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- The eigenvalues of A^k are $\lambda_1^k, \ldots, \lambda_n^k$. If $S^{-1}AS = \Lambda$, then $S^{-1}A^kS = \Lambda^k$.
- If A is invertible, then the eigenvalues of A^{-1} are $\lambda_1^{-1}, \ldots, \lambda_n^{-1}$.
- The eigenvectors of A, A^k, and A^{-1} are identical.
- If A and B are diagonalizable, they have the same eigenvector matrix S if and only if $AB = BA$.
  - $\Rightarrow$: Suppose there exists S such that $S^{-1}AS = \Lambda_A$ and $S^{-1}BS = \Lambda_B$. $AB = S\Lambda_A S^{-1} S\Lambda_B S^{-1} = S\Lambda_A\Lambda_B S^{-1} = S\Lambda_B\Lambda_A S^{-1} = S\Lambda_B S^{-1} S\Lambda_A S^{-1} = BA$.
  - $\Leftarrow$: Suppose $AB = BA$. Let x be an eigenvector of A, i.e., $Ax = \lambda x$. **Case 1**: $Bx = 0$. Then, x is an eigenvector of B. **Case 2**: $Bx \neq 0$. $ABx = BAx = \lambda Bx$. A has distinct eigenvalues, each eigenspace of A is one-dimensional. Hence, $Bx = \mu x$ for some scalar μ, i.e., Bx is a multiple of x. In other words, x is an eigenvector of B as well.
- **Theorem**: AB and BA have the same eigenvalues.
- **Theorem**: Let A be an $n \times n$ matrix over F. Assume that the characteristic polynomial of A has solutions in F. Then, for each eigenvalue of A, geometric multiplicity $\leq$ algebraic multiplicity.
- **Recurrence relation (Difference equation)**: Using Fibonacci sequence as an example: $a_{k+2} = a_{k+1} + a_k$, where $a_0 = 0, a_1 = 1$.
  - Let $u_k = (a_{k+1}, a_k)$. We have $u_0 = (1, 0)$ and $u_k = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} u_{k-1}$.
  - Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. We have $u_k = A^k u_0 = S\Lambda^k S^{-1} u_0 = S\Lambda^k c$ where $c = S^{-1} u_0$.
- **Markov chain**:
  - **Transition matrix (Markov matrix)**: a square matrix used to describe the transitions of a Markov chain.
  - Essential assumptions:
    - The total # of instances stays fixed, i.e., columns sum to 1.
    - The number of instances is never negative, i.e., all entries are non-negative.
    - Current state depends only on the last state, i.e., u_k depends only on u_{k-1} (no history).
  - Let u_k be the distribution of instances at time k. We have $u_k = Au_{k-1}$ and $u_k = A^k u_0 = S\Lambda^k S^{-1} u_0 = S\Lambda^k c$ where $c = S^{-1} u_0$.
  - Eigenvectors of a transition matrix A are $\lambda_1 = 1, \lambda_2, \ldots, \lambda_n$ which satisfies $1 > \lambda_2 \geq \ldots \geq \lambda_n > 0$.
  - No matter what the initial distribution may have been, the steady state u_∞ satisfies $u_\infty = Au_\infty$, i.e., u_∞ is the eigenvector corresponding to $\lambda = 1$.
- The **conjugate transpose** of a matrix A is denoted by A^{H} ($= \overline{A^\top}$). **Note**: $(AB)^{\mathrm{H}} = B^{\mathrm{H}}A^{\mathrm{H}}$.
- For $x \in \mathbb{C}^n$, the length of x, i.e., $\|x\|$, is defined as $\|x\|^2 = |x_1|^2 + \ldots + |x_n|^2 = x^{\mathrm{H}}x = \overline{x}^\top x$.
- The inner product of complex vector $\langle x, y \rangle = x^{\mathrm{H}}y = \overline{x}^\top y = y^\top \overline{x}$. **Note**: (1) $\langle x, y \rangle = \overline{\langle y, x \rangle}$; (2) $\langle cx, y \rangle = c\langle x, y \rangle$; (3) $\langle x, cy \rangle = \overline{c}\langle x, y \rangle$.
- x is orthogonal to y if $x^{\mathrm{H}}y = 0$.
- An $n \times n$ matrix A is called a **Hermitian matrix** if and only if $A^{\mathrm{H}} = A$.

- If $A^{\mathrm{H}} = A$, then $x^{\mathrm{H}}Ax$ is real for all $x \in \mathbb{C}^n$.
    - $(x^{\mathrm{H}}Ax)^{\mathrm{H}} = x^{\mathrm{H}}A^{\mathrm{H}}x = x^{\mathrm{H}}Ax$.
- Every eigenvalue of a Hermitian matrix is real.
    - Let λ be an eigenvalue of A and $A^{\mathrm{H}} = A$.
    - There exists $x \neq 0$ such that $Ax = \lambda x$.
    - $x^{\mathrm{H}}Ax = \lambda x^{\mathrm{H}}x \Rightarrow \lambda = x^{\mathrm{H}}Ax/x^{\mathrm{H}}x$, which is a real number.
- The eigenvectors corresponding to distinct eigenvalues of a Hermitian matrix are orthogonal to each other.
    - Suppose $A^{\mathrm{H}} = A$ and $Ax_1 = \lambda_1 x_1$, $Ax_2 = \lambda_2 x_2$, where $\lambda_1 \neq \lambda_2$, $x \neq 0$, $y \neq 0$.
    - $\lambda_1 x_1^{\mathrm{H}} x_2 = (\lambda_1 x_1)^{\mathrm{H}} x_2 = (Ax_1)^{\mathrm{H}} x_2 = x_1^{\mathrm{H}} A^{\mathrm{H}} x_2 = x_1^{\mathrm{H}} A x_2 = x_1^{\mathrm{H}}(Ax_2) = x_1^{\mathrm{H}}(\lambda_2 x_2) = \lambda_2 x_1^{\mathrm{H}} x_2$.
    - $(\lambda_1 - \lambda_2)x_1^{\mathrm{H}} x_2 = 0$ and $\lambda_1 \neq \lambda_2$. Hence, $x_1^{\mathrm{H}} x_2 = 0$, i.e., eigenvectors are orthogonal.