# MATH 427 Project

## Transcendental Numbers

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# Introduction

- A transcendental number is not a solution for a polynomial with integer coefficients.
- Euler and Lambert proved $e$, $\pi$ irrational, respectively.
- Liouville showed transcendental numbers exist.
- $e$, $\pi$ proven transcendental by Hermite, Lindemann, respectively.
- Gelfond-Schneider solved Hilbert's 7th problem.

# Definition

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*A complex number $\xi$ is called an* ***algebraic number*** *if it is a solution to some polynomial equation $P(x) = 0$ where $f(x)$ is a polynomial over $\mathbb{Q}$. Equivalently, an* ***algebraic number*** *is a root for some polynomial $P(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$ where $a_i \in \mathbb{Z}$ for all $i = 0, 1, 2, \ldots, n$.*

## Example

For all $r \in \mathbb{Q}$, $r$ is algebraic.
$\sqrt{2}$ is algebraic.

# Definition

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*A complex number $z$ is said to be **transcendental** if it is not algebraic, i.e. there does not exists a polynomial $P(x)$ with integer coefficients such that $P(z) = 0$.*

## Example

$\pi$, $e$ and $\beta = \sum_{j=1}^{\infty} 10^{-j!} = 0.110001000....$

*Note.* All transcendental numbers are irrational, but not all irrational numbers are transcendental.

# Proposition

## Proposition

*The number* $\beta = \sum_{j=1}^{\infty} 10^{-j!} = 0.110001000\ldots$ *is transcendental.*

# Proof

## Proof

Assume, on the contrary, that $\beta = \sum_{j=1}^{\infty} 10^{-j!}$ is algebraic, then it satisfies some polynomial equation $P(x) = \sum_{i=0}^{n} a_i x^i = 0$ where $a_i \in \mathbb{Z}$ for all $i = 0, 1, 2, \ldots, n$. Let $\beta_k = \sum_{j=1}^{k} 10^{-j!}$, then:

$$\beta - \beta_k = \sum_{j=1}^{\infty} 10^{-j!} - \sum_{j=1}^{k} 10^{-j!} = \sum_{j=k+1}^{\infty} 10^{-j!} < 2 \sum_{j=k+1}^{k+1} 10^{-j!} = 2 \cdot 10^{-(k+1)!} \tag{1}$$

By the mean value theorem, there exists a number $c$ between $\beta$ and $\beta_k$ such that $|P(\beta) - P(\beta_k)| = |P(\beta_k)| = |\beta - \beta_k||P^{'}(c)|$. $P^{'}(x) = \sum_{i=1}^{n} i a_i x^{i-1}$. For $x \in (0, 1)$, we have:

$$|P^{'}(x)| = |\sum_{i=1}^{n} i a_i x^{i-1}| < \sum_{i=1}^{n} |i a_i| \tag{2}$$

# Proof

## Proof

Combining (1) and (2), we get:

$$|\beta - \beta_k||P^{'}(c)| < 2 \cdot 10^{-(k+1)!} \cdot \sum_{i=1}^{n} |ia_i|$$

Since $P(x)$ has only $n$ zeros, we can choose $k$ sufficiently large so that $P(\beta_k) \neq 0$. Doing so, we get:

$$|P(\beta_k)| = |\sum_{i=0}^{n} a_i \beta_k^i| \geq 10^{-n \cdot (k)!}$$

Moreover, we observe that for large $k$, we have

$$10^{-n \cdot (k)!} > 2 \cdot 10^{-(k+1)!} \cdot \sum_{i=1}^{n} |ia_i|$$

# Proof

## Proof

This implies $|P(\beta_k)| > |\beta - \beta_k||P^{'}(c)|$, a contradiction. Thus, $\beta = \sum_{j=1}^{\infty} 10^{-j!} = 0.110001000...$ is transcendental. $\square$

# Lemma

## Lemma

*For* $n \in \mathbb{N} \cup \{0\}$*, the set* $P_n = \{\, a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n : a_i \in \mathbb{Z}\ \forall i = 0, 1, 2, \ldots, n\}$ *of polynomials of degree less than or equal to n is countable.*

# Theorem

**Theorem**

*The set of all algebraic numbers is countable.*

# Proof

## Proof

The set of all polynomials with integer coefficients is

$$P = \bigcup_{n=0}^{\infty} P_n$$

By the previous Lemma, the set $P_n$ is countable for all $n \in \mathbb{N} \cup \{0\}$. Hence, $P$ is a countable union of countable sets and therefore countable. Because every polynomial of degree $n$ has at most $n$ roots, the set of all possible roots of all polynomials with integer coefficients is a countable union of finite sets. Thus, the set of all algebraic numbers is countable $\square$

# Corollary

## Corollary

*The set of transcendental numbers is uncountable.*

# Gelfond Schneider Theorem

## Gelfond-Schneider Theorem

*If $\alpha$ and $\beta$ are algebraic numbers with $\alpha \neq 0$ and $\alpha \neq 1$ and if $\beta$ is not a rational number, then any value of $\alpha^{\beta}$ is transcendental.*

## Example

$2^{\sqrt{2}}$, $\sqrt{2}^{\sqrt{2}}$ and $2^{i}$.

# Theorem

An alternative form of Gelfond Schneider Theorem

## Theorem

*If $\alpha$ and $\gamma$ are non-zero algebraic numbers and if $\alpha \neq 1$, then $(ln\gamma)/(ln\alpha)$ is either rational or transcendental.*

# Transcendence of $e$

## Theorem

*$e$ is transcendental*

Let $f(x)$ be a real polynomial with degree $m$. Define the following integral

$$I(t) := \int_0^t e^{t-u} f(u)\,du \tag{3}$$

Where $t$ is a complex variable and the integral is taken over the segment joining 0 and $t$. If we apply the integration by parts theorem we get

$$I(t) = e^t \sum_{j=0}^{m} f^{(j)}(0) - \sum_{j=0}^{m} f^{(j)}(t) \tag{4}$$

# Transcendence of $e$ Cont.

Let us denote by $\bar{f}(x)$ the polynomial $f(x)$ but we replace its coefficients by their absolute values. Consider the following estimate

$$|I(t)| \leq \int_0^t |e^{t-u} f(u)| du \leq |t| e^{|t|} \bar{f}(|t|) \tag{5}$$

Now, suppose to the contrary that $e$ is an algebraic number

$$q_0 + q_1 e + q_2 e^2 + \dots q_n e^n = 0 \tag{6}$$

Where $q_0 \neq 0$ and $n > 0$ and all the $q$'s are integers. Define the following quantity

$$J := q_0 I(0) + q_1 I(1) + \cdots + q_n I(n) \tag{7}$$

also

$$f(x) = x^{p-1} (x-1)^p \dots (x-n)^p \tag{8}$$

# Transcendence of $e$ Cont.

Where $p$ is a large prime. Using (4) and (6) to substitute in (7) results in

$$J = -\sum_{j=0}^{m}\sum_{k=0}^{n} q_k f^{(j)}(k) \tag{9}$$

The degree of $f(x)$ is $m = (n+1)p - 1$. Notice that when either $j < p$ , $k > 0$ or $j < p - 1$, $k = 0$ then we can conclude that $f^{(j)}(k) = 0$ and for all $j$ and $k$ different from $j = p - 1$, $k = 0$ and different from the previous conditions, $f^{(j)}(k)$ would be divisible by $p!$. We will now use Leibniz rule on $f(x)$

$$(fg)^n = \sum_{k=0}^{n} \binom{n}{k} f^{n-k} g^k \tag{10}$$

Where here caution must be taken because we used our previous notation to state the rule.

# Transcendence of $e$ Cont.

Applying this on (8) and letting $g \to x^{p-1}$ while $f$ should be substituted for all the other factors in (8). From this we obtain

$$f^{(p-1)}(0) = (p-1)!(-1)^{np}(n!)^p \tag{11}$$

Notice that $(p-1)! \mid f^{(p-1)}(0)$ and $p!$ does not divide $f^{(p-1)}(0)$ whenever $p > n$. This implies that if $p > |q_0|$ then $J$ is divisible by $(p-1)!$ which means

$$|J| \geq (p-1)! \tag{12}$$

But using the estimate $\bar{f}(k) \leq (2n)^m$ along with (5) we can establish the following inequality

$$|J| \leq |q_1|e\bar{f}(1) + \cdots + |q_n|ne^n\bar{f}(n) \leq c^p \tag{13}$$

where $c$ is some constant that is independent of $p$. Now, if we were to choose an appropriate $c$ together with a sufficiently large $p$ then (12) and (13) would be inconsistent. Therefore, $e$ must be transcendental.

# Squaring the Circle

One of the problems related to transcendental numbers is called squaring the circle. It is the problem of trying to construct a square that has the same area as the area of a unit circle. This would mean that the square we have to construct is of side length $\sqrt{\pi}$. Now, the difficulty in the problem arises because we want to do this under the rules of straightedge and compass construction.

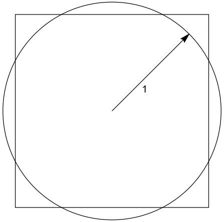


Figure 1: Squaring the Circle

## Squaring the Circle Cont

One can build complex constructions from the following simple ones:

- drawing a segment passing through two points.
- drawing a circle with a certain radius centered at a certain point.
- intersecting two lines.
- intersecting a line with a circle.
- intersecting two circles.

# Squaring the Circle Cont.

## Example

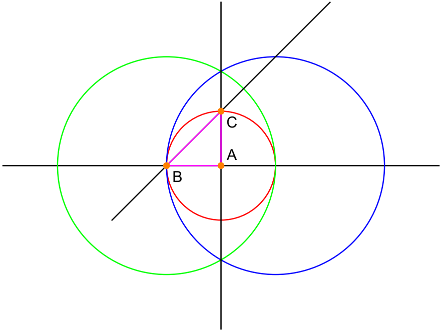


Figure 2: construction of a right triangle with hypotenuse length equal to $\sqrt{2}$.

# Squaring the Circle Cont.

## Example

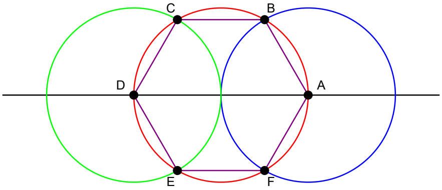


Figure 3: Construction of hexagon.

# Squaring the Circle Cont.

**Theorem (Constructible Numbers)**

*any number $\alpha$ that can be constructed using ruler and compass have the following two properties*

1. *$\alpha$ is an algebraic number*
2. *The degree of the minimal polynomial of $\alpha$ is of a power of two.*

By minimal here, we mean an irreducible polynomial $p(x)$ with leading coefficient 1 and lowest degree for which $p(\alpha) = 0$. This means that $\sqrt[3]{2}$ which satisfies $x^3 - 2 = 0$ is not a constructible number since the degree of the minimal polynomial is 3 which is not of degree that is a power of 2.

## Squaring the Circle Cont.

The following theorem was given by *Lindemann* in 1882 and improved by *Weierstrass*. The following formulation of the theorem is due to *Baker*.

### Theorem

*(Lindemann-Weierstrass theorem) for all $\alpha_1, \alpha_2, \ldots \alpha_n$ that algebraic and distinct and for all $\beta_1, \beta_2, \ldots, \beta_n$ that are non-zero and algebraic we have*

$$\beta_1 e^{\alpha_1} + \cdots + \beta_n e^{\alpha_n} \neq 0 \tag{14}$$

## Squaring the Circle Cont.

### Theorem

*$\pi$ is transcendental*

Using Euler's identity

$$e^{i\pi} + 1 = 0 \tag{15}$$

Assume to the contrary that $\pi$ is algebraic, then $i\pi$ is also algebraic. From this we can see that the left side of the identity is a linear combination that must satisfy the condition of Lindemann-Weierstrass theorem, but the right side is 0 and hence a contradiction. Therefore $\pi$ is transcendental.

Since $\pi$ is transcendental, then by the constructible numbers theorem it cannot be constructable. This means that squaring the circle is impossible to solve. The impossibility of this problem was established after two millennium from the time it was posed.

Thank you for listening !

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