Sliding Mode Control Design Example: Single-link Robot Position Control

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1 The Underlying Principle

Variable structure control that produces a true sliding mode employs a discontinuous plant input (control) to force the system state onto (in a geometric sense) a surface. In the state space, the surface must represent a stable dynamic. Physically, sliding mode control can be likened to using a jackhammer in construction work, e.g. an accurate model of the problem is not required, so long as the tool is sufficiently powerful. See Fig. 1. Design of a sliding mode controller for a single-link robotic arm is presented next as an example.

Figure 1: Industrial jackhammer - an analogy to sliding mode control.

2 The Model

Consider the inverted pendulum arm shown in Fig. 2. Angular position θ is referenced from the vertical dashed line. An external torque τ is applied at the pivot point.

The plant is modeled by the nonlinear differential equation:

$$J\ddot{\theta} + B\dot{\theta} = \tau + WL\sin\theta \tag{1}$$

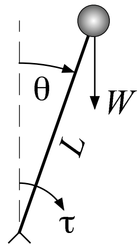

Figure 2: An inverted pendulum

Table 1: Nomenclature and parameter values

parameter	Description	Value
J	moment of inertia	1
B	friction coefficient	1
W	arm weight	10
L	distance from pivot to center of gravity	1
θ	angular position	
τ	input torque	

Choose the state variables $x_1 = \theta - \theta_{ref}, x_2 = \dot{x}_1$. Then, the nonlinear state variable model is given by:

$$\dot{x}_1 = x_2 \tag{2}$$

$$\dot{x}_2 = \frac{1}{J}(WL \sin x_1 - Bx_2 + \tau) \tag{3}$$

3 Linear Analysis and Design

The approximate, linear state variable model (linearized about the origin) is:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ WL/J & -B/J \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/J \end{bmatrix} \tau \tag{4}$$

For the given parameters (Table 1), the linear model has eigenvalues at $s_1 = 2.7, s_2 = -3.7$. The plant is unstable. A linear state feedback control

$$\tau = -30x_1 - 8x_2 \tag{5}$$

yields the closed loop eigenvalues $s_1 = -4, s_2 = -5$.

4 Sliding Mode Controller Design

4.1 Switching Surface

Choose a linear switching equation as:

$$\sigma(x) = 4x_1 + x_2 \tag{6}$$

If the system state is on the surface defined by $\sigma(x) = 0$, then the two state variables are related by:

$$x_2 = -4x_1 \tag{7}$$

Substituting (7) into the model (2) yields the "sliding mode," a reduced order dynamic given by:

$$\dot{x}_1 = -4x_1 \tag{8}$$

The sliding mode is stable, with a time constant of 1/4.

4.2 Controller Design

To produce a sliding mode, dynamics that are near the surface (7) should satisfy the relationship $\sigma\dot{\sigma} < 0$

$$\begin{aligned}
\sigma\dot{\sigma} &= \sigma(\nabla_x \sigma)\dot{x} \\
&= \sigma[4 \quad 1]\begin{bmatrix} x_2 \\ 10\sin x_1 - x_2 + \tau \end{bmatrix} \\
&= \sigma(4x_2 + 10\sin x_1 - x_2 + \tau) \\
&= \sigma 10 \sin x_1 + \sigma 3x_2 + \sigma\tau \\
&< 0
\end{aligned} \tag{9}$$

A candidate solution is to choose the input as:

$$\tau = k_1 + k_2 x_2 \tag{10}$$

where k_1 and k_2 are variable gains (switching their values). Substituting (10) into (9) yields the problem:

$$\sigma\dot{\sigma} = \sigma(k_1 + 10\sin x_1) + \sigma x_2(k_2 + 3) < 0 \tag{11}$$

One set of k_1 and k_2 values that can solve the problem is given by switching:

$$k_1 = \begin{cases} -12 & , \quad \sigma > 0 \\ 12 & , \quad \sigma < 0 \end{cases} \tag{12}$$

$$k_2 = \begin{cases} -4 & , \quad \sigma x_2 > 0 \\ 0 & , \quad \sigma x_2 < 0 \end{cases} \tag{13}$$

Notice that uncertainty or variations in the plant parameters can be overcome by sufficiently large large k_1 and k_2.

5 Simulation Results

The SIMULINK diagram for the control system is shown in Fig. 3. The linear switching equation (6) is implemented as a matrix gain block. The switching gains k_1 and k_2 are represented by relay blocks. To reduce the simulation time, lower bounds for the simulation step size are used. The drawback is that chatter will be introduced because of inaccurate computation of the relays' zero crossing times.

The simulated position error responses are plotted in Fig. 4. The reference motion is a 90° rotation of the arm, from vertical to horizontal position. In the dashed curve, one can observe the steady state error that results from using linear state feedback (5).

The response in a phase portrait response is shown in Fig. 5. Reaching and sliding modes are evident. Chatter of the state trajectory around the surface $\sigma(x) = 0$ can be observed upon close inspection.

Behavior of the sliding mode switching equation (6) is plotted in Fig. 6. The time periods for reaching and sliding are evident, as is the chatter around $\sigma = 0$.

The plant input is plotted in Fig. 7. Again, chatter is evident. The chatter is not significant at the output variable (solid curve in Fig. 4), due to the lowpass nature of the plant.

6 Practical Issues

In this example, the plant input is mechanical torque τ. In practice it is not feasible to create a switching torque. Mechanical actuators such as motors introduce additional dynamics. A more practical approach is to consider a higher order design model. For example, the model (1) can be augmented by equations that describes the motor dynamics, using voltage applied to the motor as the input variable. Switching the motor input voltage is a common practice in high performance, high efficiency motor drive electronics, so the problem of control chatter is greatly reduced.

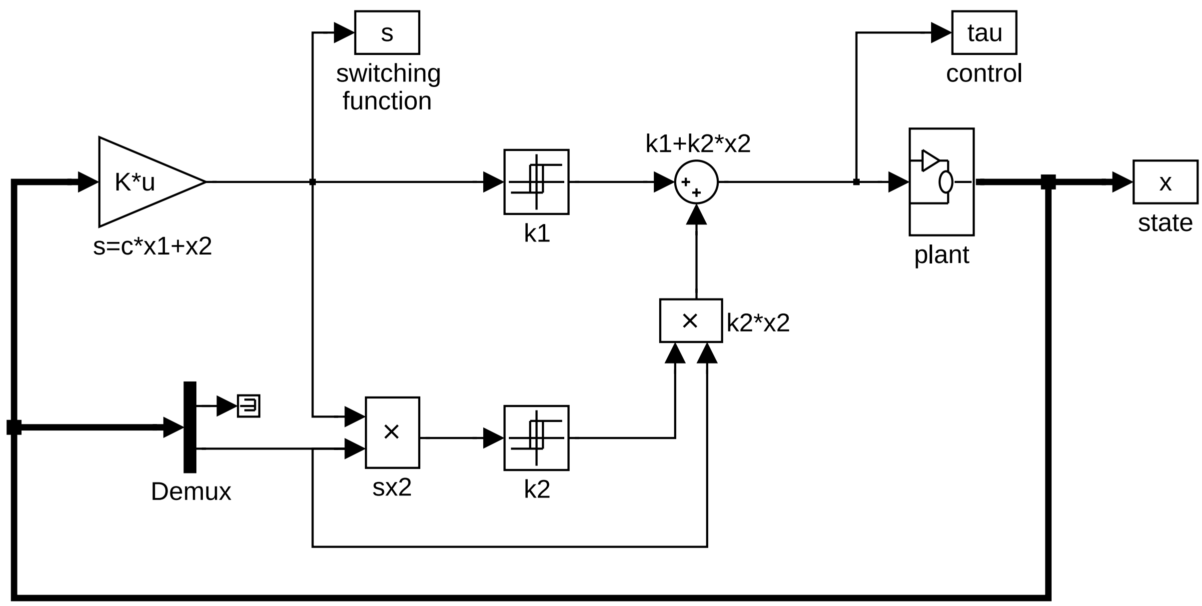

Figure 3: Variable structure system block diagram

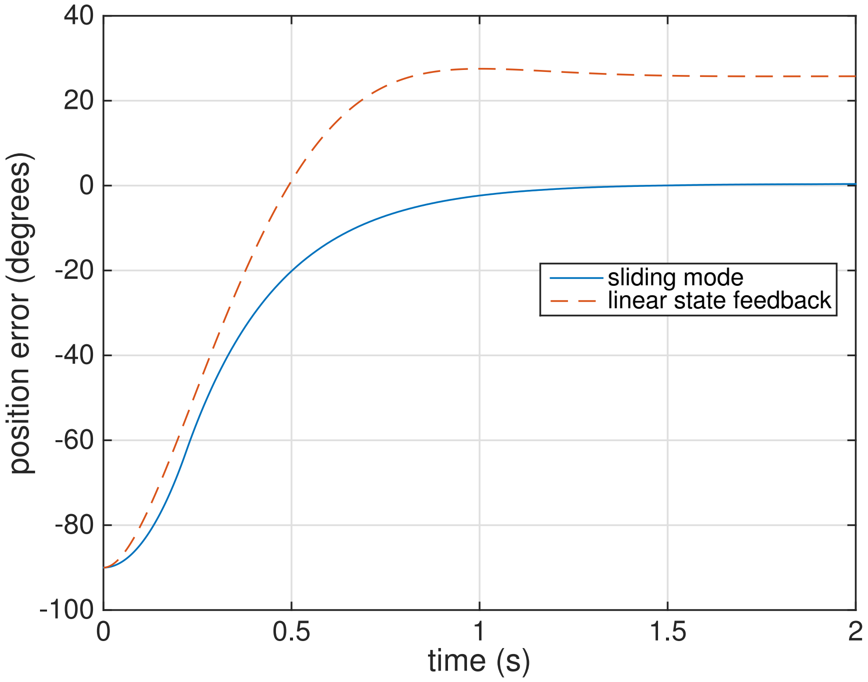

Figure 4: Position error responses to linear state feedback and sliding mode controls

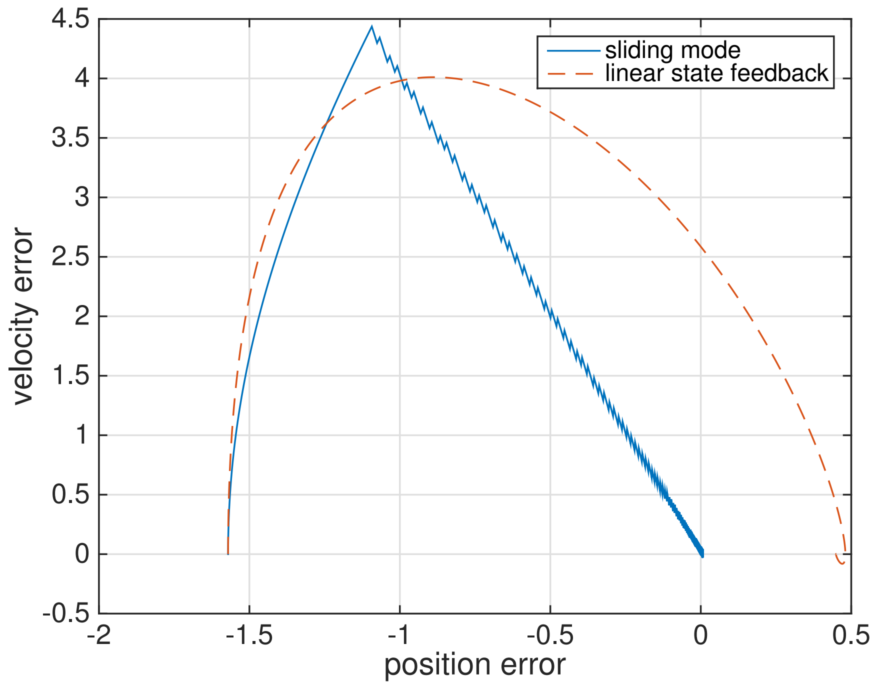

Figure 5: Comparison of phase portraits

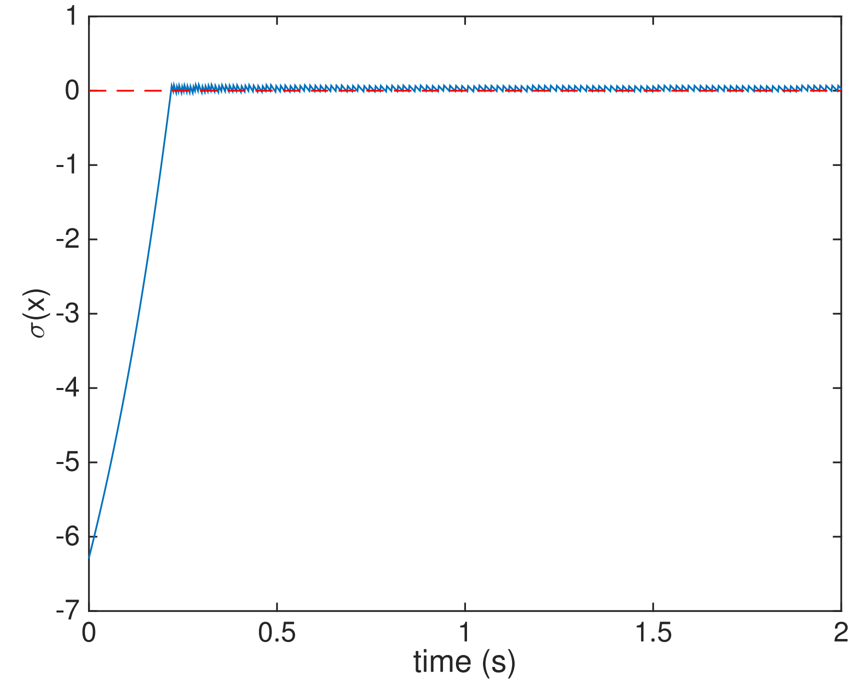

Figure 6: Switching "surface" function $\sigma(x)$

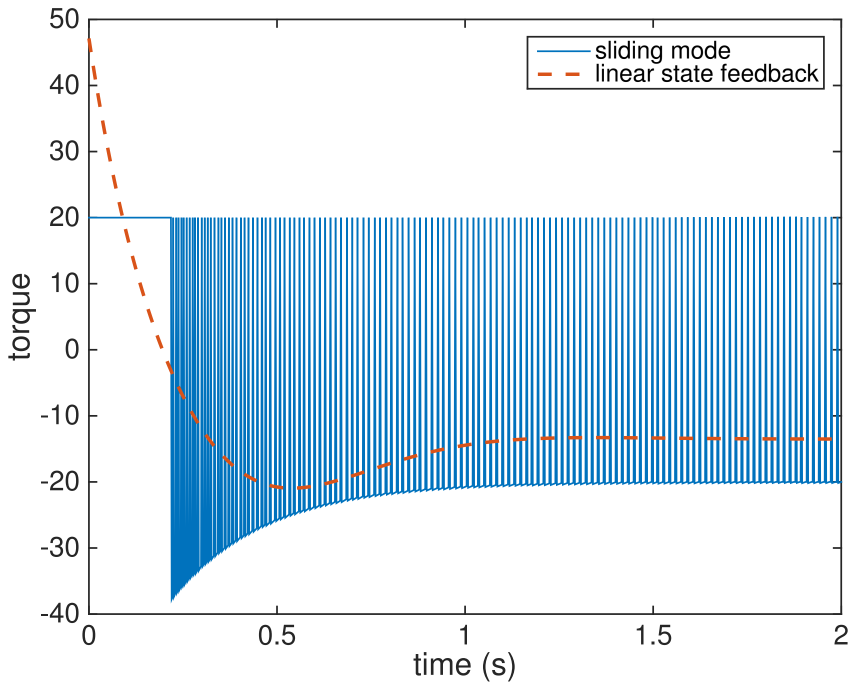

Figure 7: Plant inputs