\end{cases}$. Therefore,
  $P(f(x)=4)=\frac{6}{36}, P(f(x)=-1)=\frac{30}{36}$. The expected value $E[X]=4\cdot\frac{6}{36}-1\cdot\frac{30}{36}=-\frac{1}{6}$

2.3-11

1. **Mean**: $M'(t)=\frac{dM(t)}{dt}=\frac{2}{5}e^t+\frac{2}{5}e^{2t}+\frac{6}{5}e^{3t}$. The first moment about 0 is $M'(0)=\frac{2}{5}+\frac{2}{5}+\frac{6}{5}=2$. The mean of X is the first moment about 0, i.e. $\mu=2$.
2. **Variance**: $M''(t)=\frac{dM(t)}{dt^2}=\frac{2}{5}e^t+\frac{4}{5}e^{2t}+\frac{18}{5}e^{3t}$. The second moment about 0 is $M''(0)=\frac{2}{5}+\frac{4}{5}+\frac{18}{5}=\frac{24}{5}$. The variance of X is the difference of the second moment and the square of the first moment about 0, i.e. $\sigma^2=M''(0)-(M'(0))^2=\frac{24}{5}-2^2=\frac{4}{5}$.
3. **PMF**: The definition of $M(t)=E[e^{tx}]$. Therefore, $M(t)=\frac{2}{5}e^t+\frac{1}{5}e^{2t}+\frac{2}{5}e^{3t}$ implies that the pmf of X is $P(X=1)=\frac{2}{5}, P(X=2)=\frac{1}{5}, P(X=3)=\frac{2}{5}$, where $X\in S=\{1,2,3\}$. The pmf of X also satisfies $P(S)=\frac{2}{5}+\frac{1}{5}+\frac{2}{5}=1$.

2.4-20

a. $M(t)=(0.3+0.7e^t)^5$
   i. $M(t)=(0.3+0.7e^t)^5=\sum_{x=0}^{5}e^{tx}\binom{5}{x}0.7^x0.3^{5-x}$, which is a **binominal distribution** $P(X=x)=f(x)=\binom{5}{x}0.7^x0.3^{5-x}$, where $X\in S=\{0,1,2,3,4,5\}$.
   ii. **Mean**: $M'(t)=3.5e^t(0.3+0.7e^t)^4$. The first moment about 0 is $M'(0)=3.5(0.3+0.7)^4=3.5$. The mean $\mu=M'(0)=3.5$.
   Variance: $M''(t)=3.5e^t(0.3+0.7e^t)^4+9.8e^{2t}(0.3+0.7e^t)^3$. The second moment about 0 is

$M''(0) = 3.5(0.3+0.7)^4 + 9.8(0.3+0.7)^3 = 13.3$. The variance $\sigma^2 = M''(0) - (M'(0))^2 = 13.3 - 3.5^2 = 1.05$.

iii. $P(1 \le X \le 2) = \binom{5}{1}0.7^1 0.3^{5-1} + \binom{5}{2}0.7^2 0.3^{5-2} = 0.16065$.

b. $M(t) = \frac{0.3e^t}{1-0.7e^t}, t < -\ln(0.7)$

i. $M(t) = \frac{0.3e^t}{1-0.7e^t} = 0.3e^t(1 + 0.7e^t + 0.7^2e^{2t} + \ldots) = 0.3(e^t + 0.7e^{2t} + 0.7^2e^{3t} + \ldots) = \sum_{x=1}^{\infty}$, which is a **geometric distribution** $P(X = x) = f(x) = 0.3 \cdot 0.7^{x-1}$, where $X \in S = \{1, 2, 3, \ldots\}$.

ii. **Mean**: $M'(t) = \frac{0.3e^t}{1-0.7e^t} + \frac{0.21e^{2t}}{(1-0.7e^t)^2}$. The first moment about 0 is $M'(0) = \frac{0.3}{0.3} + \frac{0.21}{0.3^2} = \frac{10}{3}$. The mean $\mu = M'(0) = \frac{10}{3}$.

Variance: $M''(t) = \frac{0.3e^t}{1-0.7e^t} + \frac{0.21e^{2t}}{(1-0.7e^t)^2} + \frac{0.42e^{2t}}{(1-0.7e^t)^2} + \frac{0.294e^{3t}}{(1-0.7e^t)^3} = \frac{0.3e^t}{1-0.7e^t} + \frac{0.63e^{2t}}{(1-0.7e^t)^2} + \frac{0.294e^{3t}}{(1-0.7e^t)^3}$. The variance $\sigma^2 = M''(0) - (M'(0))^2 = \frac{0.3}{0.3} + \frac{0.63}{0.3^2} + \frac{0.294}{0.3^3} - (\frac{10}{3})^2 = \frac{70}{9}$.

iii. $P(1 \le X \le 2) = 0.3 + 0.3 \cdot 0.7 = 0.51$.

c. $M(t) = 0.45 + 0.55e^t$

i. $M(t) = 0.45 + 0.55e^t = \sum_{x=0}^{1} e^{tx}0.55^x 0.45^{1-x}$, which is a **Bernoulli distribution** $P(X = x) = f(x) = 0.55^x 0.45^{1-x}$, where $X \in S = \{0, 1\}$

ii. **Mean**: $M'(t) = 0.55e^t$. The first moment about 0 is $M'(0) = 0.55$. The mean $\mu = M'(0) = 0.55$.

Variance: $M''(t) = 0.55e^t$. The second moment about 0 is $M''(0) = 0.55$. The variance $\sigma^2 = M''(0) - (M'(0))^2 = 0.55 - 0.55^2 = 0.2475$.

iii. $P(1 \le X \le 2) = 0.55 + 0 = 0.55$.

d. $M(t) = 0.3e^t + 0.4e^{2t} + 0.2e^{3t} + 0.1e^{4t}$

i. $M(t) = 0.3e^t + 0.4e^{2t} + 0.2e^{3t} + 0.1e^{4t}$, whose pmf is $P(X = 1) = 0.3, P(X = 2) = 0.4, P(X = 3) = 0.2, P(X = 4) = 0.1$.

ii. **Mean**: $M'(t) = 0.3e^t + 0.8e^{2t} + 0.6e^{3t} + 0.4e^{4t}$. The first moment about 0 is $M'(0) = 0.3 + 0.8 + 0.6 + 0.4 = 2.1$. The mean $\mu = M'(0) = 2.1$.

Variance: $M''(t) = 0.3e^t + 1.6e^{2t} + 1.8e^{3t} + 1.6e^{4t}$. The second moment about 0 is $M''(0) = 0.3 + 1.6 + 1.8 + 1.6 = 5.3$. The variance $\sigma^2 = M''(0) - (M'(0))^2 = 5.3 - 2.1^2 = 0.89$.

iii. $P(1 \le X \le 2) = 0.3 + 0.4 = 0.7$

e. $M(t) = \sum_{x=1}^{10}(0.1)e^{tx}$

i. $M(t) = \sum_{x=1}^{10}(0.1)e^{tx}$, which is **uniform distribution** $P(X = x) = f(x) = 0.1$, where $X \in S = \{1, 2, \ldots, 10\}$.

ii. **Mean**: $M'(t) = \sum_{x=1}^{10} 0.1xe^{tx}$. The first moment about 0 is $M'(0) = 0.1(1 + 2 + \ldots + 10) = 5.5$.

Variance: $M''(t) = \sum_{x=1}^{10} 0.1x^2e^{tx}$. The second moment about 0 is $M''(0) = 0.1(1^2 + 2^2 + \ldots + 10^2) = 38.5$. The variance $\sigma^2 = M''(0) - (M'(0))^2 = 38.5 - 5.5^2 = 8.25$.

iii. $P(1 \le X \le 2) = 0.1 + 0.1 = 0.2$.