Input-State Linearization

Two Examples

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Example 1: A 2nd Order Problem

Consider the plant

$$\dot{x}_1 = a \sin x_2 \tag{1}$$
$$\dot{x}_2 = -x_1^3 + u \tag{2}$$

or

$$\dot{x} = f(x) + g(x)u$$

where

$$f(x) = \begin{bmatrix} a \sin x_2 \\ -x_1^3 \end{bmatrix}, \qquad g(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The Problem

Find a transformation $z = T(x)$ such that:

- the plant dynamics appear as:

$$\dot{z} = Az + B(\text{nonlinear function including } u) \tag{3}$$

  where (A, B) are in controllable canonic form.
- Then the input u can be designed to cancel undesirable terms of the nonlinear function.

Set Up the Problem

The problem is second order, so the transformation $T(x)$ consists of two functions:

$$T(x) = \begin{bmatrix} T_1(x) \\ T_2(x) \end{bmatrix}$$

Since (A, B) are in controllable canonic form, the transformation must satisfy these conditions:

$$\begin{aligned} T_2(x) &= \mathcal{L}_f T_1(x) \\ &= \nabla_x T_1(x) f(x) \qquad (4) \\ \mathcal{L}_g T_1(x) &= \nabla_x T_1(x) g(x) \\ &= 0 \qquad (5) \\ \mathcal{L}_g T_2(x) &= \mathcal{L}_g \mathcal{L}_f T_1(x) \\ &\neq 0. \qquad (6) \end{aligned}$$

Study condition (5)

$$
\begin{aligned}
\mathcal{L}_g T_1(x) &= 0 \\
\nabla_x T_1(x) g(x) &= 0 \\
\begin{bmatrix} \frac{\partial T_1}{\partial x_1} & \frac{\partial T_1}{\partial x_2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} &= 0 \\
\frac{\partial T_1}{\partial x_2} &= 0 \qquad (7)
\end{aligned}
$$

Therefore, the function $T_1(x)$ must be independent of the variable x_2.

Study condition (6)

$$
\begin{aligned}
\mathcal{L}_g \mathcal{L}_f T_1(x) &\neq 0 \\
\mathcal{L}_g \left(\nabla_x T_1(x) f(x) \right) &\neq 0 \\
\mathcal{L}_g \left(\begin{bmatrix} \frac{\partial T_1}{\partial x_1} & \frac{\partial T_1}{\partial x_2} \end{bmatrix} \begin{bmatrix} a \sin x_2 \\ -x_1^3 \end{bmatrix} \right) &\neq 0 \\
\mathcal{L}_g \left(\frac{\partial T_1}{\partial x_1} a \sin x_2 \right) &\neq 0 \\
\nabla_x \left(\frac{\partial T_1}{\partial x_1} a \sin x_2 \right) \begin{bmatrix} 0 \\ 1 \end{bmatrix} &\neq 0 \\
\frac{\partial}{\partial x_2} \left(\frac{\partial T_1}{\partial x_1} a \sin x_2 \right) &\neq 0 \\
\frac{\partial T_1}{\partial x_1} a \cos x_2 &\neq 0
\end{aligned}
\tag{8}
$$

Summary of conditions for $T_1(x)$

From (7) and (8):

1. $T_1(x)$ must not be a function of x_2.
2. $T_1(x)$ should not be zero when $\cos x_2$ is nonzero.

An acceptable solution is $T_1(x) = x_1$.
Another solution is $T_1(x) = \sin x_1$. Solutions are not unique!

Finish the transformation $T(x)$

The function $T_2(x)$ is computed after choosing a function $T_1(x)$, using the condition (4):

$$\begin{aligned} T_2(x) &= \mathcal{L}_f T_1(x) \\ &= \begin{bmatrix} \frac{\partial T_1}{\partial x_1} & \frac{\partial T_1}{\partial x_2} \end{bmatrix} f(x) \\ &= [1 \quad\quad 0] \begin{bmatrix} a \sin x_2 \\ -x_1^3 \end{bmatrix} \\ &= a \sin x_2 \end{aligned}$$

The complete transformation is given by:

$$z = T(x) = \begin{bmatrix} T_1(x) \\ T_2(x) \end{bmatrix} = \begin{bmatrix} x_1 \\ a \sin x_2 \end{bmatrix}. \tag{9}$$

Plant dynamics in new coordinate z

$$
\begin{aligned}
\dot{z}_1 &= \dot{x}_1 \\
&= a \sin x_2 \\
&= z_2 && (10) \\
\dot{z}_2 &= a \cos x_2 \dot{x}_2 \\
&= a \cos x_2 (-x_1^3 + u) && (11)
\end{aligned}
$$

Indeed,

$$\dot{z} = Az + B(\text{nonlinear function including } u)$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The control u

From (11), the control u that linearizes the plant dynamics (in z coordinates) can be determined. Furthermore, a new input variable v can be introduced. The control is given by:

$$u = x_1^3 + \frac{1}{a \cos x_2} v \tag{12}$$

The input-state linearized dynamics (in z coordinates) are described by:

$$\dot{z} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} z + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v$$

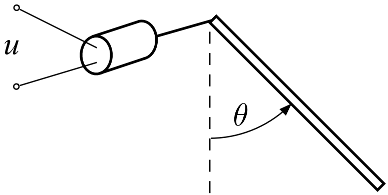
Linear system techniques can now be used to design a linear feedback law for v, so that the closed loop dynamics of are asymptotically stable!

Input-State Linearization Comments

- The math can be quite complicated.
- The linearization may not be global – see the singularity in (12)
- The transformed coordinates z may not be physically meaningful, which means state feedback may not be feasible. A state estimator becomes necessary.

Example 2: A 3rd Order Problem

Robot arm driven by electric motor



Nomenclature

θ: arm angle, u: motor input voltage

J: moment of inertia, B: friction coefficient

L: motor winding inductance, R: winding resistance

K: motor voltage and torque constant, g: gravity constant

Robot arm model

Plant dynamics are given by:

$$J\ddot{\theta} + B\dot{\theta} + MgL\sin\theta = Ki$$
$$L\frac{di}{dt} + Ri + K\dot{\theta} = u$$

State variable model

Let $x_1 = \theta$, $x_2 = \dot{\theta}$, $x_3 = i$. Then

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{MgL}{J}\sin x_1 - \frac{B}{J}x_2 + \frac{K}{J}x_3 \\
\dot{x}_3 &= -\frac{K}{L}x_2 - \frac{R}{L}x_3 + \frac{1}{L}u
\end{aligned}$$

or (the simplified form):

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\sin x_1 - x_2 + x_3 \\
\dot{x}_3 &= -x_2 - x_3 + u
\end{aligned} \tag{13}$$

State variable model (continued)

In the form $\dot{x} = f(x) + g(x)u$

$$f(x) = \begin{bmatrix} x_2 \\ -\sin x_1 - x_2 + x_3 \\ -x_2 - x_3 \end{bmatrix}, \quad g(x) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The nonlinear coordinate transformation

$$z = T(x) = \begin{bmatrix} T_1(x) \\ T_2(x) \\ T_3(x) \end{bmatrix}$$

$T_1(x)$ must satisfy:

$$\begin{aligned} \mathcal{L}_g T_1 &= 0 \\ \mathcal{L}_g \mathcal{L}_f T_1 &= 0 \\ \mathcal{L}_g \mathcal{L}_f^2 T_1 &\neq 0 \end{aligned}$$

After finding T_1, the remaining functions are given by:

$$T_2 = \mathcal{L}_f T_1, \qquad T_3 = \mathcal{L}_f T_2 = \mathcal{L}_f^2 T_1$$

Study conditions on $T_1(x)$

From the first condition on T_1:

$$\mathcal{L}_g T_1 = 0 \quad \rightarrow \quad \frac{\partial T_1}{\partial x_3} = 0$$

Conclusion: T_1 must not be a function of x_3.

Continue search for $T_1(x)$

From the second condition on T_1:

$$\begin{aligned} \mathcal{L}_g \mathcal{L}_f T_1 = 0 \quad &\rightarrow \quad \mathcal{L}_g \left[\begin{array}{ccc} \frac{\partial T_1}{\partial x_1} & \frac{\partial T_1}{\partial x_2} & 0 \end{array} \right] f(x) = 0 \\ &\rightarrow \quad \frac{\partial}{\partial x_3} \left(\frac{\partial T_1}{\partial x_1} x_2 + \frac{\partial T_1}{\partial x_2} (-\sin x_1 - x_2 + x_3) \right) = 0 \end{aligned}$$

Oh, this looks really tough!
Another way to look at the second condition:

$$\mathcal{L}_g T_2 = 0 \quad \rightarrow \quad \frac{\partial T_2}{\partial x_3} = 0$$

So, $T_2(x)$ must not be a function of x_3.

Solving the conditions

Suggestion:

- Make some simplifying choices, and
- Check that all conditions are satisfied

For example, consider letting

$$\frac{\partial T_1}{\partial x_2} = 0.$$

Then the second condition simplifies to

$$\frac{\partial}{\partial x_3}\left(\frac{\partial T_1}{\partial x_1} x_2\right) = 0$$

One solution is $T_1(x) = x_1$ (the arm angle).

Finishing the coordinate transformation $T(x)$

$$
\begin{aligned}
z_1 &= T_1 = x_1 \\
z_2 &= T_2 = \mathcal{L}_f T_1 = x_2 \\
z_3 &= T_3 = \mathcal{L}_f T_2 = -\sin x_1 - x_2 + x_3
\end{aligned}
$$

or

$$
z = \begin{bmatrix} \text{arm position (angle)} \\ \text{angular rate} \\ \text{angular acceleration} \end{bmatrix}
$$

Plant dynamics in new coordinate z

$$\begin{aligned}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= z_3 \\
\dot{z}_3 &= -(\cos x_1)\dot{x}_1 - \dot{x}_2 + \dot{x}_3
\end{aligned}$$

which has the desired form

$$\dot{z} = Az + B(\text{nonlinear function including } u)$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Try these exercises...

1. Complete the problem by finding the nonlinear $u(x)$ that
   - cancels the nonlinear dynamics, and
   - introduces a new input variable v.
2. The output of the original model is

$$y = x_1 \qquad \text{(arm angle).}$$

   Show that input-output linearization yields the same result for this application.

Comments

- Input-state linearizing solutions are generally not unique for a given problem. In other words, there are many transformations $T(x)$ that are mathematically satisfactory. The challenge is to find a solution that meets engineering constraints.
- Input-output linearization may be easier to perform than input-state linearization, but internal dynamics (zero dynamics) must be checked.
- If the relative degree equals the system order, then input-output linearization is also achieves input-state linearization.

Gain Scheduling interpretation

- These techniques can be interpreted as “continuous” gain scheduling:
  - The linear feedback algorithm for v is “fixed.”
  - The nonlinear control u represents a “scheduled” modification of v, performed smoothly over all values of the scheduling variable(s).
- One difference: Classical gain scheduling rules are based on *auxiliary* variables, not feedback variables.
  - Thought: Treat auxiliary variables as feedback variables – just a matter of interpretation?
  - Then classical gain scheduling is a type of feedback linearization, with possibly discrete changes.