MATH 453 Project

The Khalimsky Topology

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Definition

Definition 1.1 (Khalimsky Topology)

The **Khalimsky topology** $(\mathbb{Z}, \mathscr{K})$ is the topology generated by $\{\{2n-1, 2n, 2n+1\} : n \in \mathbb{Z}\}$.
Equivalently, $O \in \mathscr{K}$ iff for all even integers $n \in O$, we have that the adjacent integers of n are in O, i.e. $n-1, n+1 \in O$.

Proposition

Proposition 1.2

Let $U \subseteq 2\mathbb{Z}+1$, *then* U *is open in* $(\mathbb{Z}, \mathcal{K})$.

Proof

We have $U \in \mathcal{K}$ iff for all even integers $n \in U$, $n-1, n+1 \in U$. Since U has no even integers, the condition holds vacuously. $\square$

Definition

Definition 1.3

A **Khalimsky interval** is an interval $[a, b]_{\mathbb{Z}} = [a, b]_{\mathbb{R}} \cup \mathbb{Z}$ equipped with the topology induced by the Khalimsky topology on $\mathbb{Z}$.

Definition 1.4

The **Khalimsky Plane** is the Cartesian product of two Khalimsky lines, and, more generally, **Khalimsky space** is the Cartesian product of n copies of $\mathbb{Z}$.

Proposition

Proposition 1.5

Let $(\mathbb{Z}, \mathscr{K})$ denote the ***Khalimsky topology*** *and let $n \in \mathbb{Z}$. Then the following properties hold:*

(i) $\overline{\{2n\}} = \{2n\}$.

(ii) $\overline{\{2n+1\}} = \{2n, 2n+1, 2n+2\}$.

(iii) *The smallest open set containing $2n$ is $\{2n-1, 2n, 2n+1\}$.*

(iv) *The smallest open set containing $2n+1$ is $\{2n+1\}$.*

Definition

Definition 1.6 (Alexandroff Space)

A topological space such that the intersection of any family of open sets is open is called a **Smallest-Neighborhood Space** or an **Alexandroff Space**.

Proposition

Proposition 1.7

The Khalimsky topology $(\mathbb{Z}, \mathcal{K})$ is an Alexandroff Space.

Proof

Let $\{O_i, i \in I\}$ be a family of open sets in $(\mathbb{Z}, \mathcal{K})$ and let $O = \bigcap_{i \in I} O_i$ and assume $O \neq \emptyset$. If $O \subseteq 2\mathbb{Z}+1$, then $O \in \mathcal{K}$. If not, then there exists an even integer $x \in O$. So, $x \in O_i$ for all $i \in I$. Hence, $x-1, x+1 \in O_i$ for all $i \in I$. Therefore, $x-1, x+1 \in O$. Thus, $O \in \mathcal{K}$.

Locally Connectedness of Alexandroff Spaces

Definition 1.8 (Locally Connectedness)

A topological space (X, τ) is said to be **locally connected** if for every point in X, there exists a fundamental system of neighborhoods (FSN) consisting of connected sets.

Proposition 1.9

Every Alexandroff Space is locally connected.

Proof

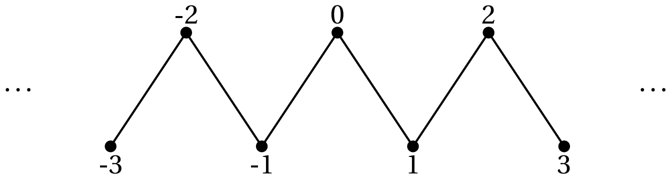
Let (X, τ) be an Alexandroff Space. For $x \in X$, let $V_\tau(x)$ be the smallest open set containing x. We show that $V_\tau(x)$ is connected. If $V_\tau(x) \subseteq U \cup V$ where U and V are open disjoint sets, then $x \in U$ or $x \in V$. If $x \in U$, then $V_\tau(x) \subseteq U$. If $x \in V$, then $V_\tau(x) \subseteq V$. Hence, $V_\tau(x)$ is connected. $\mathscr{B} = \{V_\tau(x)\}$ is an FSN of connected neighborhoods of x. Thus, (X, τ) is locally connected.

Ordered Set

Ordered Set

The **Khalimsky topology** $(\mathbb{Z}, \mathcal{K})$ can be represented as an ordered set:

$$x \leq_{\mathcal{K}} y \text{ iff } y \in \overline{\{x\}}^{\mathcal{K}}$$



In the same sense, we can see that $(\mathbb{Z}, \leq_{\mathcal{K}})$ is a poset as if $x, y \in \mathbb{Z}$ such that $x \leq_{\mathcal{K}} y$ and $y \leq_{\mathcal{K}} x$, then $y \in \overline{\{x\}}^{\mathcal{K}}$ and $x \in \overline{\{y\}}^{\mathcal{K}}$. Hence, $x = y$.

Lemma

Lemma 1.10

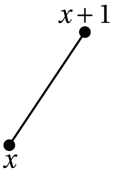
For all $x \in \mathbb{Z}$, $\{x, x+1\}$ *is a path in* $(\mathbb{Z}, \mathcal{K})$.

Proof

Case 1

Case 1: x is odd

Let $\varphi : ([0,1], \mathscr{U}) \longrightarrow (\mathbb{Z}, \mathscr{K})$ be given by $\varphi(t) = x$ for $t \in [0,1)$ and $\varphi(1) = x+1$, where x is an odd integer. Then $\varphi^{-1}(\{x\}) = [0,1)$ which is open in $([0,1], \mathscr{U})$. Thus, φ is continuous.

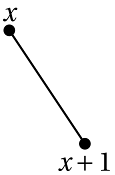


Proof

Case 2

Case 2: x is even

Let $\varphi : ([0,1], \mathscr{U}) \longrightarrow (\mathbb{Z}, \mathscr{K})$ be given by $\varphi(t) = x+1$ for $t \in [0,1)$ and $\varphi(1) = x$, where x is an even integer. Then $\varphi^{-1}(\{x+1\}) = [0,1)$ which is open in $([0,1], \mathscr{U})$. Thus, φ is continuous.



Theorem

Theorem 1.11

*The **Khalimsky topology** $(\mathbb{Z}, \mathscr{K})$ is path-connected and, therefore, connected.*

Compactness

Definition 1.12 (Compactness)

A space X is said to be **compact** if every open covering $\mathscr{A}$ of X, there exists a finite subcovering.

Proposition 1.13

The **Khalimsky topology** $(\mathbb{Z}, \mathscr{K})$ is Lindelöf compact but not compact.

Proof

Let $V(2n) = \{2n-1, 2n, 2n+1\}$, then $V(2n)$ is open for all $n \in \mathbb{Z}$ and $\mathbb{Z} = \bigcup_{n\in\mathbb{Z}} V(2n)$.
Thus, $\{V(2n) : n \in \mathbb{Z}\}$ is an open covering which does not have a finite subcover.
It is Lindelöf compact as every open cover has a countable subcover.

Pseudocompactness

Definition 1.14 (Pseudocompactness)

A space (X, τ) is said to be **pseudocompact** if every continuous function $f : (X, \tau) \longrightarrow (\mathbb{R}, \mathscr{U})$ is bounded.

Proposition 1.15

The **Khalimsky topology** $(\mathbb{Z}, \mathscr{K})$ is pseudocompact.

Proof

Let $f : (\mathbb{Z}, \mathscr{K}) \longrightarrow (\mathbb{R}, \mathscr{U})$ be a continuous map. As $\mathscr{K}$ is connected, $f(\mathbb{Z})$ is connected, so $f(\mathbb{Z})$ is an interval. Moreover, $f(\mathbb{Z})$ is countable as $\mathbb{Z}$ is countable. Therefore, $f(\mathbb{Z}) = \{c\}$ where $c \in \mathbb{R}$. Hence, the image of $(\mathbb{Z}, \mathscr{K})$ under any continuous function to $(\mathbb{R}, \mathscr{U})$ is bounded. Thus, $(\mathbb{Z}, \mathscr{K})$ is pseudocompact.

Thank you for listening !

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