Probability Homework 4 B00401062 羅文斌

4.1-8

a. The joing pmf of X and Y will follow multinomial distribution.

$P(X=x, Y=y) = f(x,y) = \frac{7!}{x!y!(7-x-y)!}(0.78)^x(0.01)^y(0.21)^{7-x-y}$, where $0 \le x+y \le 7$.

b. Marginal pmf $P(X=x) = f_X(x) = \sum_y f(x,y) = \sum_{y=0}^{7} \frac{7!}{x!y!(7-x-y)!}(0.78)^x(0.01)^y(0.21)^{7-x-y}$

$= \frac{7!}{x!(7-x)!}(0.78)^x \sum_{y=0}^{7} \frac{(7-x)!}{y!(7-x-y)!}(0.01)^y(0.21)^{7-x-y}$

$= \binom{7}{x}(0.78)^x \sum_{y=0}^{7} \binom{7-x}{y}(0.01)^y(0.21)^{7-x-y} = \binom{7}{x}(0.78)^x(0.01+0.21)^{7-x}$

$= \binom{7}{x}(0.78)^x(0.22)^{7-x}$, which is a binomial distribution $\sim B(7, 0.78)$, where $x = 0, 1, \dots, 7$.

4.2-3

a. $\mu_X = \frac{1+2+3+4}{4} = \frac{5}{2}$

$\mu_Y = \frac{5}{2} \times 2 = 5$

$\sigma_X^2 = \frac{1^2+2^2+3^2+4^2}{4} - (\frac{5}{2})^2 = \frac{5}{4}$

$\sigma_Y^2 = (2^2 \times \frac{1}{16} + 3^2 \times \frac{2}{16} + 4^2 \times \frac{3}{16} + 5^2 \times \frac{4}{16} + 6^2 \times \frac{3}{16} + 7^2 \times \frac{2}{16} + 8^2 \times \frac{1}{16}) - 5^2 = \frac{5}{2}$

$\sigma_{XY} = E(XY) - \mu_X\mu_Y = (\frac{1\times(2+3+4+5)}{16} + \frac{2\times(3+4+5+6)}{16} + \frac{3\times(4+5+6+7)}{16} + \frac{4\times(5+6+7+8)}{16}) - \frac{5}{2} \times 5 = \frac{5}{4}$

$\rho = \sigma_{XY}/(\sigma_X\sigma_Y) = \frac{5}{4}/(\sqrt{\frac{5}{4}}\sqrt{\frac{5}{2}}) = \frac{1}{\sqrt{2}}$

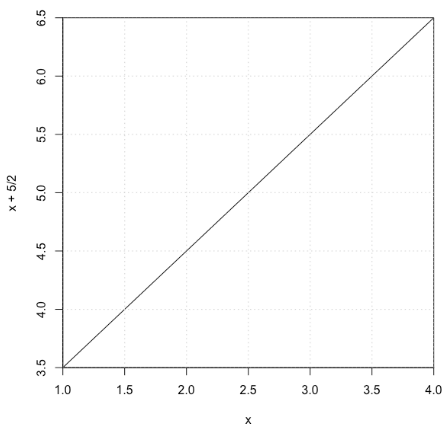
b.

The least squares regression line:

$y = \mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(x - \mu_X)$

$= 5 + 1 \times (x - \frac{5}{2}) = x + \frac{5}{2}$

The line does make sense. The regression line tells that $Y - X$ is 2.5, which is exactly the expected value of the second roll.



4.3-7

$E[Y|1] = \frac{2+3+4+5}{4} = 3.5, E[Y|2] = \frac{3+4+5+6}{4} = 4.5, E[Y|3] = \frac{4+5+6+7}{4} = 5.5, E[Y|4] = \frac{5+6+7+8}{4} = 6.5.$

$[x, E(Y|x)]$ for $x = 1, 2, 3, 4$ all lie on the best-fitting line.

4.4-16

Let $\textbf{Exp}(\lambda)$ denote the exponential distrubition.

$f_X(x) = \int_x^\infty 2^{-x-y}dy = -2e^{-x-y}]_{y=x}^\infty = 2e^{-2x} \sim \textbf{Exp}(2)$.

$f_Y(y) = \int_0^y 2^{-x-y}dx = -2e^{-x-y}]_{x=0}^y = 2e^{-y} - 2e^{-2y} \sim 2 \cdot \textbf{Exp}(1) - \textbf{Exp}(2)$

$\mu_X = 1/2$

$\mu_Y = 2(1/1) - (1/2) = 3/2$

$E(XY) = \int_{0\le x\le y<\infty} 2xye^{-x-y}dxdy = 2\int_0^\infty xe^{-x}(\int_x^\infty ye^{-y}dy)dx = 2\int_0^\infty xe^{-x}(-(y+1)e^{-y}]_{y=x}^\infty)dx$

$= 2\int_0^\infty x(x+1)e^{-2x}dx = 2\int_0^\infty x^2e^{-2x}dx + 2\int_0^\infty xe^{-2x}dx = -x^2e^{-2x}]_{x=0}^\infty + 4\int_0^\infty xe^{-2x}dx = 1$

$\sigma_{XY} = E(XY) - \mu_X\mu_Y = 1 - (1/2)(3/2) = 1/4$

$\sigma_X^2 = 1/2^2 = 1/4$

$\sigma_Y^2 = \int_0^y 2y^2(e^{-y} - e^{-2y})dy - (\mu_Y)^2 = \int_0^y 2y^2e^{-y}dy - \int_0^y 2y^2e^{-2y}dy - (3/2)^2$

$= 4\int_0^\infty ye^{-y}dy + \int_0^\infty 2ye^{-2y}dy - (9/4) = 4 - (1/2) - (9/4) = 5/4$

$\therefore \rho = \sigma_{XY}/(\sigma_X\sigma_Y) = \frac{1}{4}/(\sqrt{\frac{1}{4}}\sqrt{\frac{5}{4}}) = \frac{1}{\sqrt{5}}$

4.5-10

$\sigma_X = \sqrt{110.25} = 10.5$, and $\sigma_Y = \sqrt{2.89} = 1.70$

a. $P(2.80 \le Y \le 5.35) = \Phi(\frac{5.35-2.80}{1.70}) - \Phi(\frac{2.80-2.80}{1.70}) = \Phi(1.50) - \Phi(0) = 0.9332 - 0.5000 = 0.4332$

b. The least squares regression line is $y = 2.80 + (-0.57)\frac{1.7}{10.5}(x - 72.30)$.

$E(Y|X = 82.3) = 2.80 + (-0.57)\frac{1.7}{10.5}(82.3 - 72.30) = 1.877$

$\textbf{Var}(Y|X = 82.3) = \sigma_Y^2(1 - \rho^2) = 2.89(1 - (-0.57)^2) = 1.951$

$P(2.76 \le y \le 5.34|X = 82.3) = \Phi(\frac{5.34-1.877}{\sqrt{1.951}}) - \Phi(\frac{2.76-1.877}{\sqrt{1.951}})$

$= \Phi(2.479) - \Phi(0.632) = 0.9934 - 0.7363 = 0.2571$