Higher-order Sliding Mode Example

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Abstract

Higher order sliding mode is one technique that can be used when the plant input cannot switch. The idea is to augment the plant model with a filter, and then include the filter model in the overall design model. The filter input becomes the new control input, and the filter output produces the plant input. In this example, a filter is a single integrator.

1 The Plant

Consider the plant with state x and input u:

$$\dot{x} = Ax + Bu$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -a_1 & -a_2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

2 The Augmented Plant

Suppose the input u is not suited for switching control. Then augment the plant by placing a filter before the input. Call the filter input w. For example, the filter can be an integrator. The new augmented model becomes:

$$\dot{x} = Ax + Bu \tag{1}$$

$$\dot{u} = w. \tag{2}$$

For analysis and design, a state transformation can be helpful. For this example, let the new state be defined as:

$$z_1 = x_1 \tag{3}$$

$$z_2 = \dot{x}_1 = x_2 \tag{4}$$

$$z_3 = \ddot{x}_1 = \dot{x}_2 = -a_1 x_1 - a_2 x_2 + u \tag{5}$$

Therefore, the augmented model (1), (2) in new coordinates is in controllable canonic form:

$$\dot{z} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -a_1 & -a_2 \end{bmatrix} z + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} w \tag{6}$$

3 Variable Structure Control Design

The idea is to design a variable structure controller for (6), using w as the control input. In this way, the switching control will be filtered to produce the plant input u.

3.1 Choose switching function

The augmented state equation is in controllable canonic form. Let the switching function be described by:

$$\sigma(z) = c_1 z_1 + c_2 z_2 + z_3. \tag{7}$$

Then on the surface $\sigma = 0$, the state z is constrained as:

$$z_3 = -c_1 z_1 - c_2 z_2. \tag{8}$$

The sliding mode dynamic model is found by substituting (8) into (6). The resulting dynamic is reduced in order, and is described by the state equations

$$\dot{z}_1 = z_2 \tag{9}$$

$$\dot{z}_2 = -c_1 z_1 - c_2 z_2. \tag{10}$$

The characteristic equation is given by:

$$s^2 + c_2 s + c_1 = 0. \tag{11}$$

Therefore, coefficients c_1 and c_2 can be designed to yield the desired sliding mode dynamic characteristic.

3.2 Designing the variable structure control

The sliding mode can be produced by any control that satisfies the reaching condition:

$$\sigma\dot{\sigma} < 0.$$

In this example, the equivalent control approach is demonstrated. Let the control w be the sum of two terms:

$$w = w_{eq} + w_{sw}$$

where w_{eq} is the smooth, equivalent control component and w_{sw} is a switching control component.

3.2.1 Equivalent control part

The equivalent control satistfies the equation

$$\dot{\sigma} = 0,$$

or

$$(\nabla_z \sigma)\dot{z} = 0.$$

Applying the switching function and augmented state equation in the above yields:

$$\begin{bmatrix} c_1 & c_2 & 1 \end{bmatrix} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = 0$$

or

$$c_1 z_2 + c_2 z_3 - a_1 z_2 - a_3 z_3 + w = 0.$$

Therefore, the equivalent control is given by:

$$w_{eq} = (a_1 - c_1)z_2 + (a_2 - c_2)z_3. \tag{12}$$

3.2.2 Switching control part

The switching control will apply whenever the switching function $\sigma(z)$ is nonzero. A simple approach is to base the control on the sign of σ:

$$w_{sw} = -M \operatorname{sgn}(\sigma) \tag{13}$$

where $M > 0$ is a constant.

Table 1: Simulation parameters

variable	description	value
a_1	plant A-matrix coefficient 1	-1
a_2	plant A-matrix coefficient 2	0
c_1	switching function coefficient 1	50
c_2	switching function coefficient 2	10
M	switching control gain	75
$x_1(0)$	initial x_1	-1
$x_2(0)$	initial x_2	1

4 Simulation Study

The SIMULINK model is shown in Fig. 1. Signal dimensions are indicated by the small numbers on the signal lines, and bus signals use wider lines. At the center are the plant and filter (integrator). Plant state x and input u are gathered by a three-input bus creator, and a function block is used to construct the transformed state variable z_3 from the bussed variables. The function implements the transformation (5). The transformed variables z_1, z_2 are x_1, x_2, respectively. A second three-input bus creator is used to gather all of the variables in the state vector z. On the left-hand side, the bus signal (vector) z is used to create the switching function and the equivalent control. The switching function (7) is constructed by a matrix multiplication block, and values are stored the workspace variable 's'. The equivalent control (12) is constructed using a function block, and values are stored in the workspace variable 'w_eq'. A relay block is used to define the switching control, and the values are stored in the workspace variable 'w_sw'.

Simulation parameters are given in Table 1. The plant A-matrix coefficients correspond to an unstable plant with eigenvalues $s = \pm 1$. The switching function coefficients yield a sliding mode with characteristic equation

$$(s + 5)^2 + 5^2 = s^2 + 10s + 50 = 0.$$

Plotted in Fig. 2 is the response of the plant state x. The switching function response over time is plotted in Fig. 3. Note that the sliding mode starts after time $t \approx 0.55$. Shown in subfigures of Fig. 4 are the equivalent control w_{eq}, the switching control w_{sw}, control w, and the filtered plant input u. A three-dimensional plot of the state trajectory z is shown in Fig. 5.

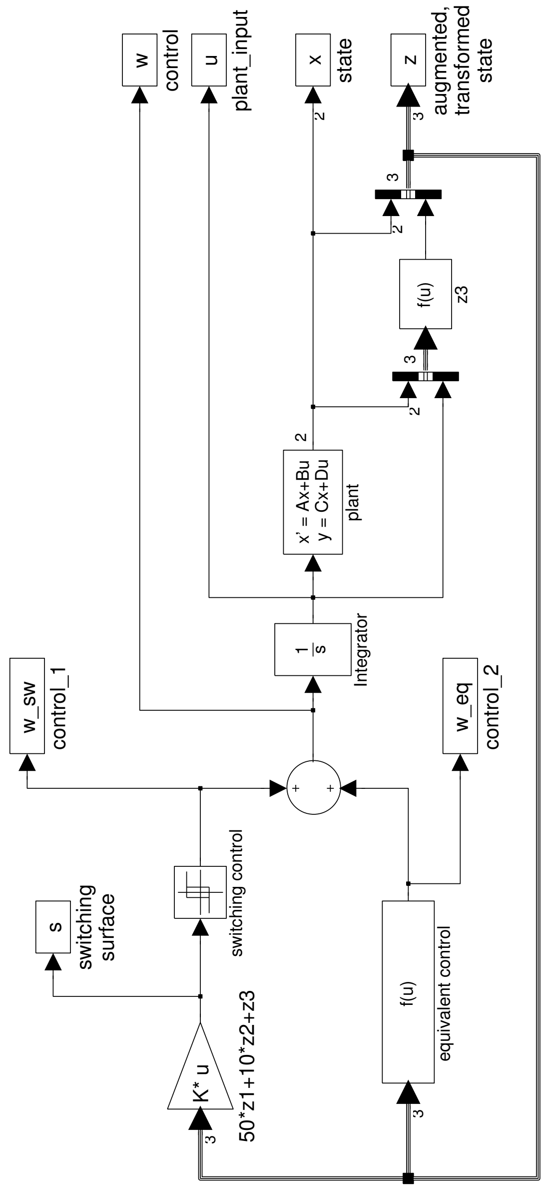

Figure 1: SIMULINK model of the higher order sliding model example.

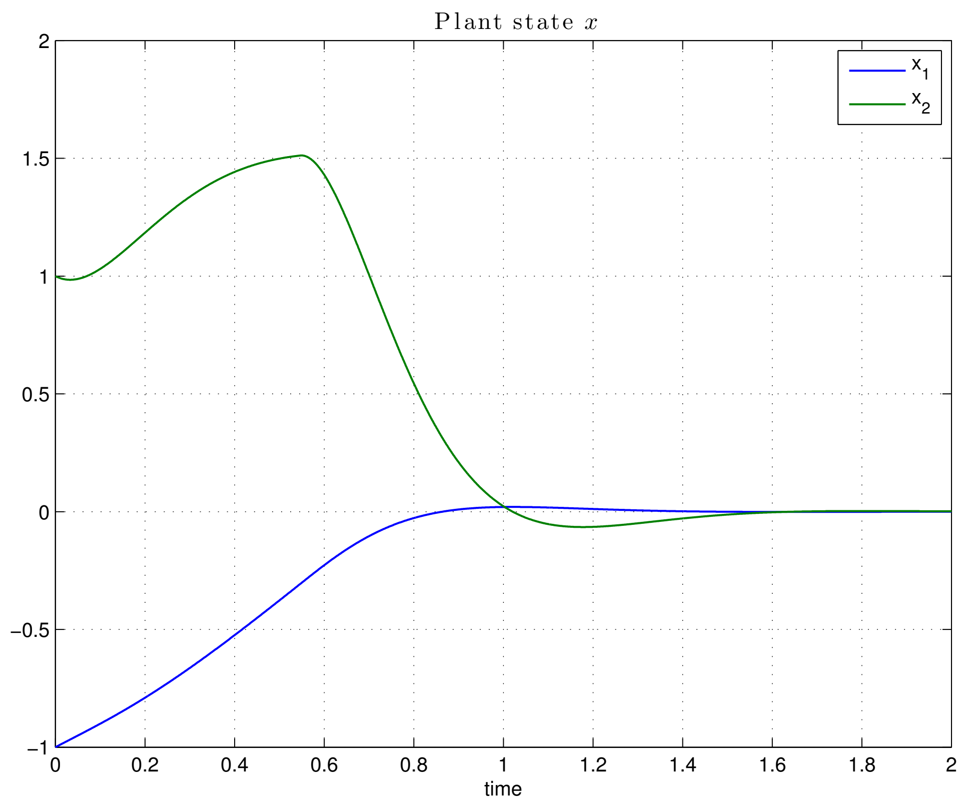

Figure 2: State response x_1 and x_2 vs. time.

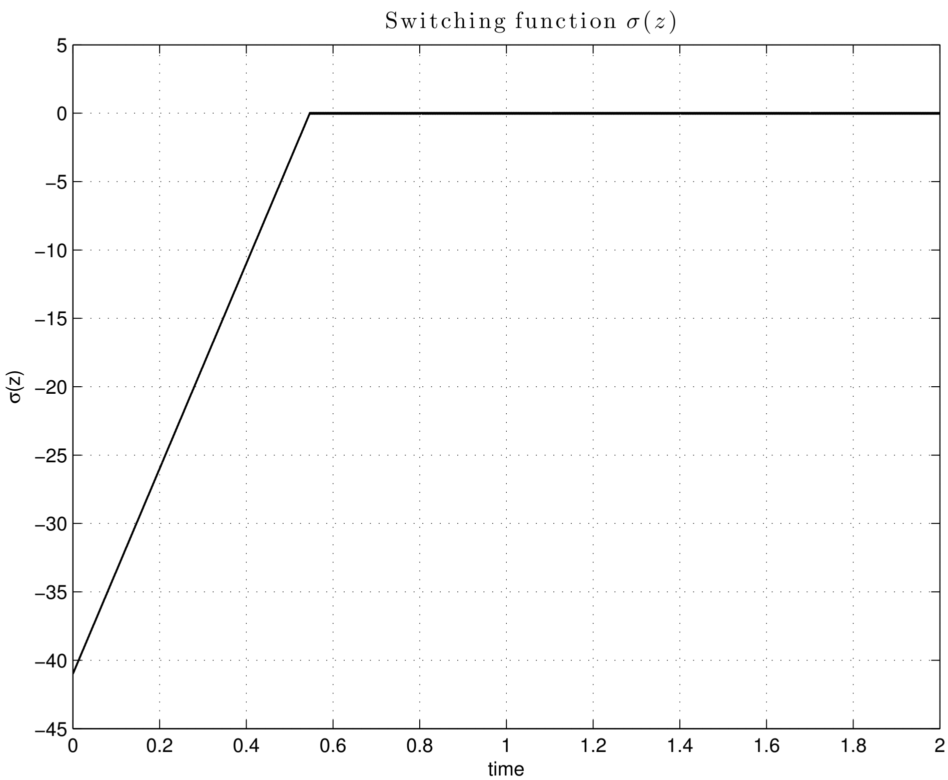

Figure 3: Switching function response $\sigma(z)$ vs. time.

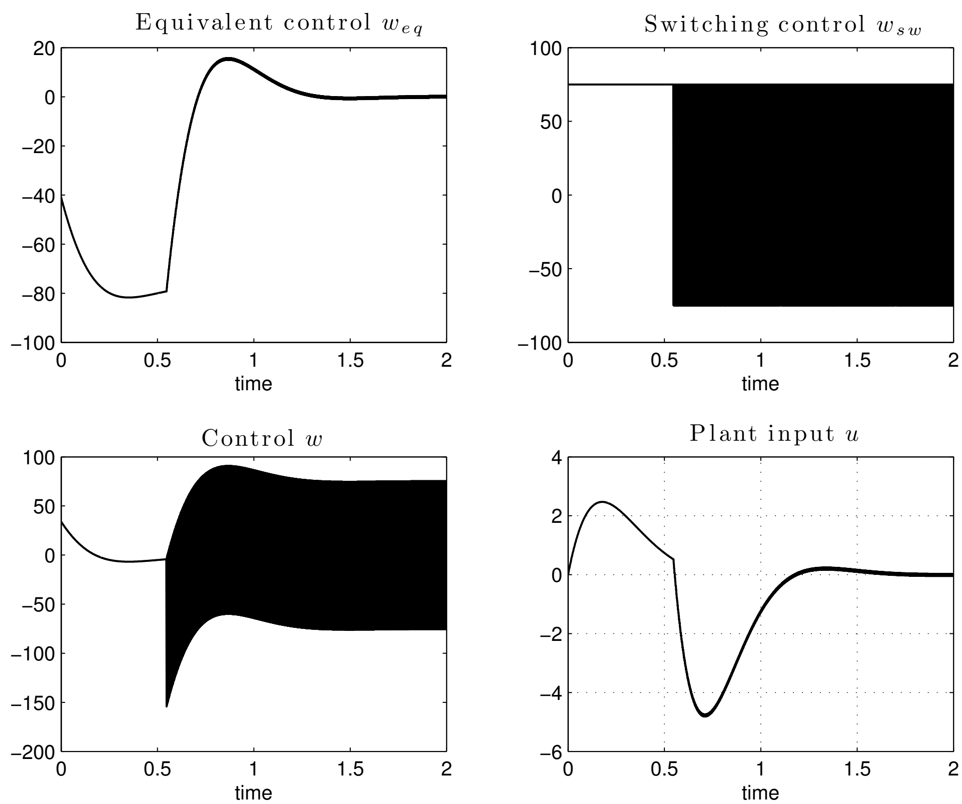

Figure 4: Equivalent control w_{eq} (upper left), switching control w_{sw} (upper right), control w (lower left), and filtered plant input u (lower right).

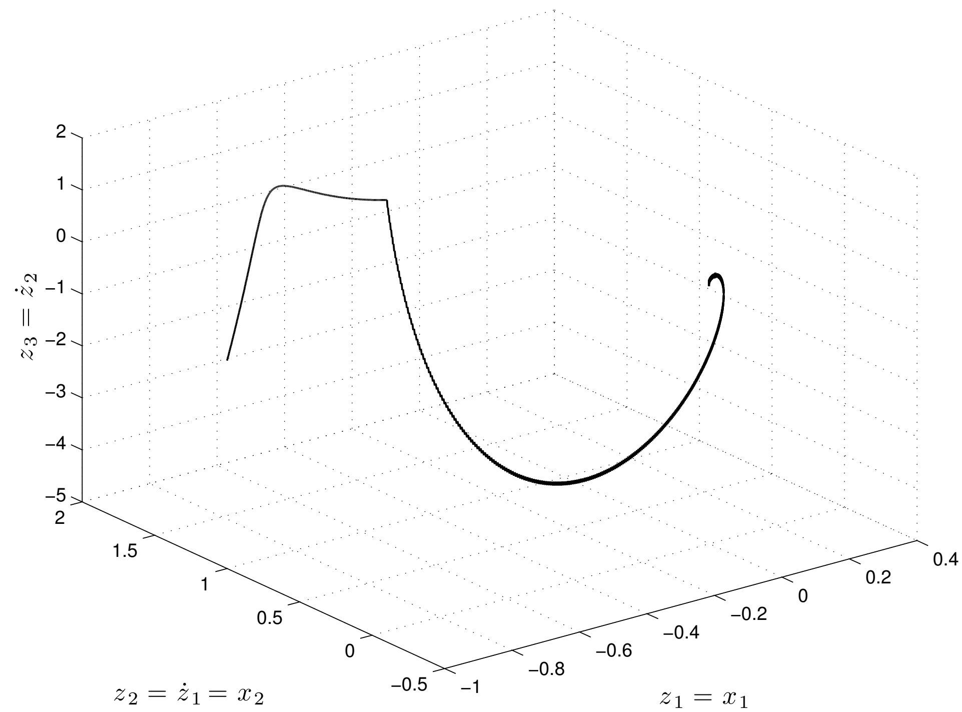

Figure 5: State trajectory z.

Appendix: MATLAB script

(Use with the SIMULINK model file 'higher_order_sliding_1'.)

```
% Higher order sliding mode example
% J. Hung, Spring 2013

clear all

% Plant model xdot = Ax + Bu, y = Cx + Du
a1 = -1;
a2 = 0;
A=[0 1; -a1 -a2]  % system matrix
e=eig(A)          % system eigenvalues
B=[0; 1]          % input matrix
C=eye(2,2)        % output matrix
D=zeros(2,1)      % feedthrough matrix

% initial state
x0=[-1 1];

% Switch surface s = c1*z1 + c2*z2 + z3
%  or s = Kz where K is [c1 c2 1] and
%  z is the 3x1 transformed state vector
% Characteristic equation has roots at -5 +/- j5
c1 = 50
c2 = 10
K=[c1 c2 1]

% Magnitude of switching component
M=75

% Simulate the model
sim('higher_order_sliding_1')

% Plot results
figure(1)
plot(t,x,'linewidth',1), grid
title('Plant state $x$','interpreter','latex','fontsize',12)
xlabel('time')
```

```
legend('x_1','x_2')

figure(2)
plot(t,s,'linewidth',1), grid
title('Switching function $\sigma(z)$','interpreter','latex','fontsize',12)
xlabel('time')
ylabel('\sigma(z)')

figure(3)
subplot(2,2,1)
plot(t,w_eq,'linewidth',1)
title('Equivalent control $w_{eq}$','interpreter','latex','fontsize',12)
xlabel('time')

subplot(2,2,2)
plot(t,w_sw,'linewidth',1)
title('Switching control $w_{sw}$','interpreter','latex','fontsize',12)
xlabel('time')

subplot(2,2,3)
plot(t,w,'linewidth',1)
title('Control $w$','interpreter','latex','fontsize',12)
xlabel('time')

subplot(2,2,4)
plot(t,u,'linewidth',1), grid
title('Plant input $u$','interpreter','latex','fontsize',12)
xlabel('time')

figure(4)
plot3(z(:,1),z(:,2),z(:,3),'linewidth',1)
grid
xlabel('$z_1 = x_1$','interpreter','latex','fontsize',14)
ylabel('$z_2 = \dot{z}_1 = x_2$','interpreter','latex','fontsize',14)
zlabel('$z_3 = \dot{z}_2$','interpreter','latex','fontsize',14)
```