Chapter 2 :

Ordinary Differential Equations of Order 1

Motivation.

In modern times, differential equations have played a central role in modeling and understanding dynamic phenomena across a wide range of scientific and engineering disciplines. Developments in these fields have been made possible by the use and solution of various types of differential equations. In image processing, the introduction of differential equations allows images to be seen as continuous signals.

2.1. Definitions and Properties.

Definitions 1.

- For $n \in \mathbb{N}^*$ and $\in \mathbb{R}$, we call **" Ordinary differential equation "** of order n every equation contains an unknown function $y(x)$ and its derivatives:

$$F(x, y, y'(x), y"(x), \dots, y^n(x)) = 0$$

- The degree of the highest derivative is called the **" Order "** of the differential equation.
- The function $y(x)$ is called **"Solution"** of the differential equation on an open interval I.
- Solve a differential equation, that is to say we look for all the solutions $y(x)$.
- We note $y, y', \dots, y^n$ instead of $y'(x), y"(x), \dots, y^n(x)$.
- We can add initial conditions to the differential equation like : $y(x_0) = y_0$.
- If the solution $y(x)$ depends on a parameter we say that it is a " **General Solution** ", if $y(x)$ it does not correspond to a parameter we say that it is a " **Particular Solution** ".

Example 1.

- We can easily verify that the function $y(x) = kx^2 + x^4$ is a solution of the differential equation of order 1 : $xy' - 2y - x^3 = 0$.
- We can also verify that the function $y(x) = kx^3 + x^2$ is a solution of the differential equation of order 2 : $x^2y" - 6y + 4x^2 = 0$.

Definition s 2.

- A differential equation **of order 1** is in the form: $F(x, y, y'(x)) = 0$ where $y'(x) = F(x, y)$.
- The solution of an ODE of order 1 on an open interval I is a function $y : I \longrightarrow \mathbb{R}$ defined and differentiable on I which satisfies the equation.

Remark. Solving the differential equation $y' = f(x)$ returns to finding a primitive of the function f. This is why we sometimes say "integrate the differential equation".

Example 2.

- For the differential equation $y' - 3x^2 = 0$, we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 \Leftrightarrow \mathrm{d}y = 3x^2\mathrm{d}x \Leftrightarrow \int \mathrm{d}y = \int 3x^2\mathrm{d}x \Leftrightarrow y = x^3 + c \quad , \quad c \in \mathbb{R}$$

- For the differential equation $y' + \sin x = 0$, we have

$$\mathrm{d}y = -\sin x \,\mathrm{d}x \Leftrightarrow \int \mathrm{d}y = \int -\sin x \,\mathrm{d}x \Leftrightarrow y = \cos x + c \quad , \quad c \in \mathbb{R}$$

The particular case $y = 0$ is a **singular** solution.

Definitions 3. (Differential equations with separate or separable variables)

Differential equations with separate variables are given by the form:

$$\psi(y)\, y' = \varphi(x)$$

With $\psi$is a function depends on $y$and $\varphi$is a function depends on x.

To solve this equation we write:

$$\psi(y)\,\frac{\mathrm{d}y}{\mathrm{d}x} = \varphi(x) \Leftrightarrow \int \psi(y)\mathrm{d}y = \int \varphi(x)\mathrm{d}x$$

It remains to look for the primitives of $\psi(y)$and $\varphi(x)$.

Example 3. Consider the ordinary differential equation of order 1:

$$xy' - 2x^2y = 0 \;\ldots\ldots\ldots\ldots\ldots (E3)$$

It is an ODE with separate variables, to solve this equation we have

$$(E3) \Leftrightarrow \frac{1}{y}\mathrm{d}y = 2x\mathrm{d}x \Leftrightarrow \int \frac{1}{y}\mathrm{d}y = \int 2x\mathrm{d}x \Leftrightarrow \ln|y| = x^2 + \mathrm{c} \Leftrightarrow y = \pm k e^{x^2} \quad , \quad k = e^c \in \mathbb{R}_+^*$$

This solution is set to $I = \mathbb{R}^*$. The particular case is a $y = 0$**singular** solution .

Definitions 4. (Homogeneous differential equations)

We say that the first order differential equation $y'(x) = F(x,y)$is **homogeneous** of degree α if the function $F(x,y)$is homogeneous of degree α. That is to say if it verifies the relation:

$$F(tx,ty) = t^\alpha F(x,y)$$

In practice, we study homogeneous degree equations $\alpha = 0$given by the form:

$$y' = f\left(\frac{y}{x}\right) \ldots\ldots\ldots\ldots\ldots (\boldsymbol{H_0})$$

With f is a function depends on $\frac{y}{x}$.

To solve this equation we put $y(x) = x\,z(x)$, we will have$y' = z + xz'$ from where:

$$(\boldsymbol{H_0}) \Leftrightarrow z + xz' = f(z) \Leftrightarrow \frac{1}{f(z) - z}\mathrm{d}z = \frac{1}{x}\mathrm{d}x$$

The differential equation has a separate variable.

- If $f(z)-z=0$, for a value $z_0$then the solution $y_0(x)=x\,z_0$ is called **singular solution** .

Example 4. Consider the ordinary differential equation of order 1:

$$x^2y'-2xy+y^2=0 \;\ldots\ldots\ldots\ldots\ldots (E4)$$

We have

$$(E4)\Leftrightarrow y'=\frac{2xy-y^2}{x^2}$$

The function $F(x,y)=\frac{2xy-y^2}{x^2}$is homogeneous of degree 0, because:

$$F(tx,ty)=\frac{2t^2xy-t^2y^2}{t^2x^2}=F(x,y)$$

To solve the equation we put $y(x)=x\,z(x)$, we will have$y'=z+xz'$ from where:

$$(E4)\Leftrightarrow z+xz'=2z-z^2\Leftrightarrow \frac{1}{2z-z^2}\mathrm{dz}=\frac{1}{x}\mathrm{dx}\Leftrightarrow \int\frac{1}{2z-z^2}\mathrm{dz}=\int\frac{1}{x}\mathrm{dx}$$

For the first integral, we have $z\notin\{0,2\}$:

$$\int\frac{1}{2z-z^2}\mathrm{dz}=\frac{1}{2}\int\left(\frac{1}{z}+\frac{1}{2-z}\right)\mathrm{dz}=\frac{1}{2}\ln|z(2-z)|$$

From where

$$(E5)\Leftrightarrow\frac{1}{2}\ln|z(2-z)|=\ln|x|+c\Leftrightarrow|z(2-z)|=kx^2\quad,\quad k=e^c\in\mathbb{R}_+^*$$

For $z\in\,]0\,,2[$the solution check the equation:

$$z^2-2z+kx^2=0$$

By replacing with $y=xz$, we find that the general solution of $(E4)$is the set of equation curves:

$$y^2-2xy+kx^4=0$$

For $z=0$ the value $y=0$ is a singular solution. For $z=2$ the function $y=2x$ is not a singular solution of the differential equation.

2.2. Linear Differential Equations.

Definitions 5.

We say that the first order differential equation is **linear** if it has the form:

$$y'+a(x)y+b(x)=0$$

With $a(x),b(x)$are defined and continuous functions on an open interval $I\subset\mathbb{R}$.

- The equation $y'+a(x)y=0$ is called equation without second member or homogeneous equation. It is an equation with separable variables. In fact, we have:

$$y'+a(x)y=0\Leftrightarrow\frac{1}{y}\mathrm{dy}=-a(x)\mathrm{dx}\Leftrightarrow y_h=y=ke^{A(x)}$$

With $A(x)=\int-a(x)\,\mathrm{dx}$is the primitive of " $-a(x)$".

To find the solution of the linear differential equation with second member we apply the " **variation of the constant " method** :

We write $y=k(x)e^{A(x)}$ from where $y'=k'(x)e^{A(x)}-k(x)a(x)e^{A(x)}$ (since $A'(x)=a(x)$).

Then the equation becomes : $k'(x)=-b(x)e^{-A(x)}$. So,

$$k(x) = -\int b(x)e^{-A(x)}dx$$

We obtain a particular solution $y_p = -e^{A(x)} \int b(x)e^{-A(x)}dx$.

The general solution is the sum of the two solutions: $y_g = y_p + y_h$

Example 5. Consider the linear differential equation of order 1:

$$y' + y = e^x + 1 \quad \dots\dots\dots\dots\dots (E5)$$

The solutions of the homogeneous equation $y' + y = 0$ are : $y_h = ke^{-x}$.

We apply the method of variation of the constant, we set : $y = k(x)e^{-x}$.

We have : $k'(x) = e^{2x} + e^x$. From where $k(x) = \frac{1}{2}e^{2x} + e^x$.

The particular solution is : $y_p = \frac{1}{2}e^x + 1$.

The general solution is the sum of the two solutions:

$$y_g = \frac{1}{2}e^x + 1 + ke^{-x} \quad , \quad k \in \mathbb{R}$$

Definitions 6. (Integral curves)

An **integral curve** of a differential equation is the graph of a solution to that equation. For a first order linear differential equation, we have for each point (x_0, y_0) passing a single integral curve.

Example 6. Consider the linear differential equation of order 1:

$$y' + y = x \quad \dots\dots\dots\dots\dots (E6)$$

The solutions to this equation are : $y(x) = x - 1 + ke^{-x} \quad , \quad k \in \mathbb{R}$.

For the point $(0,1)$, there is only one solution which checks $y(0) = 1$, it is the function:

$$y(x) = x - 1 + 2e^{-x}$$

The graph of this solution is the integral curve which passes through $(0,1)$.