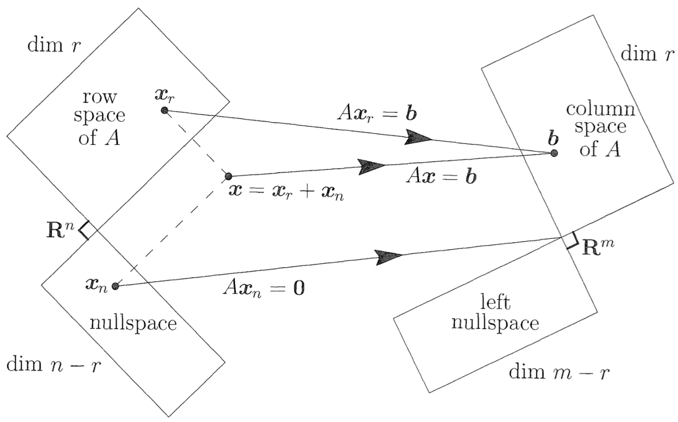
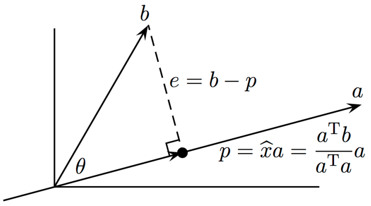
2017-11-21

- $\{e_1, e_2, \ldots, e_n\}$ is an orthonormal set (basis) for $\mathbb{R}^n$.
- **Orthogonal subspace**: Let W_1 and W_2 be subspaces of an inner product space V. W_1 is orthogonal to W_2 ($W_1 \perp W_2$) if $\forall w_1 \in W_1, \forall w_2 \in W_2, \langle w_1, w_2 \rangle = 0$.
- Note: In $\mathbb{R}^3$, xy-plane and yz-plane are *not* orthogonal to each other.
- The subspace spanned by u is orthogonal to the subspace spanned by v if $\langle u, v \rangle = 0$.
- *Theorem*: Given $A_{m \times n}$. The row space is orthogonal to the nullspace in $\mathbb{R}^n$. The column space is orthogonal to the left nullspace in $\mathbb{R}^m$.
- Note: The nullspace contains every vector orthogonal to the row space.
- *Proposition*: Let V be an inner product space, and W be a subspace of V. Define $U = \{v \in V | \forall w \in W, \langle v, w \rangle = 0\}$. Then, U is a subspace of V.
- **Orthogonal complement**: The subspace U is called the orthogonal complement of W in V, denoted by $W^\perp$, if U contains all vectors orthogonal to W.
- Examples:
  - $N(A) = C(A^\top)^\perp$: The nullspace is the orthogonal complement of the row space in $\mathbb{R}^n$.
  - $N(A^\top) = C(A)^\perp$: The left nullspace is the orthogonal complement of the column space in $\mathbb{R}^m$.
- The equation $Ax = b$ is solvable iff $b^\top y = 0$ where $A^\top y = 0$.
- Solvability of $Ax = b$:
  - Direct approach: b must be a combination of columns of A.
  - Indirect approach: b must be orthogonal to every vector that is orthogonal to the columns of A.
- V and W can be orthogonal without being complements.
- Splitting $\mathbb{R}^n$ into orthogonal parts will split every vector into $x = v + w$.



- The mapping from row space to column space is actually invertible. Every matrix A transforms its row space to its column space.
- When A^{-1} fails to exist, we can see a natural substitute, which is call **pseudoinverse**, denoted by A^+.
  - $\forall x \in C(A^\top), A^+ A x = x$.
  - $\forall y \in N(A^\top), A^+ y = 0$.
- The cosine of the angle between any two vectors a & b is $\cos\theta = \frac{a^\top b}{\|a\|\|b\|}$. If we consider the relationship between $\|a\|$, $\|b\|$ and $\|b - a\|$, then we have $\|b - a\|^2 = \|b\|^2 + \|a\|^2 - 2\|a\|\|b\|\cos\theta$.

- The projection of b onto the line a through O is $p = \frac{a^\top b}{a^\top a} a$



- Any two vectors in the inner product space satisfy the **Cauchy-Shwart Inequality**: $|a^\top b| \le \|a\|\|b\|$.
  - $\|b - p\|^2 = \|b - \frac{a^\top b}{a^\top a} a\|^2 \ge 0$. Hence, $|a^\top b| \le \|a\|\|b\|$.
  - The equality holds $\Leftrightarrow \|b - p\|^2 = 0 \Leftrightarrow \exists \alpha \in \mathbb{R}, b = p = \alpha a$.
- What is the projection matrix P of the linear transformation that maps b to p?
  - $p = \frac{a^\top b}{a^\top a} a = \frac{aa^\top}{a^\top a} b = Pb$.
  - $P = \frac{aa^\top}{a^\top a}$, which is a projection of rank one.
  - P is singular and symmetric.