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- A set S of vectors in V/F is linearly independent iff for each finite subset of S is linearly independent.
- The nullspace of A is $\{0\}$ iff columns of A are linearly independent.
- Given a row echelon matrix U:
  - Nonzero rows of U are linearly independent.
  - Columns containing the pivots are linearly independent.
- Example: In $\mathbb{R}^n$, $e_1, e_2, \ldots, e_n$ are linearly independent.
- To check linear dependency of any set of vectors $v_1, v_2, \ldots, v_n \in \mathbb{R}^m$, then define $A = [v_1, v_2, \ldots, v_n]$ and solve $Ax = 0$:
  - If there exists nonzero solution, then v's are linearly dependent.
  - If there are no free variables (i.e. rank $= n$), then v's are linearly independent.
  - If rank $< n$, then v's are linearly dependent.
  - If $m < n$, then v's are linearly dependent.
- Define S be a set of vectors in V/F. The subspace spanned by $S = \{v_1, v_2, \ldots, v_n\}$ is the intersection W of all subspaces of V containing S. W is denoted as $\langle v_1, v_2, \ldots, v_n \rangle$ or $\langle S \rangle$.
- Theorem: The subspace W spanned by a nonempty set S of vectors in V/F is the set T of all linear combinations of vectors in S.
  - Every linear combinations of vectors in S is in $W \Rightarrow T \subseteq W$.
  - T is a subspace containing S by definition of subspace. $\Rightarrow W \subseteq T$.
- $C(A)$ is the space spanned by columns of A.
- A **basis** for a vector space V/F is a set of vectors that satisfies it is linearly independent and it spans the vector space. If the basis of V/F is finite, then V/F is finite-dimensional.
- There's *one and only one* way to write every $v \in V/F$ as a linear combination of the basis elements.
- The **standard basis** $\{e_1, e_2, \ldots, e_n\}$ is a basis for $\mathbb{R}^n$.
- For any nonsingular matrix $A_{n\times n}$, the columns/rows of A are a basis of $\mathbb{R}^n$.
- The columns of row echelon matrix U that contains pivots are a basis for the column space of U.
- Theorem: Any two bases for V contain the same number of vectors. The number is called the **dimension** of V.
- Theorem: For any vector space, there exists a basis.
- Theorem: Given a finite-dimensional vector space V. Any linearly independent set in V can be *extended* to a basis. Any spanning set of V can be *reduced* to a basis.
- There are 4 fundamental subspaces associated to $A_{m\times n}$: **column space** $C(A) \leq \mathbb{R}^m$, **nullspace** $N(A) \leq \mathbb{R}^n$, **row space** $C(A^\mathrm{T}) \leq \mathbb{R}^n$, and **left nullspace** $N(A^\mathrm{T}) \leq \mathbb{R}^m$.
- If a row echelon matrix U of A has r pivots, then the row space of A, or $C(A^\mathrm{T})$, has dimension r. Also, $C(A^\mathrm{T})$ and $C(U^\mathrm{T})$ have the same basis, i.e. $C(A^\mathrm{T}) = C(U^\mathrm{T})$. Note that $C(A) \neq C(U)$.
- Given $A_{m\times n}$. The nullspace $N(A)$ is of dimension $n - r$. A basis of $N(A)$ can be constructed by reducing to $Ux = 0$ which has $n - r$ free variables. Let each free variable be 1, in turn, and others be 0, and solve $Ux = 0$. The $n - r$ vectors produced in this manner will be a basis for $N(A)$. $N(A)$ is

also called the **kernel** of A and its dimension is called the **nullity** of A.