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- Solving $Ax = b$ is equivalent to solving $Lc = b$ and then $Ux = c$, where $A = LU$.
- How can we undo the steps of Gaussian elimination? By multiplying $E^{-1}(= L)$ with $Ux = c$.
- The entries below the diagonal of L are the multipliers from each step of Gaussian elimination.
- **Triangular factorization** $A = LU$ with no exchanges of rows.
  - L is the **lower triangular matrix**, with 1s on the diagonal. The multipliers l_{ij} (taken from elimination) are below the diagonal.
  - U is the **upper triangular matrix** which appears after forward elimination. The diagonal entries of U are the *pivots*.
- The triangular factorization can be written $A = LDU$, where L and U have 1s on the diagonal and D is the diagonal matrix of pivots.
- There is some freedom, and there is a "Crout algorithm" that arranges the calculations of LU decomposition in a slightly different way. There is no freedom in the final LDU decomposition.
- **Permutation matrix**: P_{ij} is (1) *row exchange* of row i and row j, or (2) *column exchange* of column i and column j.
  - PA performs row exchange of A.
  - AP performs column exchange of A.
  - There is a single "1" in every row and column.
  - Multiplication of permutation matrices is *not* commutative.
  - P^{-1} is always the same as P^{T}.
- In the *nonsingular* case, if there is a permutation matrix P that reorders the rows of A to avoid zeros in the pivot positions, then $Ax = b$ has a unique solution by solving $PAx = Pb$.
- In the *singular* case, no P can produce a full set of pivots: elimination fails.
- If A is invertible, then the matrix B satisfying $AB = BA = I$ is unique, and B is denoted as A^{-1}. (Proof by contradiction)
- Not all matrices have inverses. An inverse is impossible when $Ax = 0$ and x is nonzero. (Proof by contradiction)
- A product AB of invertible matrices is inverted by $B^{-1}A^{-1}$ (in reverse order).
- How to calculate A^{-1}? The **Gauss-Jordan method**: $[A|I] \to [U|L^{-1}] \to [I|A^{-1}]$
- A matrix is invertible iff it satisfies all the following equivalent criteria (1) independent columns/rows, (2) nonzero pivots, (3) nonzero determinant, (4) nonzero eigenvalues.
- Suppose A has a full set of n pivots. $AA^{-1} = I$ gives n separate systems $Ax_i = e_i$ for the columns of A^{-1}. They can be solved by elimination or by Gauss-Jordan. Row exchanges may be needed, but the

columns of A^{-1} are uniquely determined.

- In general, three types of elementary matrices are allowed in Gauss-Jordan method:
  - E_{ij} : to subtract a multiple l of row j from row i.
  - P_{ij} : to exchange rows i and j.
  - D: to multiply all rows by their pivots.