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- Let $\{q_1, \ldots, q_n\}$ be orthonormal basis. $\forall b \in \mathbb{R}^n$, $\exists x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $b = x_1 q_1 + \ldots + x_n q_n$.
  - Let $Q = [q_1, \ldots, q_n]$. Then, $x = Q^{-1} b = Q^\top b$.
  - The coefficients $x_i = q_i^\top b$. Hence, $b = (q_1^\top b) q_1 + \ldots + (q_n^\top b) q_n$.
  - $(q_i^\top b) q_i$ is the projection of b onto q_i.
- Let $Q_{m \times n}$ $(m > n)$ be a matrix with orthogonal columns. $Q^\top Q = I_{n \times n}$. $Q^\top$ is the left inverse of Q.
- *Proposition*: Let $Q_{m \times n}$ $(m > n)$ be a matrix with orthogonal columns. The least square solution to the normal equation $Qx = b$ is $\tilde{x} = Q^\top b$. The projection matrix $P = QQ^\top$.
- Norm (length) of a vector $x \in \mathbb{R}^n$: $\|x\|_1 = \sum |x_i|$. $\|x\|_2 = \sqrt{\sum x_i^2}$. $\|x\|_\infty = \max |x_i|$.
- *Theorem*: Let V be an inner product space and $S = \{x_1, \ldots, x_n\}$ be an orthogonal set of nonzero vectors. If $y = \sum a_i x_i$, then $a_i = \frac{\langle x_i, y \rangle}{\langle x_i, x_i \rangle}$.
- Let $\{y_1, y_2\}$ be a linearly independent set. What is the orthogonal set $\{x_1, x_2\}$ that spans the same space as $\{y_1, y_2\}$? $x_1 = y_1$. $x_2 = y_2 - \frac{\langle y_1, y_2 \rangle}{\langle y_1, y_1 \rangle} y_1 = (I - \frac{y_1 y_1^\top}{y_1^\top y_1}) y_2$.
- **Gram-Schmidt orthogonalization process**: Let V be an inner product space and $S = \{y_1, \ldots, y_n\}$ be a linearly independent set of nonzero vectors. The orthogonal set $S' = \{x_1, \ldots, x_n\}$ that spans the same space as S: $x_1 = y_1$. $x_i = y_i - \sum_{k=1}^{i-1} \frac{\langle x_k, y_i \rangle}{\langle x_k, x_k \rangle} x_k$.
- *Theorem*: Every $A_{m \times n}$ w/ linearly independent columns can be factored into $A = Q_{m \times n} R_{n \times n}$. The columns of Q are orthonormal and R is an invertible upper triangular matrix. When $m = n$ and all matrices are square, Q is orthogonal.
  - Let $a_1, \ldots, a_n$ be the columns of A.
  - By Gram-Schmidt orthogonalization process, we can construct orthonormal vectors $q_1, \ldots, q_n$ s.t. for $i = 1, \ldots, n$, $\text{span}\{q_1, \ldots, q_i\} = \text{span}\{a_1, \ldots, a_i\}$.
  - For $i = 1, \ldots, n$, let $q_i' = a_i - \sum_{k=1}^{i-1} (q_k^\top a_i) q_k$, and $q_i = \frac{q_i'}{\|q_i'\|}$.
  - $a_i = (q_1^\top a_i) q_i + \ldots + (q_{i-1}^\top a_i) q_{i-1} + \|q_i'\| q_i$, which is linear combination of $q_1, \ldots, q_i$.
  - Hence, $A = QR = [q_1, \ldots, q_n] \begin{bmatrix} \|q_1'\| & \ldots & q_1^\top a_n \\ 0 & \ddots & \vdots \\ 0 & 0 & \|q_n'\| \end{bmatrix}$
- An inconsistent system $Ax = b$, where A has linearly independent columns, can be transformed into a consistent one. $Ax = b \rightarrow A^\top A \tilde{x} = A^\top b \rightarrow R^\top R \tilde{x} = R^\top Q^\top b \rightarrow R\tilde{x} = Q^\top b$.
- Let $x = r\cos\theta$, $y = r\sin\theta$, $z = z$. Then, $\begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} dr \\ d\theta \\ dz \end{bmatrix}$. Hence, $dV = dxdydz = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} drd\theta dz = rdrd\theta dz.$