2017-10-05

- Solving recurrences: *substitution method*, *recursion-tree method*, *master theorem*.
- Useful tricks for substitution method:
  - Strengthen the inductive hypothesis by subtracting a low-order term. Ex: $T(n) = 4T(n/2) + O(n)$.
  - Change of variables. Ex: $T(n) = 2T(\sqrt{n}) + \log n$.
- **Master theorem**: $T(n) = \begin{cases} O(1) \text{ if } n = 1 \\ aT(\frac{n}{b}) + f(n) \text{ if } n \geq 1 \end{cases}$
  - If $\exists \epsilon > 0$ s.t. $f(n) = O(n^{\log_b a - \epsilon})$, then $T(n) = \Theta(n^{\log_b a})$.
  - If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \lg n)$.
  - If $\exists \epsilon > 0$ s.t. $f(n) = \Omega(n^{\log_b a + \epsilon})$, and $\exists c < 1$ s.t. $af(\frac{n}{b}) \leq cf(n)$, then $T(n) = \Theta(f(n))$.
- Examples using master theorem:
  - $T(n) = aT(\frac{n}{b}) + f(n^{\log_b a} \lg^k n) \implies T(n) = \Theta(n^{\log_b a} \lg^{k+1} n)$
- Matrix multiplication:
  - Lower bound: $\Omega(n^2)$
  - Upper bound by Strassen's algorithm: $O(n^{\lg 7})$
- **Strassen's algorithm**:
  - Total arithmetic operations: 12 addition, 6 subtraction, 7 multiplicaiton.
  - Time complexity: $T(n) = 7T(\frac{n}{2}) + \Theta(n^2) \implies T(n) = \Theta(n^{\lg 7})$.
- Selection problem: $\Theta(n)$
  - Lower bound: $\Omega(n)$
  - Upper bound by sorting: $O(n \log n)$
  - Upper bound by D&C: $O(n)$
- Selection problem by D&C:
  - Divide the n inputs into n/r groups each of r (odd) numbers.
  - Collect the median of each group $\implies O(n)$.
    - Choose the median of medians $\implies T(\frac{n}{r})$.
    - Discard at least $\frac{r+1}{4r}$ inputs $\implies T(\frac{(3r-1)n}{4r})$.
  - Time complexity: $T(n) = T(\frac{n}{r}) + T(\frac{(3r-1)n}{4r}) + \Theta(n) \implies T(n) = \Theta(n)$.
- Closest-pair problem:
  - Lower bound: $\Omega(n)$
  - Upper bound by brute-force algorithm: $O(n^2)$
  - Upper bound by D&C: $O(n \log n)$

- Closest-pair problem by D&C:
  - Divide points evenly along x-coordinate.
  - Find closest pair in each side recursively.
  - At most 8 points in a $\delta \times 2\delta$ block, where δ is the smaller distance from two sides.