École polytechnique

Report
Stochastic simulation, and rare event simulation, MAP473D.

Simulating rare events
Systemic risk

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Abstract

The financial crisis of 2008 has shown that failures of financial institutions may lead to a systemic crisis which can be defined as a macro-level risk which can impair the stability of the entire financial system. It can occur as a consequence of an aggregate negative shock affecting all institutions in the system. This crisis has shed light on the importance of contagion phenomena.

In this paper we will try to simulate numerically the cascades of insolvency and to assess contagion and systemic risk in a network of interlinked institutions. We propose a stochastic modeling using graphs for the financial system.

First, we will start by a simple model of independent exposures and fixed matrix of interlinks. Using Monte-Carlo stochastic algorithms, we will try to compute probabilities of insolvency in a simple financial network. We will also define a loss I(T) which describes contagion's effect and study its distributions and characteristics(Expected value, Variance, Value-At-Risk...) in order to comprehend quantitatively the phenomena.

Second, we will extend our results to random financial networks using a homogeneous graphs modeling. This will lead us to define some basic measure like contagion Index and Average loss and will confirm some results proved in literature numerically.

Third, we will study the cascades of insolvency and detect the dangerous institutions/links that lead to failure in the network as well as the most probable scenario of contagion. This will help us to understand deeply the role of network structure and exposure in contagion.

Fourth, we will change this model and try to study the same characteristic using the long term balance model and the dynamic modeling. We will show the importance of taking this points into account because of their remarkable effects.

Finally, we will apply this model to a real world situation and compute the most likely scenario of contagion.

1 Probabilistic modeling of contagion phenomena

The financial system is naturally modelled by a counterparties graph G(V,E) where vertices are institutions and edges represent the exposition of one institution to another.

Our objective is to study the solvency of our system at a predefined time T. If the capital of an institution i is less than a predefined value c_i, then this institution is not solvable. When a bank is no longer solvable, his network is also affected, each institution linked to it will lose a fraction 1-R of his exposition to the unsolvable bank. This will led to the contagion phenomena.

At a date t, the institution i has a capital $X_i(t)$, which is a safety mattress for the company creditors to absorb the potential losses. The balance sheet of the company is a balance between its assets, the properties it owns which have a positive economic value, and its liabilities, which are the debts of the company or the properties it owns which have a negative economic value.

Assets	Liabilities
Interbank assets $\sum_{j=1}^{n} e_{i,j}(t)$	Interbank liabilities $\sum_{j=1}^{n} e_{j,i}(t)$
Other assets $A_i(t) = 0$	Capital $X_i(t)$

Table 1: Assets - liabilities balance of the company i at date t

Between the times t_k and t_{k+1} where $t_k = k\delta(\delta > 0)$ the capital of a company undergoes fluctuations due to the market, and the dynamic of these fluctuations is given by :

$$X(t_{k+1}) = e^{-\lambda\delta}X(t_k) + \mu(1 - e^{-\lambda\delta}) + \sigma\sqrt{\delta}W_k$$

where W_k is a sequence of independent and identically distributed gaussian variables and t_k represents time discretion. the parameter μ is interpreted as the mean balance of the capital, which means when X(t) departs from this value, the parameter λ acts as a restoring force towards μ, and the parameter σ >0 is the standard deviation of the fluctuations of the capital. $X(t_k)$ is a gaussian process.

At the date $T = N\delta$ a solvency balance sheet takes place to conclude on the financial strength of the institution i. If the capital of the institution i is beneath a deterministic critical threshold c_i it is insolvent. A bank is solvent if its remaining capital at date T is greater than the critical threshold $X_i(t_k) > c_i$. An insolvent bank defaults and becomes liquidated, and all its creditors lose a fraction 1-R of their exposition to this defaulting bank. This loss is subtracted from their remaining capital and can in turn make these creditors become insolvent. This insolvency cascade depends strongly on the recovery rate R of the defaulting bank. To simplify, we will suppose all banks have the same recovery rate $R = 5\%$.
We define the set $D_0^T = \{i \in \{1, ..., n\} : X_i(T) < c_i\}$ of the financial institutions of the network initially defaulting at date T. The insolvency cascade is the sequence of sets $D_0^T \subset D_1^T \subset ... \subset D_{n-1}^T$, defined as :

$$D_k^T = D_{k-1}^T \bigcup \{j \notin D_{k-1}^T : X_j(T) - \sum_{p \in D_{k-1}^T} (1 - R)e_{j,p} < c_i\}, k \geq 1$$

In a financial network of size n, this cascade finishes after at most n-1 steps. At the step k, D_k^T is the set of insolvent financial institutions (therefore defaulting) because of their counter party exposure to banks from the set D_{k-1}^T that have just defaulted in the previous step.
To quantify the systemic risk and the contagion effect, we define a default impact $I(t)$ at date T due to the default cascade at time T, which is the sum of the losses generated by the banks defaulting at time T.

$$I(T) = \sum_{j \in D_{n-1}^T} \{X_j(T) + \sum_{p \notin D_{n-1}^T} (1-R)e_{p,j}\}$$

We make the assumption in a first time that the capitals change independently, which means the sequence of variables $(W_k)_{k\geq 0}$ for each institution are mutually independent.
We will start by computing the default probabilities for a simple network of 5 banks, where the inter-bank links are given by the following matrix :

$$E = \begin{pmatrix} 0 & 3 & 0 & 0 & 6 \\ 3 & 0 & 0 & 0 & 0 \\ 3 & 3 & 0 & 0 & 0 \\ 2 & 2 & 2 & 0 & 2 \\ 0 & 2 & 3 & 3 & 0 \end{pmatrix}$$

The time horizon T is one year, and observation dates $(t_k)_k$ are every month or every trimester for example. The other parameters are fixed as follows :

$$X_i(0) = \mu = 15, \sigma = 8, \lambda = 20, c_i = 10, i = 1, ..., 5$$

Simplification of the expression of the capitals

Before simulating the evolution of the capitals given by the formula of the Markov chain of the following model :

$$X(t_{k+1}) = e^{-\lambda\delta} X(t_k) + \mu(1 - e^{-\lambda\delta}) + \sigma\sqrt{\delta} W_k \tag{1}$$

we observe that the random vectors $X(t_k)$ are a linear combination of independent gaussian vectors, therefore X(T) is also a gaussian vector whose mean and variance we can compute using the recurrence relation.

$$X(t_k) \sim N(\mu_k, \sigma_k^2)$$

According to the recurrence relation :

$$\mu_{k+1} = e^{-\lambda\delta}\mu_k + \mu(1 - e^{-\lambda\delta})$$

$$\sigma_{k+1}^2 = e^{-2\lambda\delta}\sigma_k^2 + \sigma^2\delta$$

with $\mu_0 = \mu$ and $\sigma_0^2 = 0$. Therefore X_T is a Gaussian vector of mean μ_T and variance σ_T^2 such that :

$$X(T) \sim N(\mu_T, \sigma_T^2)$$

$$\mu_T = \mu$$

$$\sigma_T = \sigma\sqrt{\delta \frac{1 - e^{-2\lambda}}{1 - e^{-2\lambda\delta}}}$$

2 Monte Carlo method for the estimation of the number of insolvent counter-parties

The first algorithm we implemented to estimate the probability that the number of insolvent institutions is $k = 0, 1, 2, 3, 4, 5$ is the naive Monte Carlo method. We started by implementing in python a function D_n which computes an array of 5 values, that corresponds to whether a bank is in the set D_n^T. This function takes in entry the capitals X_T, the threshold c, the recovery rate R and the exposures matrix E.

Algorithm 1 Computing D_n^T

Require: Initial capitals X_i
  Exposures matrix E
  Threshold c
  Recovery rate R
Ensure: Set D_n^T : array of size n of boolean values, that shows if bank n°i is insolvent (1) or not (0)
  n is the number of banks in the network
  m is the size of the sample
  D= zeros ([m, n, n])
  D[:,0,:] = (X<c)
  N = (X<c)
  for k=1...n **do**
    $X \leftarrow X - (1-R)(E.N^\top)^\top(1 - D[:, k-1, :])$
    $N \leftarrow (X < c) - D[:, k-1, :]$
    $D[:, k-1, :] = X < c$
  end for
  return D

This function returns an array D_n, where $D_n[i]$ equals 1 if the bank i is insolvent, and 0 otherwise.
For the naive Monte Carlo method, we draw a sample of size m, composed of m vectors X, where each vector X is of size 5 and represents the capital of institution i at time T.
we compute the probability

$$\widehat{P_n^k} = \frac{1}{m}\sum_{i=1}^{m} 1(\text{k banks have defaulted in the simulation n°i})$$

We use the central limit theorem to compute a confidence interval for the previous estimated probabilities using the formula

$$P^k \in [\widehat{P_n^k} - 1.96\frac{\widehat{\sigma^n}}{\sqrt{n}}, \widehat{P_n^k} + 1.96\frac{\widehat{\sigma^n}}{\sqrt{n}}]$$

where $\widehat{\sigma^n}$ is the non-biased estimator of the standard deviation of the estimator $\widehat{P_n^k}$.
In our code we used a value of $m = 10^6$, although the previous function D_n is very optimized and can work for a matrix X of up to 10^8 samples.

The estimation of I(T) given by the naive Monte Carlo method is :

$$\widehat{I(T)} = 2.0829$$

The confidence interval of 95% is : $IC = [2.0667, 2.0992]$

Number	Probability	Precision	std
0	0.9189	0.00053	0.2728
1	0.0499	0.00042	0.2178
2	0.0131	0.00022	0.1141
3	0.0071	0.00016	0.0844
4	0.0052	0.00014	0.0719
5	0.0054	0.00014	0.0738

Table 2: Estimation of the probability of number of defaulting banks for a sample size of $m = 10^5$

3 Adaptative algorithm of optimal mean in the case of multidimensional gaussian variables

As mentioned above, the probabilties we are trying to compute can be expressed as the expected value of a function f of the gaussian vector X_T of dimension d=5.
As $X_T = \mu + \sigma W_T$ where W_T is a standard normal gaussian, the probabilities we are trying to compute can be viewed as a function of normal standard gaussian variables W_T. Therefore, we can use a change of probability method using a parameter $\theta \in R^d$.
We know that

$$E[f(X)] = E\left[f(X+\theta)\, e^{-\theta X - \|\theta\|^2/2}\right],$$

The objective is to learn, using an adaptative method, the optimal value of the parameter θ.
The sequence of random variables[1]

$$M_n(\theta) = \frac{1}{n}\sum_{k=1}^{n} f(X_k+\theta)\; e^{-\theta X_k - \|\theta\|^2/2}, \qquad \{X_k, k \geq 0\} \text{ v.a. i.i.d. de loi } N(0,1) \tag{2}$$

is a non-biased estimator of the objective $E[f(X)]$.
Its variance is $n^{-1}\left(V(\theta) - ([f(X)])^2\right)$ where

$$V(\theta) = \left[f^2(X)e^{-\theta X + \|\theta\|^2/2}\right].$$

Our goal is to use the estimator defined in (2) with an optimal value $\theta*$ of θ which minimizes the variance $V(\theta)$.
The function V is of class C^2, strictly convex and has a unique minimum $\theta*$, and its gradient and hessian matrix are :

$$\begin{aligned} G(\theta) &= \left[f^2(X)\,(\theta - X)\,e^{-\theta X + \|\theta\|^2/2}\right] \in^{d\times 1}, \\ H(\theta) &= \left[f^2(X)\,(I_d + (\theta - X)(\theta - X)')\,e^{-\theta X + \|\theta\|^2/2}\right] \in^{d\times d}, \end{aligned}$$

To approximate $\theta*$, we use the Newton method : $\theta*$ will be approached as the limit of a sequence $(\theta_n)_n$ defined as follows :

$$\theta_{k+1} = \theta_k - (H_n(\theta_k))^{-1}\; G_n(\theta_k) \tag{3}$$

where $H_n(t), G_n(t)$ are the Monte Carlo approximations of the functions H(t) and G(t) defined above.

[1]Robust adaptive importance sampling for normal random vectors article in projecteuclid.org/euclid.aoap/1255699541

$$G_n(\theta) = \frac{1}{n}\sum_{k=1}^{n} f^2(X_k)(\theta - X_k)e^{-\theta X_k + \|\theta\|^2/2},$$

$$H_n(\theta) = \frac{1}{n}\sum_{k=1}^{n} f^2(X_k)\left(I_d + (\theta - X_k)(\theta - X_k)'\right)e^{-\theta X_k + \|\theta\|^2/2}.$$

We initialize the sequence $(\theta_n)_n$ with a value θ_0, and given a threshold $\epsilon > 0$, we iterate the relation (3) as long as $\|G_n(\theta_k)\| > \varepsilon$.

3.1 Results of the optimal mean algorithm

For an initial choice of parameter $\theta = (0, 0, 0, 0, 0)$ the optimal mean algorithm returns more precise values of the probabilities (smaller standard deviation of the estimators) than the naive Monte Carlo method on a same sample, which is the plus value of this algorithm.

Number	Optimal mean	std	Monte Carlo	std
0	0.9160	0.0053	0.9147	0.0054
1	0.0517	0.0039	0.0539	0.0044
2	0.0165	0.0023	0.0155	0.0024
3	0.0067	0.0013	0.0058	0.0015
4	0.0046	0.0011	0.0047	0.0013
5	0.0055	0.0010	0.0054	0.0014

Table 3: Comparison between the optimal mean algorithm and the naive Monte Carlo method on a sample size of $m = 10^4$ using initial parameter $\theta_0 = (0, 0, 0, 0, 0)$

4 Splitting methods using first order auto-regression

Since the probability of having 0 insolvent bank is far greater than the others. The naive Monte-Carlo method may seem not accurate especially to calculate the last 5 probabilities. Hence, we have also implemented two versions of the splitting method in order to generate more samples which verify N>0

4.1 First version

4.1.1 Model and algorithm

Let's denote by N the random variable which determines the number of unsolvable banks after the contagion cascade. We can express this quantity as $N = f(Z)$ where $Z \sim \mathcal{N}(\mu_T, \sigma_T I_n)$. Our objective is to estimate $P(N = k)$ for $k \in 0, 1, 2, 3, 4, 5$. This is equivalent to estimate $P(N \geq k)$ for $k \in 1, 2, 3, 4, 5$ because

$$P(N = k) = P(N \geq k) - P(N \geq k + 1)$$

for $k \in 1, 2, 3, 4$ and we have $P(N = 0) = 1 - P(N \geq 1), P(N = 5) = P(N \geq 5)$

Given this reformulation, a first natural splitting method can be implemented. Let's denote $A_k = [k, 5]$ hence $[0, 5] = A_0 \supset A_1 \supset \cdots \supset A_k = A$. We can compute $P(N \in A)$ by the formula :

$$P(N \in A) = \prod_{\ell=1}^{k} P(N \in A_\ell \,|\, N \in A_{\ell-1})$$

In order to compute these conditional probabilities, we will define a Markov chain by generating an initial sample of gaussian vector X_T as described above, we call it X_0^A. We put, for $i \geq 1$ and $\rho \in [0, 1]$

$$X_i^A := \begin{cases} \rho(X_{i-1}^A - \mu) + \sqrt{1-\rho^2}Y_i + \mu & \text{if } \rho(X_{i-1}^A - \mu) + \sqrt{1-\rho^2}Y_i + \mu \in f^{-1}(A), \\ X_{i-1}^A & \text{otherwise,} \end{cases}$$

where $(Y_i)_{i\geq 1}$ is an i.i.d. random variable $\mathcal{N}(0, \sigma I_n)$, independent of X_0^A.

This defines a Markov chain in A (And then in A_{l-1} in each step of the algorithm) verifying the Ergodic theorem's conditions. Precisely, we have for each $l \in 1, 2, 3, 4, 5$

$$\frac{1}{n}\sum_{i=1}^{n} 1(X_i^A{}_{l-1} \in f^{-1}(A_l)) \underset{n\to\infty}{\longrightarrow} P(N \in A_l \,|\, N \in A_{l-1}) = P(N \geq l \,|\, N \geq l-1)$$

where $N = f(X_T)$,$X_T \sim \mathcal{N}(\mu, \sigma I_n)$

Here is the allure of the algorithm (See algorithm2)

Algorithm 2 Splitting1

Require: $X_T \sim \mathcal{N}(\mu, \sigma I_n)$,Y$\sim \mathcal{N}(0, \sigma I_n), m \geq 0, \rho \in [0, 1]$
Ensure: $Probs = [P(N \geq l \,|\, N \geq l-1), l = 1...n], AccRates = [AcceptanceRate(l), l = 1...n]$

```
  ρ_bar ← √(1 − ρ²)
  X ← [X_T]
  Probs ← array of size n
  AccRate ← array of size n
for l=1...n do
    startingPoint ← X[N(X) ≥ l − 1][0]
    X ← [StartingPoint]
    Y ← m samples of Z∼ N(0, σI_n)
    acc ← 0
    for k=1...m do
      newX ← ρX[size(X)] + ρ_bar Y[k] + μ(1 − ρ)
      if N(newX) ≥ l − 1 then
        X ← [X, newX] and
        acc ← acc + 1
      else
        X ← [X, X[size(X)]]
      end if
    end for
    acc ← acc/(m+1)
    AccRate[l] ← acc
    Probs[l] ← 1/(m+1) Σ_{i=1}^{m+1} 1(N(X) ≥ l)
end for
```

4.1.2 Fine-tuning the model

Generally, the trick in splitting's methods is the choice of ρ which determines how much the defined Markov chain fluctuates and most importantly the accuracy of the algorithm. In the literature, an acceptance rate between 0.25 and 0.30 leads to a good splitting's algorithms. Hence, we had plotted the acceptance rates (for the five probabilities) in function of ρ (See Figure 2). More precisely, we lunched a macro-run of the algorithm (100 times) and then we calculate both standard deviation of our estimated probabilities and the average acceptance rates defined in the above algorithm2.

Note that there are no central limit theorem applicable in this case. Therefore, we should lunch a macro-run to have an idea about the algorithm's accuracy.

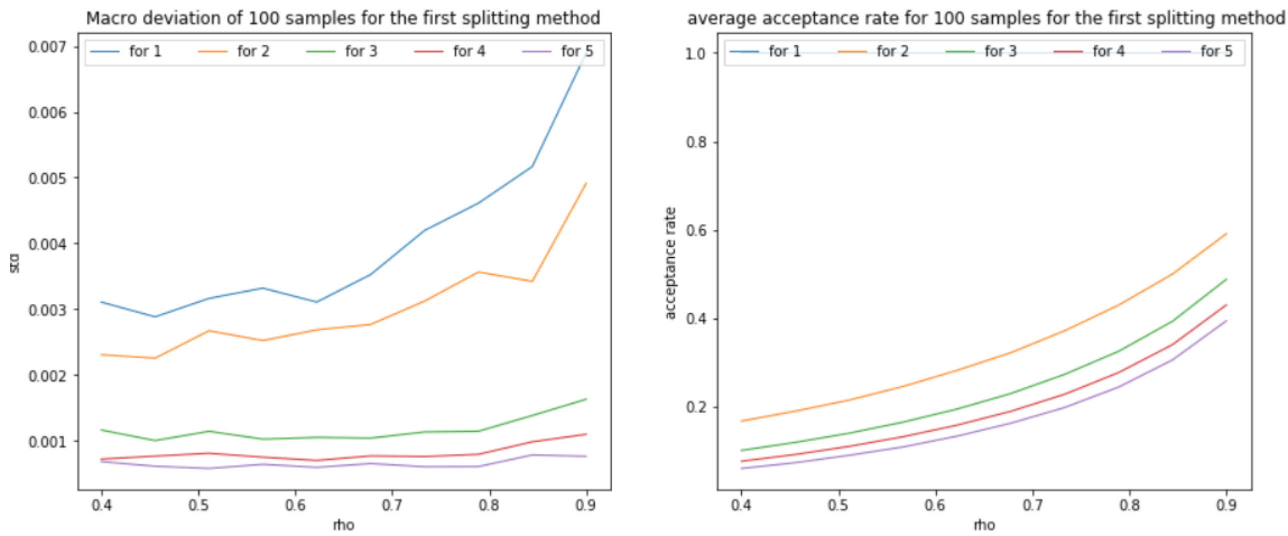

Figure 1: Right: The average std of results for a macro-run of 100 samples using m=10^4 in function of ρ Left: The average acceptance rate for the same parameters in function of ρ

4.1.3 Comments on our results

- We can see in the left graph that standard deviation std of the computed probabilities in splitting method is an increasing function of ρ for $P(N = 0)$ and $P(N = 1)$. This is normal because as we said previously ρ determines the fluctuations (when ρ increases the chain tends to fluctuate less) of the Markov chain and then the chain stuck in small set of values included in the large set N=0 U N=1 which depends on the samples token and then the more ρ is big the more the algorithm is less precise to calculate these two probabilities.
- For $\rho \leq 0.8$ it seems that the standard deviation is stable and small for the other probabilities. It may be seen as a proof of the robustness of the algorithm. Also, the sets of N=k for k=2,3,4,5 are relatively small and then the phenomena described in the last point justify in fact these small values of std which confirms here the accuracy of splitting's methods in calculating small probabilities.
- According to these graphs and the fact that the acceptance rate should be between 0.25 and 0.3 : $\rho = 0.8$ seems a good value for this first version of splitting.

4.1.4 Results and comparisons

Now that we have found an acceptable ρ. We have lunched the algorithm on 100 macro-run using $m = 10^4$ samples [See Table 4]. We can see easily that this method performs far more better than the naive monte-carlo method using the same method to compute std (100 macro-run with 10^4 samples).

Number	Probability	std	Naive Probability	std naive
0	0.9188	0.0049	0.9177	0.0299
1	0.0502	0.0037	0.0502	0.0213
2	0.0128	0.0012	0.0137	0.0113
3	0.0073	0.0008	0.0073	0.0081
4	0.0052	0.0006	0.0059	0.0076
5	0.0054	0.0006	0.0052	0.0078

Table 4: Results of the first version of splitting compared with the naive monte-carlo. Parameters: $\rho = 0.8$, $m = 10^4$, 100 micro-runs to compute the standard deviations

4.2 Second version

4.2.1 Model and algorithm

The idea of this second version of Splitting comes from the observation that the weakness of the naive monte-carlo method comes from the few samples for which N>0 it can generate. The main idea of this method is to generate a lot of samples in the set N(X)>0 using Markov Chain and then compute the conditional probabilities $P(N \geq k \mid N > 0)$ k=1...n using a simple Monte-Carlo estimator.

Note that $P(N > 0) = 1 - P(X \geq c)^5 = 0.0854$ (using scipy library in Python) where $X \sim N(\mu_T, \sigma_T^2)$. We should just multiply this quantity by the output of the algorithm to obtain the insolvency probabilities.

This leads naturally to the following algorithm[See algorithm 3]. You may remark a small difference with the implementation we did [See Scripts]. In fact, we had replaced the While loop by a single drawing of m samples of X_T and then we delete those with N=0. This improvement increases the samples generated by the algorithm(It can be seen as a merge with the naive Monte carlo method).

4.2.2 Fine-tuning the model

Using the same method as described above for the first version, we will use 100 macro-run using $m = 10^4$ in order to plot the variations of both std of results and average acceptance rate in function of ρ [See Figure 3]. $\rho = 0.65$ seems a good value according to the previous analysis. The same comments may be told fot these graphs too.

Algorithm 3 Splitting2

Require: $X_T \sim \mathcal{N}(\mu, \sigma I_n)$,Y$\sim \mathcal{N}(0, \sigma I_n), m \geq 0, \rho \in [0, 1]$
Ensure: $Probs = [P(N = l \,|\, N > 0), l = 1...n], AccRate$

```
ρ_bar ← √(1 − ρ²)
X ← X_T
while N(X)=0 do
    X ← N(μ, σI_n)
end while
Probs← array of size n
AccRate ← 0
Y ←m samples of Z∼ N(0, σI_n)
for k=1...m do
    X_1 ← X[N(X) > 0][0]
    newX ← ρX_1 + ρ_bar Y[k] + μ(1 − ρ)
    if N(newX) > 0 then
        X ← [X, newX] and
        AccRate ← AccRate + 1
    else
        X ← [X, X_1]
    end if
end for
AccRate ← AccRate / m
Probs[l] ← 1/(m+1) Σ_{i=1}^{m+1} 1(N(X) = l)
```

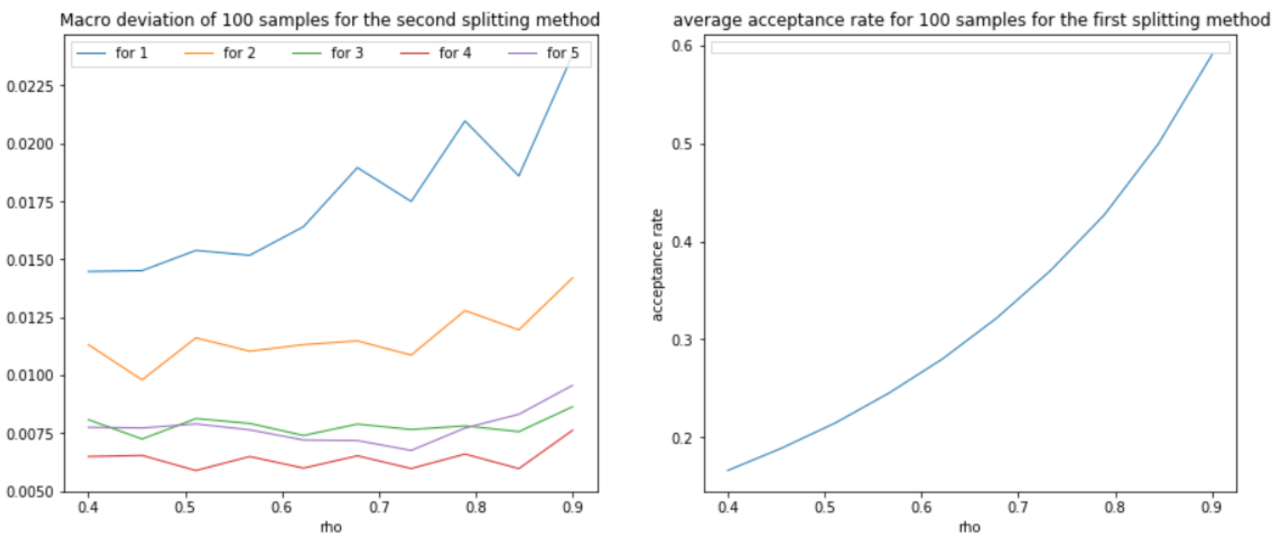

Figure 2: Second splitting method. Right: The average std of results for a macro-run of 100 samples using m=10^4 in function of ρ Left: The average acceptance rate for the same parameters in function of ρ

4.2.3 Results and comparisons

Now that we have found an acceptable ρ. We have lunched the algorithm on 100 macro-run using $m = 10^4$ samples. [See Table 5] We can see easily that this method performs far more better than the naive monte-carlo method using the same method to compute std (100 macro-run with 10^4 samples). In comparison with the first version(using the same samples), this method seems to perform a little bit better than the first. The average acceptance rate obtained is 0.300633.

Number	Probability sp2	std sp2	Probability sp1	std sp1
1	0.0524	0.0014	0.05035	0.0038
2	0.01379	0.0009	0.01305	0.0015
3	0.0078	0.0007	0.0072	0.0008
4	0.0055	0.0004	0.0052	0.0007
5	0.0057	0.0006	0.005	0.0007

Table 5: Results of the second version of splitting compared with the first version using same initial samples. Parameters: $\rho = 0.65$, $m = 10^4$, 100 micro-runs to compute the standard deviations

5 Distribution of default impact and Value At Risk

To quantify the systemic risk and the contagion effect, we defined a default impact $I(T)$ at date T due to the default cascade at time T, which is the sum of the losses generated by the banks defaulting at time T.

$$I(T) = \sum_{j \in D_{n-1}^T} X_j(T) + \sum_{p \notin D_{n-1}^T} (1-R) e_{p,j}$$

In this section, we will study the distribution of $I(T)|N(X) > 0$ since the case where I(T)=0 is not interesting (There is no contagion). As described in the above description of the second splitting method, generating samples naively leads basically to I(T)=0 since the probability of having 0 insolvent banks is really high.
Hence, we extract naturally from algorithm2 another algorithm generating randomly samples $X_T \sim \mathcal{N}(\mu, \sigma I_n)$ verifying I(T)>0 (Called GENI, see algorithm4). This will help us to increase the precision of our results.Here is the algorithm used (Algorithm4)

5.1 Distributions of default Impact

5.1.1 Distribution of $I(T)|I > 0$

Let's start by generating 107889 samples using our function and plotting the histogram of conditional I(T).[Figure 4]

Algorithm 4 GenI : Generator of samples verifying N>0

Require: $X_T \sim \mathcal{N}(\mu, \sigma I_n)$,Y$\sim \mathcal{N}(0, \sigma I_n), m \geq 0, \rho = 0.65$
Ensure: X=an array containing vectors X_T verifying N>0
$\rho_{bar} \leftarrow \sqrt{1-\rho^2}$
$X \leftarrow X_T$
while N(X)=0 **do**
    $X \leftarrow \mathcal{N}(\mu, \sigma I_n)$
end while
Y $\leftarrow$m samples of Z$\sim \mathcal{N}(0, \sigma I_n)$
for k=1...m **do**
    $X_1 \leftarrow X[N(X) > 0][0]$
    $newX \leftarrow \rho X_1 + \rho_{bar} Y[k] + \mu(1-\rho)$
    if $N(newX) > 0$ **then**
        $X \leftarrow [X, newX]$

    else
        $X \leftarrow [X, X_1]$
    end if
end for

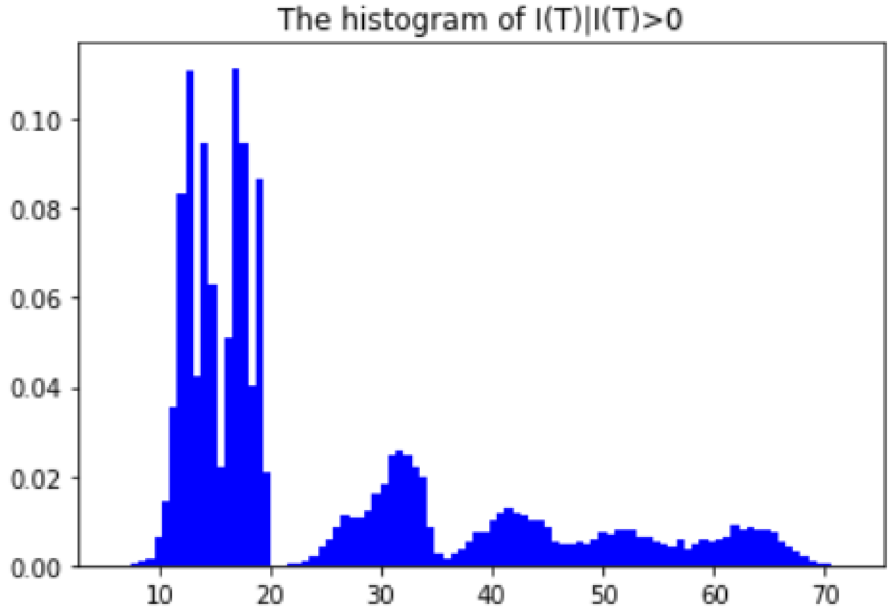

Figure 3: The histogram of I(T)|I(T)>0. Number of samples: 107889 generated by algorithm4 with $\rho = 0.65$

The result here is straightforward. We can clearly distinguish 5 blocks, each one corresponds to N=k in an increasing order. We can interpret this blocks by the fact that $N = f(X)$ is integer but $X \in R^n$ and then $I(T)$ is real.

5.1.2 Distribution of $I(T)|$"Systemic contagion"

We will suppose now that we already have systemic contagion and see the distribution of I(T) under this condition.
By using the algorithm above, we have managed to draw 7343 samples verifying N=5. It seems

sufficient to plot a histogram[See figure 5] (Note that one can also derive another generator of samples verifying N=5 from algorithm 3 just as we did before with algorithm 2).

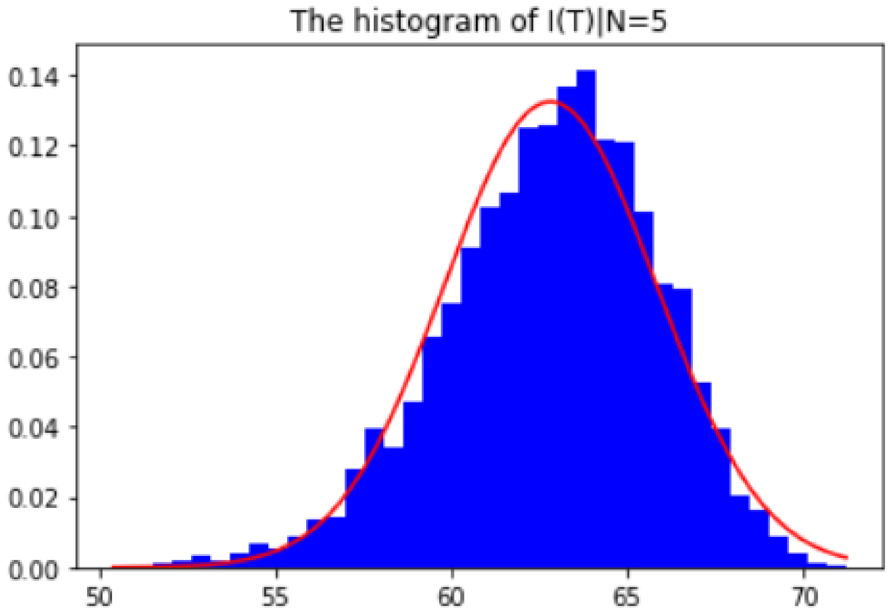

Figure 4: The histogram of I(T)|N=5. Number of samples: 7343 generated by algorithm 4 with $\rho = 0.65$

5.1.3 Maximum of Likelihood Estimation

The histogram we obtained in the last section show that the distribution of I(T) under the condition "Systemic Condition" is almost Gaussian. This will lead us naturally to assume that $I(T) \sim \mathcal{N}(\mu, \sigma^2)$. Hence, we are aiming to estimate these parameters μ and σ. We will use the maximum of likelihood method to estimate them. In particular, assume that $(I_1(T), ...I(T)_n)$ are an i.i.d random realizations of $I(T)$.
Let's define the two unbiased and consistent estimators :
- Estimator of the mean: $\overline{I_n(T)} = \frac{1}{n}\sum_{i=1}^{n} I_n(T)$
- Estimator of the variance: $\overline{S_n^2} = \frac{1}{n-1}\sum_{i=1}^{n}(I_i(T) - \overline{I_n(T)})^2$
We denote by t_α^n and $\chi_\alpha^2(n)$ respectively the quantiles of a Student and a Chi2 laws with n liberty degrees
We can demonstrate [2] that:

$$P\Big(\overline{I_n(T)} - S_n t_{1-\alpha/2}^{n-1}/\sqrt{n} \leq \mu \leq \overline{I_n(T)} + S_n t_{1-\alpha/2}^{n-1}/\sqrt{n}\Big) = 1 - \alpha$$

$$P\Big((n-1)S_n^2/\chi_{n,1/3}^2(n-1) \leq \sigma^2 \leq (n-1)S_n^2/\chi_{n,2/3}^2(n-1)\Big) = 1 - \alpha$$

Hence, we can easily estimate our parameters and compute the confidence interval. In the simulation, we will put $\alpha = 0.05$

Results

Here are our results using these estimators (Table 6)

[2]See Gersende Fort, Matthieu Lerasle, Eric Moulines, Jaouad Mourtada,Inférence statistiqueApplied Mathematics Department of the Ecole Polytechnique 2018 for more details

Estimated Mean	62.8220
Confidence Interval for μ	[62.800 , 62.845]
Estimated std	3.111
Confidence interval for std	[3.111 , 3.113]
Assymmetry coefficient	-0.3337
Kurtosis	3.0530

Table 6: Results Maximum of Likelihood estimation.Samples: 7196 generated by algorithm 4 with Parameters: $\rho = 0.65$, $m = 10^4$. $\alpha = 0.05$ for the confidence intervals

5.1.4 Normality test

We used a Q-Q plot of the distribution of I(T) to test the normality of its distribution.[See Figure 6]

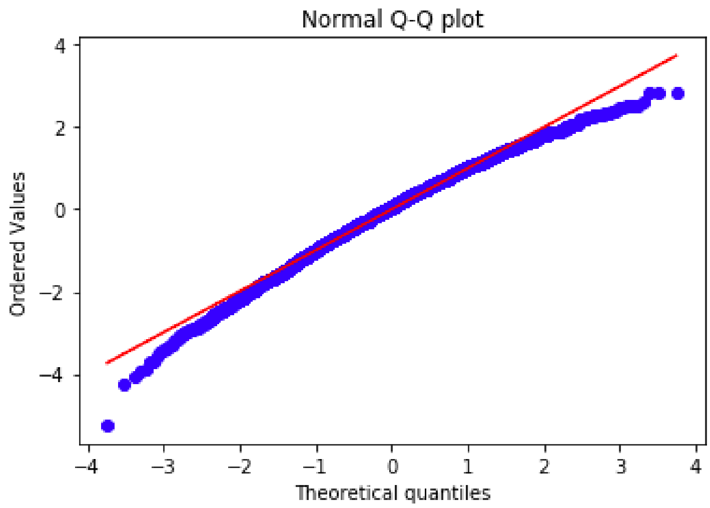

Figure 5: Q-Q plot of I(T)

The points are aligned along the line y=x for the most part, which suggests the data could be normally distributed, which is coherent with the fact that the kurtosis of the distribution is close to 3 (kurtosis of a normal distribution is 3). The curved pattern suggests that the central quantiles are more closely spaced than the far ones, which suggests that the tail of the distribution is a bit far off that of a normal distribution.

5.2 Value At Risk and empirical quantiles

In this section, we will try to estimate the Value-At-Risk of I(T). Since we do not know explicitly the law of this quantity, we will try an empirical approximation using the Glivenko-Cantelli theorem. We will simulate a large number of capitals $X_T \sim \mathcal{N}(\mu_T, \sigma_T^2)$ and then compute the corresponding I(T).
The VaR (Value-At-Risk) can be defined as follows:

$$VaR(\alpha) = Q(1-\alpha) \qquad 0 < \alpha < 1,$$

where the values of α are approximately zero and Q is the unknown repartition function of I(T). This function is usually used to mesure the Risk. For example, $VaR(0.01)$ represents the

value of I(T) which can be exceeded in only 1% of cases. Assume now that we have generated an i.i.d random variables $(I_1(T), ...I_m(T))$
For each m, We denote by $(I_{(m,i)}(T))_{1\leq i\leq m}$ the order statistic of $(I_i(T))_{1\leq i\leq m}$:

$$I_{(m,1)}(T) \leq I_{(m,2)}(T) \leq \cdots \leq I_{(m,i)}(T) \leq \cdots \leq I_{(m,m)}(T)$$

Let's define now the empirical repartition function:

$$F_m(x) := \frac{1}{m}\sum_{i=1}^{m} 1_{I_i(T)\leq x}, \qquad x \in R.$$

5.2.1 Application of Glivenko-Cantelli Theorem

The Glivenko-Cantelli Theorem affirms that almost surely F_m converges uniformly to Q when m tends to infinity. Therefore, we will use this theorem to plot both the empirical repartition functions of I(T) and I(T)|I>0 using respectively 10^5 and 107889 samples. We obtain these graphs[Figure 7]

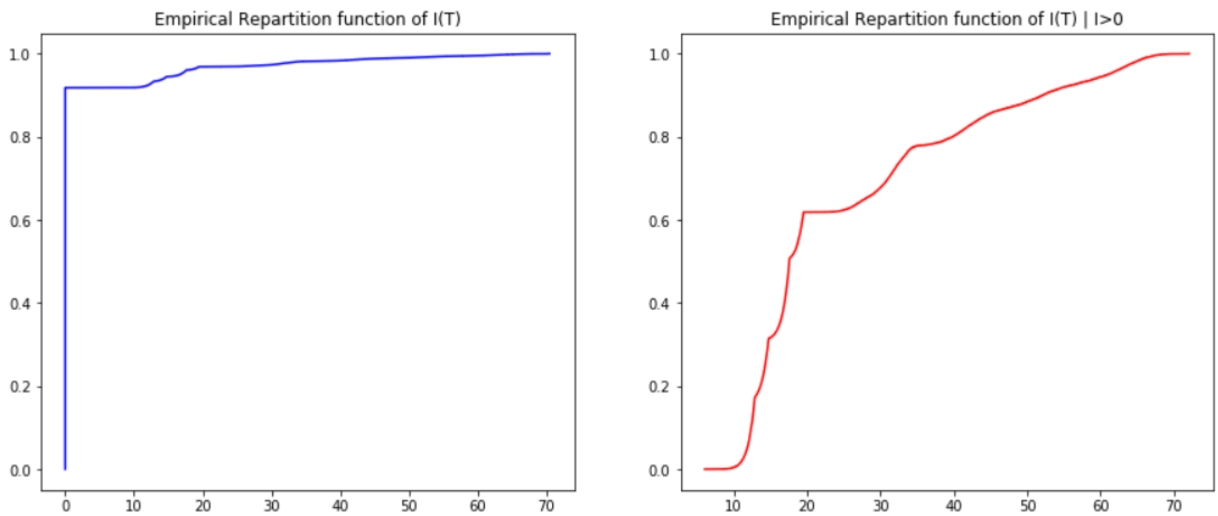

Figure 6: Repartition functions. Left: I(T) for 10^5 samples generated naively. Right: I(T)|I>0 for 107889 samples generated by algorithm3 with $\rho = 0.65$ and $m = 10^4$

5.2.2 Comments and discussion

- The repartition function of I(T) have a infinite derivative for 0. This is a consequence of the fact that the probability having 0 insolvent banks and then I(T)=0 is very high.
- The repartition function of I(T)|I>0 is more "regular" and representative. Although, its derivative is discontinuous in some points which means that there are some values x of I(T) verifying $P(I(T) = x) \neq 0$. It's an intuitive result, let's imagine the case where bank 1 becomes insolvent with a capital, let's assume, $X_1 = 9 < c$ and that $X_2 = X_3 = ... = X_n = 15$. For all X=(x,X$_2, ..., X_n)with$ x≤ 9 (this is a non negligible event!) we will have the same value of I(T) which proves the existence of such a points.

5.2.3 Value-At-Risk

For $0 < u < 1$, we can demonstrate easily that the empirical quantile $Q_m(u) := \inf\{x \in R : u \leq F_m(x)\}$ is equal to $Q_m(u) = I_{(m,\lceil mu \rceil)}(T)$, where $\lceil y \rceil$ designs the superior integer portion of

a real number y.
According to the Glivenko-Cantelli theorem and the definition of F and F_m we have:

$$|F(Q_m(u))-F(Q(u))| \leq |F(Q_m(u))-F_m(Q_m(u))|+|F_m(Q_m(u))-F(Q(u))| \leq ||F-F_m||_\infty+\left|\frac{\lceil nu \rceil}{m}-u\right| \to 0$$

By definition of F_m, $F_m(Q(u))$ is the empirical mean of m i.i.d variables of Bernoulli with the parameter $P(I(T) \leq Q(u)) = u$. By the CLT (Central Limit Theorem), we have :

$$\sqrt{m}(F_m(Q(u)) - u) \longrightarrow \mathcal{N}(0, u(1-u))$$

This leads naturally to define a confidence interval for our estimated value of $Q(u)$. We can deduce immediately from the last convergence that:

$$P\left(u - 1.96\frac{\sqrt{u(1-u)}}{\sqrt{m}} \leq F_m(Q(u)) < u + 1.96\frac{\sqrt{u(1-u)}}{\sqrt{m}}\right) \longrightarrow 0.95.$$

and then :

$$P\big(Q(u) \in [Q_m(u_m^-), Q_m(u_m^+)]\big) \longrightarrow 0.95.$$

where $u_m^{\pm} = u \pm 1.96\frac{\sqrt{u(1-u)}}{\sqrt{m}}$ with $u = 1-\alpha$
$[Q_m(u_m^-), Q_m(u_m^+)]$ is then a confidence interval of threshold 95% to compute $VaR(\alpha)$

5.2.4 Results

These are our results about values of VaR for different α

Alpha	VaR	CI	Conditional VaR	CI
0.99	48.57	[46.83,49.92]	66.04	[65.98,66.17]
0.999	65.65	[65.61,65.69]	68.87	[68.75,69.00]
0.9999	68.64	[68.58,68.68]	70.69	[70.30,71.24]
0.99999	70.29	[70.18,70.42]	72.10	[71.39,72.10]
0.999999	71.72	[71.48,71.96]	72.10	[72.10,72.10]

Table 7: Estimated VaR and Conditional VaR and confidence intervals using respectively 10^7 samples for VaR generated naively and 107889 generated by algorithm3 with Parameters: $\rho = 0.65$, $m = 10^4$. $\alpha = 0.05$

Comments

- The Naive-Monte Carlo simulation required 10^7 samples to provide "acceptable" results. The probability of the event we are trying to simulate is 10^{-6}. It's a rare event. Fortunately, the main algorithm which computes the cascades of insolvency was optimal and allow us to compute in real time 10^7 samples .
- In other side, the estimation of conditional VaR especially for VaR(0.999999) seems not accurate. It's normal because we are using just 10^5 samples generated by the algorithm4. The for loop in this algorithm takes a lot of time and we can't simulate larger numbers of samples in real time.

5.2.5 Splitting method and stratified sampling to estimate VaR

Regarding our last results of VaR, it's necessary to implement a new algorithm in order to generate more samples verifying I(T) large so that we can calculate VaRs values accurately. The idea is to compute iteratively these values using splitting. In fact, we can easily notice that for each $n \in N$ we have:

$$P(I(T) > VaR(1 - 10^{-(n+1)})|\, I(T) > VaR(1 - 10^{-n}) = 0.1.$$

and $P(I(T) > VaR(0.99))$ can be computed accurately by a naive monte-carlo simulation-using 10^5 samples for example. This leads us to design a splitting algorithm to simulate these rare events (See Scripts function splitting3). This algorithm is similar to those discussed before. Although, since the value of ρ is very important here, we use a specific value for ρ for each iteration. We get these results of fine-tuning (Same principle as above). The vector $\rho = [0.73, 0.918, 0.962, 0.985]$ gives acceptance rates around 30% in each iteration. [See Figure 8]

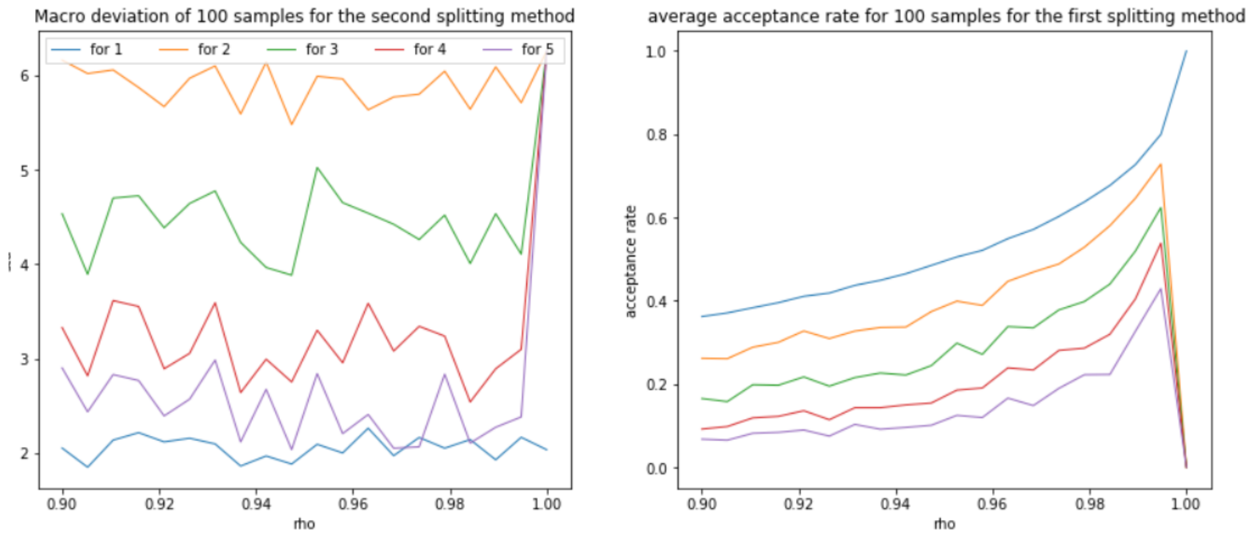

Figure 7: Third splitting method to estimate VaR. Right: The average std of results for a macro-run of 100 samples using m=10^5 in function of ρ Left: The average acceptance rate for the same parameters in function of ρ

Results

The following table presents our results:

Alpha	VaR	acceptance rate
0.99	49.2791	1
0.999	63.3275	0.17212
0.9999	66.5663	0.2105
0.99999	67.3562	0.20483
0.999999	68.0700	0.32987

Table 8: Estimated VaRs using 10^5 samples for third splitting algorithm with Parameters: $\rho = [0.73, 0.918, 0.962, 0.985]$

Note that these values will be far more accurate if we macro-run the algorithm many times and take the average of both VaRs and acceptance rates.

6 Insolvency cascades in the inter-bank network

For a given number of defaults in the network (for example 3 banks become insolvent), we compute the most likely scenarios that lead to that number of insolvent banks. We have used a sample size of $m = 10^6$. We first extract the scenarios of contagion which leads to k insolvent banks in the end of the contagion, and then we compute the scenario with the highest probability.

- In the case of one insolvent bank, it is the bank n°4 that defaults with the highest probability. There are 50007 scenarios where there is only one insolvent bank among the $m = 10^6$ ones we ran, and 13090 of which it is the bank n°4 that defaults. The probability of this scenario conditionally to a number of defaults equal to 1 is 26.18%.
- In the case of two insolvent banks, the most likely scenario is $5 \to 1$. There are 12881 scenarios where exactly 2 banks default, and 4626 of which correspond to the scenario $5 \to 1$. The probability of this cascade is 35.91% conditionally to two banks defaulting.
- In the case of 3 insolvent banks, the most likely scenario is $5 \to 1 \to 4$. This scenario occurs 1385 out of the 7411 cases where there are exactly 3 defaults. The probability of this cascade is 18.69%.
- In the case of 4 insolvent banks, the scenario with the highest probability is $5 \to 1 \to 3 \to 4$. This cascade occurs 431 times out of 5052 cases where we have 4 defaults, which corresponds to a probability of 8.53%.
- In the final case of 5 insolvent banks, the most likely scenario is $2 \to 3 \to 5 \to 1, 4$. This scenario happens 228 out of 5432 times, which corresponds to a probability of 4.19%.

7 Identification of dangerous counter-party links

Given the previous contagion cascades, if the bank 5 defaults, it causes the bank 1 to default as well. The bank 5 triggers the contagion, because if it defaults, it strongly exposes bank n°1 which in turn exposes other banks. This could be explain given the exposures matrix :

$$E = \begin{pmatrix} 0 & 3 & 0 & 0 & 6 \\ 3 & 0 & 0 & 0 & 0 \\ 3 & 3 & 0 & 0 & 0 \\ 2 & 2 & 2 & 0 & 2 \\ 0 & 2 & 3 & 3 & 0 \end{pmatrix}$$

The exposure of bank 1 to bank 5 is $e_{1,5} = 6$, which is the highest exposure in this financial network. The link $5 \to 1$ is therefore a very dangerous link in the network.

If we decide to delete this link, we define a new exposures matrix :

$$E_2 = \begin{pmatrix} 0 & 3 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 \\ 3 & 3 & 0 & 0 & 0 \\ 2 & 2 & 2 & 0 & 2 \\ 0 & 2 & 3 & 3 & 0 \end{pmatrix}$$

The VaR of the default impact I(T) of the original financial network for a threshold $\alpha = 0.999999$ is 71.46045.

We compute the default impact I(T) using the new exposures matrix E_2, and we find a new value for the VaR of the default impact for the same threshold $\alpha = 0.999999$: $VaR = 69.9599$ We can conclude that the removal of the link $5 \to 1$ almost reduced the value at risk, and therefore removed a part of the risk of the financial network.

In the case of one bank defaulting, the most likely scenario is the bank n°4 defaulting. This can also be explained by the exposures matrix. The bank n°4 is very exposed to all the other banks, but doesn't expose many banks. It absorbs all the contagion cascades, and plays the role of a central bank.

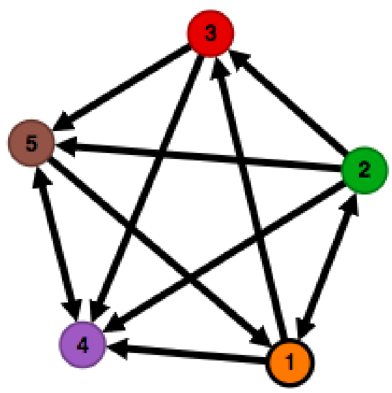

Figure 8: Directed graph of the network, showing the central role of bank 4 in the network

If we decide to make the bank n°4 less central by reducing its exposures to other banks or removing them, the financial network becomes less risky. We define a third exposures matrix :

$$E_3 = \begin{pmatrix} 0 & 3 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 \\ 3 & 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 2 & 3 & 3 & 0 \end{pmatrix}$$

The value at risk of the default impact for a threshold $\alpha = 0.999999$ is $VaR = 64.347671$, which means the VaR has decreased by 9.953%.

To sum up, we can increase the safety of the network by :

- Reducing or removing the exposure of bank 1 to bank 5.
- Decentralizing bank n°4 by reducing its exposure to other banks.

8 Effects of connectivity and size with random financial Networks

The previous results were computed for the specific matrix E. In reality, networks between institutions are more complex. Precisely, Inter-banks networks in developed countries contain several thousands of nodes (For example, The federal Deposit Insurance Corporation insured 7969 institutions as of 3/18/2010). Given a description of the large-scale structure of realistic networks, it seems natural to model our network by a random graph, whose statistical properties correspond to these observations. In [1], they have taken the example of the Brazilian inter-banks network.

In the next section, we will try to see the effect of both size and connectivity in the contagion phenomena.

8.1 Too connected to fail ?

A recurrent question in the literature on financial networks is the impact of connectivity on resilience to contagion. Previous studies showed that resilience of a network cannot be simply examined by a measure of connectivity such as the average degree of links. The resilience of a network to contagion is determined inherently by the structure of the network and his exposures. In fact, concentrated and heterogeneous networks are found to be more resilient to contagion.

8.1.1 Homogeneous graphs

The relation between contagion and connectivity is non-monotonous. Although, in the case of homogeneous graphs with randomly chosen exposures. It seems very intuitive that there is a positive correlation. Hence, we have chosen to consider random binomial graphs. That is, we create a link between two edges with a probability p. After the creation of edges, we affect a randomly uniform exposure between 0 and 8 to the edges. This choices are justified by the parameters of the model (c,σ,μ...).
Using this graph's structure, we tried to plot the average I(T) (over 100 graphs of size n=20; Note that $m = 10^4$ samples of X_T are used to estimate I(T) for each graph using the naive Monte-Carlo estimator) in function of p the probability of creating an edge in these graphs (aka the connectivity of our graphs).
Our results are showed below.

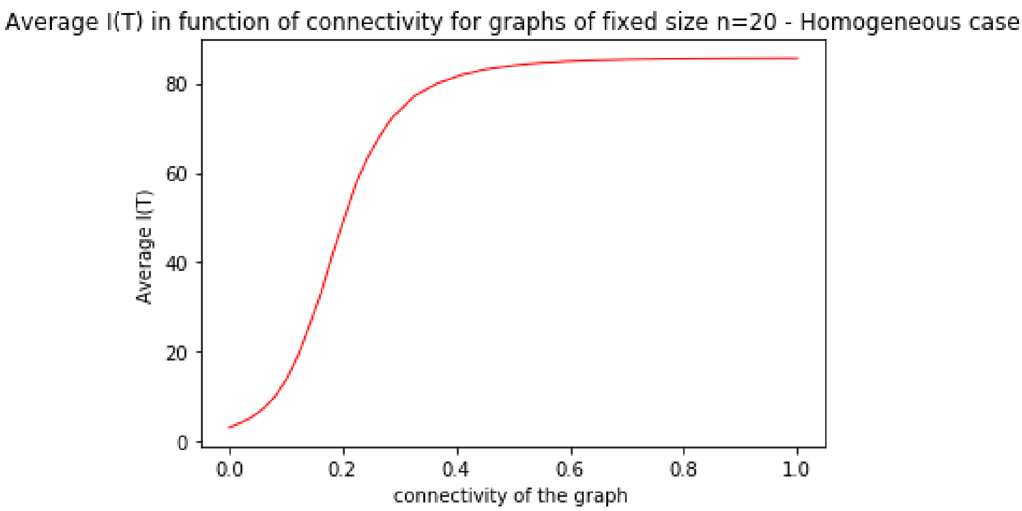

Figure 9: Average I(T) in function of connectivity for binomial graphs of size 20. Samples of X_T are generated naively ($m = 10^4$ for each graph). Average are calculated over 100 graphs with same connectivity

8.1.2 Comments and improvements

-In a homogeneous graph, the correlation between contagion and connectivity seems to be positive.
-This modeling seems to be unreal. In fact, real financial networks are heterogeneous in a sense that connectivity is not uniformly distributed. There are institutions(The big ones) which are far more connected to the network than others.
- Another approach suggested in literature is to model the financial network with Free-large scale networks using the preferential attachment model. In this model, given a sequence of out-degrees, an arbitrary out-going edge is assigned to an end-node i with probability proportional to the power $d_n^+(i)^\alpha$($d_n^+(i)$ is the number of out-going edges of node i) where α>0. This leads to positive correlation between in-degrees and out-degrees. Precisely, the distribution of out-degree is a Pareto law and the distribution of in-degree conditional to out-degrees is a Poisson distribution. This gives a more accurate modeling of real-financial networks.[3]

8.2 Size effects and contagion index

Another important question is the correlation between networks size and contagion. If we measure contagion by I(T), it seems very intuitive that this relation is strongly positive. Yet in this case I(T) is not a good indicator and we should take "averaged" magnitudes to study this correlation. Thus, we define the contagion Index $\alpha_n(E, X) = \frac{|D_{n-1}(E,X)}{n}$ where n is the size of our network (aka the number of institutions). Another time, we will use the homogeneous graphs as a model(Even if it's not the best modeling of real financial networks), simulate random graphs of different sizes (100 samples from each size) and compute average I(T) and contagion Index. We plotted our results below:

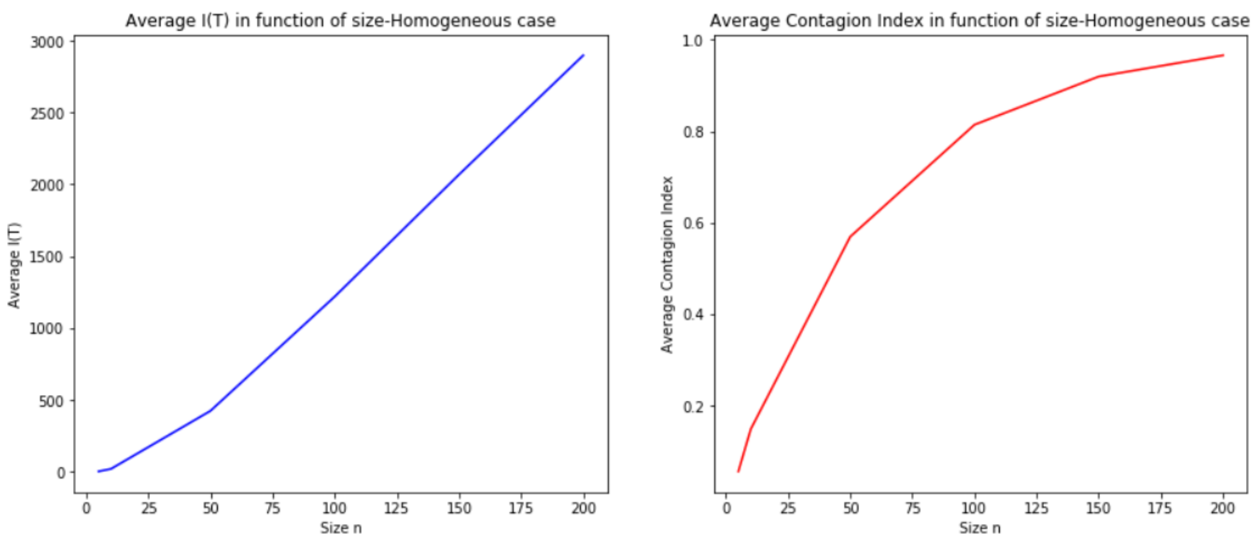

Figure 10: Correlation between size and contagion. Left: Average I(T) (Computed over 100 graphs of same size using 10^5 samples of X) in function of size.Right: Average Contagion Index for the same parameters

[3]More details are given in Andreea Minca. Modélisation mathématique de la contagion de défaut. Mathématiques [math]. Université Pierre et Marie Curie - Paris VI, 2011. Français. <tel-00624419>

Comments

- The contagion index seems to be exponential in function of size. In other terms, the presence of a large number of banks facilitate the contagion phenomena. In fact, the probability of having at least one insolvent bank at the beginning of the cascade is simply:

$$P(N>0) = 1 - P(N=0) = 1 - (P(X_T > c))^n = 1 - (1 - P(X_T < c))^n \simeq 1 - \exp nP(X_T < c)$$

using the fact that $P(X_T < c)$ is too small. This gives a simple (but not complete) explication of the graph's allure.
- Another point is that starting from n=50 the correlation between I(T) and n seems to be linear. In fact, we have plotted Average I(T) and guess the slope of the graph ... It's approximately μ!
- This graph modeling stills far from reality but it gives some good intuitions about the propagation of contagion phenomena...

9 Long term balance model

A model commonly used for default risk simulations is an balance model with one idiosyncratic factor and one systemic factor. In this model, all the capitals of the institutions have a common factor $(Z_k)_{k\geq 0}$.
This new factor governs the long term balance of the financial network. This long term capital is a gaussian process, with a dynamic given by the following equation :

$$Z(t_{k+1}) = e^{-\lambda_e \delta} Z(t_k) + \sigma_e \sqrt{\delta} \bar{W}_k$$

where $(\bar{W}_k)_{k\geq 0}$ is a new sequence of independent and identically distributed random variables following a standard normal gaussian distribution, and $\lambda_e = 10$, $\sigma_e = 3$.
This long term balance capital is added to the capital $X_i(t)$ of each institution i.

9.1 New total capitals

We observe that the random vectors $Z(t_k)$ are a linear combination of independent gaussian vectors, therefore Z(T) is also a gaussian vector whose mean and variance we can compute using the recurrence relation.

$$Z(t_k) \sim N(\mu_k^e, \sigma_k^{e2})$$

According to the recurrence relation :

$$\mu_{k+1}^e = e^{-\lambda_e \delta} \mu_k$$

$$\sigma_{k+1}^e{}^2 = e^{-2\lambda_e \delta} \sigma_k^{e2} + \sigma_e^2 \delta$$

with $\mu_0 =$ and $\sigma_0^2 = 0$. Therefore Z_T is a Gaussian vector of mean μ_T^e and variance $\sigma_T^e{}^2$ such that :

$$Z(T) \sim N(\mu_T^e, \sigma_T^e{}^2)$$

$$\mu_T^e = 0$$

$$\sigma_T^e = \sigma^e \sqrt{\delta \frac{1 - e^{-2\lambda_e}}{1 - e^{-2\lambda_e \delta}}}$$

Therefore the total capital is $X(T) = \mu + W(T) + Z(T)$ where $W(T) \sim N(0, \sigma_T^2)$ and $Z(T) \sim N(0, \sigma_T^e{}^2)$, thus :

$$X(T) \sim N(\mu_T^T, \sigma_T^{T2})$$

where :

$$\mu_T^T = \mu$$

and

$$\sigma_T^T = \sqrt{\sigma^{e2}\delta\frac{1-e^{-2\lambda_e}}{1-e^{-2\lambda_e\delta}} + \sigma^2\delta\frac{1-e^{-2\lambda}}{1-e^{-2\lambda\delta}}}$$

The variance of the simple model we used previously is $V(X) = 5.5306$, and the new variance of the long term balance model is $V(X) = 6.4552$.

9.2 Probability of the number of defaults using naive Monte Carlo method

We use the previous Monte Carlo method on the new model on a sample of size $m = 10^5$.

Number	Probability	Precision	std
0	0.88759	0.00195	0.3158
1	0.05259	0.0013	0.2232
2	0.01534	0.0007	0.1229
3	0.01148	0.0006	0.1065
4	0.01312	0.0007	0.1137
5	0.01988	0.0008	0.1395

Table 9: Estimation of the probability of number of defaulting banks for a sample size of $m = 10^5$ using the long term balance model

We also computed an estimation of the default impact I(T) as well as a 95% confidence interval:

$$\widehat{I(T)} = 3.5625$$

$$IC = [3.4899, 3.6352]$$

The results show that the probability of at least one bank defaulting is 11.728%, whereas it was of 8.023% in the previous model. This can be explained by the fact that the standard deviation of the gaussian process has increased, and therefore there is a greater probability for the capitals to be below the threshold c. There is also a common factor Z to the capitals of all 5 banks, which means there is a correlation between them, that is further transported by the contagion phenomenon, and which causes for more banks to default.

9.3 Estimation of the VaR and distribution of I(T)

As described above we have estimated the VaR for different values of α using 10^7 samples and the first approach described above. We obtained these results:

9.3.1 Value-At-Risk

These are our results about values of VaR for different α

Alpha	VaR	CI
0.99	60.17	[60.14,60.20]
0.999	66.05	[66.02,66.08]
0.9999	68.62	[68.57,68.67]
0.99999	70.21	[70.14,70.40]
0.999999	71.61	[71.40,71.95]

Table 10: Estimated VaR and confidence intervals using 10^7 samples for VaR generated naively. For IC we use $\alpha = 0.05$

9.3.2 Distribution of I(T)|I>0

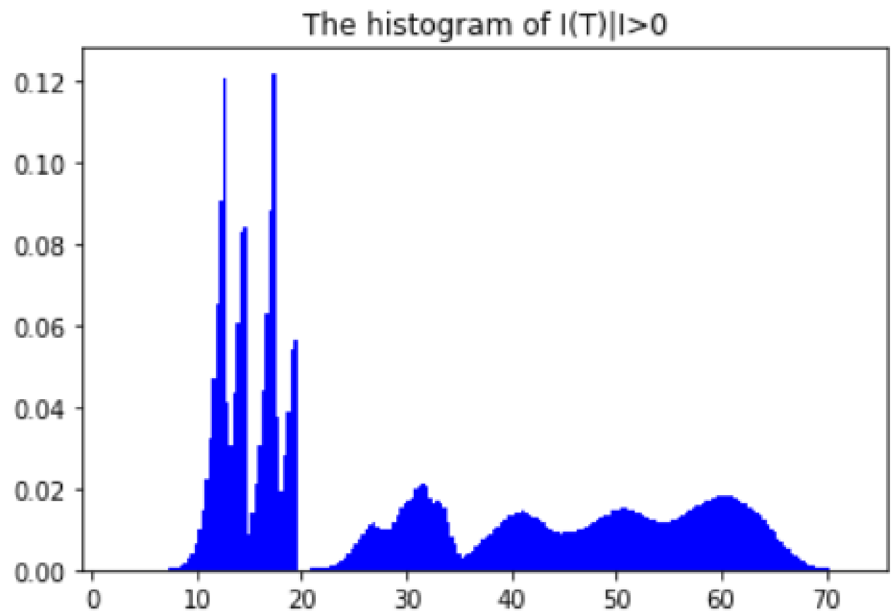

Figure 11: The histogram of I(T)|I(T)>0 using the long term model. Number of samples: 1114142 generated naively

9.3.3 Distribution of I(T)|"Systemic Contagion"

This time, the distribution of I(T) under the condition of systemic contagion is clearly a gaussian distribution as showed below.

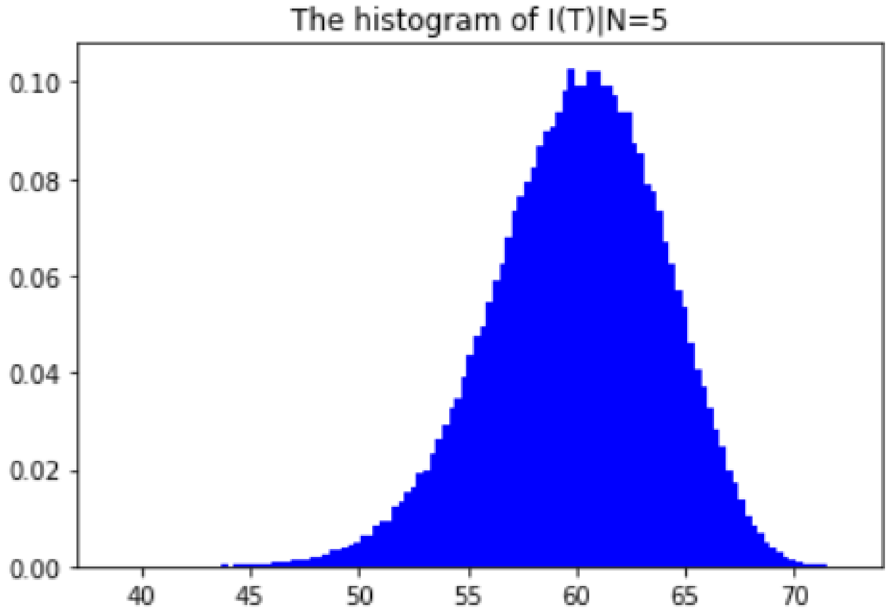

Figure 12: The histogram of I(T)|"Systemic contagion using the long term model. Number of samples: 1114142 generated naively

9.4 Insolvency cascades

We have computed the most likely insolvency cascades with respect to the number of defaulting banks like in a previous paragraph.

Number of insolvent banks	Insolvency cascade	Conditional probability of occurrence
1	4	25.29%
2	$5 \to 1$	36.96%
3	$5 \to 1 \to 4$	18.22%
4	$3 \to 5 \to 1 \to 4$	8.72%
5	$2 \to 3 \to 4 \to 5 \to 1$	4.21%

Table 11: Insolvency cascades of the long term balance model on a sample of size $m = 10^5$

We observe that bank n°4 is still the one that defaults first, and is still considered dangerous and very exposed in this model as well. The link $5 \to 1$ is also still dangerous in this model as well.

10 Dynamic contagion model

In fact, the insolvency of an institution may occur in any time before our horizon T. Hence, it's interesting to take the contagion risk in a dynamic way. This leads us to define the dynamic impact of risk $I(T_k), k = 1, ..., N$ and hence the total impact by summing all our default impacts between 0 and T.

$$I(T) = \sum_{i=1}^{N} I(t_k) \tag{4}$$

We will then see the implication of this changing on the model. First, the computation of the insolvency cascade becomes far more complicated (And the complexity is also multiplied by 12) because we should update the values of capitals each month using the formula:

$$X(t_{k+1}) = e^{-\lambda\delta}X(t_k) + \mu(1 - e^{-\lambda\delta}) + \sigma\sqrt{\delta}W_k$$

We will skip directly to results. You can see the details of implementation on our scripts.

10.1 Probabilities over the number of insolvent banks

We consider the same parameters as above (E,c,σ...) and we use a naive monte-carlo estimator to compute prbabilities. These are our results:

Number	Probability	Precision
0	0.3785	0.0030
1	0.2560	0.0027
2	0.1462	0.0021
3	0.0951	0.0018
4	0.0664	0.0015
5	0.0576	0.0014

Table 12: Estimation of the probability of number of defaulting banks for a sample size of $m = 10^5$ using the dynamic contagion model

We also computed an estimation of the default impact I(T) using the same method as well as a 95% confidence interval:

$$\widehat{I(T)} = 17.0034$$

$$IC = [16.9022, 17.1045]$$

Comments

- The probabilities of insolvency of 1 and 2 banks increase dramatically (respectively by factors of 7 and 14) but the others probabilities didn't change a lot. This can be explained by the fact that the probability of a bank becoming insolvent (simply due to fluctuations) between 0 and T is greater than the simple probability of being insolvent at horizon T.
- The average value of I(T) increases which reflects the intensity of dynamic contagion.

10.2 Distribution of I(T)|I>0

We have plotted here the distribution of I(T)|I>0 using 10^5 samples generated naively. We can already see the increasing in I(T) values comparing to the last two models. We notice also that the five separated blocks we had seen are no more as separable as before.

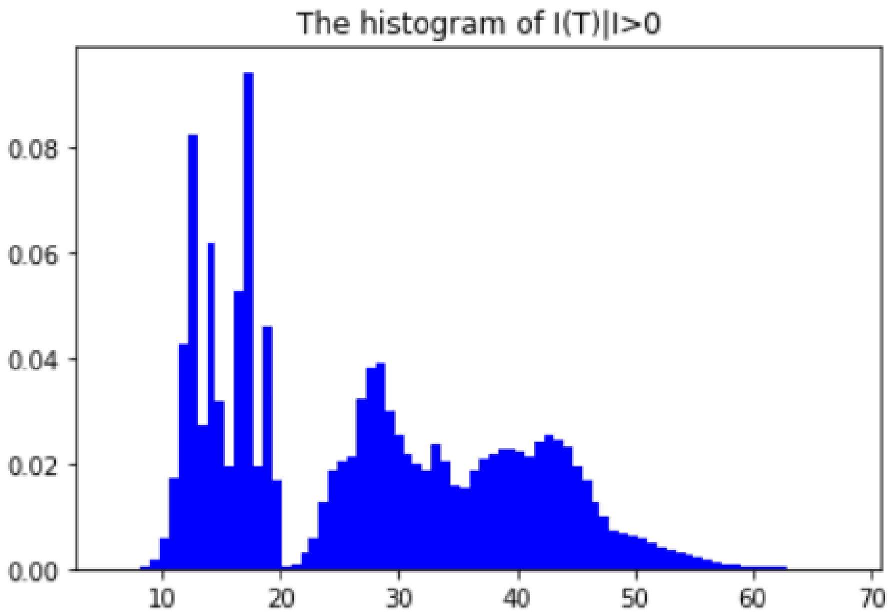

Figure 13: The histogram of I(T)|I(T)>0 using the long term model. Number of samples: 1114142 generated naively

10.3 Distribution of I(T)|"Systemic Contagion"

We plotted also the distribution of I(T) under the systemic contagion. This time, this distribution seems to be a superposition of Gaussians. It is also more probable and dangerous than before which the importance of taking contagion in dynamic way.

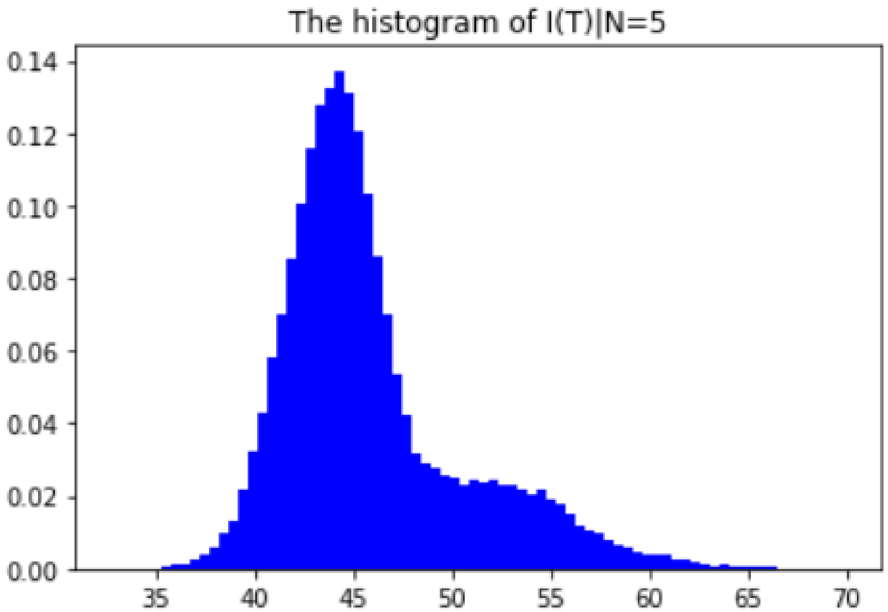

Figure 14: The histogram of I(T)|"Systemic contagion" using the dynamic contagion model. Number of samples used 57718

10.4 Variation of I(T) within time

An important question is the evolution of I(T) over time. To illustrate it, we have plotted the averaged I(T) (We launched the algorithm on 100 macro-run and compute average I(T) for each month. This variation seems to be linear, but in reality it's a very smoothed exponential and we can see it if plot just $I(t_k)$ generated each month by contagion phenomena.

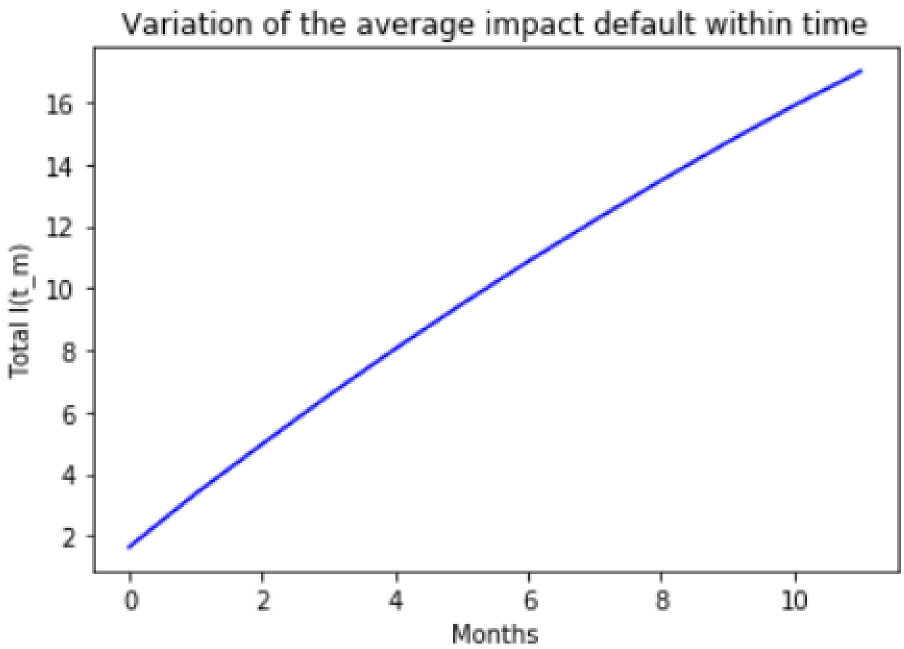

Figure 15: The averaged total I(T) within time using the dynamic model. Number of macro-runs:100. Number of samples used to compute contagion: 10^5.

10.5 Value-At-Risk

We have also computed the value of VaR for different α. Results are presented below.

Alpha	VaR	CI
0.99	52.03	[51.96,52.11]
0.999	58.64	[58.51,58.79]
0.9999	63.20	[62.96,63.61]
0.99999	66.53	[65.98,67.56]
0.999999	69.93	[67.80,69.93]

Table 13: Estimated VaR and confidence intervals using 10^6 samples for dynamic contagion model. For IC we use $\alpha = 0.05$

10.6 Insolvency cascade

Once again, we were interested in the insolvency cascades. We compute the group of banks wich defaults with the highest probability of every number of defaults.

Number of insolvent banks	Insolvent group	Conditional probability of occurrence
1	4	26.35%
2	1 , 5	21.86%
3	1, 4 , 5	31.25%
4	1, 3, 4, 5	44.65%

Table 14: Insolvency cascades of the long term balance model on a sample of size $m = 10^5$

We observe that the insolvent groups are the same in this model as in the previous one, and the probabilities of each scenario are very close to the previous ones. This is due to the exposures matrix and the link $5 \to 1$ which is still considered dangerous.

11 On the road to a new world financial crisis?

The aim of this section is to apply the previous model to a real world situation, and to compute the most likely insolvency cascades.
We will apply the model on a network of 10 countries with bilateral exposures to one another, and we will compute the contagion cascade. The exposures matrix for the countries considered is the following :

	US	GB	DE	FR	IT	FI	TR	SE	GR	CL
US	0	1095468	491136	529732	35143	379	4781	35410	5434	1558
GB	520953	0	465682	281450	46050	2324	2790	43516	13441	64
DE	175515	175165	0	260927	257408	2557	4152	77171	5185	55
FR	202470	269083	189401	0	42440	3434	1578	12794	2089	58
IT	35071	66387	162285	392577	0	711	439	1094	584	24
FI	10400	3943	13773	7860	1198	0	1	121991	2	2
TR	19745	24682	18306	23873	4017	0	0	234	30451	5
SE	22778	19033	34244	13119	2745	3405	166	0	24	8
GR	7320	14060	26059	56740	4085	27	107	145	0	0
CL	7846	2948	4542	3318	813	0	0	100	0	0

Table 15: Countries bilateral exposures. Source : Bank for International Settlements - ultimate risk basis dataset.

The initial capitals of each country are not the same as previously considered, but are different from country to country to best describe the reality. We chose initial capitals proportional to every country's GDP.

US	GB	DE	FR	IT	FI	TR	SE	GR	CL
21410230	3022580	4416800	3060070	2261460	304133	961655	628802	235836	295844

Table 16: Initial capitals

The threshold c is also chosen variable in this case, to be proportional to every country's initial capital. The recovery rate is R=0.05. Under these hypotheses, the probability of at least one default is :

$$P(\#D_T \geq 1) = 4, 1.10^{-6}$$

We then computed the most likely insolvency cascade :
Greece
↓
Great Britain, Germany, France, Turkey
↓
US, Italy, Finland, Sweden, Chile

Although this scenario has a very small probability (210^{-6}, it is the most likely scenario where all countries default in the end, and it shows where the contagion would start from and how it will propagate in the financial network, and therefore it will allow us to find solution to increase the safety of the network.

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Disclaimer : The models used in this paper is not ours, and all errors remain ours.

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