Chapter 2 :

Complex Numbers

Motivation.

Several equations do not admit solutions in $\mathbb{R}$. For example, the solution to the equation $x^2 + 1 = 0$ is equal to $\sqrt{-1}$, which does not belong to $\mathbb{R}$. So, we must construct another set which contains the real numbers, as well as the solutions of the algebraic equations in question.

“ *Complex numbers find their interest in Fourier analysis which is widely used in many fields, such as signal processing. Another example in electromagnetism is alternating current : since the voltage of such a circuit oscillates , it can be represented as a complex number via Euler's formula :* $V = V_0(\cos \omega t + i \sin \omega t)$. » **(Wikipedia)**

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2.1. Definitions and properties

Definition 1.

- **The set of complex numbers**, denoted $\mathbb{C}$, is an extension of the set of real numbers $\mathbb{R}$. It is a commutative field equipped with the operations $+$, $\times$.
- A complex number is a pair of real numbers (a, b) .
- We denote by ***i*** the couple $(0,1)$, we have : $i^2 = -1$ hence $i = \sqrt{-1}$.
- We identifying a real x by the couple $(x, 0)$.

Examples. $\mathfrak{z} = (2, 2\sqrt{3}) = 2 + 2\sqrt{3}\, i$, $\mathfrak{z} = \left(0, \frac{1}{4}\right) = \frac{1}{4}\, i$, $\mathfrak{z} = (12,0) = 12$, $\mathfrak{z} = (1,5) = 1 + 5i$

Definition 2.

- We call the algebraic form of a complex number $\mathfrak{z}$ the writing : $\mathfrak{z} = a + ib$.
- The number "a" is called the real part, denoted $\mathcal{Re}(\mathfrak{z})$.
- The number "b" is called the imaginary part, denoted $\mathcal{Im}(\mathfrak{z})$.
- The module of $\mathfrak{z}$, denoted $|\mathfrak{z}|$, is a positive real number defined by : $|\mathfrak{z}| = \sqrt{a^2 + b^2}$.
- The argument of $\mathfrak{z}$, denoted $\arg \mathfrak{z}$, is the angle $\theta \in [0, 2\pi[$ defined by : $\cos \theta = \frac{a}{|\mathfrak{z}|}$, $\sin \theta = \frac{b}{|\mathfrak{z}|}$.

Definition 3.

- We call the trigonometric form of a complex number $\mathfrak{z}$ the writing :

$$\mathfrak{z} = |\mathfrak{z}|(\cos \theta + i \sin \theta) .$$

- The conjugate of $\mathfrak{z}$, denoted $\bar{\mathfrak{z}}$, is given by : $\bar{\mathfrak{z}} = a - ib$.

Example. For $\mathfrak{z} = 2 + 2\sqrt{3}i$, we have:

$|\mathfrak{z}| = \sqrt{2^2 + \left(2\sqrt{3}\right)^2} = 4,\ \ \cos\theta = \frac{2}{4} = \frac{1}{2}$ and $\sin\theta = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2}$, therefore $\arg\mathfrak{z} = \theta = \frac{\pi}{3}$.

The trigonometric form is given by : $\mathfrak{z} = 4\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$.

- ✓ $\bar{\mathfrak{z}} = 2 - 2\sqrt{3}i$, $|\bar{\mathfrak{z}}| = 4,\ \ \cos\varphi = \frac{1}{2}$ and $\sin\varphi = -\frac{\sqrt{3}}{2}$, therefore $\arg\bar{\mathfrak{z}} = \varphi = -\frac{\pi}{3}$.

Properties. For $\mathfrak{z}, \mathfrak{z}' \in \mathbb{C}$, we have the following formulas:

1) $|\bar{\mathfrak{z}}| = |\mathfrak{z}|$, $|\mathfrak{z}.\mathfrak{z}'| = |\mathfrak{z}|.|\mathfrak{z}'|$.
2) $\bar{\mathfrak{z}} + \overline{\mathfrak{z}'} = \overline{\mathfrak{z} + \mathfrak{z}'}$, $\bar{\mathfrak{z}}.\overline{\mathfrak{z}'} = \overline{\mathfrak{z}.\mathfrak{z}'}$.
3) $\frac{1}{\mathfrak{z}} = \frac{\bar{\mathfrak{z}}}{|\mathfrak{z}|^2}$, $\frac{1}{\bar{\mathfrak{z}}} = \overline{1/\mathfrak{z}}$.
4) $\mathcal{Re}(\mathfrak{z}) = \frac{\mathfrak{z}+\bar{\mathfrak{z}}}{2}$, $\mathcal{Im}(\mathfrak{z}) = \frac{\mathfrak{z}-\bar{\mathfrak{z}}}{2i}$.
5) For $\mathfrak{z} = a + ib$, we have: $\mathfrak{z}.\bar{\mathfrak{z}} = |\mathfrak{z}|^2 = a^2 + b^2$.

Example. Let $\mathfrak{z} = 5 + 6i$ and $\mathfrak{z}' = 1 - 2i$, we have:

- ✓ $\mathfrak{z} + \mathfrak{z}' = (5 + 4i) + (1 - 2i\,) = 6 + 4i$.
- ✓ $\mathfrak{z} \times \mathfrak{z}' = (5 + 4i) \times (1 - 2i\,) = 13 - 6i$.

Example. In a previous example, for $\mathfrak{z} = 2 + 2\sqrt{3}i$, we have $\bar{\mathfrak{z}} = 2 - 2\sqrt{3}i$, therefore:

$\mathcal{Re}(\mathfrak{z}) = \frac{\mathfrak{z}+\bar{\mathfrak{z}}}{2} = \frac{1}{2}\left(2 + 2\sqrt{3}i + 2 - 2\sqrt{3}i\right) = 2$, $\mathcal{Im}(\mathfrak{z}) = \frac{\mathfrak{z}-\bar{\mathfrak{z}}}{2i} = \frac{1}{2i}\left(2 + 2\sqrt{3}i - 2 + 2\sqrt{3}i\right) = 2\sqrt{3}$.

Definition 4.

- We call "complex exponential", denoted $e^{i\theta}$, the complex number of module 1 and argument θ, i.e : $e^{i\theta} = \cos\theta + i\sin\theta$.
- We call « exponential forme » of a complex number $\mathfrak{z}$ of argument θ, the writing : $\mathfrak{z} = |\mathfrak{z}|e^{i\theta}$.
- For $\mathfrak{z} = a + ib$, on a : $e^{\mathfrak{z}} = e^{a+ib} = e^{a}.e^{ib} = e^{a}(\cos b + i\sin b)$

Examples.

1) For $\mathfrak{z} = 2 + 2\sqrt{3}i$, we have: $\mathfrak{z} = 4\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ therefore $\mathfrak{z} = 4e^{i\frac{\pi}{3}}$.
2) $e^{i0} = 1$, $e^{i\pi} = -1$, $e^{i\frac{\pi}{2}} = i$.

Properties. For every $\theta, \theta' \in \mathbb{R}$, we have:

- $\frac{1}{e^{i\theta}} = e^{-i\theta}$, $\overline{e^{i\theta}} = e^{-i\theta}$.
- $e^{i\theta}.e^{i\theta'} = e^{i(\theta+\theta')}$, $\frac{e^{i\theta}}{e^{i\theta'}} = e^{i(\theta-\theta')}$.
- $\left(e^{i\theta}\right)^n = e^{in\theta}$, $\forall n \in \mathbb{Z}$.
- **Moivre formula** : $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$.
- $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$, $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2}$.

Proposition 1. (Square root)

Every complex number has two square roots.

Proof. For $\mathfrak{z} = a + ib$, we search $r = x + iy$ such that $r^2 = \mathfrak{z}$. So, we have the following system:

$$\begin{cases} x^2 - y^2 = a \\ x^2 + y^2 = \sqrt{a^2 + b^2} \\ 2xy = b \end{cases} \Leftrightarrow \begin{cases} x = \pm \dfrac{1}{\sqrt{2}} \sqrt{\sqrt{a^2 + b^2} + a} \\ y = \pm \dfrac{1}{\sqrt{2}} \sqrt{\sqrt{a^2 + b^2} - a} \\ \text{signe}(xy) = \text{signe}(b) \end{cases}$$

We study the following cases:

- ✓ If $b > 0$, then x, y have a same sign. So :

$$r = \pm \frac{1}{\sqrt{2}} \left(\sqrt{\sqrt{a^2 + b^2} + a} + i \sqrt{\sqrt{a^2 + b^2} - a} \right)$$

- ✓ If $b < 0$, then x, y have a different sign. So :

$$r = \pm \frac{1}{\sqrt{2}} \left(\sqrt{\sqrt{a^2 + b^2} + a} - i \sqrt{\sqrt{a^2 + b^2} - a} \right)$$

- ✓ If $b = 0$ and $a \geq 0$, therefore : $r = \pm\sqrt{a}$.
- ✓ If $b = 0$ and $a \leq 0$, therefore : $r = \pm i\sqrt{a}$.

Examples.

For $\mathfrak{z} = i$ (ie . $a = 0$, $b = 1$), we pose $r = x + iy$ such that $r^2 = \mathfrak{z}$, we have:

$$\begin{cases} x^2 - y^2 = 0 \\ x^2 + y^2 = 1 \\ 2xy = 1 \end{cases} \Leftrightarrow \begin{cases} 2x^2 = 1 \\ 2y^2 = 1 \\ 2xy = 1 \end{cases} \Leftrightarrow \begin{cases} x = \pm \dfrac{1}{\sqrt{2}} \\ y = \pm \dfrac{1}{\sqrt{2}} \\ xy \geq 0 \end{cases}$$

We observe that $b > 0$, hence x, y have a same sign. So the square roots of i are the following:

$$r_1 = \frac{1}{\sqrt{2}}(1 + i) \quad , \quad r_2 = -\frac{1}{\sqrt{2}}(1 + i)$$

Proposition 2. (Quadratic equation)

Let the quadratic equation : $a\mathfrak{z}^2 + b\mathfrak{z} + c = 0$, with $a, b, c \in \mathbb{C}$.

This equation admits two complex solutions $\mathfrak{z}_1$, $\mathfrak{z}_2$ given by :

$$\mathfrak{z}_1 = \frac{-b + \delta}{2a} \quad , \quad \mathfrak{z}_2 = \frac{-b - \delta}{2a}$$

with δ is one of the square roots of the discriminant $\Delta = b^2 - 4ac$.

Remark. If $\Delta = 0$, then $\mathfrak{z}_1 = \mathfrak{z}_2 = \frac{-b}{2a}$ (the solution is called **double**).

Example. Let the equation : $(1+i)\mathfrak{z}^2 - (5+i)\mathfrak{z} + 6 + 4i = 0$.

The discriminant is $\Delta = 16 - 30i = \delta^2$, hence $\delta = 5 - 3i$.

So the solutions of the equation are:

$$\mathfrak{z}_1 = \frac{(5+i)+(5-3i)}{2(1+i)} = 2 - 3i \quad , \quad \mathfrak{z}_2 = \frac{(5+i)-(5-3i)}{2(1+i)} = 1 + i$$

Corollary. (Equation with real coefficients)

If the coefficients a, b, c of the equation $a\mathfrak{z}^2 + b\mathfrak{z} + c = 0$ are real. Then, $\Delta \in \mathbb{R}$ and we have three cases :

- $\Delta > 0$, the equation admits two real solutions :

$$\mathfrak{z}_1 = \frac{-b+\sqrt{\Delta}}{2a} \quad , \quad \mathfrak{z}_2 = \frac{-b-\sqrt{\Delta}}{2a}$$

- $\Delta = 0$, the equation admits a real double solution $\mathfrak{z}_0 = -\frac{b}{2a}$.
- $\Delta < 0$, the equation admits two complex solutions $\mathfrak{z}_1$, $\mathfrak{z}_2$ (not real) :

$$\mathfrak{z}_1 = \frac{-b+i\sqrt{-\Delta}}{2a} \quad , \quad \mathfrak{z}_2 = \frac{-b-i\sqrt{-\Delta}}{2a}$$

Example. Let the equation : $\mathfrak{z}^2 + \mathfrak{z} + 1 = 0$.

The discriminant $\Delta = -3 = 3i^2$, hence $\sqrt{\Delta} = i\sqrt{-\Delta} = i\sqrt{3}$.

So, the solutions of the equation are:

$$\mathfrak{z}_1 = \frac{-1+i\sqrt{3}}{2} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i \quad , \quad \mathfrak{z}_2 = \frac{-1-i\sqrt{3}}{2} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

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References.

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2) G. Laffaille and C. Pauly . *Analysis* Course1 . Cote d'Azur University, Canada, 2006.