Solutions to Exercise #5

(範圍: Relations)

1. P. 252: 4. (10%)

Sol: $A \times B = B \times A$, when A or B is empty, or $(a, b) \in A \times B$ iff $(a, b) \in B \times A$. The latter means that $a \in A$ iff $a \in B$ and $b \in B$ iff $b \in A$, i.e., $A = B$.

2. P. 252: 12. (10%)

Sol: Since $4096 = 2^{|A \times B|} = 2^{|A| \times |B|} = 2^{|A| \times 3}$, we have $|A| = 4$.

3. P. 364: 3. (10%)

Sol: It suffices to show that $\Re$ is reflexive, antisymmetric, and transitive.

reflexive: for all $(a, b) \in A \times B$, $a \Re_1 a$ and $b \Re_2 b \Rightarrow (a, b) \Re (a, b)$.

antisymmetric: for all $(a, b), (x, y) \in A \times B$,

$$\begin{aligned} & (a, b) \Re (x, y) \text{ and } (x, y) \Re (a, b) \\ \Rightarrow\ & a \Re_1 x,\ b \Re_2 y,\ x \Re_1 a, \text{ and } y \Re_2 b \\ \Rightarrow\ & a = x \text{ and } b = y \\ \Rightarrow\ & (a, b) = (x, y). \end{aligned}$$

transitive: for all $(a, b), (p, q), (x, y) \in A \times B$,

$$\begin{aligned} & (a, b) \Re (p, q) \text{ and } (p, q) \Re (x, y) \\ \Rightarrow\ & a \Re_1 p,\ b \Re_2 q,\ p \Re_1 x, \text{ and } q \Re_2 y \\ \Rightarrow\ & a \Re_1 x \text{ and } b \Re_2 y \\ \Rightarrow\ & (a, b) \Re (x, y). \end{aligned}$$

4. P. 365: 6 (only for (a) and (c)). (10%)

Sol:

(a)
$$\begin{array}{c c} & \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \\ e \end{array} & \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \end{array}$$

(c) a, b, c, d, e or a, c, b, d, e.

5. P. 370: 8. (10%)

Sol: (a) It suffices to show that $\Re$ is reflexive, symmetric and transitive.

reflexive: for all $x \in A$, $3 \mid (x-x) \Rightarrow (x, x) \in \Re$.

symmetric: for all $x, y \in A$, $(x, y) \in \Re \Rightarrow 3 \mid (x-y) \Rightarrow 3 \mid (y-x) \Rightarrow (y, x) \in \Re$.

transitive: for all $x, y, z \in A$, $(x, y) \in \Re$ and $(y, z) \in \Re \Rightarrow 3 \mid (x-y)$ and $3 \mid (y-z)$
$\Rightarrow 3 \mid ((x-y)+(y-z)) \Rightarrow 3 \mid (x-z) \Rightarrow (x, z) \in \Re$.

(b) The equivalence classes are $\{1, 4, 7\}$, $\{2, 5\}$, and $\{3, 6\}$.
The partition of A induced by $\Re$ is $\{\{1, 4, 7\}, \{2, 5\}, \{3, 6\}\}$.

6. P. 371: 14 (only for (a), (b), (d), (f)). (20%)

Sol: (a) Since $\Re$ is reflexive, we have $|\Re| \geq 7$. So, it is impossibe to have $\Re$ with $|\Re| = 6$.

(b) $\Re = \{(x, x) \mid \text{for all } x \in A\}$.

(d) $\Re = \{(x, x) \mid \text{for all } x \in A\} \cup \{(y, z), (z, y)\}$, where $y \in A$, $z \in A$ and $y \neq z$ (for example, $\Re = \{(x, x) \mid \text{for all } x \in A\} \cup \{(1, 2), (2, 1)\}$).

(f) Since $\Re$ is symmetric, we have $|\Re| = 7 + 2k$, an odd value, where k is the number of pairs of symmetric two-tuples (i.e., (x, y) and (y, x)) contained in $\Re$. So, it is impossibe to have $\Re$ with $|\Re| = 22$, an even value.

7. P. 66: 2. (10%)

Sol:

p	q	$p \wedge q$	$p \vee (p \wedge q)$
0	0	0	0
0	1	0	0
1	0	0	1
1	1	1	1

8. P. 66: 4. (10%)

Sol: $[(p \wedge q) \wedge r] \vee [(p \wedge q) \wedge \neg r] \Leftrightarrow (p \wedge q) \wedge (r \vee \neg r) \Leftrightarrow (p \wedge q) \wedge T \Leftrightarrow p \wedge q$.
$[(p \wedge q) \vee \neg q] \Leftrightarrow (p \vee \neg q) \wedge (q \vee \neg q) \Leftrightarrow (p \vee \neg q) \wedge T \Leftrightarrow p \vee \neg q$.
Therefore, the given statement simplifies to $(p \vee \neg q) \rightarrow s$ or $(q \rightarrow p) \rightarrow s$.

9. Prove that if $3 \mid n^2$, then $3 \mid n$, where n is a positive integer, by the methods of

(a) $p \to q \Leftrightarrow \neg q \to \neg p$; (5%)

(b) contradiction. (5%)

Sol: (a) If $n = 3k + 1$, then $n^2 = (3k + 1)^2 = 3k' + 1$.

If $n = 3k + 2$, then $n^2 = (3k + 2)^2 = 3k' + 1$.

(b) Suppose $3 \mid n^2$ and $n = 3k + 1$.

$n = 3k + 1 \Rightarrow n^2 = (3k + 1)^2 = 3k' + 1$, a contradiction to $3 \mid n^2$.

Similarly, there is a contradiction, if $3 \mid n^2$ and $n = 3k + 2$.