Chapter 1. Mathematical logic.

1.1 Proposition

Definition 1. We call a proposition, every declarative sentence that has a meaning and which may be true or false. If it is true, it takes the value 1, and if it is false, it takes the value 0. We denote the propositions with the letters P, Q, R,...

Examples 1.

- Jijel is a coastal city, is a true proposition.

- 3 is an even number, is a false proposition.

- the sky is clear, this expression is not a proposition, because we cannot judge it.

Negation of a proposition: The negation of a proposition P, is a proposition denoted by $\bar{P}$, ˜P, or ¬P, which we read: the negation of P or not P. If P is true, then its negation is false and vice versa.

For example, the negation of the false proposition: 25<10 is the correct proposition: 25≥10.

Truth Table: The truth table for the proposition P and its negation $\bar{P}$ is given by:

P	$\bar{P}$
1	0
0	1

1.2 Logical connectors.

- **Conjunction.** The conjunction of two propositions P and Q (or more) is the proposition denoted **P∧Q**, which is false if at least one of them is false.
  For example the proposition 3>2 and 5 is a positive number, a false proposition.
- **Disjunction.** The Disjunction of two propositions P and Q (or more) is the proposition denoed by **P∨Q**, which is true if at least one of them is true.
  For example the proposition 1>2 or (-3) is a positive number, is a true proposition.
- **Implication.** The logical implication of the two propositions P and Q, denoted by **P⇒Q**, is the proposition $\bar{P}$**∨Q**.
  The implication **Q⇒P** is called the inverse implication of the implication P⇒Q.
  For example the implication: 1 >2 ⇒2 is an odd number, is a true proposition.
- **Equivalence**. The equivalence of the two propositions P and Q, denoted by **P⇔Q** is the proposition **(P⇒Q)∧ (Q⇒P)**.
  For example $x^2 \geq 0 \Leftrightarrow x^2+1 \geq 1$, is a true proposition.

Remark. If two propositions Q and P are equivalent, then they are both true or both false at the same time.

The truth table for all these propositions is given by:

P	Q	$\bar{P}$	$\bar{Q}$	P∧Q	P∨Q	P⇒Q	Q⇒P	P⇔Q
1	1	0	0	1	1	1	1	1
1	0	0	1	0	1	0	1	0
0	1	1	0	0	1	1	0	0
0	0	1	1	0	0	1	1	1

Theorem 1. Let P, Q and R three propositions, we have

1- $\overline{\overline{P}} \Leftrightarrow P$ ‧$P\wedge P \Leftrightarrow P$ and $P\vee P \Leftrightarrow P$.
2- $P \wedge Q \Leftrightarrow Q \wedge P$ and $P \vee Q \Leftrightarrow Q \vee P$.
3- $P \wedge (Q \wedge R) \Leftrightarrow (P \wedge Q) \wedge R$ and $P \vee (Q \vee R) \Leftrightarrow (P \vee Q) \vee R$.
4- $P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$ and $P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$.
5- $\overline{P \wedge Q}$) $\Leftrightarrow \overline{P} \vee \overline{Q}$ and $(\overline{P \vee Q}) \Leftrightarrow \overline{P} \wedge \overline{Q}$.

Proof. We suffice by proving propositions 4) and 5, and we use the truth table for that

For 4) we have the true table below

P	Q	R	$P\wedge Q$	$P\wedge R$	$Q\vee R$	$P\wedge(Q\vee R)$	$(P\wedge Q)\vee(P\wedge R)$
1	1	1	1	1	1	1	1
1	1	0	1	0	1	1	1
1	0	1	0	1	1	1	1
1	0	0	0	0	0	0	0
0	1	1	0	0	1	0	0
0	1	0	0	0	1	0	0
0	0	1	0	0	1	0	0
0	0	0	0	0	0	0	0

Which confirms that $P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$.

In the same way, we prove $P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$.

For 5) we have the true table below

P	Q	$\neg P$	$\neg Q$	$P\wedge Q$	$\neg(P\wedge Q)$	$\neg P\vee\neg Q$	$P\vee Q$	$\neg(P\vee Q)$	$\neg P\wedge\neg Q$
1	1	0	0	1	0	0	1	0	0
1	0	0	1	0	1	1	1	0	0
0	1	1	0	0	1	1	1	0	0
0	0	1	1	0	1	1	0	1	1

Which confirms that $(\overline{P \wedge Q}) \Leftrightarrow \overline{P} \vee \overline{Q}$ and $(\overline{P \vee Q}) \Leftrightarrow \overline{P} \wedge \overline{Q}$.

The opposite contrast implication. The opposite contrast of an implication $P \Rightarrow Q$ is the implication $\boldsymbol{\overline{Q} \Rightarrow \overline{P}}$.

We have $(\boldsymbol{\overline{Q} \Rightarrow \overline{P}}) \Leftrightarrow (P \Rightarrow Q)$. Indeed. $(\overline{Q} \Rightarrow \overline{P}) \Leftrightarrow (\overline{\overline{Q}} \vee \overline{P}) \Leftrightarrow (Q \vee \overline{P}) \Leftrightarrow (\overline{P} \vee Q) \Leftrightarrow (P \Rightarrow Q)$.

Remark that

- The propositions $\overline{P \wedge Q}$ and $\overline{P \vee Q}$ called respectively **NOTAND** and **NOTOR** and also denoted respectively by $P \uparrow Q$ and $P \uparrow Q$.
- The proposition $P \oplus Q$ is called **XOR** is defined by $(P \oplus Q) \Leftrightarrow (\overline{P \Leftrightarrow Q})$
- The connectors **NOTAND**, **NOTOR** and **XOR** ($\uparrow$, $\uparrow$ and $\oplus$) are used in electronics and computing.

1.3 **Predicate and quantifiers**

- A **Predicate** is a mathematical statement whose validity depends on one or more variables belonging to a set E.

For example: 1) $n \in N$, n even (for n=2 is true and for n=5 is false).

2) $x \in R$, $y \in R$ $x+y=10$. (for x=2 et y=8 is true and for x=1 and y=3 is false).

- **Universal quantifier** is denoted by $\forall$ and the expression **$\forall x\in E; P(x)$**, which can be read "**for all x in E, P(x)**". is a proposition (because we can judge it to be true or false).

For example, 1) $\forall x\in R; x^2+1 > 0$. Is a true proposition.

2) $\forall x, y\in]0\ 1[: xy\in]0\ 1[$, is a true proposition

3) $\forall x, y\in]0\ 1[: x+y\in]0\ 1[$ Is a false proposition

- **The existential quantifier** is denoted by $\exists$ and the expression **$\exists x\in E; P(x)$**, which can be read "**There exists an element x in E such that P(x)**". is a proposition (because we can judge it to be true or false).

2) $\exists n\in\mathbb{N} : n$ even, is a true proposition

3) $\forall x\in\mathbb{R}, \exists y\in\mathbb{R} : x+y=10$, is a true proposition.

4) $\exists x\in\mathbb{R}, \forall y\in\mathbb{R} : x+y=10$, s a false proposition.

5) $\exists x, \exists y\in]0\ 1[: x+y\in]0\ 1[$ is a true proposition.

Statement	Whene true	Whene false
$\forall x\in E; P(x)$,	P(x) is true for all x,	There is an x for which P(x) is false
$\exists x\in E; P(x)$	There is an x for which P(x) is true	P(x) is false for all x,

Remark.

- The negation of : $\forall x\in E, P(x)$: is the proposition $\exists x\in E : \overline{P(x)}$.
- The negation of : $\exists x\in E, P(x)$: is the proposition $\forall x\in E : \overline{P(x)}$.
- The negation of : $\forall x\in E, \exists y\in E, P(x,y)$: is the proposition $\exists\ x\in E, \forall y\in E, \overline{P(x,y)}$.
- The negation of : $\exists x\in E, \forall y\in E, P(x,y)$: is the proposition $\forall x\in E, \exists y\in E, \overline{P(x,y)}$.

For example.

The sequence (U_n) converges towards $l \Leftrightarrow \forall\varepsilon>0, \exists n_0\in\mathbb{N}: \forall n\in\mathbb{N}: n \geq n_0 \Rightarrow | U_n - l| < \varepsilon$.

Its negation is

The sequence (U_n) is not convergent to $l \Leftrightarrow \exists\varepsilon>0, \forall n_0\in\mathbb{N}: \exists n\in\mathbb{N}: n \geq n_0$ and $| U_n - l| \geq \varepsilon$

1.4 **Reasoning methods**

There are six reasoning methods:

1.4.1 **Direct reasoning:** To prove that Q is true, we assume that P is true and prove the validity of the entailment $P \Rightarrow Q$.

For example. We have $(x+1/2)^2 \geq 0$ is true and $(x+1/2)^2 \geq 0 \Rightarrow (x+1/2)^2+3/4 \geq 3/4$ is a true statement, and thus: $x^2+x+1=(x+ 1/2)^2+3/4 \geq 3/4$ is true.

1.4.2 Backward reasoning: If we want to prove the validity of a proposition, we assume that it is false and try to find a contradiction or a false statement.

Example: If "a" is a natural number such that ($a^3 + a^2 + a$) is an odd number, then show that "a" is an odd number. We assume that "a" is even, so "a^2" is even and "a^3" is even, and from it ($a^3 + a^2 + a$) is even, and this contradicts the assumption.

1.4.3 Reasoning by separation of cases: If we have: $P \Rightarrow R$ and $Q \Rightarrow R$ are true, then we can prove that $[P \vee Q] \Rightarrow R$ is true.

Example. Prove that for all $x > 0$: $x+1/\sqrt{x}>1$.

Since $x>0$ we have two cases:

The first case: $x>1$: then $1/\sqrt{x} > 0$ and therefore $x+1/\sqrt{x}>1$.

The second case: $0<x\leq 1$: then $1/\sqrt{x}\geq 1$ and therefore $x+1/\sqrt{x}>1$.

1.4.4 Reasoning by opposite contrast: We know that the implication $P\Rightarrow Q$ is equivalent to the opposite of its opposite $\overline{Q}\Rightarrow \overline{P}$, so to prove the validity of $P\Rightarrow Q$, we are satisfied with proving the validity of $\overline{Q}\Rightarrow \overline{P}$.

Example. Prove the next proposition $\forall x\in\mathbb{R}$: $x^2 > 4 \Rightarrow x>2 \vee x<-2$, it suffices to prove the implication contrast opposite $x \leq 2 \wedge x > -2 \Rightarrow x^2 \leq 4$?

$(x \leq 2 \wedge x \geq -2) \Leftrightarrow (-2 \leq x \leq 2) \Leftrightarrow |x| \leq 2 \Rightarrow |x|^2 \leq 4$, so $(x \leq 2 \wedge x \geq -2) \Rightarrow |x|^2 \leq 4$.

1.4.5 Reasoning by counter-example

To prove that the proposition $\forall x\in E, P(x)$ is false, it is enough to show that the proposition $\exists x\in E, \overline{P(x)}$ is true.

Example. Prove that every positive integer number is the sum of three squares. For all positive integer x, $\exists$ positive integers a, b, c such that: $x=a^2 + b^2 + c^2$.

We prove that this statement is false by giving a counter-example. For x=7, the only case for a, b, c are a=0, b=1 and c=2 and $a^2 + b^2 + c^2=5 \neq 7$.

1.4.6 Reasoning by regression: To prove the validity of the proposition $[\forall\, n \geq n_0 ;P(n)]$ where n and n_0 are natural numbers, we follow the following:

- Prove that $P(n_0)$ is true.

- We assume that for $n \geq n_0$: P(n) is true and we prove that P(n +1) is true.

Example. Prove that : $\forall n\in\mathbb{N}$: $1+3+5+.....+(2n+1)=(n+1)^2$.

- The property is true for n=0, because $1=(0+1)^2$.
- We assume that for $n \geq 0$: P(n) is true, i.e. $1+3+5+.....+(2n+1)=(n+1)^2$, then we prove that : $1+3+5+.....+(2n+3)\overset{?}{=}(n+2)^2$?

  We have

  $$1+3+5+.....+(2n+3)=[1+3+5+....+(2n+1)]+(2n+3)$$
  $$=(n+1)^2+(2n+3)=n^2+4n+4=(n+2)^2$$