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Overview of Transcendental Numbers

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Abstract

Transcendental numbers are numbers that cannot be a solution to a polynomial with integer coefficients. In this report, we will define transcendental numbers and then prove that the so-called Liouville number and e are transcendental numbers. In addition, we show that there are too many transcendental numbers in the sense that all but countably many real numbers are transcendental and hence almost all real numbers are transcendental. Moreover, we discuss one of Hilbert's problems related to transcendental numbers, which classifies a broad class of numbers as transcendental. Finally, we discuss the problem of squaring the circle and its relation to the transcendence of π.

1 Introduction

In analytic number theory, transcendental numbers are defined as those numbers that do not constitute a solution for polynomials with integer coefficients. The first person, most likely, to first define this notion was *Leonhard Euler* in the 18th century. At that time, specifically in 1744, Euler was the first to prove that e was an irrational number. Following his steps, *Lambert* proved that π is an irrational number in 1761. People had intuition at that time that transcendental numbers exist but could not prove their existence. The first serious work that showed their existence is attributed to Joseph Liouville, who in 1844 gave a necessary criterion that the algebraic numbers must satisfy. This criterion is now called Liouville's theorem. The problem with the result that Liouville had produced was that it effectively showed the existence of transcendental numbers that are somewhat contrived, i.e. they are not typically encountered in other areas of mathematics and do not arise naturally like e or π. These constants that arise in analysis proved to be difficult to tackle, and only in 1873 the first prove of the transcendence of e was established by *Charles Hermite*. In 1882, roughly a decade after Hermite's proof, *Lindemann* was able to show that π is a transcendental number which had a direct implication on the long-standing geometric question about squaring the circle which proved to be impossible to solve. *Hilbert*, in 1900, provided a list of 23 questions to be solved in the future. The seventh problem on this list stated the following: if a and b are algebraic numbers with a being different from 0 and 1 and b is irrational, then is it the case that a^b is a transcendental number? the question was answered positively by the two mathematicians *Aleksandr Gelfond* and *Theodor Schneider* in 1934. From this, one can see that the field of transcendental numbers is a rich subject that has ties with other areas like analysis and geometry. In this report, we will try to discuss and prove (and sometimes state) some of the most important results and theorems related to this subject.

2 Transcendental Numbers and Their Existence

Definition 2.1. *A complex number ξ is called an **algebraic number** if it is a solution to some polynomial equation $P(x) = 0$ where $f(x)$ is a polynomial over $\mathbb{Q}$. Equivalently, an **algebraic number** is a root for some polynomial $P(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$ where $a_i \in \mathbb{Z}$ for all $i = 0, 1, 2, ..., n$.*

Clearly, any rational number r is an algebraic number. Taking $P(x) = x - r$, we can see that $f(r) = 0$. Also, $\sqrt{2}$ is an algebraic number as it is a root to $Q(x) = x^2 - 2$. Moreover, the set of algebraic numbers is closed under addition, multiplication, and division (by non-zero number).

Definition 2.2. *A complex number z is said to be **transcendental** if it is not algebraic, i.e. there does not exists a polynomial $P(x)$ with integer coefficients such that $P(z) = 0$.*

We can observe that a transcendental number is irrational as all rational numbers are algebraic. However, not all irrational numbers are transcendental since $\sqrt{2}$ is algebraic. The most famous examples of transcendental numbers are the familiar constants π and e.

We shall prove the existence of transcendental numbers by showing that the number $\beta = \sum_{j=1}^{\infty} 10^{-j!} = 0.110001000...$ is not algebraic. This was one of the numbers used by Liouville in 1851 in the first proof of the existence of transcendental numbers.

Proposition 2.1. *The number $\beta = \sum_{j=1}^{\infty} 10^{-j!} = 0.110001000...$ is transcendental.*

Proof. Assume, on the contrary, that $\beta = \sum_{j=1}^{\infty} 10^{-j!}$ is algebraic, then it satisfies some polynomial equation $P(x) = \sum_{i=0}^{n} a_i x^i = 0$ where $a_i \in \mathbb{Z}$ for all $i = 0, 1, 2, ..., n$. Let $\beta_k = \sum_{j=1}^{k} 10^{-j!}$, then:

$$\beta - \beta_k = \sum_{j=1}^{\infty} 10^{-j!} - \sum_{j=1}^{k} 10^{-j!} = \sum_{j=k+1}^{\infty} 10^{-j!} < 2 \sum_{j=k+1}^{k+1} 10^{-j!} = 2 \cdot 10^{-(k+1)!} \tag{1}$$

By the mean value theorem, there exists a number c between β and β_k such that $|P(\beta) - P(\beta_k)| = |P(\beta_k)| = |\beta - \beta_k||P'(c)|$. $P'(x) = \sum_{i=1}^{n} i a_i x^{i-1}$. For $x \in (0, 1)$, we have:

$$|P'(x)| = |\sum_{i=1}^{n} i a_i x^{i-1}| < \sum_{i=1}^{n} |i a_i| \tag{2}$$

Combining (1) and (2), we get:

$$|\beta - \beta_k||P'(c)| < 2 \cdot 10^{-(k+1)!} \cdot \sum_{i=1}^{n} |i a_i|$$

Since $P(x)$ has only n zeros, we can choose k sufficiently large so that $P(\beta_k) \neq 0$. Doing so, we get:

$$|P(\beta_k)| = |\sum_{i=0}^{n} a_i \beta_k^i| \geq 10^{-n \cdot (k)!}$$

Moreover, we observe that for large k, we have

$$10^{-n \cdot (k)!} > 2 \cdot 10^{-(k+1)!} \cdot \sum_{i=1}^{n} |i a_i|$$

This implies $|P(\beta_k)| > |\beta - \beta_k||P'(c)|$, a contradiction. Thus, $\beta = \sum_{j=1}^{\infty} 10^{-j!} = 0.110001000...$ is transcendental. □[1]

In fact, almost all real numbers are transcendental. This is because there are uncountably infinite transcendental numbers and countably infinite algebraic numbers as we shall see in the next section.

3 Uncountability of Transcendental Numbers

In 1874, Georg Cantor proved that the set of algebraic numbers is countable and the set of real numbers is uncountable. We will show that the set of transcendental numbers is uncountable by showing that the set of algebraic number is countable.

Lemma 3.1. *For $n \in \mathbb{N} \cup \{0\}$, the set $P_n = \{\ a_0 + a_1 x + a_2 x^2 + ... + a_n x^n : a_i \in \mathbb{Z}\ \forall i = 0, 1, 2, ..., n\}$ of polynomials of degree less than or equal to n is countable.*

Proof. If $n = 0$, then $P_n = P_0 = \mathbb{Z}$, which is countable. Let $n \in \mathbb{N}$ and define

$$f : P_n \longrightarrow \mathbb{Z}^{n+1}$$

$$a_0 + a_1 x + a_2 x^2 + ... + a_n x^n \mapsto (a_0, a_1, a_2, ..., a_n)$$

Clearly, f is a bijection. Hence, the set P_n is countable as $\mathbb{Z}^{n+1}$ is countable. □

Theorem 3.1. *The set of all algebraic numbers is countable.*

Proof. For each fixed integer $n \in \mathbb{N} \cup \{0\}$, let $P_n = \{\ a_0 + a_1 x + a_2 x^2 + ... + a_n x^n : a_i \in \mathbb{Z}\ \forall i = 0, 1, 2, ..., n\}$ be the set of polynomials of degree n or less. The set of all polynomials with integer coefficients is

$$P = \bigcup_{n=0}^{\infty} P_n$$

By Lemma 3.1, the set P_n is countable for all $n \in \mathbb{N} \cup \{0\}$. Hence, P is a countable union of countable sets and therefore countable. Because every polynomial of degree n has at most n roots, the set of all possible roots of all polynomials with integer coefficients is a countable union of finite sets. Thus, the set of all algebraic numbers is countable □

It is an immediate result of Theorem 3.1 that the set of transcendental numbers is uncountable.

Corollary 3.1. *The set of transcendental numbers is uncountable.*

Proof. Let $\mathbb{A}$ and $\mathbb{T}$ denote the set of algebraic numbers and the set of transcendental numbers, respectively. By the definition of transcendental numbers, $\mathbb{T} = \mathbb{R} - \mathbb{A}$. Assume $\mathbb{T}$ is countable, then $\mathbb{R} = \mathbb{A} \cup \mathbb{T}$ is countable, a contradiction. Thus, the set of transcendental numbers is uncountable. □

4 Transcendence of e

The Liouville number that we proved its transcendence in proposition 2.1 may seem to be contrived, after all we do not encounter this number naturally in analysis or other fields. Here, we want to give a proof for the transcendence of e. The proof that e is transcendental was first given by *Charles Hermite* in 1873.

Theorem 4.1. *e is transcendental [2]*

Proof. Let $f(x)$ be a real polynomial with degree m. Define the following integral

$$I(t) := \int_0^t e^{t-u} f(u) du \tag{3}$$

Where t is a complex variable and the integral is taken over the segment joining 0 and t. If we apply the integration by parts theorem repeatedly leaving e^t outside the integral

$$\begin{aligned} \int_0^t e^{-u} f(u) du &= -e^{-u} f(u)\Big|_0^t + \int_0^t e^{-u} f'(u) du \\ &\vdots \qquad\qquad\qquad \vdots \\ \int e^{-u} f^{(m)}(u) du &= -e^{-u} f^{(m)}(u)\Big|_0^t + \int e^{-u} f^{(m+1)}(u) du \end{aligned} \tag{4}$$

noticing that the last term is zero and substituting backwardly we obtain

$$I(t) = e^t \sum_{j=0}^{m} f^{(j)}(0) - \sum_{j=0}^{m} f^{(j)}(t) \tag{5}$$

Let us denote by $\bar{f}(x)$ the polynomial $f(x)$ but we replace its coefficients by their absolute values. Consider the following estimate

$$|I(t)| \leq \int_0^t |e^{t-u} f(u)| du \leq |t| e^{|t|} \bar{f}(|t|) \tag{6}$$

Now, suppose to the contrary that e is an algebraic number

$$q_0 + q_1 e + q_2 e^2 + \ldots q_n e^n = 0 \tag{7}$$

Where $q_0 \neq 0$ and $n > 0$ and all the q's are integers. Define the following quantity

$$J := q_0 I(0) + q_1 I(1) + \cdots + q_n I(n) \tag{8}$$

also

$$f(x) = x^{p-1}(x-1)^p \ldots (x-n)^p \tag{9}$$

Where p is a large prime. Using (5) and (7) to substitute in (8) results in

$$J = -\sum_{j=0}^{m} \sum_{k=0}^{n} q_k f^{(j)}(k) \tag{10}$$

The degree of $f(x)$ is $m = (n+1)p-1$. Notice that when either $j < p$, $k > 0$ or $j < p-1$, $k = 0$ then we can conclude that $f^{(j)}(k) = 0$ and for all j and k different from $j = p-1$, $k = 0$ and different from the previous conditions, $f^{(j)}(k)$ would be divisible by $p!$. We will now use Leibniz rule on $f(x)$

$$(fg)^n = \sum_{k=0}^{n} \binom{n}{k} f^{n-k} g^k \tag{11}$$

Where here caution must be taken because we used our previous notation to state the rule. Applying this on (9) and letting $g \to x^{p-1}$ while f should be substituted for all the other factors in (9). From this we obtain

$$f^{(p-1)}(0) = (p-1)!(-1)^{np}(n!)^p \tag{12}$$

Notice that $(p-1)! \mid f^{(p-1)}(0)$ and $p!$ does not divide $f^{(p-1)}(0)$ whenever $p > n$. This implies that if $p > |q_0|$ then J is divisible by $(p-1)!$ which means

$$|J| \geq (p-1)! \tag{13}$$

But using the estimate $\bar{f}(k) \leq (2n)^m$ along with (6) we can establish the following inequality

$$|J| \leq |q_1| e \bar{f}(1) + \cdots + |q_n| n e^n \bar{f}(n) \leq c^p \tag{14}$$

where c is some constant that is independent of p. Now, if we were to choose an appropriate c together with a sufficiently large p then (13) and (14) would be inconsistent. Therefore, e must be transcendental. □

5 Gelfond–Schneider Theorem

In 1900, David Hilbert announced a list of twenty-three outstanding unsolved problems in mathematics. The seventh problem was settled in 1934 by A. O. Gelfond, which was followed by an independent proof by Th. Schneider in 1935. It concerns the irrationality and transcendence of certain numbers.[3]

Theorem 5.1 (Gelfond–Schneider Theorem)**.** *If α and β are algebraic numbers with $\alpha \neq 0$ and $\alpha \neq 1$ and if β is not a rational number, then any value of α^β is transcendental.*

The theorem establishes the transcendence of numbers such as $2^{\sqrt{2}}$, $\sqrt{2}^{\sqrt{2}}$ and 2^i. In general, $\alpha^\beta = e^{\beta ln \alpha}$ is multiple-valued, and this is the reason for the phrase "any value of" in the statement of Theorem 4.1.

We will state an alternative form of Theorem 4.1 and prove their equivalence.[3]

Theorem 5.2. *If α and γ are non-zero algebraic numbers and if $\alpha \neq 1$, then $(ln\gamma)/(ln\alpha)$ is either rational or transcendental.*

Theorem 4.1 $\Longrightarrow$ Theorem 4.2: Assume the hypotheses and conclusion of Theorem 4.1, let $\beta = (ln\gamma)/(ln\alpha)$, so $\gamma = \alpha^\beta$. Suppose that α and γ are non-zero algebraic numbers and, on the contrary, β is algebraic but not rational. Then, by the conclusion of Theorem 4.1, γ is transcendental, a contradiction. Thus, Theorem 4.1 implies Theorem 4.2.

Theorem 4.1 $\Longleftarrow$ Theorem 4.2: Assume the hypotheses and conclusion of Theorem 4.2. Suppose that α and β are algebraic numbers, $\alpha \neq 0$, $\alpha \neq 1$, and β is not a rational number, but a value of α^β is algebraic. Since $\alpha^\beta \neq 0$, we apply the conclusion of Theorem 4.2. Let $\gamma = \alpha^\beta$, then $(ln\gamma)/(ln\alpha) = (ln\alpha^\beta)/(ln\alpha) = \beta$ is either rational or transcendental, a contradiction. Thus, Theorem 4.2 implies Theorem 4.1.

To conclude this section, we show the importance that α and β must be algebraic numbers.

Example 5.1. Let $\alpha = \sqrt{2}^{\sqrt{2}}$ and $\beta = \sqrt{2}$. Observe that $\alpha^\beta = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^2 = 2$ is algebraic. The reason for this is that α is transcendental and not algebraic, contrary to the hypotheses of Theorem 4.1.

6 Squaring the Circle

One of the problems related to transcendental numbers is called squaring the circle. It is the problem of trying to construct a square that has the same area as the area of a unit circle. This would mean that the square we have to construct is of side length $\sqrt{\pi}$. Now, the difficulty in the problem arises because we want to do this under the rules of straightedge and compass construction. One of the rules of straghtedge and compass construction is that the ruler you use has no scale and thus you cannot go from 0 to $\sqrt{\pi}$ because it is not marked on your ruler. We will digress a little bit to explain this type of construction and then explain how it is related to transcendental numbers.

6.1 Ruler and compass Construction

In the straightedge and compass or ruler and compass construction, one will have a ruler of infinite length and a compass that can draw circles of any radius. We will assume that the compass is non-collapsing which means that after drawing a circle, we can take the compass somewhere else and draw the same circle with same radius (it is equivalent to a collapsing compass in terms of what they can construct). Typically, one can build complex constructions from the following simple ones:

- drawing a segment passing through two points.
- drawing a circle with a certain radius centered at a certain point.
- intersecting two lines.
- intersecting a line with a circle.
- intersecting two circles.

We will talk about two examples to show how these constructions should work by exploiting our simple constructions. The first example is about constructing a triangle with hypotenuse of length $\sqrt{2}$ and the second is about constructing a hexagon.

Example 6.1. let us examine how can we obtain a triangle of hypotenuse length $\sqrt{2}$. The construction is shown in Fig.1. We start by drawing a long line and mark on it the points A and B. Now, we center our compass on A and draw a circle around it of length $|AB| = 1$ (notice that we cannot measure lengths but here we are normalizing everything with respect to AB and hence it is okay to assume its length is 1). This would be the red circle in Fig.1. Now, draw a circle at point B of length $2|AB| = 2$, this will be the green circle. Do the same by centering your compass on the point at the right of A away from it by a unit length and draw a circle of radius $2|AB| = 2$ this is the blue circle in Fig.1. Now, we draw a vertical line from the upper point of intersection and passing through A. This will mark the point C on the red circle. Then, pass a line through the points C and B and with this we obtain the triangle $\triangle ABC$ colored in magenta and with hypotenuse of length $\sqrt{2}$.

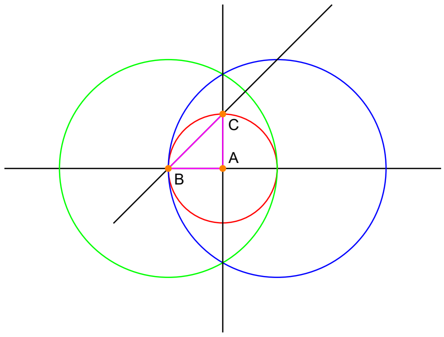

Figure 1: construction of a right triangle with hypotenuse length equal to $\sqrt{2}$.

Example 6.2. In this example we will construct a hexagon like the one in Fig.2. Start by drawing a horizontal line. On that line, draw a circle of diameter AD (this is an arbitrary length, no measurement has been done). This is the red circle in Fig.2. Now, center your compass on A and draw a circle of diameter AD, this is the blue circle. Do the same at point D and you will obtain the green circle. We notice that the blue and green circle intersect the red one at B, C, E, F. make the segments BC and EF by connecting these points. After this, make the following segments AB , CD, DE , FA by connecting A, B and C, D and D, E and F,A. This completes our construction and we obtain the hexagon $ABCDEF$ in purple in Fig.2

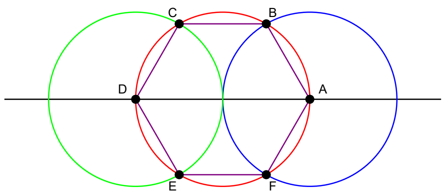

Figure 2: Construction of hexagon.

We now return to transcendental numbers and state an important property about constructable numbers (numbers obtained by ruler and compass construction) proved by Pierre Wantzel in 1837.

Theorem 6.1 (Constructible Numbers [4])**.** *any number α that can be constructed using ruler and compass have the following two properties*

1. *α is an algebraic number*
2. *The degree of the minimal polynomial of α is of a power of two.*

By minimal here, we mean an irreducible polynomial $p(x)$ with leading coefficient 1 and lowest degree for which $p(\alpha) = 0$. This means that $\sqrt[3]{2}$ which satisfies $x^3 - 2 = 0$ is not a constructible number since the degree of the minimal polynomial is 3 which is not of degree that is a power of 2.

The following theorem was given by *Lindemann* in 1882 and improved by *Weierstrass*. The following formulation of the theorem is due to *Baker*.

Theorem 6.2. *(Lindemann-Weierstrass theorem)[2] for all $\alpha_1, \alpha_2, \ldots \alpha_n$ that algebraic and distinct and for all $\beta_1, \beta_2, \ldots, \beta_n$ that are non-zero and algebraic we have*

$$\beta_1 e^{\alpha_1} + \cdots + \beta_n e^{\alpha_n} \neq 0 \tag{15}$$

Lindemann-Weierstrass theorem together with theorem 6.1 enable us to make a conclusion about the problem of squaring the circle.

inconstructibility of π

We want to show that π is transcendental. Using Euler's identity

$$e^{i\pi} + 1 = 0 \tag{16}$$

Assume to the contrary that π is algebraic, then $i\pi$ is also algebraic. From this we can see that the left side of the identity is a linear combination that must satisfy the condition of Lindemann-Weierstrass theorem, but the right side is 0 and hence a contradiction. Therefore π is transcendental.

Since π is transcendental, then by theorem 6.1 it cannot be constructable. This means that squaring the circle is impossible to solve. The impossibility of this problem was established after two millennium from the time it was posed. It should be noted that it is necessary for a constructable number to be algebraic but being algebraic is not sufficient to guarantee constructability is we showed with $\sqrt[3]{2}$.

7 Conclusion

In conclusion, we explored the transcendental numbers, briefly tracing their historical development and establishing their theoretical basis. We began by defining algebraic and transcendental numbers and showing the existence of transcendental numbers through Liouville's construction and also through Hermite's proof for the transcendence of e. Moreover, we showed that while the set of algebraic numbers is countable, the set of transcendental numbers is uncountable, indicating that almost all real numbers are transcendental.

We then studied the Gelfond–Schneider theorem, a result that solved Hilbert's seventh problem and extended our understanding of the transcendence of certain numbers. Furthermore, we connected transcendental numbers to geometry by analysing the problem of squaring the circle. Using the Lindemann–Weierstrass theorem, we showed the transcendence of π, and by Wantzel's results on constructible numbers, we concluded that squaring the circle with only a straightedge and compass is impossible.

However, there are still many unsolved questions in the study of transcendental numbers. For example, we do not know, at present, whether $\pi + e, \pi e, \pi/e, \pi^e$ are transcendental or even irrational. We know that at least one of $\pi + e, \pi e$ must be irrational. Also, Euler's constant γ is not known to be transcendental or algebraic. There are many conjectures regarding transcendental numbers that are still unsolved.[5]

In summary, despite the significant progress made in the study of transcendental numbers over the past two centuries, especially in proving the transcendence of certain numbers, much remains unknown.

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