# 2017-09-20

- **Contradictions ($\bot$)**: sentences of the form $\varphi \land \neg\varphi$ or $\neg\varphi \land \varphi$.
- Proof rules for connectives (continued):
  - **Negation introduction ($\neg$i)**: $\dfrac{\boxed{\begin{matrix}\phi \\ \vdots \\ \bot\end{matrix}}}{\neg\phi}\ \neg\mathrm{i}$
  - **Negation elimination ($\neg$e)**: $\dfrac{\phi \quad \neg\phi}{\bot}\ \neg\mathrm{e}$
- **Contradiction elimination ($\bot$e)**: $\dfrac{\bot}{\phi}\ \bot\mathrm{e}$
- **Reductio ad absurdum (RAA)**: $\neg p \Rightarrow \bot \vdash p$ (RAA)
- **Law of the excluded middle (LEM)**: $\vdash p \lor \neg p$ (LEM)
- Proof rules for natural deduction:
  - Fundamental: $\land$i, $\land$e, $\lor$i, $\lor$e, $\neg$i, $\neg$e, $\neg\neg$e, $\implies$ i, $\implies$ e, $\bot$e
  - Derived: $\neg\neg$i, MT, RAA, LEM
- **Provable equivalence ($\dashv\vdash$)**: $\phi$ and $\psi$ are **provably equivalent** ($\phi \dashv\vdash \psi$) if both $\phi \vdash \psi$ and $\psi \vdash \phi$.
- **Indirect proofs** (or **proof by contradiction**):
  - An argument for a proposition that shows its negation to be incompatible with a previously accepted or established premise.
  - *Non-constructive*. We do not show why $\phi$ holds; we only know $\neg\phi$ is impossible.
  - *Intuitionistic logicians* are averse to prove indirectly.
  - Examples of proof rules: $\neg\neg$e, RAA, LEM.
- **Well-formedness**:
  - A **well-formed** formula is constructed by applying the following rules finitely many times: atom, $\neg$, $\land$, $\lor$, $\implies$ .
  - Backus Naur form (BNF): $\phi ::= p|(\neg\phi)|(\phi \land \phi)|(\phi \lor \phi)|(\phi \Rightarrow \phi)$
  - **Inversion principle**: the construction process of a well-formed formula can always be inverted.
  - Subformulae are the well-formed formulae corresponding to its parse tree.
- A **valuation** or **model** of a formula $\phi$ is an assignment from each proposition atom in $\phi$ to a truth value.
- **Semantic sequent**: $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$
  - **Holds** if for every valuations where $\phi_1, \phi_2, \ldots, \phi_n$ are true, $\psi$ is also true.
  - Reads $\phi_1, \phi_2, \ldots, \phi_n$ **semantically entail** $\psi$ (**semantic entailment**).

- **Soundness theorem**: If $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ is valid, then $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$ holds.
- Proof for soundness: proof by introduction.
- **Completeness theorem**: If $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$ holds, then $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ is valid.
- Proof for completeness:
    - Assume $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$ holds.
    - $\vDash \phi_1 \implies (\phi_2 \implies (\ldots(\phi_n \implies \psi)))$ holds.
    - $\vdash \phi_1 \implies (\phi_2 \implies (\ldots(\phi_n \implies \psi)))$ is valid.
    - $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ is valid.
- The natural deduction proof system is both *sound* and *complete*, i.e. $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ is *valid* iff $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$ *holds*.
- **Semantic equivalence ($\equiv$)**: $\phi$ and $\psi$ are **semantically equivalent** ($\phi \equiv \psi$) if both $\phi \vDash \psi$ and $\psi \vDash \phi$.
- A sentence (formula) $\phi$ such that $\vDash \phi$ is called a **tautology**, and $\phi$ is a **valid** formula.