Homework 2

1. $E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}, E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}, U = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$

2. $\begin{bmatrix} a & r & r & r \\ a & b & s & s \\ a & b & c & t \\ a & b & c & d \end{bmatrix} \xrightarrow{E_1} \begin{bmatrix} a & r & r & r \\ 0 & b-r & s-r & s-r \\ 0 & b-r & c-r & t-r \\ 0 & b-r & c-r & d-r \end{bmatrix} \xrightarrow{E_2} \begin{bmatrix} a & r & r & r \\ 0 & b-r & s-r & s-r \\ 0 & 0 & c-s & t-s \\ 0 & 0 & c-s & d-s \end{bmatrix} \xrightarrow{E_3} \begin{bmatrix} a & r & r & r \\ 0 & b-r & s-r & s-r \\ 0 & 0 & c-s & t-s \\ 0 & 0 & 0 & d-t \end{bmatrix}.$

   Four conditions for 4 pivots are $a \neq 0, b \neq r, c \neq s, d \neq t$.

3. $A_1^{-1} = \begin{bmatrix} 0 & 1/3 \\ 1/2 & 0 \end{bmatrix}, A_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}, A_3^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$

4. (a) True. (b) False. Identity matrices are invertible. (c) True. (d) True.

5. A is not invertible when $\begin{vmatrix} 2 & c & c \\ c & c & c \\ 8 & 7 & c \end{vmatrix} = -c(c-2)(c-7) := 0 \implies c = 0, 2, 7.$

6. (a) A^2 and B^2 are symmetric, so $A^2 - B^2$ is symmetric. (b) $(A+B)(A-B) = A^2 + BA - AB - B^2$. Neither BA nor AB is guaranteed to be symmetric, so $(A+B)(A-B)$ is not guaranteed to be symmetric. (c) $(ABA)^{\mathrm{T}} = A^{\mathrm{T}}B^{\mathrm{T}}A^{\mathrm{T}} = ABA$, so ABA is symmetric. (d) $(ABAB)^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}}B^{\mathrm{T}}A^{\mathrm{T}} = BABA \neq ABAB$, so $ABAB$ is not guaranteed to be symmetric.

7. $\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -2 & 3 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -4 & 5 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -6 & 7 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 5 & 0 & 8/3 & 4/3 & 1 & 0 \\ 0 & 0 & 0 & 7 & 16/5 & 8/5 & 6/5 & 1 \end{array}\right] \to$

   $\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2/3 & 1/3 & 0 & 0 \\ 0 & 0 & 1 & 0 & 8/15 & 4/15 & 1/5 & 0 \\ 0 & 0 & 0 & 1 & 16/35 & 8/35 & 6/35 & 1/7 \end{array}\right] \implies A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2/3 & 1/3 & 0 & 0 \\ 8/15 & 4/15 & 1/5 & 0 \\ 16/35 & 8/35 & 6/35 & 1/7 \end{bmatrix}$

   $(I+B)^{-1} = I + B^{-1} = I + ((I+A)^{-1}(I-A))^{-1} = I + (I-A)^{-1}(I+A) = I + (I - A^{-1})(I+A) = I + I - A^{-1} + A - I = A - A^{-1} + I$

   $= \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 3 & 0 & 0 \\ 0 & -4 & 5 & 0 \\ 0 & 0 & -6 & 7 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2/3 & 1/3 & 0 & 0 \\ 8/15 & 4/15 & 1/5 & 0 \\ 16/35 & 8/35 & 6/35 & 1/7 \end{bmatrix} + I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -8/3 & 11/3 & 0 & 0 \\ -8/15 & -64/15 & 29/5 & 0 \\ -16/35 & -8/35 & -216/35 & 55/7 \end{bmatrix}.$

8. a. $B = A^{-1}CA$

   b. Suppose A is not invertible $\Leftrightarrow |A| = 0 \implies |AB| = |A||B| = 0 \Leftrightarrow AB$ is not invertible, which is a contradiction to the assumption that AB is invertible. Therefore, if AB is invertible, then A is invertible, and $A^{-1} = B(AB)^{-1}$.

9. A_1 is essentially $(E_{31}E_{21})^n$ with $l_{31} = -m$ and $l_{21} = -n$. Therefore, we hypothesize $A_1 = \begin{bmatrix} 1 & 0 & 0 \\ nl & 1 & 0 \\ nm & 0 & 1 \end{bmatrix}$. In the base case where $n = 1$, $\begin{bmatrix} 1 & 0 & 0 \\ l & 1 & 0 \\ m & 0 & 1 \end{bmatrix}$ holds. By inductive hypothesis, $\begin{bmatrix} 1 & 0 & 0 \\ l & 1 & 0 \\ m & 0 & 1 \end{bmatrix}^{n+1} = \begin{bmatrix} 1 & 0 & 0 \\ l & 1 & 0 \\ m & 0 & 1 \end{bmatrix}^n \begin{bmatrix} 1 & 0 & 0 \\ l & 1 & 0 \\ m & 0 & 1 \end{bmatrix}$

   $= \begin{bmatrix} 1 & 0 & 0 \\ nl & 1 & 0 \\ nm & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ l & 1 & 0 \\ m & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ (n+1)l & 1 & 0 \\ (n+1)m & 0 & 1 \end{bmatrix}$. Therefore, $A_1 = \begin{bmatrix} 1 & 0 & 0 \\ nl & 1 & 0 \\ nm & 0 & 1 \end{bmatrix}$.

   $A_2 = \begin{bmatrix} 1 & 0 & 0 \\ -l & 1 & 0 \\ -m & 0 & 1 \end{bmatrix}$. $A_3 = \begin{bmatrix} 1 & 0 & 0 \\ -l & 1 & 0 \\ lm & -m & 1 \end{bmatrix}$