The Input-State Linearization Solvability

Mathematical Details

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Nomenclature

x, y, Z	scalars
$x(y), F(z)$	scalar-valued functions with scalar argument
$\mathbf{x}, \mathbf{y}, \mathbf{z}$	vectors
$T(\mathbf{x})$	scalar-valued function with vector argument
$\mathbf{f}(\mathbf{x}), \mathbf{T}(\mathbf{x})$	vector-fields: vector-valued functions with vector argument
$\mathcal{D}$	a set of vector fields (its span is called a *distribution*)
$\mathbf{A}, \mathbf{F}$	matrices
R^n	vector space of real-valued variables in n dimensions
$\mathcal{L}_{\mathbf{f}} T(\mathbf{x})$	Lie derivation of scalar $T(\mathbf{x})$ with respect to vector $\mathbf{f}$

What is Desired?

Given the process dynamics in terms of state $\mathbf{x} \in \mathrm{R}^n$:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})u \tag{1}$$

one desires a coordinate transformation:

$$\mathbf{z} = \mathbf{T}(\mathbf{x})$$

so that process dynamics in terms of new state $\mathbf{z}$ take this form:

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{b}\beta(\mathbf{z})[u + \alpha(\mathbf{z})]. \tag{2}$$

- functions $\alpha(\mathbf{z}), \beta(\mathbf{z})$ are scalar-valued nonlinearities
- pair $\mathbf{A}, \mathbf{b}\beta(\mathbf{z})$ is a type of *Brunovsky form*

Summary of the state transformation problem

First, find a $T_1(\mathbf{x})$ satisfying:

$$\begin{aligned} \mathcal{L}_{\mathbf{g}} T_1(\mathbf{x}) &= 0 \\ \mathcal{L}_{\mathbf{g}} \mathcal{L}_{\mathbf{f}} T_1(\mathbf{x}) &= 0 \\ &\vdots \\ \mathcal{L}_{\mathbf{g}} \mathcal{L}_{\mathbf{f}}^{n-2} T_1(\mathbf{x}) &= 0 \\ \mathcal{L}_{\mathbf{g}} \mathcal{L}_{\mathbf{f}}^{n-1} T_1(\mathbf{x}) &\neq 0 \end{aligned}$$

Then, recursively compute $\mathbf{T}(\mathbf{x})$:

$$\begin{aligned} T_2(\mathbf{x}) &= \mathcal{L}_{\mathbf{f}} T_1(\mathbf{x}) \\ T_3(\mathbf{x}) &= \mathcal{L}_{\mathbf{f}}^2 T_1(\mathbf{x}) \\ &\vdots \\ T_n(\mathbf{x}) &= \mathcal{L}_{\mathbf{f}}^{n-1} T_1(\mathbf{x}) \end{aligned}$$

Q: Can $T_1(\mathbf{x})$ be found?
A: Depends on $\mathbf{f}(\mathbf{x})$ and $\mathbf{g}(\mathbf{x})$!

Some mathematical preliminaries

1. The Lie bracket
2. Jacobi identity

The Lie Bracket

Given two vectors fields $\mathbf{f}(\mathbf{x})$, $\mathbf{g}(\mathbf{x}) \in \mathcal{R}^n$, the *Lie bracket* is a third vector field defined by:

$$\underbrace{[\mathbf{f}, \mathbf{g}]}_{n\times 1} = \underbrace{\nabla_{\mathbf{x}} \mathbf{g}(\mathbf{x})}_{n\times n} \cdot \underbrace{\mathbf{f}(\mathbf{x})}_{n\times 1} - \underbrace{\nabla_{\mathbf{x}} \mathbf{f}(\mathbf{x})}_{n\times n} \cdot \underbrace{\mathbf{g}(\mathbf{x})}_{n\times 1}.$$

Another notation is $ad_{\mathbf{f}}\mathbf{g}$. With this notation, orders of Lie brackets are recursively defined:

$$\begin{aligned} ad_{\mathbf{f}}^0 \mathbf{g} &= \mathbf{g}(\mathbf{x}) \\ &\vdots \\ ad_{\mathbf{f}}^n \mathbf{g} &= ad_{\mathbf{f}} \left(ad_{\mathbf{f}}^{n-1} \mathbf{g} \right) \\ &= [\mathbf{f}, ad_{\mathbf{f}}^{n-1} \mathbf{g}] \end{aligned}$$

Example: Lie brackets applied to the LTI state equation

$$\dot{\mathbf{x}} = \underbrace{\mathbf{Ax}}_{\mathbf{f}} + \underbrace{\mathbf{b}}_{\mathbf{g}} u.$$

Then,

$$\begin{aligned}
ad_{\mathbf{f}}^0 \mathbf{g} &= \mathbf{b} \\
ad_{\mathbf{f}} \mathbf{g} &= [\mathbf{Ax}, \mathbf{b}] = \nabla_{\mathbf{x}} \mathbf{b} \cdot \mathbf{Ax} - \nabla_{\mathbf{x}} \mathbf{Ax} \cdot \mathbf{b} \\
&= -\mathbf{Ab} \\
ad_{\mathbf{f}}^2 \mathbf{g} &= [\mathbf{Ax}, -\mathbf{Ab}] = \nabla_{\mathbf{x}}(-\mathbf{Ab}) \cdot \mathbf{Ax} - \nabla_{\mathbf{x}} \mathbf{Ax} \cdot -\mathbf{Ab} \\
&= \mathbf{A}^2 \mathbf{b} \\
&\vdots \\
ad_{\mathbf{f}}^{n-1} \mathbf{g} &= (-1)^n \mathbf{A}^{n-1} \mathbf{b}
\end{aligned}$$

Compare to the linear controllability matrix for $(\mathbf{A}, \mathbf{b})$!

Jacobi identity

Given:

- two vectors fields $\mathbf{f}(\mathbf{x})$ and $\mathbf{g}(\mathbf{x})$,
- a scalar-valued function $h(\mathbf{x})$

Then:

$$\mathcal{L}_{ad_{\mathbf{f}}\mathbf{g}} h = \mathcal{L}_{\mathbf{f}} \mathcal{L}_{\mathbf{g}} h - \mathcal{L}_{\mathbf{g}} \mathcal{L}_{\mathbf{f}} h.$$

Interpretation: The left-hand side is the directional derivative of $h(\mathbf{x})$ along the direction of the vector defined by the Lie bracket $[\mathbf{f}, \mathbf{g}]$.

Jacobi identity and the state transformation problem

Recall $T_1(\mathbf{x})$ must satisfy:

$$
\begin{aligned}
\mathcal{L}_{\mathbf{g}} T_1(\mathbf{x}) &= 0 \\
\mathcal{L}_{\mathbf{g}}\mathcal{L}_{\mathbf{f}} T_1(\mathbf{x}) &= 0 \\
&\vdots \\
\mathcal{L}_{\mathbf{g}}\mathcal{L}_{\mathbf{f}}^{n-2} T_1(\mathbf{x}) &= 0 \\
\mathcal{L}_{\mathbf{g}}\mathcal{L}_{\mathbf{f}}^{n-1} T_1(\mathbf{x}) &\neq 0
\end{aligned}
\quad \longrightarrow
$$

Apply each equation to the Jacobi identity's 2nd term (red):

$$\mathcal{L}_{ad_{\mathbf{f}}\mathbf{g}} h = \mathcal{L}_{\mathbf{f}}\mathcal{L}_{\mathbf{g}} h - \mathcal{L}_{\mathbf{g}}\mathcal{L}_{\mathbf{f}} h.$$

where $h = T_1$ . . .

The first equation:

$$\mathcal{L}_{ad_{\mathbf{f}}^{0}\mathbf{g}} T_1 = \mathcal{L}_{\mathbf{g}} T_1 = 0$$

The next $n-2$ equations:

$$\begin{aligned}\mathcal{L}_{ad_{\mathbf{f}}\mathbf{g}} T_1 &= \mathcal{L}_{\mathbf{f}}\mathcal{L}_{\mathbf{g}} T_1 - \mathcal{L}_{\mathbf{g}}\mathcal{L}_{\mathbf{f}} T_1 \\ &= \mathcal{L}_{\mathbf{f}} 0 - \underbrace{\mathcal{L}_{\mathbf{g}} T_2}_{0} = 0\end{aligned}$$

$$\vdots$$

$$\mathcal{L}_{ad_{\mathbf{f}}^{n-2}\mathbf{g}} T_1 = 0$$

The n-th equation:

$$\mathcal{L}_{ad_{\mathbf{f}}^{n-1}\mathbf{g}} T_1 \neq 0$$

After applying the Jacobi identity...

Original conditions on $T_1(\mathbf{x})$:

$$\begin{aligned}
\mathcal{L}_{\mathbf{g}} T_1(\mathbf{x}) &= 0 \\
\mathcal{L}_{\mathbf{g}} \mathcal{L}_{\mathbf{f}} T_1(\mathbf{x}) &= 0 \\
&\vdots \\
\mathcal{L}_{\mathbf{g}} \mathcal{L}_{\mathbf{f}}^{n-2} T_1(\mathbf{x}) &= 0 \\
\mathcal{L}_{\mathbf{g}} \mathcal{L}_{\mathbf{f}}^{n-1} T_1(\mathbf{x}) &\neq 0
\end{aligned}$$

$$\longrightarrow$$

Are equivalent to:

$$\begin{aligned}
\mathcal{L}_{ad_{\mathbf{f}}^0 \mathbf{g}} T_1 &= 0 \\
\mathcal{L}_{ad_{\mathbf{f}} \mathbf{g}} T_1 &= 0 \\
&\vdots \\
\mathcal{L}_{ad_{\mathbf{f}}^{n-2} \mathbf{g}} T_1 &= 0 \\
\mathcal{L}_{ad_{\mathbf{f}}^{n-1} \mathbf{g}} T_1 &\neq 0
\end{aligned}$$

Equivalent equations re-written using matrix notation

Conditions on $T_1(\mathbf{x})$:

$$
\begin{aligned}
\mathcal{L}_{ad_{\mathbf{f}}^0\mathbf{g}} T_1 &= 0 \\
\mathcal{L}_{ad_{\mathbf{f}}\mathbf{g}} T_1 &= 0 \\
&\vdots \\
\mathcal{L}_{ad_{\mathbf{f}}^{n-2}\mathbf{g}} T_1 &= 0 \\
\mathcal{L}_{ad_{\mathbf{f}}^{n-1}\mathbf{g}} T_1 &\neq 0
\end{aligned}
$$

Rewritten as a $1 \times n$ row:

$$
\Rightarrow \quad \underbrace{(\nabla_{\mathbf{x}} T_1)}_{1\times n} \underbrace{\begin{bmatrix} \mathbf{g} & ad_{\mathbf{f}}\mathbf{g} & \cdots & ad_{\mathbf{f}}^{n-1}\mathbf{g} \end{bmatrix}}_{n\times n} = \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 0 \\ \neq 0 \end{bmatrix}'}_{1\times n}
$$

Two problems in one!

$$(\nabla_{\mathbf{x}} T_1)\left[\mathbf{g} \;\; ad_{\mathbf{f}}\mathbf{g} \;\; \cdots \;\; ad_{\mathbf{f}}^{n-1}\mathbf{g}\right] = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \neq 0 \end{bmatrix}'$$

1. The first $n-1$ rows must equal zero. This is an under-determined problem.
2. The full n rows characterize a nonzero solution $(\nabla_{\mathbf{x}} T_1)$.

Solving the under-determined problem

The first problem:

$$(\nabla_{\mathbf{x}} T_1)\left[\mathbf{g} \;\; ad_{\mathbf{f}}\mathbf{g} \;\; \cdots \;\; ad_{\mathbf{f}}^{n-1}\mathbf{g}\right] = \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}'}_{n-1}$$

is under-determined, and has a solution locally if and only if the set of vector fields

$$\left\{\mathbf{g}, ad_{\mathbf{f}}\mathbf{g}, \cdots, ad_{\mathbf{f}}^{n-2}\mathbf{g}\right\}$$

satisfy an integrability condition called *involutivity*.

What is a test for Involutivity?

A set of vectors

$$\mathcal{D} = \{\mathbf{v_1}, \mathbf{v_2}, \ldots\}$$

is involutive if and only if

$$[\mathbf{v_i}, \mathbf{v_j}] \in \mathcal{D} \text{ for all } i, j.$$

In other words, the Lie bracket of any two vectors must lie in the span of the set.

Examples of involutivity test

Example 1: involutive

$$\{\mathbf{v_1}, \mathbf{v_2}\} = \left\{ \begin{bmatrix} 2x_3 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -x_1 \\ -2x_2 \\ x_3 \end{bmatrix} \right\}$$

$$[\mathbf{v_1}, \mathbf{v_2}] = \begin{bmatrix} -4x_3 \\ 2 \\ 0 \end{bmatrix}$$

$$= -2\mathbf{v_1} + 0\mathbf{v_2}.$$

Example 2: Not involutive

$$\{\mathbf{v_1}, \mathbf{v_2}\} = \left\{ \begin{bmatrix} 2x_1 \\ -1 \\ x_3 \end{bmatrix}, \begin{bmatrix} -x_1 \\ x_2 \\ x_3 \end{bmatrix} \right\}$$

$$[\mathbf{v_1}, \mathbf{v_2}] = \begin{bmatrix} 0 \\ -1 \\ -2x_3 \end{bmatrix}$$

$[\mathbf{v_1}, \mathbf{v_2}]$ is independent of $\{\mathbf{v_1}, \mathbf{v_2}\}$

Solving the second problem

The full, non-zero problem:

$$(\nabla_{\mathbf{x}} T_1) \underbrace{\begin{bmatrix} \mathbf{g} & ad_{\mathbf{f}}\mathbf{g} & \cdots & ad_{\mathbf{f}}^{n-1}\mathbf{g} \end{bmatrix}}_{\mathcal{D}} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \neq 0 \end{bmatrix}'$$

has a solution if and only if matrix $\mathcal{D}$ is full rank. The Lie brackets

$$\mathbf{g}, ad_{\mathbf{f}}\mathbf{g}, \cdots, ad_{\mathbf{f}}^{n-1}\mathbf{g}$$

must be independent.

Summary

Input-state linearization necessary & sufficient conditions

A nonlinear system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})u, \ \mathbf{x} \in \mathrm{R}^n$$

can be input-state linearized to a linear controllable canonic form, if and only if:

- The set of vector fields

$$\left\{\mathbf{g}, ad_{\mathbf{f}}\mathbf{g}, \cdots, ad_{\mathbf{f}}^{n-2}\mathbf{g}\right\}$$

  is involutive, and
- The Lie brackets

$$\mathbf{g}, ad_{\mathbf{f}}\mathbf{g}, \cdots, ad_{\mathbf{f}}^{n-1}\mathbf{g}$$

  are independent.