Homework 3

1. 1. Reject. $q_0 \vdash q_{\text{rej}}$
   2. Accept. $q_0 011 \vdash \lhd\, p_0 11 \vdash \lhd\, 0 p_1 1 \vdash \lhd\, 01 p_1 \vdash \lhd\, 01 s 1 \vdash \lhd\, 0 s 10 \vdash \lhd\, s 000 \vdash t \lhd 100 \vdash q_{\text{acc}} \lhd 100$
   3. Reject. $q_0 100 \vdash \lhd\, p_1 00 \vdash \lhd\, 1 p_0 0 \vdash \lhd\, 10 p_0 \vdash \lhd\, 10 q_{\text{rej}} 0$
   4. Accept. $q_0 111 \vdash \lhd\, p_1 11 \vdash \lhd\, 1 p_1 1 \vdash \lhd\, 11 p_1 \vdash \lhd\, 11 s 1 \vdash \lhd\, 1 s 10 \vdash \lhd\, s 100 \vdash s \lhd 000 \vdash \lhd\, r_1 000$ $\vdash \lhd\, 1 r_0 00 \vdash \lhd\, 10 r_0 0 \vdash \lhd\, 100 r_0 \vdash \lhd\, 10 t 00 \vdash \lhd\, 1 t 000 \vdash \lhd\, t 1000 \vdash t \lhd 1000 \vdash q_{\text{acc}} \lhd 1000$
2. Let M_2 be an NTM and have $\Sigma_2 = \Sigma \cup \Gamma \cup Q$. On input w:
   1. **If** w has any symbol other than Σ_2: Reject.
   2. **If** w has no state symbol or has more than one state symbol from Q: Reject.
   3. Move head position to the state symbol $w_i = q \in Q$.
   4. **If** $q = q_0$: Accept.
   5. Consider a two-tuple $(q^+, w^+) \in Q \times \Gamma$ and choose one of the three following steps nondeterministically :
      1. **If** there is (q^+, w^+) such that $\delta(q^+, w^+) = (q, w_{i+1}, \text{Stay})$: Overwrite $q w_{i+1}$ with $q^+ w^+$. **Otherwise**: Reject
      2. **If** there is (q^+, w^+) such that $\delta(q^+, w^+) = (q, w_{i+2}, \text{Left})$: Overwrite $q w_{i+1} w_{i+2}$ with $w_{i+1} q^+ w^+$. **Otherwise**: Reject.
      3. **If** there is (q^+, w^+) such that $\delta(q^+, w^+) = (q, w_{i-1}, \text{Right})$: Overwrite $w_{i-1} q$ with $q^+ w^+$. **Otherwise**: Reject.
   6. **Repeat** step 1-5 until accept or reject.
3. Consider a three-tape Turing machine.
   1. Put input w on tape 1.
   2. Scan w from left to right and put head position on the state symbol $w_i = q$.
   3. Copy the state symbol $w_i = q$ and the next input symbol w_{i+1} to tape 2.
   4. Match (q, w_{i+1}) on tape 2 to the corresponding transition relation $\delta(q, w_{i+1}) = (q^+, w^+, d)$, where $d \in \{\text{Left,Right,Stay}\}$.
   5. Copy the instruction (q^+, w^+, d) to tape 3.
   6. **If** $d = \text{Stay}$: Overwrite $q w_{i+1}$ with $q^+ w^+$.
   7. **Else if** $d = \text{Left}$: Overwrite $w_{i-1} q w_{i+1}$ with $q^+ w_{i-1} w^+$
   8. **Else if** $d = \text{Right}$: Overwrite $q w_{i+1}$ with $w^+ q^+$.
   9. **Return** output on tape 1.

4.
- Let a decider for L_{fin} be R := On input $\lfloor M \rfloor$: Accept if $L(M)$ is finite. Reject if $L(M)$ is infinite.
- Define $M'(M, w)$:= On input x: Accept if M accepts w.
- Construct a decider S := On input (M, w): Construct $M'(M, w)$. Run R on $\lfloor M' \rfloor$. Accept if H rejects. Reject if R accepts.
- Run S on input (M, w) and consider the following two cases:
  - $w \in L(M)$: $M'(M, w)$ accepts everything, i.e., $L(M')$ is infinite. Hence, R rejects and S accepts.
  - $w \notin L(M)$: $M'(M, w)$ accepts nothing, i.e., $|\text{L(M')}| = 0$, which is finite. Hence, R accepts and S rejects.
- Conclusion: If R decides L_{fin}, then S decides $A_{\text{TM}} = \{(M, w) | w \in L(M)\}$. However, A_{TM} is undecidable (as proved in class), so there is no such R that decides L_{fin}. L_{fin} is undecidable.

5.
- Let $\text{EQ}_{\text{CFG,DFA}} = \{(G, A) | G$ is a CFG, A is a DFA, $L(G) = L(A)\}$.
- Let a decider for $\text{EQ}_{\text{CFG,DFA}}$ be R := On input (G, A): Accept if $L(G) = L(A)$. Reject if $L(G) \neq L(A)$.
- Construct a decider S := On input (G): Construct a DFA A that accepts Σ^*. Run R on (G, A). Accept if R accepts. Reject if R rejects.
- Run S on input G and consider the following two cases:
  - $L(G) = \Sigma^*$: $L(A) = \Sigma^*$. Hence, R accepts (G, A), and S accepts.
  - $L(G) \neq \Sigma^*$: $L(A) = \Sigma^*$. Hence, R rejects (G, A), and S rejects.
- Conclusion: If R decides $\text{EQ}_{\text{CFG,DFA}}$, then S decides $\text{ALL}_{\text{CFG}} = \{G | G$ is a CFG, $L(G) = \Sigma^*\}$. However, ALL_{CFG} is undecidable (as proved in class), so there is no such R that decides $\text{EQ}_{\text{CFG,DFA}}$. $\text{EQ}_{\text{CFG,DFA}}$ is undecidable.

6.
- Let $\text{CFL}_{\text{TM}} = \{\lfloor M \rfloor | L(M)$ is context free$\}$.
- Let a decider for CFL_{TM} be R := On input $\lfloor M \rfloor$: Accept if $L(M)$ is context free. Reject if $L(M)$ is not.
- Define $M'(M, w)$:= On input x: Accept if x has the form $0^n 1^n 0^n$. Otherwise, run M on w. Accept if M accepts w.
- Construct a decider S := On input (M, w): Construct $M'(M, w)$. Run R on $\lfloor M' \rfloor$. Accept if R accepts. Reject if R rejects.
- Run S on input (M, w) and consider the following two cases:
  - $w \in L(M)$: $M'(M, w)$ accepts everything, i.e., $L(M') = \Sigma^*$, which is context free. Hence, R accepts, and S accepts.
  - $w \notin L(M)$: $M'(M, w)$ accepts input of the form $0^n 1^n 0^n$, which is not context free. Hence, R rejects, and S rejects.
- Conclusion: If R decides CFL_{TM}, then S decides $A_{\text{TM}} = \{(M, w) | w \in L(M)\}$. However, A_{TM} is undecidable (as proved in class), so there is no such R that decides CFL_{TM}. CFL_{TM} is undecidable.