Chapter 4_part 2

Functions with a real variable

(Comparison - Continuity)

4.3. Comparison of functions

Let f, gtwo functions be defined in the neighborhood of a point a.

Definition 11. **(Equivalence)**

We say that f is equivalent to g in the neighborhood of a, we note $\boldsymbol{f \sim g}$, if:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = 1 .$$

Important examples. Equivalences in the neighborhood of $a = 0$:

$$e^x \sim x + 1 \quad , \quad \ln(x+1) \sim x \quad , \quad \sin x \sim x \quad , \quad \cos x \sim 1 + \frac{x^2}{2} \quad , \quad \tan x \sim x$$

Theorem 3.

1) If $f \sim g$ in the neighborhood of a, then: $\lim_{x \to a} f(x)$ existe $\Leftrightarrow \lim_{x \to a} g(x)$ existe.

In this case the limits are equal.

2) If $f_1 \sim g_1$and $f_2 \sim g_2$in the neighborhood of a, then: $f_1 \times f_2 \sim g_2 \times g_1$ and $\frac{f_1}{f_2} \sim \frac{g_2}{g_1}$.

3) If $\lim_{x \to b} \varphi(x) = a$ and $f \sim g$ in the neighborhood of a, then: $fo\varphi \sim go\varphi$ in the neighborhood of b.

Remark. The sum of equivalent functions is not always equivalent.
For example: $x^2 + x \sim -x$and $x \sim x$in the vicinity of $a = 0$, on the other hand $x^2 \nsim 0$.

Definition 11. **(Negligible)**

We say that f is negligible in g the neighborhood of a, we note $\boldsymbol{f = o(g)}$, if:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = 0 .$$

Important examples. In the neighborhood of $a = 0$, we have: $e^x - 1 = o\,(x+1) \quad , \quad \ln x = o\left(\frac{1}{x}\right)$

In the neighborhood of $a = +\infty$, we have: $\quad x = o\,(e^x) \quad , \quad \ln x = o\,(x)$

Exercise: Demonstrate the general case for $\alpha, \beta, \gamma \in \mathbb{R}$

- $|\ln x|^\alpha = o\left(\frac{1}{x^\beta}\right) \quad , \quad$ au voisinage de $a = 0$.
- $(\ln x)^\alpha = o\left(x^\beta\right) \quad , \quad$ au voisinage de $a = +\infty$
- $x^\beta = o\left(e^{\gamma x}\right) \quad , \quad$ au voisinage de $a = +\infty$

Proposition 7 . We have the folowing properties:

1) If$f = o(g)$ so $= o(h)$what : $f = o(h)$.
2) If $f_1 = o(g_1)$and $f_2 = o(g_2)$, then : $f_1 \times f_2 = o(g_1 g_2)$.
3) If $f_1 = o(g)$and $f_2 = o(g)$, then : $f_1 + f_2 = o(g)$.
4) If $= o(g)$then : $\frac{1}{g} = o(\frac{1}{f})$.

Noticed. The sum and division of negligible functions is not always negligible.

For example: $x^2 = o(x)$and $-x^3 = o(-x + x^2)$in the neighborhood of $a = 0$, on the other hand $x^2 - x^3 \neq o(x^2)$.

Definition 12. **(Dominated – Landou notation)**

- We say that f is dominated by $g$in the neighborhood of , we note $\boldsymbol{f = \mathrm{O}(g)}$, if:

$$\exists d, K \in \mathbb{R}_+^*: \ |x - a| < d \Longrightarrow |f(x)| \leq K|g(x)|.$$

- It is said that f is dominated by g in $+\infty$, if:

$$x > N \Longrightarrow |f(x)| \leq C|g(x)|$$

Examples in computer science.

In analyzing an **algorithm** , we can find that the time (counted as he number of steps) necessary in order to of solve a problem issue of size n East given by

$$T(n) = 4\,n^2 - 2\,n + 2.$$

In ignorant the constants (this Who East based because they depend of material particular on which THE program executes) and the terms Who grow the more slowly , we could say

"$T(n)$ grows as n^2" Or "$T(n)$ East of the *order* of n^2»

And We we would write : $T(n) = O(n^2)$.

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is a list of categories of functions that are used in algorithm analyzes . They are listed in order of growth from slowest to fastest.

rating	**complexity**
$O(1)$	constant
$O(\log n)$	logarithmic
$O((\log n)^c)$	poly logarithmic
$O(n)$	linear
$O(n \log n)$	"quasi-linear"
$O(n^2)$	quadratic
$O(n^c)$	polynomial
$O(c^n)$	exponential
$O(n!)$	factorial

4.4. Continuity.

Definition 13. Let be an interval I of $\mathbb{R}$, a point $a \in I$and a function $f: I \longrightarrow \mathbb{R}$.

- We say that f is **continuous** in a if :

$$\lim_{x \to a} f(x) = f(a)$$

In other words :

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in D \ : \ |x-a| < \delta \Longrightarrow \ |f(x) - f(a)| < \varepsilon$$

- We say that f is continuous on I iff f is continuous at every point of I.
- We note by $\mathcal{C}(I, \mathbb{R})$ all the functions defined and continued on I In $\mathbb{R}$.

Example. The function $f$defined on $\mathbb{R}$ by

$$f(x) = \begin{cases} x \sin\left(\dfrac{1}{x}\right) & , \quad x \neq 0 \\ 0 & , \quad x = 0 \end{cases}$$

Is continuous at the point $a = 0$. In fact, we have: $|f(x) - f(a)| = \left|x \sin\left(\frac{1}{x}\right)\right| \leq |x|$.

So you just have to choose $\delta = \varepsilon$.

Remarks.

1) We can replace the limit in the definition with the following: $\lim_{h \to 0} f(a+h) = f(a)$
2) The sum, product and quotient of continuous functions is a continuous function.
3) A function that is not continuous is called " **discontinuous** ".

Example.

- The usual functions are continuous over the definition domain:x^n , $\ln x$, e^x , $\cos x$, $\sin x$, $\tan x$ ....
- The integer part function $E(x)$is not continuous at integer points $a \in \mathbb{Z}$. It is continuous at every point $a \in \mathbb{R} \setminus \mathbb{Z}$.

Proposition 8. If f is continuous in$a \in I$ and if $f(a) \neq 0$, then:$\exists \delta > 0, \forall x \in]a-\delta, a+\delta[\ t.q \ \ f(x) \neq 0$

Proposition 9. Let two intervals I, J of $\mathbb{R}$, a point $a \in I$ and two functions $f: I \longrightarrow J$, $g: J \longrightarrow \mathbb{R}$.

If f is continuous in a and g is continuous in $f(a)$, then gof is continuous in a.

Definition 14. Let be an interval$I$ of $\mathbb{R}$, a point $a \in I$and a function $f: I \longrightarrow \mathbb{R}$.

- We say that$f$ is **continuous to the right** in$a$ if :$\lim_{\substack{> \\ x \to a}} f(x) = f(a)$

In other words :$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in D \ : \ \ 0 < x - a < \delta \Longrightarrow \ |f(x) - f(a)| < \varepsilon$

- We say that$f$ is **continuous to the** left$a$ if :$\lim_{\substack{< \\ x \to a}} f(x) = f(a)$

In other words :$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in D \ : \ \ \delta < x - a < 0 \Longrightarrow \ |f(x) - f(a)| < \varepsilon$

Proposition 10. f is continuous in $a \in I \Leftrightarrow \ f$ continuous to the right and to the left in a.

Example. The function f defined on $\mathbb{R}$ by

$$f(x)=\begin{cases} x-1 & , \quad x\geq 0 \\ x+1 & , \quad x<0 \end{cases}$$

is continuous to the right at $a=0$, but is not continuous to the left so it is not continuous at "0".

In fact, we have:

$$\lim_{\substack{x\to 0\\ >}} f(x)=-1=f(0) \qquad , \qquad \lim_{\substack{x\to 0\\ <}} f(x)=1\neq f(0)$$

Example. The integer part function $E(x)$ is not right continuous at all integer $a\in\mathbb{Z}$ points , but it is not left continuous at these points.

Definition 15. Let the interval I of $\mathbb{R}$, $a\in I$ and $f: I\setminus\{a\}\longrightarrow\mathbb{R}$. If f admits a finite limit ℓ in a, we call **extension by continuity** of f in a the function $\tilde{f}$ defined by:

$$\tilde{f}(x)=\begin{cases} f(x) & \text{si} \quad x\neq a \\ \ell & \text{si} \quad x=a \end{cases}$$

In this case, the function $\tilde{f}$ is continuous in I.

Example. For $f(x)=x\sin\left(\frac{1}{x}\right)$ which is defined on $\mathbb{R}^*$, we have $\lim_{x\to 0} f(x)=1$.
Therefore f is extendable by continuity in "0", and its extension is given by:

$$\tilde{f}(x)=\begin{cases} x\sin\left(\dfrac{1}{x}\right) & , \quad x\neq 0 \\ 0 & , \quad x=0 \end{cases}$$

The function $\tilde{f}$ is continuous in $a=0$ because $\lim_{x\to 0}\tilde{f}(x)=\tilde{f}(0)$.

Proposition 11. Let $f: I\longrightarrow\mathbb{R}$ a function and $a\in I$. So f is continuous in a iff for every sequence $(x_n)_{n\in\mathbb{N}}$ converges to , we have $(f(x_n))_{n\in\mathbb{N}}$ converged to $f(a)$. ie .

$$\lim_{x\to a} f(x)=f(a) \quad \Leftrightarrow \quad \forall (x_n)_{n\in\mathbb{N}}\subset D \text{ tel que } \lim_{n\to+\infty} x_n=a \text{ on a } \lim_{n\to+\infty} f(x_n)=f(a)$$

Noticed. To show that a function is not continuous at , it suffices to find a sequence $(x_n)_{n\in\mathbb{N}}$ which converges to a but $\lim_{n\to+\infty} f(x_n)\neq f(a)$.

Example. Let the function f be defined $\mathbb{R}$ by:

$$f(x)=\begin{cases} \sin\left(\dfrac{1}{x}\right) & , \quad x\neq 0 \\ 0 & , \quad x=0 \end{cases}$$

The function f is not continuous in $a=0$. Indeed, we have for the sequence of general terms $x_n=\frac{2}{(2n+1)\pi}$ which tend towards 0:

$$\lim_{n\to+\infty} f(x_n)=\lim_{n\to+\infty}\sin\left((2n+1)\frac{\pi}{2}\right)=1\neq 0=f(0)$$

Intermediate Value Theorems (IVT):

Theorem 4. Consider f a function defined and continuous on an interval $[a,b]$. Then, For every y between $f(a)$ and $f(b)$, there exists $c \in [a,b]$ such that $f(c) = y$.

Corollary 1. If f is a continuous and strictly monotonic function on $[a,\ b]$, then for all real k numbers between $f(a)$ and $f(b)$, the equation $f(x) = k$ has a unique solution in $[a,\ b]$.

Corollary 2. If f is continuous on $[a,\ b]$ and $f(a).f(b) < 0$, then it exists $c \in [a,b]$ such that $f(c) = 0$. If moreover f is strictly monotonic on $[a,\ b]$ the number "c" is unique.

Corollary 3. If f is a continuous function on an interval I , then $f(I)$ is an interval.

Theorem 5. If f is a continuous function on an interval $\left[a,\ b\right]$ then:

$$f\left([a,\ b]\right) = \left[m,\ M\right] \quad , \quad \text{such that} \quad m = \min f\ ,\ M = \max f\ .$$

Example. The function $f(x) = x^3 - 2x + 2$ is defined and continues on $\mathbb{R}$, therefore on the interval $[-2,1]$. On the other hand, we have: $f(-2).f(1) = -2 < 0$. Then the equation $f(x) = 0$ admits at least one solution on $[-2,1]$. To calculate an approximate value of this solution we apply the dichotomy method, we find that $c = -1.76929$.

Corollary 4. If f is a continuous function on an interval $I = \left[a,\ b\right]$ we have:

- If f is increasing, then $\left([a,\ b]\right) = [f(a),\ f(b)]$.
- If f is decreasing, then $\left([a,\ b]\right) = [f(b),\ f(a)]$.

Approximation of continuous functions.

The Stone-Weierstrass theorem makes it possible to uniformly approximate continuous functions over a segment by simpler functions (polynomials, staircase functions, piecewise affine functions).

Theorem 7. (Stone-Weierstrass)

Let be $f\colon \left[a\ ,\ b\right] \to \mathbb{R}$ a continuous function and let $\varepsilon > 0$. Then there exists a polynomial P such that:

$$\forall x \in [a,b] : \quad |f(x) - P(x)| \le \varepsilon$$

In other words, any continuous function is uniform limit of polynomials.

Theorem 8. Let be $f\colon \left[a\ ,\ b\right] \to \mathbb{R}$ a continuous function and $\varepsilon > 0$. Then there exists a staircase function $h\colon \left[a\ ,\ b\right] \to \mathbb{R}$ such as: $\forall x \in [a,b] : \quad |f(x) - h(x)| \le \varepsilon$

Theorem 9. Let be $f\colon \left[a\ ,\ b\right] \to \mathbb{R}$ a continuous function and $\varepsilon > 0$. Then there exists an affine function $g\colon \left[a\ ,\ b\right] \to \mathbb{R}$ such as: $\forall x \in [a,b] : \quad |f(x) - g(x)| \le \varepsilon$

Theorem of reciprocal functions:

Theorem 6. Let f a function defined on an interval I.Si f be **continuous** and **strictly monotonic** on I, then:

1) The function f is bijective from I in the interval $J = f(I)$. So it admits an inverse function defined on $J = f(I)$.

2) reciprocal f^{-1} function is continuous and strictly monotonic on J and it has the same sense of monotonicity as .

Noticed. In practice, if f is not monotonic on I we divide the interval I in to subintervals on which the function f is strictly monotonic.

Example . The restriction on $\mathbb{R}^+$ the function $x \to x^n$ is continuous and strictly increasing on $\mathbb{R}^+$, the image of zero is zero and the limit $+\infty$is $+\infty$.
So the inverse function is defined by $\mathbb{R}^+ \to \mathbb{R}^+$: $y \to \sqrt[n]{y}$

$$\begin{cases} y \in \mathbb{R}^+ \\ x = \sqrt[n]{y} \end{cases} \Leftrightarrow \begin{cases} x \in \mathbb{R}^+ \\ y = x^n \end{cases}$$

Example. The function $f \to x^2 + 3$ is a bijection of $]-\infty, 0]$on $[3, +\infty[$ and has a reciprocal application that we seek to determine by solving, for$y$ in $[3, +\infty[$, the equation $x^2 + 3 = y$, or again $x^2 = y - 3$.
Since $y \geq 3$, this equation has two solutions, only one of which belongs to the interval $]-\infty, 0]$ it is $x = -\sqrt{y-3}$. So the reciprocal of f is f^{-1} defined by $f^{-1}(y) = -\sqrt{y-3}$.

Table of usual reciprocal functions.

Function$f(x)$	Departure and arrival	Function reciprocal	Departure and arrival	Notes
$f(x) = x^n$	$[0, +\infty[\to [0, +\infty[$	$f^{-1}(x) = \sqrt[n]{x}$	$[0, +\infty[\to [0, +\infty[$	$n \in \mathbb{N}^*$
$f(x) = e^x$	$\mathbb{R} \to [0, +\infty[$	$f^{-1}(x) = \ln x$	$]0, +\infty[\to \mathbb{R}$	
$f(x) = a^x$	$\mathbb{R} \to [0, +\infty[$	$f^{-1}(x) = \log x$	$]0, +\infty[\to \mathbb{R}$	$a \in \mathbb{R}^+$
$f(x) = x^\alpha$	$]0, +\infty[\to]0, +\infty[$	$f^{-1}(x) = x^{\frac{1}{\alpha}}$	$]0, +\infty[\to]0, +\infty[$	$\alpha \in \mathbb{R}^*$
$f(x) = \sin x$	$\left[-\frac{\pi}{2}, +\frac{\pi}{2}\right] \to [-1, 1]$	$f^{-1}(x) = \arcsin(x)$	$[-1, 1] \to \left[-\frac{\pi}{2}, +\frac{\pi}{2}\right]$	
$f(x) = \cos x$	$[0, \pi] \to [-1, 1]$	$f^{-1}(x) = \arccos(x)$	$[-1, 1] \to [0, \pi]$	
$f(x) = \tan x$	$\left[-\frac{\pi}{2}, +\frac{\pi}{2}\right] \to \mathbb{R}$	$f^{-1}(x) = \mathrm{arctg}(x)$	$\mathbb{R} \to \left[-\frac{\pi}{2}, +\frac{\pi}{2}\right]$	