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- $T : V \to W$ over F is a linear transformation. Then, $T^{-1}(0_W)$ is called the **nullspace (kernel)** of R. $T(V)$ is called the **range (image)** of T. Dimension of $T^{-1}(0_W)$ is the **nullity**. Dimension of $T(V)$ is the **rank**.
- $T : P_3(\mathbb{R}) \to P_2(\mathbb{R})$. $T(f) = \frac{d}{dt}(f)$.
  - Basis B_3 of $P_3(\mathbb{R}) = \{1, t, t^2, t^3\}$
  - Basis B_2 of $P_2(\mathbb{R}) = \{1, t, t^2\}$
  - Differentiation matrix $[T]_{B_3}^{B_2} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$
- $T : P_2(\mathbb{R}) \to P_3(\mathbb{R})$. $T(f) = \int_0^t (f)$.
  - Basis B_2 of $P_2(\mathbb{R}) = \{1, t, t^2\}$
  - Basis B_3 of $P_3(\mathbb{R}) = \{1, t, t^2, t^3\}$
  - Integration matrix $[T]_{B_2}^{B_3} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$
- Note that $[\frac{d}{dt} \circ \int_0^t] = [\frac{d}{dt}]_{B_3}^{B_2}[\int_0^t]_{B_2}^{B_3} = I_3$. However, $[\int_0^t \circ \frac{d}{dt}] \neq [\int_0^t]_{B_2}^{B_3}[\frac{d}{dt}]_{B_3}^{B_2} \neq I_4$.
- **Rotation matrix** Q_θ through an angle θ:
  - $Q_\theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$. $Q_\theta \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$.
  - Hence, $Q_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.
  - Properties: $Q_\theta^2 = Q_{2\theta}$. $Q_\theta^{-1} = Q_{-\theta}$.
- **Projection matrix** P_θ onto the θ-line:
  - $P_\theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos^2\theta \\ \cos\theta\sin\theta \end{bmatrix}$. $P_\theta \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\theta \\ \sin^2\theta \end{bmatrix}$.
  - Hence, $P_\theta = \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$.
  - Properties: $P_\theta^2 = P_\theta$. Singular. Symmetric.
  - $P_\theta(\alpha \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}) = \alpha \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$. $P_\theta(\alpha \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.
  - Basis of $N(P_\theta)$: $\begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$.
- **Reflection matrix** H_θ through θ-line:
  - $H_\theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos 2\theta \\ \sin 2\theta \end{bmatrix}$. $H_\theta \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \sin 2\theta \\ -\cos 2\theta \end{bmatrix}$.
  - Hence, $H_\theta = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$.
  - Properties: $H_\theta^2 = I$. $H_\theta^{-1} = H$. $H_\theta = 2P_\theta - I$.

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- $H_\theta(\alpha \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}) = \alpha \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$. $H_\theta(\alpha \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}) = \alpha \begin{bmatrix} \sin\theta \\ -\cos\theta \end{bmatrix}$.

- If the 1st basis is on the θ-line, and the 2nd basis is perpendicular, then $Q^* = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. $P^* = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. $H^* = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.
- The length of a vector $x \in \mathbb{R}^n$, denoted as $\|x\|$, is $\|x\| = \sqrt{x^\top x} = \sqrt{\sum_1^n x_i^2}$.
- Given $x, y \in \mathbb{R}^n$, if $x \perp y$, then $x^\top y = y^\top x = \sum_1^n x_i y_i = 0$.
- An **inner product** on V/F is a function that assigns to every ordered pair of vectors $x, y \in V$, denoted $\langle x, y \rangle$, s.t. for any $z \in V$ and $\alpha \in F$,
  - $\langle \alpha x + y, z \rangle = \alpha \langle x, z \rangle + \langle y, z \rangle$
  - $\langle y, x \rangle = \overline{\langle x, y \rangle}$
  - If $x \neq 0$, then $\langle x, x \rangle > 0$.
- If $F = \mathbb{R}$, then $\langle x, y \rangle = \langle y, x \rangle$.
- Inner product is *linear* in the first component.
- **Standard inner product (dot product)** on $\mathbb{R}^n$: $\langle x, y \rangle = x^\top y = \sum_i^n x_i y_i$.
- Examples of inner products: Given $x, y \in \mathbb{C}^n/\mathbb{C}$. Define $\langle x, y \rangle = x^\top \overline{y}$.
- $x, y \in \mathbb{R}^n$ are perpendicular iff the standard inner product $\langle x, y \rangle = x^\top y = 0$.
- An **inner product space** is a real or complex vector space (a vector space over $\mathbb{R}$ or $\mathbb{C}$) together w/ a specified inner product on that space.
- In an inner product space V, x is **orthogonal** to y if $\langle x, y \rangle = 0$.
- A set S of vectors is called an **orthogonal set** if all pairs of distinct vectors in S are orthogonal.
- An **orthonormal set** S is an orthogonal set and $\forall v \in S$, $\langle v, v \rangle = \|v\|^2 = 1$.
- *Proposition*: An orthogonal set of nonzero vectors is linearly independent.
  - Let S be an orthogonal set of nonzero vectors $v_1, v_2, \ldots, v_n \in V$.
  - Let $c_1, c_2, \ldots, c_n \in F$, and $\sum_i^n c_i v_i = c$.
  - Consider any $v_j \in S$. $\langle c, v_j \rangle = \langle \sum_i^n c_i v_i, v_j \rangle = c_j \langle v_j, v_j \rangle$.
  - $\forall j, c = 0$ iff $c_j = 0$.
  - Hence, S is linearly independent.