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- Solving a linear system $Ax = b$ with **Gaussian elimination** involves **forward elimination** and **back substitution**.
- By definition, pivots *cannot* be zero.
- Under what circumstances could the process break down?
  - Something must go wrong in the singular case.
  - Something might go wrong in the nonsingular case.
  - We do not know whether a zero will appear until we try, by actually going through the elimination process.
- The cost of elimination:
  - One operation: one division/multiplication-subtraction.
  - Forward elimination on LHS: $\sum_{k=1}^{n} k^2 - k = \frac{n^3-n}{3}$
  - Forward elimination on RHS: $(n-1)+(n-2)+\ldots+1 = \frac{n^2-n}{2}$
  - Back substitution: $n+(n-1)+\ldots+1 = \frac{n^2+n}{2}$
  - Total operations: $\frac{n^3-n}{3} + \frac{n^2-n}{2} + \frac{n^2+n}{2} \sim O(n^3)$
  - There now exists a method that requires only $O(n^{\log_2 7})$ multiplications!
- Matrix notation:
  - $Ax = b$: A (coefficients), x (unknowns), and b (RHS).
  - $A \in \mathbb{R}^{m\times n}$: $m \times n$ is called the **dimension (size)** of A.
- Matrix arithmetics:
  - *Addition*: Suppose $A = (a_{ij})_{m\times n}, B = (b_{ij})_{m\times n}$. $A + B = (a_{ij} + b_{ij})_{m\times n}$.
  - *Multiplication*: Suppose $A = (a_{ij})_{m\times n}, B = (b_{ij})_{n\times p}$. $AB = (c_{ij})_{m\times p}$, where $c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$.
  - *Scalar multiplication*: Suppose $A = (a_{ij})_{m\times n}$. $\alpha A = (\alpha a_{ij})_{m\times n}$
- **Elementary matrix** (**elimination matrix**):
  - The elementary matrix E_{ij} subtracts l times row j from row i. This E_{ij} includes $-l$ in row i, column j.
  - Gaussian elimination is essentially a series of $E(Ax) = Eb$.
- Different ways to look at matrix multiplication:
  - Each entry of AB is the product of a row and a column: $(AB)_{ij}$ = (row i of A) times (column j of B).
  - Each column of AB is the product of a matrix and a column: column j of AB = A times (column j of B).
  - Each row of AB is the product of a row and a matrix: row i of AB = (row i of A) times B.

- Matrix multiplication is associative, distributive, but *not* commutative.
- The product of lower triangular matrices is lower triangular.