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- The $R(A)$ stands for the **range** of A, which is essentially the column space of A. Let $f(x) = A_{m \times n}x$:
  - The domain of $f = \mathbb{R}^n$
  - The range of $f = \{b \in \mathbb{R}^m | Ax = b\} = R(A) = C(A)$
  - The kernel of $f = \{x \in \mathbb{R}^n | Ax = 0\} = N(A)$
- If U is the row echelon form of A, $C(A) \neq C(U)$, but they have the same dimension. For U, the columns w/ pivots are a basis for $C(U)$. Then, the corresponding columns of A form a basis for $C(A)$. Since the two systems $Ax = 0$ and $Ux = 0$ are equivalent and have the same solutions. A nontrivial solution x means a linear combination of columns of U, hence the same linear combination of columns of A. So, if a set of columns of U is independent, then so are the corresponding column of A and vice versa. To find a basis of $C(A)$, we pick those columns of A, which corresponds to the columns of U w/ pivots.
- The dimension of the column space = the rank r, which also equals the dimension of the row space. # of independent columns = # of independent rows. Or, column rank = row rank.
- Given $A_{m \times n}$. For A, dimension of $C(A)$ + dimension of $N(A)$ = # of columns of A, i.e., $r + (n - r) = n$. For $A^\top$, dimension of $C(A^\top)$ + dimension of $N(A^\top)$ = # of columns of $A^\top$, i.e., $r + (m - r) = m$.
- To find the left nullspace of $A_{m \times n}$, we compute $L^{-1}PA = U$. The last $m - r$ rows of $L^{-1}P$ must be a basis for the left nullspace since the last $m - r$ rows of $L^{-1}P$ are independent and dimension of the left nullspace is also $m - r$.
- **Orthogonality**: $C(A) \perp N(A^\top) \leq \mathbb{R}^m$. $C(A^\top) \perp N(A) \leq \mathbb{R}^n$.
- *Existence* and *Uniqueness* of solution to $Ax = b$. Given $A_{m \times n}$:
  - *Existence*: The system $Ax = b$ has at least one solution x for every b iff the columns span $\mathbb{R}^m$ ($r = m$). In this case, $\exists$ a right inverse $R_{n \times m}$ s.t. $AR = I_m$. This is possible only if $m \leq n$.
  - *Uniqueness*: The system $Ax = b$ has at most one solution x for every b iff the columns are linearly independent ($r = n$). In this case, $\exists$ a left inverse $L_{n \times m}$ s.t. $LA = I_n$. This is possible only if $m \geq n$.
- The following statements about a matrix $A_{n \times n}$ are equivalent:
  - A is nonsingular.
  - The rows/columns of A span $\mathbb{R}^n$.
  - The rows/columns of A are linearly independent.
  - Elimination can be completed: $PA = LDU$.
  - A is invertible.
  - Determinant of $A \neq 0$.
  - 0 is not an eigenvalue of A.
  - A is positive definite.
- A transforms x into Ax:
  - Stretch: $A = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$

    - Rotation of 90°: $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
    - Reflection about $x = y$: $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
    - Projection onto x-axis: $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
- Let V and V' be vector spaces of F. A linear transformation T from V to V' is a function from V to V' that preserves the operations on V and V', i.e., $\forall x, y \in V, \forall \alpha \in F, T(\alpha x + y) = \alpha Tx + Ty$.
- Linear transformation $T = \frac{d}{dt} : P_n(\mathbb{R}) \to P_{n-1}(\mathbb{R})$.
    - Nullspace: $N(T) = P_0(\mathbb{R})$. Dimension $= 1$.
    - Range: $C(T) = P_{n-1}(\mathbb{R})$. Dimension (rank) $= n$.
- Linear transformation $T = \int_0^t : P_n(\mathbb{R}) \to P_{n+1}(\mathbb{R})$.
    - Nullspace: $N(T) = \{0\}$. Dimension $= 0$.
    - Range: $P_{n+1}(\mathbb{R}) - P_0(\mathbb{R})$. Dimension (rank) $= n + 1$.