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The Khalimsky Topology on $\mathbb{Z}$

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Abstract

The Khalimsky topology is a remarkable topology that introduces continuity in situations where one might imagine only the uninteresting discrete topology. The leading idea is to consider $\mathbb{Z}$ not as a subspace of $\mathbb{R}$ but as a quotient space. This paper investigates the definition and fundamental properties of the Khalimsky topology, including representation as an ordered poset. We establish that the topology is Alexandroff and path-connected but not compact, while showing it is pseudocompact. These results illustrate the importance of the Khalimsky topology to digital topology and discrete mathematical modeling.

1 Introduction

Definition 1.1 (Khalimsky Topology)**.** *The **Khalimsky topology** $(\mathbb{Z}, \mathcal{K})$ is the topology generated by*

$$\{\{2n-1, 2n, 2n+1\} : n \in \mathbb{Z}\}.$$

Equivalently, $O \in \mathcal{K}$ iff for all even integers $n \in O$, we have that the adjacent integers of n are in O, i.e. $n-1, n+1 \in O$.

Proposition 1.1. *Let $U \subseteq 2\mathbb{Z}+1$, then U is open in $(\mathbb{Z}, \mathcal{K})$.*

Proof. We have $U \in \mathcal{K}$ iff for all even integers $n \in U$, $n-1, n+1 \in U$. Since U has no even integers, the condition holds vacuously. □

Definition 1.2. *A **Khalimsky interval** is an interval $[a,b]_{\mathbb{Z}} = [a,b]_{\mathbb{R}} \cup \mathbb{Z}$ equipped with the topology induced by the Khalimsky topology on $\mathbb{Z}$.*

Definition 1.3. *The **Khalimsky Plane** is the Cartesian product of two Khalimsky lines, and, more generally, **Khalimsky space** is the Cartesian product of n copies of $\mathbb{Z}$.*

2 Properties of the Khalimsky Topology

Proposition 2.1. *Let $(\mathbb{Z}, \mathcal{K})$ denote the Khalimsky topology and let $n \in \mathbb{Z}$. Then the following properties hold:*

(i) $\overline{\{2n\}} = \{2n\}$.

(ii) $\overline{\{2n+1\}} = \{2n, 2n+1, 2n+2\}$.

(iii) The smallest open set containing $2n$ is $\{2n-1, 2n, 2n+1\}$.

(iv) The smallest open set containing $2n+1$ is $\{2n+1\}$.

Proof. (i) To show that $\overline{\{2n\}} = \{2n\}$, it suffices to prove that the singleton $\{2n\}$ is closed. By definition, a set is closed if and only if its complement is open.

Consider the complement:

$$\mathbb{Z} \setminus \{2n\} = \{\ldots, 2n-2, 2n-1, 2n+1, 2n+2, \ldots\}.$$

Observe that the complement contains all odd integers. Hence any even integer in $\mathbb{Z} \setminus \{2n\}$ would have their adjacent integers in $\mathbb{Z} \setminus \{2n\}$. Thus, $\mathbb{Z} \setminus \{2n\}$ is open and, therefore, $\{2n\}$ is closed, i.e. $\overline{\{2n\}} = \{2n\}$.

(ii) Recall that the closure of a set is the smallest closed set containing it, equivalently, the union of the set and all its limit points.

1. Since $2n+1 \in \{2n+1\}$, obviously $2n+1$ is in the closure.

2. Consider the points $2n$ and $2n+2$ which are even integers adjacent to $2n+1$. By the definition of the Khalimsky topology, every open neighborhood of these even points contains the odd integer in the middle.

Thus, every open neighborhood of $2n$ and $2n+2$ intersects $\{2n+1\}$, which means $2n$ and $2n+2$ are limit points of $\{2n+1\}$.

3. For any other point $x \notin \{2n, 2n+1, 2n+2\}$, there exists an open neighborhood of x disjoint from $\{2n+1\}$. For example, even integers other than $2n$ or $2n+2$ have singleton neighborhoods not containing $2n+1$. Odd integers other than $2n+1$ are separated by at least two units, so their neighborhoods do not intersect $\{2n+1\}$.

Therefore,

$$\overline{\{2n+1\}} = \{2n, 2n+1, 2n+2\}.$$

(iii) Since the smallest open neighborhood containing an even integer $2n$ must include its adjacent odd integers, the smallest open set containing $2n$ is given by:

$$\{2n-1, 2n, 2n+1\}.$$

(iv) This follows from Proposition 1.1.

□

Definition 2.1 (Alexandroff Space). *A topological space such that the intersection of any family of open sets is open is called a* ***Smallest-Neighborhood Space*** *or an* ***Alexandroff Space****.*

Proposition 2.2. *The Khalimsky topology $(\mathbb{Z}, \mathcal{K})$ is an Alexandroff Space.*

Proof. Let $\{O_i, i \in I\}$ be a family of open sets in $(\mathbb{Z}, \mathcal{K})$ and let $O = \bigcap_{i \in I} O_i$ and assume $O \neq \emptyset$. If $O \subseteq 2\mathbb{Z}+1$, then $O \in \mathcal{K}$. If not, then there exists an even integer $x \in O$. So, $x \in O_i$ for all $i \in I$. Hence, $x-1, x+1 \in O_i$ for all $i \in I$. Therefore, $x-1, x+1 \in O$. Thus, $O \in \mathcal{K}$. □

3 Connectedness of the Khalimsky Topology

Definition 3.1 (Locally Connectedness). *A topological space $(X, \mathcal{T})$ is said to be* ***locally connected*** *if for every point in X, there exists a fundamental system of neighborhoods (FSN) consisting of connected sets.*

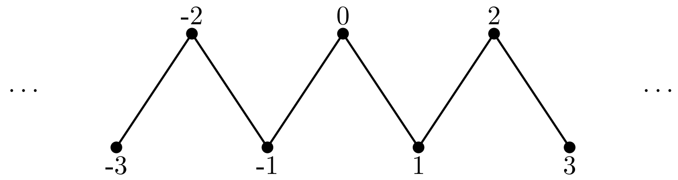
Proposition 3.1. *Every Alexandroff Space is locally connected.*

Proof. Let $(X, \mathcal{T})$ be an Alexandroff Space. For $x \in X$, let $V_{\mathcal{T}}(x)$ be the smallest open set containing x. We show that $V_{\mathcal{T}}(x)$ is connected. If $V_{\mathcal{T}}(x) \subseteq U \cup V$ where U and V are open disjoint sets, then $x \in U$ or $x \in V$. If $x \in U$, then $V_{\mathcal{T}}(x) \subseteq U$. If $x \in V$, then $V_{\mathcal{T}}(x) \subseteq V$. Hence, $V_{\mathcal{T}}(x)$ is connected. $\mathcal{B} = \{V_{\mathcal{T}}(x)\}$ is an FSN of connected neighborhoods of x. Thus, $(X, \mathcal{T})$ is locally connected. □

3.1 Representation as an Ordered Set

The Khalimsky topology $(\mathbb{Z}, \mathcal{K})$ can be represented as an ordered set:

$$x \leq_{\mathcal{K}} y \text{ iff } y \in \overline{\{x\}}^{\mathcal{K}}.$$



In the same sense, $(\mathbb{Z}, \leq_{\mathcal{K}})$ is a poset as if $x, y \in \mathbb{Z}$ such that $x \leq_{\mathcal{K}} y$ and $y \leq_{\mathcal{K}} x$, then $y \in \overline{\{x\}}^{\mathcal{K}}$ and $x \in \overline{\{y\}}^{\mathcal{K}}$. Hence, $x = y$.

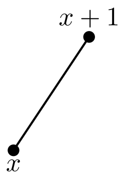
Lemma 3.1. *For all $x \in \mathbb{Z}$, $\{x, x+1\}$ is a path in $(\mathbb{Z}, \mathcal{K})$.*

Proof. **Case 1:** x is odd

Let $\varphi : ([0,1], \mathcal{U}) \longrightarrow (\mathbb{Z}, \mathcal{K})$ be given by $\varphi(t) = x$ for $t \in [0,1)$ and $\varphi(1) = x+1$, where x is an odd integer. Then

$$\varphi^{-1}(\{x\}) = [0,1)$$

which is open in $([0,1], \mathcal{U})$. Thus, φ is continuous.

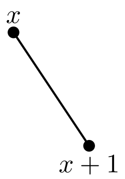


Case 2: x is even

Let $\varphi : ([0,1], \mathcal{U}) \longrightarrow (\mathbb{Z}, \mathcal{K})$ be given by $\varphi(t) = x+1$ for $t \in [0,1)$ and $\varphi(1) = x$, where x is an even integer. Then

$$\varphi^{-1}(\{x+1\}) = [0,1)$$

which is open in $([0,1], \mathcal{U})$. Thus, φ is continuous.



□

Theorem 3.1. *The **Khalimsky topology** $(\mathbb{Z}, \mathcal{K})$ is path-connected*

Proof. By lemma 3.1, every pair of consecutive integers are connected by a path, hence any integer can be connected to any other integer by concatenating these paths. □

Corollary 3.1. *The **Khalimsky topology** $(\mathbb{Z}, \mathcal{K})$ is connected.*

Proof. This follows from the fact that path connectedness implies connectedness. □

4 Compactness of the Khalimsky Topology

Definition 4.1 (Compactness)**.** *A space X is said to be **compact** if every open covering $\mathcal{A}$ of X, there exists a finite subcovering.*

Proposition 4.1. *The **Khalimsky topology** $(\mathbb{Z}, \mathcal{K})$ is not compact.*

Proof. Let

$$V(2n) = \{2n-1, 2n, 2n+1\}$$

then $V(2n)$ is open for all $n \in \mathbb{Z}$ and $\mathbb{Z} = \bigcup_{n\in\mathbb{Z}} V(2n)$. Thus, $\{V(2n) : n \in \mathbb{Z}\}$ is an open covering which does not have a finite subcover. □

Definition 4.2 (Pseudocompactness)**.** *A space $(X, \mathcal{T})$ is said to be **pseudocompact** if every continuous function $f : (X, \mathcal{T}) \longrightarrow (\mathbb{R}, \mathcal{U})$ is bounded.*

Proposition 4.2. *The **Khalimsky topology** $(\mathbb{Z}, \mathcal{K})$ is pseudocompact.*

Proof. Let $f : (\mathbb{Z}, \mathcal{K}) \longrightarrow (\mathbb{R}, \mathcal{U})$ be a continuous map. As $\mathcal{K}$ is connected, $f(\mathbb{Z})$ is connected, so $f(\mathbb{Z})$ is an interval. Moreover, $f(\mathbb{Z})$ is countable as $\mathbb{Z}$ is countable. Therefore, $f(\mathbb{Z}) = \{c\}$ where $c \in \mathbb{R}$. Hence, the image of $(\mathbb{Z}, \mathcal{K})$ under any continuous function to $(\mathbb{R}, \mathcal{U})$ is bounded. Thus, $(\mathbb{Z}, \mathcal{K})$ is pseudocompact. □

5 Conclusion

The Khalimsky topology on $\mathbb{Z}$ offers an alternative to the discrete topology, especially in the context of digital topology and image analysis.

We explored its foundational definitions and topological properties, and proved that it forms an Alexandroff space, which implies local connectedness. Moreover, we demonstrated that the Khalimsky topology is path-connected and therefore connected, but not compact—though it is pseudocompact, since continuous real-valued functions on it are bounded.

These results highlight the Khalimsky topology as a useful structure in digital topology. Further exploration might involve higher-dimensional Khalimsky spaces and applications to computational topology and digital image processing.

References

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