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- The **determinant** of A, denoted by $\det A$, is a function from $M_n(F)$ to F. Let $\det A = D(A_i, \ldots, A_n)$, where A_i is i-th row of A. The **determinant** of A satisfies the following conditions:
  - The $\det A$ is a linear function of the i-th row when the other $(n-1)$ rows are held fixed. $D(A_1, \ldots, \alpha A_i + A_i', \ldots, A_n) = \alpha D(A_1, \ldots, A_i, \ldots, A_n) + D(A_1, \ldots, A_i', \ldots, A_n)$
  - $\det A = 0$ if A has two identical rows.
  - $\det I = 1$.
  - $\det P_{ij} A$ = -$\det A$, where P_{ij} is a permutation matrix. **Proof**: $D(A_1, \ldots, A_i + A_j, \ldots, A_j + A_i, \ldots, A_n) = 0 = D(A_1, \ldots, A_i, \ldots, A_j, \ldots, A_n) + D(A_1, \ldots, A_j, \ldots, A_i, \ldots, A_n)$.
  - $\det E_{ij} A = \det A$, where E_{ij} is an elementary matrix. **Proof**: $D(A_1, \ldots, A_i + \alpha A_j, \ldots, A_j, \ldots, A_n) = D(A_1, \ldots, A_i, \ldots, A_j, \ldots, A_n) + \alpha D(A_1, \ldots, A_j, \ldots, A_j, \ldots, A_n) = D(A_1, \ldots, A_i, \ldots, A_j, \ldots, A_n)$.
  - If A has a row of zeros, then $\det A = 0$.
  - If A is triangular, then $\det A = a_{11} \ldots a_{nn}$.
  - If A is singular, then $\det A = 0$. If A is invertible, then $\det A \neq 0$.
  - $\det AB = \det A \det B$.
  - $\det A^\top = \det A$. **Proof**: If A is singular, then $A^\top$ is singular. If A is nonsingular, then $PA = LDU$. Since $\det P \det A = \det L \det D \det U = \det D$ and $\det A^\top \det P^\top = \det U^\top \det D^\top \det L^\top = \det D^\top$, hence, $\det P \det A = \det A^\top \det P^\top$. Since $PP^\top = I$, hence, $\det P \det P^\top = 1$. Hence, $\det A = \det A^\top$.
  - If A is nonsingular, then $A = P^{-1}LDU$. $\det A = \pm$ product of pivots.
- $\det A$ is the sum of $n!$ terms and for each item, every row and column contribute to it.
- $\det A = a_{i1}A_{i1} + \ldots + a_{in}A_{in}$, where the cofactor $A_{ij} = (-1)^{i+j} \det M_{ij}$, and M_{ij} is a submatrix of A by deleting row i and column j of A.
- Computation of A^{-1}: $\begin{bmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \ldots & a_{nn} \end{bmatrix} \begin{bmatrix} A_{11} & \ldots & A_{n1} \\ \vdots & \ddots & \vdots \\ A_{1n} & \ldots & A_{nn} \end{bmatrix} = \begin{bmatrix} \det A & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \det A \end{bmatrix} = (\det A) I$.
- **Cramer's rule**.
- Volume of a parallelepiped:
  - If rows of A are mutually perpendicular, then $\det A = \pm\, l_1 \ldots l_n = \pm$ volume of the parallelepiped, where l_i is the length of row i.
  - Otherwise, perform orthogonalization first.