MATH 441 Project

A Result in Game Theory Using Convex Analysis

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Definition

Definition 1.1

Let $\Omega \subset \mathbb{R}^n$.

(i) A function $f : \Omega \longrightarrow \mathbb{R}$ is said to be **convex** on Ω if for any $\lambda \in [0,1]$ and any points $x, y \in \Omega$, we have:

$$f((1-\lambda)x + \lambda y) \leq (1-\lambda)f(x) + \lambda f(y).$$

(ii) A function $f : \Omega \longrightarrow \mathbb{R}$ is said to be **strictly convex** on Ω if for any $\lambda \in (0,1)$ and any points $x, y \in \Omega$ such that $x \neq y$, we have:

$$f((1-\lambda)x + \lambda y) < (1-\lambda)f(x) + \lambda f(y).$$

Example

Examples of a Convex Function

The following functions are convex:

(i) $f(x) = \langle a, x\rangle$ for $x \in \mathbb{R}^n$, where $a \in \mathbb{R}^n$.

(ii) $f(x) = ||x||$ for $x \in \mathbb{R}^n$.

(iii) $h(x, y) = x^2 + y^2$ for $(x, y) \in \mathbb{R}^2$ (Elliptic Parabola).

Definition

Definition 1.2

A subset $\Omega \subset \mathbb{R}^n$ is **convex** if the line segment between any two points in Ω lies in Ω. That is, $\forall \vec{u}, \vec{v} \in \Omega$ and $\forall t \in [0,1]$, $t\vec{u} + (1-t)\vec{v} \in \Omega$.

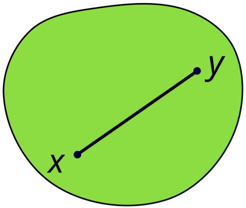

Figure 1: A convex set

Definition

Definition 1.3

Let $\Omega \subset \mathbb{R}^n$ be a convex set with $\bar{x} \in \Omega$. The **Normal Cone** to Ω at $\bar{x}$ is

$$N(\bar{x};\Omega) := \{\, v \in \mathbb{R}^n | < v, x - \bar{x} > \leq 0 \text{ for all } x \in \Omega\} \,.$$

By convention, we put $N(\bar{x};\Omega) := \emptyset$ for all $\bar{x} \notin \Omega$.

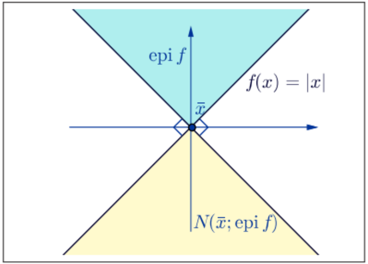

Figure 2: Epigraph of the absolute value function and the normal cone to it at the origin.

Theorems

Theorem 1.4

Let Ω be an open convex subset of $\mathbb{R}^n$ and $\varphi : \Omega \longrightarrow \mathbb{R}$. For $x, y \in \Omega$, let $g : (0,1) \longrightarrow \mathbb{R}$ be given by $g(t) = \varphi(tx + (1-t)y)$. Then the function φ is convex on Ω if and only if g is convex on $(0,1)$ for all $x, y \in \Omega$.

Theorems

Theorem 1.5

Let Ω be an open convex subset of $\mathbb{R}^n$ and let $f : \Omega \longrightarrow \mathbb{R}$ be a twice continuously differentiable function, i.e. $f \in \mathscr{C}^2(\Omega)$. Then:

(i) *f is convex iff the Hessian matrix $D^2 f(x)$ is positive* ***semidefinite*** *for all $x \in \Omega$, i.e. $\langle D^2 f(x) h, h\rangle \geq 0$ for all $h \in \mathbb{R}^n$.*

(ii) *If the Hessian is positive* ***definite****, i.e. for all $x \in \Omega$ we have: $\langle D^2 f(x) h, h\rangle > 0$ for all $h \in \mathbb{R}^n - \{0\}$, then f is strictly convex.*

This can be generalized to concavity using negative ***semidefinite*** *and negative* ***definite****.*

Theorems

Proposition 1.6

Let Ω_1 and Ω_2 be nonempty, convex subsets of $\mathbb{R}^n$ and $\mathbb{R}^p$, respectively. For $(x, y) \in \Omega_1 \times \Omega_2$, we have

$$N((x, y); \Omega_1 \times \Omega_2) = N(x; \Omega_1) \times N(y; \Omega_2).$$

Theorems

Proposition 1.7

Let $\Omega \in \mathbb{R}^n$. Assume that the function $f : \Omega \longrightarrow \mathbb{R}$ is differentiable at $\bar{x} \in \Omega$. Then the following are equivalent:

(i) $\bar{x}$ is an optimal solution to the problem: ***minimize*** *$f(x)$* ***subject to*** *$x \in \Omega$.*

(ii) $-\nabla f(\bar{x}) \in N(\bar{x}; \Omega)$

Theorems

Proposition 1.8

Let Ω be a nonempty, convex subset of $\mathbb{R}^n$ and let $\bar{\omega} \in \Omega$. Then we have $\bar{\omega} \in \Pi(\bar{x};\Omega)$ iff $\langle \bar{x} - \bar{\omega}, \omega - \bar{\omega} \rangle \leq 0$ for all $\omega \in \Omega$.

Theorems

Proposition 1.9

Let Ω be a nonempty, closed, convex subset of $\mathbb{R}^n$. Then the projection is Lipschitz continuous with constant $l = 1$. That is for any $x, y \in \mathbb{R}^n$, we have

$$\|\Pi(x;\Omega) - \Pi(y;\Omega)\| \leq \|x - y\|$$

Theorems

Theorem 1.10 (Brouwer Fixed-Point Theorem)

Let K be a convex and compact subset of $\mathbb{R}^n$ and let $f : K \longrightarrow K$ be a continuous function. Then f has a fixed point.

1. Convex Analysis
2. Theorems for Game Theory
3. Game Theory

Definitions

Definition 1.11

Let Ω_1 and Ω_2 be nonempty subsets of $\mathbb{R}^n$ and $\mathbb{R}^p$, respectively. A **Non-Cooper Active Game** in the case of two players I and II consists of two strategy sets Ω_i and two real-valued functions $u_i : \Omega_1 \times \Omega_2 \longrightarrow \mathbb{R}$ for $i = 1, 2$ called **Payoff Functions**. Then we refer to this game as $\{\, \Omega_i, u_i \,\}$ for $i = 1, 2$.

Definitions

Definition 1.12

Given a noncooperative two-person game $\{\Omega_i, u_i\}$ for $i = 1, 2$, a **Nash Equilibrium** is an element $(\bar{x}_1, \bar{x}_2) \in \Omega_1 \times \Omega_2$ satisfying the conditions

$$u_1(x_1, \bar{x}_2) \leq u_1(\bar{x}_1, \bar{x}_2) \text{ for all } x_1 \in \Omega_1,$$
$$u_2(\bar{x}_1, x_2) \leq u_2(\bar{x}_1, \bar{x}_2) \text{ for all } x_2 \in \Omega_2.$$

The conditions above mean that $(\bar{x}_1, \bar{x}_2)$ is a pair of best strategies that both players agree with in the sense that $\bar{x}_1$ is a best response of Player I when Player II chooses the strategy $\bar{x}_2$, and $\bar{x}_2$ is a best response of Player II when Player I chooses the strategy $\bar{x}_1$.

Examples

Example 1.13

		Player II	
		A	B
Player I	A	4, 4	1, 3
	B	3, 1	2, 2

Table 1: Payoff Functions

Here, $\Omega_i = \{ \text{A}, \text{B} \}$ for $i = 1, 2$ and the payoff function u_i of each player is represented in the payoff matrix above.

Examples

Example 1.13

		Player II A	Player II B
Player I	A	4, 4	1, 3
Player I	B	3, 1	2, 2

In this example, Nash equilibrium occurs when both players choose strategy A, and it also occurs when both players choose strategy B. Let us consider the case where both players choose strategy B. In this case, given that Player II chooses strategy B, Player I also wants to keep strategy B because a change of strategy would lead to a reduction of his payoff from 2 to 1. Similarly, given that Player I chooses strategy B, Player II wants to keep strategy B because a change of strategy would also lead to a reduction of his payoff from 2 to 1.

Theorems

Theorem 1.14

Consider a two-person game $\{\Omega_i, u_i\}$ *where* Ω_1 *and* Ω_2 *are nonempty, compact, convex subsets of* $\mathbb{R}^n$ *and* $\mathbb{R}^p$*, respectively. Let the payoff functions* $u_i : \Omega_1 \times \Omega_2 \longrightarrow \mathbb{R}$ *be given by:*

$$u_1(x, y) := x^T Ay, u_2(x, y) := x^T By$$

where A *and* B *are* $n \times p$ *matrices. Then this game admits a Nash equilibrium.*

Proof

Proof

It follows from the definition that an element $(\bar{x}, \bar{y}) \in \Omega_1 \times \Omega_2$ is a Nash equilibrium of the game under consideration if and only if

$$-u_1(x, \bar{y}) \geq -u_1(\bar{x}, \bar{y}) \text{ for all } x \in \Omega_1,$$
$$-u_2(\bar{x}, y) \geq -u_2(\bar{x}, \bar{y}) \text{ for all } y \in \Omega_2.$$

Moreover, by the definition of u_i, the payoff functions are both differentiable. Hence, by Proposition 1.7, this holds if and only if we have the normal cone inclusions

$$\nabla_x u_1(\bar{x}, \bar{y}) \in N(\bar{x}; \Omega_1) \text{ and } \nabla_y u_2(\bar{x}, \bar{y}) \in N(\bar{y}; \Omega_2)$$

or equivalently:

$$A\bar{y} \in N(\bar{x}; \Omega_1) \text{ and } B^T\bar{x} \in N(\bar{y}; \Omega_2)$$

that is

$$(A\bar{y}, B^T\bar{x}) \in N(\bar{x}; \Omega_1) \times N(\bar{y}; \Omega_2) = N((\bar{x}, \bar{y}); \Omega_1 \times \Omega_2) = N((\bar{x}, \bar{y}); \Omega) \quad (1)$$

where $\Omega := \Omega_1 \times \Omega_2$

Proof

Proof

Meaning,

$$\langle (A\bar{y}, B^T\bar{x}), \omega - (\bar{x}, \bar{y}) \rangle \leq 0 \text{ for all } \omega \in \Omega_1 \times \Omega_2$$

Using $\langle (A\bar{y}, B^T\bar{x}), \omega - (\bar{x}, \bar{y}) \rangle = \langle (\bar{x}, \bar{y}) + (A\bar{y}, B^T\bar{x}) - (\bar{x}, \bar{y}), \omega - (\bar{x}, \bar{y}) \rangle$ and by Proposition 1.8 we get that condition (1) is equivalent to $(\bar{x}, \bar{y}) = \Pi((\bar{x}, \bar{y}) + (A\bar{y}, B^T\bar{x}); \Omega)$. Now, define the mapping $F : \Omega \longrightarrow \Omega$ by $F(x, y) := \Pi((x, y) + (Ay, B^T x); \Omega)$. As Ω is a nonempty, closed and convex, it follows by Proposition 1.9 that F is continuous. Hence, by Theorem 1.10 (Brouwer Fixed-Point Theorem), the mapping F has a fixed point $(\bar{x}, \bar{y}) \in \Omega$. Thus, $(\bar{x}, \bar{y})$ is a Nash equilibrium of the game. $\square$

Thank you for listening !

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