WORKSHEET N°1

Exercise 1: Let the step function be defined on $[0,3]$ by :

$$\varphi(x)=\begin{cases}-1 & \text{if} \quad x=0\\ 1 & \text{if} \quad 0<x<1\\ 3 & \text{if} \quad x=1\\ -2 & \text{if} \quad 1<x<2\\ 4 & \text{if} \quad 2<x<3\end{cases}$$

1) Calculate the integral:

$$\int_0^3 \varphi(x)\mathrm{d}x$$

2) Find the expression of the function Φ, for the values of $t\in[0,3]$:

$$\Phi(t)=\int_0^t \varphi(x)\mathrm{d}x$$

3) Show that Φ is continuous but is not differentiable on $[0,3]$.

Exercise 2: Consider the family of functions φ_n , $n\in\mathbb{N}^*$ defined on $[0,1]$ by:

$$\varphi_n(x)=\begin{cases}\dfrac{p^2}{n^2} & \text{if} \quad x\in\left[\dfrac{p}{n},\dfrac{p+1}{n}\right[\\ 1 & \text{if} \quad x=1\end{cases} \quad \text{with} \quad p\in\{0,1,2,\dots,n-1\}$$

1) Write the expression for the functions $\varphi_1,\varphi_2,\varphi_3,\varphi_4$.
2) Show that $\varphi_n$is a step function.
3) Calculate the integral:

$$\int_0^1 \varphi_n(x)\mathrm{d}x$$

Exercise 3:

Let the function f defined on $[0,1]$ by : $f(x)=x^2$.

1) Give the uniform subdivision $\mathcal{S}$of the interval $[0,1]$.
2) Construct a family of step functions $(\varphi_n)_n$ **below** f on a subdivision $\mathcal{S}$.
3) Calculate the integral $K_n^-=\int_0^1 \varphi_n(x)\mathrm{d}x$, and find $\lim_{n\to+\infty} K_n^-$.
4) Construct a family of step functions $(\psi_n)_n$ **above** f on a Subdivision $\mathcal{S}$. **(leave to the students)**
5) Calculate the integral: $K_n^+=\int_0^1 \psi_n(x)\mathrm{d}x$, and find $\lim_{n\to+\infty} K_n^+$. **(leave to the students)**
6) Deduce that $\int_0^1 x^2\mathrm{d}x=\frac{1}{3}$. **(leave to the students)**

Exercise 4: Let the following integral for $a \in \mathbb{R}$:

$$K(a) = \int_0^1 (x^2 - ax)^2 \mathrm{d}x$$

1) Calculate $K(a)$.
2) Show that $K(a) = \frac{1}{3}\left(a - \frac{3}{4}\right)^2 - \frac{1}{80}$.
3) Deduce $\inf_{a\in\mathbb{R}} K(a)$.

Exercise 5: Let the following integrals:

$$I = \int_0^{\pi} \sin x \,\mathrm{d}x \quad , \qquad I_n = \int_0^{\pi} \frac{n \sin x}{x+n} \mathrm{d}x$$

4) Find a framework for $|I_n - I|$.
5) Show that $\lim_{n\to+\infty} I_n - I = 0$.
6) Deduce $\lim_{n\to+\infty} I_n$.

Exercise 6: Using the primitive table to calculate the integrals:

$$I_1 = \int_{-1}^{5} |1-x| \mathrm{d}x \quad , \quad I_2 = \int_1^2 \frac{e^x}{e^x - 7} \mathrm{d}x \quad , \quad I_3 = \int_0^1 \left(\sqrt{x^3} + \frac{1}{\sqrt{x^3}}\right) \mathrm{d}x \quad (\boldsymbol{L.E})$$

$$I_4 = \int_0^1 \frac{3x}{\sqrt{1+x^2}} \mathrm{d}x \quad (\boldsymbol{L.E}) \ , \quad I_5 = \int_0^{\frac{\pi}{4}} \sin(2x)\cos(2x)\,\mathrm{d}x \quad (\boldsymbol{L.E}) \quad , \quad I_6 = \int_{-\frac{\pi}{4}}^{0} (\sin x + \cos x)^2 \mathrm{d}x$$

Exercise 7: Calculate the average value of the following functions:

1) $f(x) = |x|$ on $[-2,2]$.
2) $f(x) = x^2 + 3x + 2$ on $[-1,0]$ then on $[-1,1]$.(***leave to the students***)
3) $f(x) = \cos x$on $[0,2\pi]$ then on $\left[0, \frac{\pi}{2}\right]$.

Reminder : the average value of an integral over an interval is given by

$$\mu = \frac{1}{b-a} \int_a^b f(x)dx$$