Problem Set 5

Constrained Optimization and Numerical Solvers

General Information

Dates:	– Issued: Friday, October 14 – Due: Thursday, October 20
Submission:	It is your responsibility to provide enough detail such that we can follow your approach and judge your solution. Please provide Matlab code and SIMULINK block diagrams as you see fit. Your solution must be submitted on *Gradescope* before 23h59.

Problem 5.1

Consider the following constrained optimization problem

$$\text{minimize} \quad ||\mathbf{x}||_2^2 \qquad \text{subject to} \quad \mathbf{a}^T\mathbf{x} = b,$$

where $\mathbf{a} \in \mathbb{R}^n, b \in \mathbb{R}$, and $\mathbf{a} \neq \mathbf{0}$. Using the optimality conditions, find an optimal solution $\mathbf{x}^*$ to the problem. Is the solution unique? What is the optimal value $||\mathbf{x}^*||_2^2$?

Problem 5.2

Recall the robot example of Lecture 9 (pages 2–3 in my handwritten notes). The goal is to move the robot along a straight line. We model the robot as a unicycle (see Figure 1) with sampled inputs v_k and ω_k at time step k, where v_k represents a *predefined*, desired forward velocity and ω_k the turn rate input. We will use the turn rate command ω_k to move the vehicle onto the line and keep it on the line.

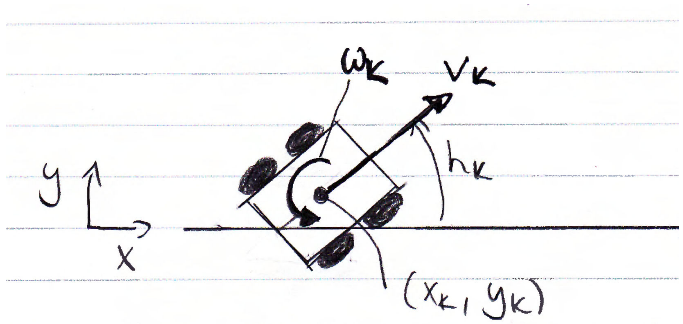

Figure 1: Unicycle-type robot traveling along a straight line.

The discrete-time dynamic equations describing the robot behavior are given by

$$\mathbf{x}_{k+1} = \begin{bmatrix} y_{k+1} \\ h_{k+1} \end{bmatrix} = \begin{bmatrix} y_k \\ h_k \end{bmatrix} + h \begin{bmatrix} v_k \sin h_k \\ \omega_k \end{bmatrix}, \tag{1}$$

where h_k is the heading of the robot at time step k, h is the discrete time step (in seconds), v_k is given, and ω_k is to be optimized. The turn rate is constrained by

$$\omega_{min} \leq \omega_k \leq \omega_{max}. \tag{2}$$

The goal of the optimization problem is to find a sequence $\{\omega_k\}, k = 0, \dots, N-1$, that minimizes the following cost function for a given initial condition (y_0, h_0) :

$$\sum_{k=1}^{N} \mathbf{x}_k^T \mathbf{Q} \mathbf{x}_k + \sum_{k=0}^{N-1} r\omega_k^2, \tag{3}$$

where $\mathbf{Q} \in \mathbb{R}^{2\times 2}$ and $r \in \mathbb{R}$ are given weights that trade-off between tracking accuracy and control effort with $\mathbf{Q} = \mathbf{Q^T} \geq 0$ and $r > 0$.

a) We make a few simplifying assumptions: *(i)* assume that angles h_k are small and $\sin h_k \approx h_k$, *(ii)* assume that the pre-scheduled forward velocity v_k is constant with $v_k = v$ for all k. Write Equation (1) in standard linear form:

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}\omega_k. \tag{4}$$

b) Based on part a), formulate the optimization problem. Assume that we optimize over $\mathbf{x}_k$, $k = 1 \dots N$, <u>and</u> ω_k, $k = 0 \dots N-1$, with the system dynamics being equality constraints. Write the equations for $N = 3$, that is, the variable to optimize is $\mathbf{x} = [\omega_0, \omega_1, \omega_2, \mathbf{x}_1^T, \mathbf{x}_2^T, \mathbf{x}_3^T]^T$. State objective function, equality and inequality constraints in terms of $\mathbf{x}$.

   Note: You will end up with large matrices and vectors.

c) Is the problem in b) convex? Provide a brief explanation.

d) Now solve the problem in b) using Matlab's `quadprog` function. Assume $N = 1000$, $h = 0.01$ [s], $v = 1$ [m/s], $\omega_{min} = -\pi/4$ [rad/s], $\omega_{max} = \pi/4$ [rad/s], $y_0 = 1.5$ [m], $h_0 = 0$ [rad], $r = 1$, and $\mathbf{Q}$ is a diagonal matrix with ones on the diagonal. Provide your Matlab code and plots of the sequences $\{y_k\}$ and $\{h_k\}$ over time.

e) How does the result in d) change as you change $\mathbf{Q}$? What happens if the diagonal entries are chosen to be different values? Provide plots to support your answer.

f) The optimal sequence $\{\omega_k\}$ found in d) is used to drive the robot with initial condition: $y_0 = 1.5$ [m], $h_0 = 0$ [rad]. However, the robot dynamics are now corrupted by an unknown disturbance such that

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}\omega_k + \begin{bmatrix} 0 \\ h\pi/25 \end{bmatrix}. \tag{5}$$

   Simulate the robot with the new dynamics in Equation (5) and the optimal sequence $\{\omega_k\}$ found in d). Provide your Matlab code and plots of the sequences $\{y_k\}$ and $\{h_k\}$ over time. What is the problem here?

g) BONUS QUESTION: We now run the optimization in a feedback loop. We solve the same optimization problem as we did in d). However, we only apply the first input value ω_0 to the robot at (y_0, h_0) and simulate the robot dynamics for one time step using Equation (5). We obtain $\mathbf{x}_1$. We use this state as new initial condition and solve another optimization problem based on this initial condition and the parameters given in part d). We again only apply the first input of the newly obtained input sequence $\{\omega_k\}$ and simulate the dynamics for one time step using Equation (5). We obtain $\mathbf{x}_2$. We continue in this fashion till we obtain $\mathbf{x}_{1000}$.

   Provide plots of the sequences $\{y_k\}$ and $\{h_k\}$ over time. How is the result different to the result in part f)?

   This feedback approach is called "Model Predictive Control". It is an advanced, optimization-based feedback controller. The benefits of this controller compared to a PID controller are: (i) it takes input and state constraints explicitly into account, and (ii) it is more intuitive to tune through changes to the weights $\mathbf{Q}$ *and* r.

Additional Practice Problems (not marked)

Problem 5.3

Consider the following optimization problem

$$\text{minimize} \;\; x_1^2 + (x_2+1)^2 \qquad \text{subject to} \;\; -1 \leq x_1 \leq 1, \;\; x_2 \geq 0.$$

Use the optimality conditions to show that the vector $(x_1, x_2) = (0, 0)$ is a unique optimal solution.

Problem 5.4

We want to design a box without a top (see figure below). The box must be able to hold a fixed volume V. The goal is to minimize the surface area in order to save material cost when producing the box. Assume $V = 1$, determine the optimal dimensions of the box (x_1^*, x_2^*, x_3^*). Show that you found a global minimum.