Homework 7

1. a. For any nonzero vectors v on the intersection line of two planes, $v \cdot v \neq 0$.
   b. The dimensions of two subspaces $\leq \mathbb{R}^5$ are both two. $5 \neq 2+2$.
   c. span{(1,2)} and span{(2,1)} meet only at 0 but are not orthogonal.

2. a. $\langle f_1, f_2 \rangle = \int_{-1}^{1} (x+1)(2x+3)dx = \frac{2}{3}x^3 + \frac{5}{2}x^2 + 3x|_{-1}^{1} = \frac{22}{3}$.
   b. $\|f_1\| = \sqrt{\langle f_1, f_1 \rangle} = \sqrt{\int_{-1}^{1}(x+1)(x+1)dx} = \sqrt{\frac{1}{3}x^3 + x^2 + x|_{-1}^{1}} = \sqrt{\frac{8}{3}}$.

3. Let $A, B \in \mathbb{R}^{2\times 2}$ and A is diagonal. $\forall A$, $\langle A, B\rangle = \mathrm{tr}(AB^\top) = a_{11}b_{11} + a_{22}b_{22} = 0$ iff $(b_{11}, b_{22}) = 0$.
   Hence, $W^\perp = \{B | B$ is a 2×2 hollow matrix (all diagonal elements are 0) $\}$.

4. $x = (2,2,4,1)$ and $\|x\| = 5$. $y = (-2,1,2,0)$ and $\|y\| = 3$.
   a. $\cos\theta = \frac{x^\top y}{\|x\|\|y\|} = \frac{6}{15} = \frac{2}{5}$. $\tan^2\theta = \frac{1}{\cos^2\theta} - 1 = \frac{21}{4}$.
   b. $\frac{yy^\top}{y^\top y}x = \frac{1}{9}\begin{bmatrix} 4 & -2 & -4 & 0 \\ -2 & 1 & 2 & 0 \\ -4 & 2 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} -12/9 \\ 6/9 \\ 12/9 \\ 0 \end{bmatrix}$.
   c. $\frac{xx^\top}{x^\top x}y = \frac{1}{25}\begin{bmatrix} 4 & 4 & 8 & 2 \\ 4 & 4 & 8 & 2 \\ 8 & 8 & 16 & 4 \\ 2 & 2 & 4 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 12/25 \\ 12/25 \\ 24/25 \\ 6/25 \end{bmatrix}$.

5. a. Disprove. Let $S = \{1\} \subset \mathbb{R}$. $S^\perp = \{0\}$. $(S^\perp)^\perp = \mathbb{R} \neq S$.
   b. Disprove. Let $S_1 = \{1\} \subset \mathbb{R}$ and $S_2 = \{2\} \subset \mathbb{R}$. $S_1^\perp = \{0\} = S_2^\perp$. But $S_1 \neq S_2$.
   c. Disprove. Let $V = W = \{0\} \subset \mathbb{R}$ s.t. $V \perp W$. $V^\perp = W^\perp = \mathbb{R}$. But $\forall v, w \neq 0 \in \mathbb{R}$, $vw \neq 0$.
   d. Disprove. Let $V = \{1\}, W = \{0\}, Z = \{1\}$ s.t. $V \perp W$ and $W \perp Z$. But $1 \cdot 1 \neq 0$.

6. Let D be a square diagonal matrix with $d_{ii} = (w_i)^{\frac{1}{2}}$ for $i = 1, \ldots, n$.
   Let $u_w = Du = ((w_1)^{\frac{1}{2}}u_1, \ldots, (w_n)^{\frac{1}{2}}u_n)$, and $v_w = Dv = ((w_1)^{\frac{1}{2}}v_1, \ldots, (w_n)^{\frac{1}{2}}v_n)$.
   By Cauchy's Inequality, we know $|u_w^\top v_w| \leq \|u_w\|\|v_w\|$. And, $u_w^\top v_w = w_1u_1v_1 + \ldots + w_nu_nv_n$.
   Hence, $|w_1u_1v_1 + \ldots + w_nu_nv_n| \leq (w_1u_1^2 + \ldots + w_nu_n^2)^{\frac{1}{2}}(w_1v_1^2 + \ldots + w_nv_n^2)^{\frac{1}{2}}$

7. a. $(1,-1,1)\cdot(2,1,-1) = 0$ and $(1,0,2)\cdot(2,1,-1) = 0$.
   b. Prove by showing $T \subseteq N(A^\top)$ and $N(A^\top) \subseteq T$.

   - Prove $T \subseteq N(A^\top)$. $\forall t \in T$, t can be expressed as $\alpha(2,1,-1)$, and $A^\top\alpha(2,1,-1) = 0$. Hence, $T \subseteq N(A^\top)$.
   - Prove $N(A^\top) \subseteq T$. We know $\mathrm{rank}(A^\top) = 2$ and $\dim(N(A^\top)) = 3 - 2 = 1$. $(2,1,-1)$ is the only basis of $N(A^\top)$. Hence, $N(A^\top) \subseteq T$.

   c. $U = \{0\}$.
   d. Let $t = (2,1,-1)$. $x_2 = \frac{tt^\top}{t^\top t}x = \frac{1}{6}(36,18,-18) = (6,3,-3)$. $x_1 = (9,2,2) - (6,3,-3) = (3,-1,5)$.