TD N°1

$(L.E)$ = **Leave to the students.**

Exercise 1:

1) To $p, q \in \mathbb{Q}$show that $p + q \in \mathbb{Q}$. (ie the sum of two rationals is a rational)
2) Is the sum of two irrational numbers an irrational number? Give a counterexample.
3) Show that $\sqrt{2} \notin \mathbb{Q}$, then show that $2 - 3\sqrt{2} \notin \mathbb{Q}$and $1 - \frac{1}{\sqrt{2}} \notin \mathbb{Q}$.$(L.E)$
4) Show by induction the following inequality, for everything $n \in \mathbb{N}^*$:

$$\sum_{k=1}^{n} \frac{1}{\sqrt{k}} < \sqrt{n} + \sqrt{n+1} - 1$$

Exercise 2:

1) Let the rational number be:$x = 0{,}234234234 \ldots$
   Compare the numbers $1000x$and x, then write $x$as frome of a fraction.
2) Write the following numbers as a fraction:$(L.E)$

$$a = 0{,}1212 \quad ; \quad b = 0{,}1\overleftrightarrow{212}\ldots \quad ; \quad c = 78{,}33\overleftrightarrow{456}\ldots$$

3) Show that decimal writing of a rational number is equivalent to fractional writing.

Exercise 3 : (Catch-up 2022/2023)

For $x \in \mathbb{R}$, we note by $\bar{E}(x)$(where $\lceil x \rceil$) the upper integer part of $x$defined by:

$\bar{E}(x) = \min\{k \in \mathbb{Z},\ \ k \geq x\}$, in this case we have: $\bar{E}(x) - 1 < x \leq \bar{E}(x)$.

1) Calculate $E(\pi)$, $E(-\frac{1}{2})$and $\bar{E}(\pi)$, $\bar{E}(-\frac{1}{2})$.
2) For any even natural $n$number, calculate: $E\left(\frac{n}{2}\right)$and $\bar{E}\left(\frac{n}{2}\right)$.
3) For any natural number m, calculate: $E\left(m + \frac{1}{2}\right)$and $\bar{E}\left(m + \frac{1}{2}\right)$.
4) By treating the even and odd cases, deduce that:

$$E\left(\frac{n}{2}\right) + \bar{E}\left(\frac{n}{2}\right) = n \quad , \quad \forall n \in \mathbb{N}$$

Exercise 4: $(L.E)$

Let $n \in \mathbb{N}^*$, $\alpha \in \mathbb{Z}$, $\beta \in \mathbb{R} \setminus \mathbb{Z}$, $x \in \mathbb{R}$. Show the following formulas:

$$E(\alpha) + E(-\alpha) = 0 \quad ; \quad E(\beta) + E(-\beta) = -1 \quad ; \quad E\left(\frac{x}{2}\right) + E\left(\frac{x+1}{2}\right) = E(x)$$

Exercise 5: $(L.E)$

Be $x, y \in \mathbb{R}$. Show the following inequalities:

1) $\big||x| - |y|\big| \leq |x + y| \leq |x| + |y| \leq |x + y| + |x - y|$.
2) $1 + |xy - 1| \leq (1 + |x - 1|)(1 + |y - 1|)$.

Exercise 6 : In each of the following cases, specify whether part A of $\mathbb{R}$admits an upper bound, a lower bound, a larger, a smaller element and determine if so:

1) $A = [0\,,1[$
2) $A = \left\{\frac{1}{2n} \quad , \; n \in \mathbb{N}^*\right\}$
3) $A = \left\{2 + \frac{(-1)^n}{n} \quad , \; n \in \mathbb{N}^*\right\}$ $(\boldsymbol{L.E})$
4) $A = \bigcup_{n\in\mathbb{N}^*} \left[0\,,1 - \frac{1}{n^2}\right]$
5) $A = \left\{\sin\frac{n\pi}{3} \quad , \; n \in \mathbb{N}\right\}$ $(\boldsymbol{L.E})$

Exercise 7 : For the following sets:

$$\left\{\frac{(-1)^n}{n} \quad , \qquad n \in \mathbb{N}^*\right\} \quad ; \quad \left\{\frac{1}{x} \;, 1 \le x \le 2\right\} \; (\boldsymbol{L.E}) \quad ; \qquad [0\,,1[\,\cup\,]2\,,3]$$

1) Is it increased? reduced?
2) Determine the set of upper bounds and the set of lower bounds.
3) Is there: the max, the sup, the min, the inf ?

Exercise 8 : **(Exam 2022/2023)**

Let the set be defined by:

$$E = \left\{(-1)^n + \frac{1}{n^2} \quad , \qquad n \in \mathbb{N}^*\right\}$$

1) Show that $E$is the union of two sets $E_1\,, E_2$.
2) Show that E_1 et $E_2$the upper bound, the lower bound, the maximum and the minimum of each set are bounded, and determine (if they exist).
3) Deduce that $E$is bounded, and determine (if they exist) the upper bound, the lower bound, the maximum and the minimum.

Indication: $\sup(A \cup B) = \max(\sup A\,, \sup B)$,$\inf(A \cup B) = \min(\inf A\,, \inf B)$

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