MATH 341 Project

Convex Functions

Mohammed Alghudiyan

Department of Mathematics
KFUPM

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Definition

Definition 1.1

Let $I \subseteq \mathbb{R}$ be an interval. A function $f : I \longrightarrow \mathbb{R}$ is said to be **convex** on I if for any t satisfying $0 \leq t \leq 1$ and any points $x_1, x_2 \in I$, we have:

$$f((1-t)x_1 + tx_2) \leq (1-t)f(x_1) + tf(x_2).$$

Examples

Example 1.2

The function $f(x) = x^2$ is **convex**. For $0 \leq t \leq 1$ and any points $a, b \in \mathbb{R}$, we have the following:
$(ta^2 + (1-t)b^2) - (ta + (1-t)b)^2 = ta^2 + (1-t)b^2 - t^2a^2 - (1-t)^2b^2 - 2abt(1-t) =$
$ta^2(1-t) + (1-t)b^2(1-(1-t)) - 2abt(1-t) = t(1-t)(a^2 + b^2 - 2ab) = t(1-t)(b-a)^2 \geq 0.$
Thus, $(ta^2 + (1-t)b^2) \geq (ta + (1-t)b)^2$

Definition

Definition 1.3

Let $f : A \longrightarrow \mathbb{R}$ be a function. The **epigraph** of f is the following set:

$$\text{epi}\, f = \{\, (x, y) \in A \times \mathbb{R} | x \in A, y \geq f(x) \,\}$$

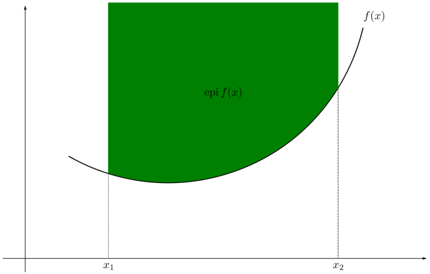

Figure 1: Epigraph of a function

Definition

Definition 1.4

A subset $\Omega \in \mathbb{R}^2$ is **convex** if the line segment between any two points in Ω lies in Ω.
That is, $\forall \vec{u}, \vec{v} \in \Omega$ and $\forall t \in [0,1]$, $t\vec{u} + (1-t)\vec{v} \in \Omega$.
Note, the empty set $\emptyset$ is a **convex** set.

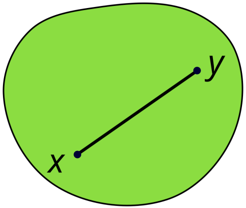

Figure 2: A convex set

1. Convex Functions
2. Epigraphs
3. Convex Sets
4. **Theorems**

Theorem 1.5

Let $f : A \longrightarrow \mathbb{R}$ be a function. Then a necessary and sufficient condition for f to be ***convex*** *is that epi f is a* ***convex*** *subset of $\mathbb{R}^2$.*

Proof of Necessary Part

Assume that f is convex and fix pairs $(x_1, t_1), (x_2, t_2) \in \operatorname{epi} f$ and a number $\lambda \in [0, 1]$. Then we have $f(x_1) \leq t_1$ and $f(x_2) \leq t_2$. Hence, the covexity of f ensures that

$$f((1-\lambda)x_1 + \lambda x_2) \leq (1-\lambda)f(x_1) + \lambda f(x_2) \leq (1-\lambda)t_1 + \lambda t_2$$

Therefore, this implies that

$$(1-\lambda)(x_1, t_1) + \lambda(x_2, t_2) = ((1-\lambda)x_1 + \lambda x_2, (1-\lambda)t_1 + \lambda t_2) \in \operatorname{epi} f$$

Thus, $\operatorname{epi} f$ is a convex subset of $\mathbb{R}^2$.

Proof of Sufficient Part

Suppose that epi f is a convex subset of $\mathbb{R}^2$ and fix $x_1, x_2 \in A$ and a number $\lambda \in [0,1]$ Then $(x_1, f(x_1)), (x_2, f(x_2)) \in \text{epi } f$. Hence,

$$((1-\lambda)x_1 + \lambda x_2, (1-\lambda)f(x_1) + \lambda f(x_2)) = (1-\lambda)(x_1, f(x_1)) + \lambda(x_2, f(x_2)) \in \text{epi } f$$

Therefore, this implies that

$$f((1-\lambda)x_1 + \lambda x_2) \leq (1-\lambda)f(x_1) + \lambda f(x_2)$$

Thus, f is a convex function.

$\square$

Theorem 1.6

Let $f : A \longrightarrow \mathbb{R}$ be a convex function and let $g : B \longrightarrow \mathbb{R}$ be an increasing convex function on a convex set B such that $f(A) \subseteq B$. Then the composition $g \circ f$ is a convex function.

Proof

Let $x_1, x_2 \in A$ and $t \in [0,1]$, then, by the convexity of f, we have

$$f((1-t)x_1 + tx_2) \leq (1-t)f(x_1) + tf(x_2)$$

Since g is an increasing convex function, it follows that

$$\begin{aligned}(g \circ f)((1-t)x_1 + tx_2) &= g(f((1-t)x_1 + tx_2)) \leq g((1-t)f(x_1) + tf(x_2)) \\ &\leq (1-t)g(f(x_1)) + tg(f(x_2)) = (1-t)(g \circ f)(x_1) + t(g \circ f)(x_2)\end{aligned}$$

Thus, the composition $g \circ f$ is a convex function.

$\square$

Thank you for listening !

Mohammed Alghudiyan
s202266040@kfupm.edu.sa