Homework 6

1. a. $\begin{bmatrix} \cos^2 0 & \sin 0 \cos 0 \\ \sin 0 \cos 0 & \sin^2 0 \end{bmatrix} \begin{bmatrix} \cos 90° & -\sin 90° \\ \sin 90° & \cos 90° \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$.

   b. $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

2. $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}$.

3. $\langle .,. \rangle$ is an inner product since it satisfies the following properties:

   a. $\langle \alpha x + y, z \rangle = 2(\alpha x_1 + y_1)z_1 + (\alpha x_2 + y_2)z_2 + 3(\alpha x_3 + y_3)z_3 =$
   $\alpha(2x_1 z_1 + x_2 z_2 + 3x_3 z_3) + (2y_1 z_1 + y_2 z_2 + 3y_3 z_3) = \alpha\langle x, z \rangle + \langle y, z \rangle$.

   b. $\langle y, x \rangle = 2y_1 x_1 + y_2 x_2 + 3y_3 x_3 = \overline{2x_1 y_1 + x_2 y_2 + 3x_3 y_3} = \overline{\langle x, y \rangle}$.

   c. $\langle x, x \rangle = 2x_1^2 + x_2^2 + 3x_3^2 = 0$ iff $x = 0$. Hence, if $x \neq 0$, then $\langle x, x \rangle \neq 0$.

4. a. $(v_1, v_3), (v_2, v_3)$

   b. $A = \begin{bmatrix} 1 & 2 & -2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$.

   All vectors orthogonal to S is $N(A) = \text{span}\{(-4, 3, 1, 0), (-1, 0, 0, 1)\}$.

5. $\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

   a. $N(A) = \text{span}\{(-2, 1, 0)\} \ni x$, e.g., $(-2, 1, 0)/\sqrt{5}$.

   b. $N(A^\top) = \text{span}\{(-1, -1, 1)\} \ni y$, e.g., $(-1, -1, 1)/\sqrt{3}$.

   c. $C(A^\top) = \text{span}\{(1, 2, 1), (3, 6, 4)\} \ni z$, e.g., $(1, 2, 1)/\sqrt{6}$.

6. a. $\begin{bmatrix} 1 & 2 & -3 \\ 2 & -3 & 1 \\ -3 & 5 & -2 \end{bmatrix}$.

   b. Impossible. $(2 - 3, 5) \cdot (1, 1, 1 = 4 \neq 0)$.

   c. Impossible. $(1, 1, 1) \in C(A)$ and $(1, 0, 0) \in N(A^\top)$, but $(1, 1, 1) \cdot (1, 0, 0) = 1 \neq 0$.

   d. Any solution to $A^2 = 0$, e.g., $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

   e. Impossible. $A^\top 1 = 0$ and $A1 = 1$. However, $1^\top A1 = 0^\top 1 = 0$ and $1^\top A1 = 1^\top 1 \neq 0$ are contradictory. Hence, there is no such A.

7. $\|3u + 4v\|^2 = (3u + 4v)^\top (3u + 4v) = 9u^\top u + 12u^\top v + 12v^\top u + 16v^\top v = 25$.