Homework 5

1. Prove. Let $A = \begin{bmatrix} 3 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. If w_1, w_2, w_3 are linearly dependent, then $\exists x \neq 0$ s.t. $[w_1 w_2 w_3]x = 0$.
   $[w_1 w_2 w_3]x = [v_1 v_2 v_3]Ax = 0$. $Ax \neq 0$ because A is fully-ranked and $x \neq 0$. However, there is a contradiction to that v_1, v_2, v_3 are linearly independent because $\exists x' = Ax \neq 0$ s.t. $[v_1 v_2 v_3]x' \neq 0$. Therefore, w_1, w_2, w_3 are linearly independent.
2. Prove. $\begin{bmatrix} 1 & 1 & 5 \\ 1 & -1 & 8 \\ 1 & -3 & 11 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 5 \\ 0 & -2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$. Therefore, $\text{rank}(S_2) = 2$, and $\text{span}(S_2)$ is a plane $\in \mathbb{R}^3$. The plane $\in \mathbb{R}^3$ is $(1,1,1) \times (1,-1,-3) \cdot (x,y,z) = x - 2y + z = 0$. Both $(1,2,3)$ and $(2,3,4)$ are in the plane, therefore, $\text{span}(S_1)$ is a subspace of $\text{span}(S_2)$.
3. a. U is an upper triangular matrix of LU decomposition of A.
   For A, $(1,1,3) \times (1,3,1) = (-8,2,2) \parallel (4,-1,-1)$, and $(1,1,0) \times (1,3,1) = (1,-1,2)$.
   $\Rightarrow C(A) = \{(x,y,z) \in \mathbb{R}^3 | 4x - y - z = 0\}$; $C(A^{\mathrm{T}}) = \{(x,y,z) \in \mathbb{R}^3 | x - y + 2z = 0\}$; $N(A) = \{(1,-1,2)t | t \in \mathbb{R}\}$; $N(A^{\mathrm{T}}) = \{(4,-1,1)t | t \in \mathbb{R}\}$.
   For U, $(1,0,0) \times (1,2,0) = (0,0,2) \parallel (0,0,1)$, and $(1,1,0) \times (0,2,1) = (1,-1,2)$.
   $\Rightarrow C(U) = \{(x,y,0) \in \mathbb{R}^3\}$; $C(U^{\mathrm{T}}) = \{(x,y,z) \in \mathbb{R}^3 | x - y + 2z = 0\}$; $N(U) = \{(1,-1,2)t | t \in \mathbb{R}\}$; $N(U^{\mathrm{T}}) = \{(0,0,z) | z \in \mathbb{R}\}$.
   b. $C(A^{\mathrm{T}}) = C(U^{\mathrm{T}})$, and $N(A) = N(U)$.
4. a. $\{(1,1,1,1)\}$
   b. $\{(-1,1,0,0), (-1,0,1,0), (-1,0,0,1)\}$
   c. $\{(-1,1,1,0), (-1,1,0,1)\}$, which span the nullspace of $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}$.
   d. $\{(1,0), (0,1)\}$, and $\{(-1,0,1,0,0), (0,-1,0,1,0), (-1,0,0,0,1)\}$
5. Basis for $p(x)$: $\{1, x, x^2, x^3\}$. Basis for the subspace with $p(1) = 0$: $\{x - 1, x^2 - 1, x^3 - 1\}$.
6. a. $C(A) : 5$; $C(A^{\mathrm{T}}) : 5$; $N(A) : 4$; $N(A^{\mathrm{T}}) : 2$.
   b. $C(A)$ is $\mathbb{R}^3$. $N(A^{\mathrm{T}}) = 0 \in \mathbb{R}^3$.
7. Row space and nullspace do not change. $(2,1,3,4)$.
8. (b) and (c) satisfy $T(v + w) = T(v) + T(w)$.
   (b) and (c) satisfy $T(cv) = cT(v)$.
   b. $T(cv + w) = (cv_1 + w_1) + (cv_2 + w_2) + (cv_3 + w_3) = c(v_1 + v_2 + v_3) + (w_1 + w_2 + w_3) = cT(v) + T(w)$.
   c. $T(cv + w) = (cv_1 + w_1, 2cv_2 + 2w_2, 3cv_3 + 3w_3) = c(v_1, 2v_2, 3v_3) + (w_1, 2w_2, 3w_3) = cT(v) + T(w)$.
   d. $T(cv) = \max cv \neq c \max v = cT(v)$.
9. $T(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} 1 & 1 \\ 3 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 8 \end{bmatrix}$. $T(\begin{bmatrix} 0 \\ 1 \end{bmatrix}) = \begin{bmatrix} 1 & 1 \\ 3 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -5 \\ -6 \end{bmatrix}$.