The Cubic Formula

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August 2018

1 Introduction

We have a complete cubic polynomial in the form $P(x) = ax^3 + bx^2 + cx + d$, that has the property of crossing the x axis at only one instance. It is our wish to determine the x value at this intersection.

1.1 Approach

To solve for the general value of x in this polynomial, we shall use a series of substitutions to first transform $P(x)$ from a complete cubic polynomial into a depressed cubic in the form $G(t) = at^3 + pt + q$, then to another polynomial in the form $H(m) = am^6 + qm^3 + u$. After that, we can transform $H(m)$ into the simple quadratic $I(n) = an^2 + qn + u$, and easily solve for the values of n that cause $I(n) = 0$, and all that is left is to back substitute until a general solution in terms of a, b, c, and d is available.

2 $P(x) \longrightarrow G(t)$

$$P(x) = ax^3 + bx^2 + cx + d \tag{1}$$

Make the substitution $x = t + \omega$, where ω is a determinable constant that is required to transform the complete cubic into a depressed one.

$$\begin{aligned} G(t) &= a(t+\omega)^3 + b(t+\omega)^2 + c(t+\omega) + d \\ &= at^3 + 3at^2w + 3atw^2 + aw^3 + bt^2 + 2bt\omega + b\omega^2 + c\omega + d \\ &= at^3 + t^2(3a\omega + b) + t(3a\omega^2 + 2b\omega + c) + (a\omega^3 + b\omega^2 + c\omega + d) \end{aligned} \tag{2}$$

It is obvious that the only way to make the cubic a depressed one is to have $\omega = -\frac{b}{3a}$, so substituting for this yields:

$$\begin{aligned} G(t) &= at^3 + t[3a(-\frac{b}{3a})^2 + 2b(-\frac{b}{3a}) + c] + [a(-\frac{b}{3a})^3 + b(-\frac{b}{3a})^2 + c(-\frac{b}{3a}) + d] \\ &= at^3 + t[\frac{b^2}{3a} - \frac{2b^2}{3a} + c] + [-\frac{b^3}{27a^2} + \frac{b^3}{9a^2} - \frac{bc}{3a} + d] \\ &= at^3 + t[\frac{3ac - b^2}{3a}] + [\frac{2b^3 + 27a^2 d - 9abc}{27a^2}] \end{aligned} \tag{3}$$

To finalize the transformation of $P(x)$ into $G(t)$, simply substitute $p = \frac{3ac-b^2}{3a}$ and $q = \frac{2b^3+27a^2d-9abc}{27a^2}$. Therefore, $G(t) = at^3 + pt + q$.

3 $G(t) \longrightarrow H(m)$

$$G(t) = at^3 + px + q \tag{4}$$

Making the substitution $t = m + \frac{\lambda}{m}$, were λ is another determinable constant that is required to transform $G(t)$ into $H(m)$, a quadratic in disguise.

$$\begin{aligned} H(m) &= a(m + \frac{\lambda}{m})^3 + p(m + \frac{\lambda}{m}) + q \\ &= am^3 + 3a\lambda m + \frac{3a\lambda^2}{m} + \frac{a\lambda^3}{m^3} + pm + \frac{p\lambda}{m} + q \\ &= am^3 + m(3a\lambda + p) + q + \frac{1}{m}(3a\lambda^2 + p\lambda) + \frac{a\lambda^3}{m^3} \end{aligned} \tag{5}$$

We multiply by m^3 in order to eliminate the rationals, and revert $H(m)$ back into polynomial form.

$$H(m) = am^6 + m^4(3a\lambda + p) + qm^3 + m^2(3a\lambda^2 + p\lambda) + a\lambda^3 \tag{6}$$

To get $H(m)$ into the desired form $am^6 + qm^3 + u$ we must eliminate the m^4 and m^2 terms, which may be done with the substitution $\lambda = -\frac{p}{3a}$, so substituting for this yields:

$$\begin{aligned} H(m) &= am^6 + qm^3 + a(-\frac{p}{3a})^3 \\ &= am^6 + qm^3 - \frac{p^3}{27a^2} \end{aligned} \tag{7}$$

Substitute $u = -\frac{p^3}{27a^2}$, and the desired form $H(m) = am^6 + qm^3 + u$ has been achieved.

4 $H(m) \longrightarrow I(n)$

Now make the substitution $n = m^3$, and it is observed that the new polynomial $I(n)$ is simply a quadratic! From here, all that is required is to solve for the zeros of the quadratic, then back substitute until the solution is in terms of a, b, c, and d.

$$n = \frac{-q \pm \sqrt{q^2 - 4au}}{2a} \tag{8}$$

5 Back Substitution

Finally, we can begin to substitute backwards, beginning with $m = \sqrt[3]{n}$.

$$\begin{aligned}
m &= \sqrt[3]{\frac{-q \pm \sqrt{q^2 - 4au}}{2a}} \\
&= \sqrt[3]{\frac{-(\frac{2b^3+27a^2d-9abc}{27a^2}) \pm \sqrt{(\frac{2b^3+27a^2d-9abc}{27a^2})^2 - 4a(-\frac{p^3}{27a^2})}}{2a}} \\
&= \sqrt[3]{\frac{\frac{9abc-2b^3-27a^2d}{27a^2} \pm \sqrt{\frac{(2b^3+27a^2d-9abc)^2}{729a^2} + \frac{4(\frac{3ac-b^2}{3a})^3}{27a}}}{2a}} \\
&= \sqrt[3]{\frac{\frac{9abc-2b^3-27a^2d}{27a^2} \pm \sqrt{\frac{(2b^3+27a^2d-9abc)^2+4(3ac-b^2)^3}{729a^2}}}{2a}} \\
&= \sqrt[3]{\frac{9abc - 2b^3 - 27a^2d \pm \sqrt{(2b^3 + 27a^2d - 9abc)^2 + 4(3ac - b^2)^3}}{54a^3}} \\
&= \frac{\sqrt[3]{9abc - 2b^3 - 27a^2d \pm \sqrt{(2b^3 + 27a^2d - 9abc)^2 + 4(3ac - b^2)^3}}}{3a\sqrt[3]{2}}
\end{aligned} \tag{9}$$

Substitute $t = m + \frac{\lambda}{m}$, $\gamma = \sqrt[3]{9abc - 2b^3 - 27a^2d \pm \sqrt{(2b^3 + 27a^2d - 9abc)^2 + 4(3ac - b^2)^3}}$:

$$
\begin{aligned}
t &= m + \frac{\lambda}{m} \\
&= (\frac{\gamma}{3a\sqrt[3]{2}}) + \frac{(-\frac{p}{3a})}{(\frac{\gamma}{3a\sqrt[3]{2}})} \\
&= \frac{\gamma}{3a\sqrt[3]{2}} - \frac{\sqrt[3]{2}(\frac{3ac-b^2}{3a})}{\gamma} \\
&= \frac{\gamma}{3a\sqrt[3]{2}} - \frac{\sqrt[3]{2}(3ac - b^2)}{3a\gamma}
\end{aligned}
\tag{10}
$$

Recall that $x = t + \omega$, so:

$$
\begin{aligned}
x &= t + \omega \\
&= (\frac{\gamma}{3a\sqrt[3]{2}} - \frac{\sqrt[3]{2}(3ac - b^2)}{3a\gamma}) + (-\frac{b}{3a}) \\
&= \frac{\sqrt[3]{4}\gamma^2 - \sqrt[3]{16}(3ac - b^2) - 2b\gamma}{6a\gamma}
\end{aligned}
\tag{11}
$$

The above statement, when all pertinent a, b, c, and d values are plugged in, will provide the x value for which the cubic polynomial $P(x)$ intersects the x axis. Further analysis into the roots of unity may be required in order to determine the imaginary roots of said $P(x)$, however that is a proof left as an exercise to the reader.