Integrator Backstepping

John Y. Hung

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1 The Introductory Explanation

Backstepping is a recursive approach for nonlinear control system design. The system to be controlled is modeled as a nested system, with each level of the nest containing integrators (state variables). The design method starts with the construction of a stabilizing controller for the innermost nest, treating a state variable from a higher level nest as a pseudo-input ("pseudo-" means the input is not the real one, but is fictitious – used only for design bookkeeping). The backstepping process builds up the controller with terms that stabilize progressively more levels of the system model.

In the next section, a second-order linear example illustrates basic principles of the recursive process. Notation and principles for the nonlinear case are then presented in Sec. 3, followed by a nonlinear design example in Sec. 4.

2 A linear example

Consider the linear system shown in Fig. 1. A state variable model is given by:

$$\dot{x}_1 = x_2 \tag{1}$$

$$\dot{x}_2 = u. \tag{2}$$

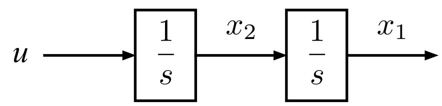

Figure 1: A second-order linear system.

2.1 Designing a pseudo-input

To start the design, a stabilizing control for the subsystem (1) is constructed. From Fig. 1, the input to the right-hand integrator is x_2, so treat the variable x_2 as the pseudo-input or pseudo-control (not a real input) of subsystem (1). Linear system design suggests a stabilizing pseudo-input to be:

$$x_2 = -a_1 x_1 \tag{3}$$

A design based on Lyapunov stability theory can arrive at the same result, and is useful practice before applying the backstepping method to a non-linear system. Consider a Lyapunov function of the single state variable x_1:

$$V_1(x_1) = \frac{1}{2}x_1^2.$$

The derivative with respect to time is given by:

$$\begin{aligned} \dot{V}_1(x_1) &= x_1\dot{x}_1 \\ &= x_1 x_2. \end{aligned}$$

Lyapunov stability theory states that the variable x_1 is asymptotically stable if $\dot{V}_1(x_1)$ is negative definite. Treating the variable x_2 as the subsystem input, and choosing it as (3) satisfies the sufficiency condition for asymptotic stability of x_1:

$$\dot{V}_1 = -a_1 x_1^2 < 0.$$

2.2 Injecting the pseudo-input to the system

Next, introduce the pseudo-input signal (3) to the system model. The idea is illustrated in Fig. 2, with the pseudo-input "entering" through a summing point. The new block diagram is useful for keeping track of signals during the design process. Notice that the block diagram introduces a new variable w:

$$w = x_2 + a_1 x_1. \tag{4}$$

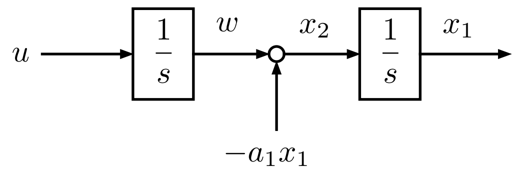

Figure 2: Introducing the first pseudo-control.

The output of the left-hand integrator was originally x_2, but is now called w with the introduction of the pseudo-input. If $w = 0$, then $x_2 = -a_1x_1$, which is the pseudo-control (3).

2.3 Backstepping: differentiating the pseudo-input

The pseudo-input (3) is fictitious in the sense that it cannot be physically injected as suggested by Fig. 2. The effect of the control must enter the system via input u. Therefore, the backstepping concepts suggests redrawing Fig. 2 by moving the pseudo-input to the input side of the integrator whose output is w.

The backstepping process propagates the pseudo-input (3) "backwards" through the integrator whose output is now called w, hence the name "integrator backstepping." The backstepping process is illustrated by the two block diagrams in the Fig. 3. Notice that the output of the left-hand integrator is no longer x_2, but has become the variable w. The backstepping process is mathematically equivalent to differentiating the new variable (4), so the input to the left-hand integrator is equivalent to:

$$\begin{aligned}\dot{w} &= \dot{x}_2 - a_1\dot{x}_1\\ &= u - a_1x_2.\end{aligned}$$

One cycle of the backstepping design process is now complete.

2.4 Start another iteration

For the next cycle, consider the problem of stabilizing dynamics of x_1 and the new variable w. The variable $\dot{w}$ is considered the new pseudo-input for the design cycle. Since a pseudo-input (3) already stabilizes x_1, a Lyapunov function of the two variables x_1 and w is constructed by building upon the Lyapunov function $V_1(x_1)$ as follows:

$$V_2(x_1, w) = V_1(x_1) + \frac{1}{2}w^2.$$

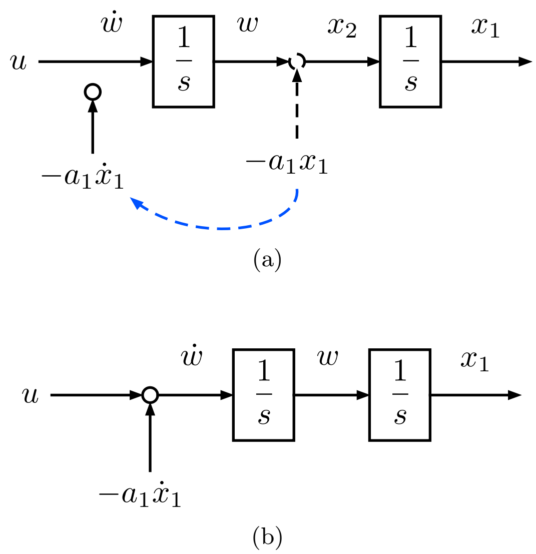

Figure 3: Backstepping the first pseudo-input: (a) move the pseudo-input to the other side of the integrator, (b) completed.

Differentiating V_2 with respect to time yields:

$$\begin{aligned}\dot{V}_2(x_1, v) &= \dot{V}_1(x_1) + w\dot{w} \\ &= -a_1 x_1^2 + w\dot{w}.\end{aligned}$$

Choosing the new, second pseudo-control as:

$$\dot{w} = -a_2 w \tag{5}$$

yields:

$$\dot{V}_2(x_1, w) = -a_1 x_1^2 - a_2 w^2$$

which is negative definite.

The new pseudo-input (5) is introduced to the system, yielding the new block diagram shown in Fig. 4.

2.5 Deriving the true input signal

After the pseudo-input (5) is introduced, there is no need for another cycle of integrator backstepping. All that remains is to derive the true input signal u in terms of the pseudo-inputs that have been backstepped or propagated

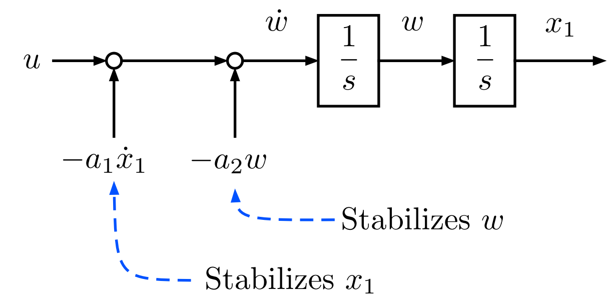

Figure 4: Introducing the second pseudo-control.

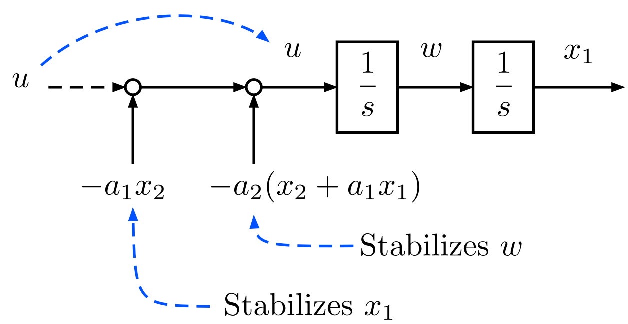

Figure 5: Deriving the true input.

back through the integrators. Fig. 4 shows the two pseudo-inputs. The equivalent true input signal u is determined by equating the input u to the backstepped pseudo-inputs, as illustrated in Fig. 5. Therefore, the true input is given by:

$$\begin{aligned} u &= -a_1\dot{x}_1 - a_2 w \\ &= -a_1 x_2 - a_2(x_2 + a_1 x_1) \\ &= -a_1 a_2 x_1 - (a_1 + a_2) x_2. \end{aligned}$$

The equivalent feedback system is shown in Fig. 6. The closed loop system has the characteristic equation given by:

$$s^2 + (a_1 + a_2)s + a_1 a_2 = 0.$$

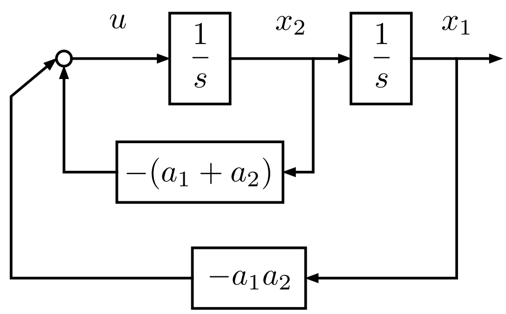

Figure 6: The completed linear design example.

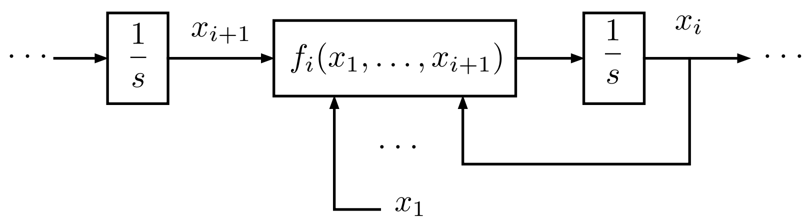

Figure 7: A nonlinear system in strict feedback form.

3 Backstepping: The general nonlinear case

Backstepping principles can be directly applied to nonlinear system that are in so-called "strict feedback" form

$$\begin{aligned}
\dot{x}_1 &= f_1(x_1, x_2) \\
\dot{x}_2 &= f_2(x_1, x_2, x_3) \\
&\vdots \\
\dot{x}_i &= f_i(x_1, \ldots, x_{i+1}) \\
&\vdots \\
\dot{x}_n &= f_n(x_1, \ldots, x_n, u)
\end{aligned} \tag{6}$$

In the strict feedback form, nonlinear functions f_i have dependence only upon state variables $x_1, \ldots, x_{i+1}$. In a block diagram (see Fig. 7), a state variable x_i appears only in feedback paths to states x_j, $j > i$, or to the nonlinear function f_{i-1} at the input to integrator x_{i-1}.

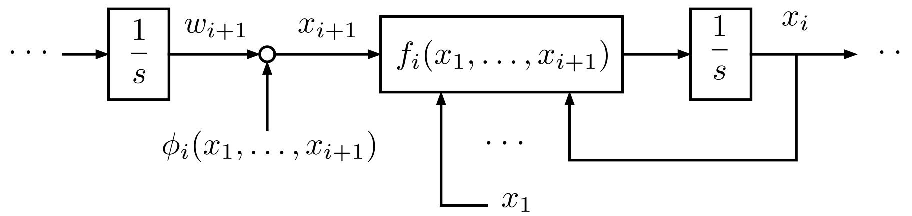

Figure 8: Injecting a pseudo-input signal ϕ_i.

The backstepping design method is a recursive approach, in which each cycle has the following steps:

1. Construct a Lyapunov function by building upon the Lyapunov function from the earlier cycle. A possible solution is:

$$V_i(x_1, \ldots, x_i) = V_{i-1}(x_1, \ldots, x_{i-1}) + \frac{1}{2}x_i^2. \tag{7}$$

2. Examine the time derivative $\dot{V}_i(x_1, \ldots, x_i)$, and design a control law that satisfies the Lyapunov criteria for asymptotic stability. In other words, the goal in this design cycle is that $\dot{V}_i$ be negative definite:

$$\begin{aligned}\dot{V}_i &= \dot{V}_{i-1} + x_i\dot{x}_i \\ &= \dot{V}_{i-1} + x_i f_i(x_1, \ldots, x_{i+1}) \\ &< 0.\end{aligned}$$

   In the given design cycle, the variable x_{i+1} is treated as the pseudo-input. To satisfy the Lyapunov stability criteria, the pseudo-input can be described by a nonlinear state feedback function:

$$x_{i+1} = \phi_i(x_1, \ldots, x_i) \tag{8}$$

3. Injecting the pseudo-control (8) is illustrated in Fig. 8. The pseudo-input introduces a new variable:

$$w_{i+1} = x_{i+1} - \phi_i(x_1, \ldots, x_i).$$

4. Perform the backstep, which is simply reconstructing an equivalent of the new variable on the input side of the integrator whose output

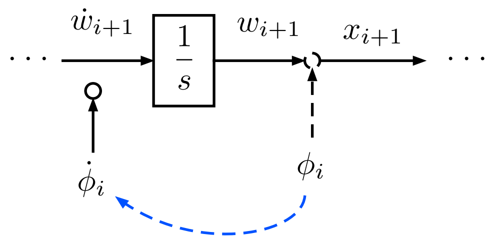

Figure 9: Backstepping the pseudo-input signal ϕ_i.

is w_{i+1}. In other words, components of the pseudo-input ϕ_{i+1} are differentiated as follows:

$$\dot{w}_{i+1} = \dot{x}_{i+1} - \dot{\phi}_i(x_1, \ldots, x_i) \tag{9}$$

and applied to the input of the integrator whose output is w_{i+1}. The output of the integrator becomes the new variable w_{i+1}. The nonlinear backstep in illustrated in Fig. 9.

5. The process repeats, starting at Step (1).

Backstepping terminates when no integrators remain. The control law for true input u is found by substituting pseudo-inputs (8) into the final backstep result (9), and solving algebraically for input u. An example is presented next.

4 A nonlinear example

Consider the system

$$\dot{x}_1 = \sin(x_1)x_2 \tag{10}$$
$$\dot{x}_2 = \cos(x_1)u \tag{11}$$

The variable x_2 is selected as the pseudo-input to subsystem (10). A candidate Lyapunov function is:

$$V_1(x_1) = \frac{1}{2}x_1^2. \tag{12}$$

Differentiating (12) yields:

$$\dot{V}_1(x_1) = x_1 \sin(x_1)x_2. \tag{13}$$

The pseudo-input x_2 is designed so that (13) is negative definite. One possible choice is:

$$x_2 = \phi_1(x_1) = -x_1 \sin(x_1). \tag{14}$$

Inject the pseudo-control at the input of the integrator whose output is x_1, and define the new variable:

$$w_2 = x_2 - \phi_1(x_1) = x_2 + x_1 \sin(x_1). \tag{15}$$

The output of the integrator that was labeled x_2 now changes to a new variable w_2, and the integrator input is equivalent to:

$$\dot{w}_2 = \dot{x}_2 + \frac{d}{dt}(x_1 \sin(x_1)) = \cos(x_1)u + \frac{d}{dt}(x_1 \sin(x_1)). \tag{16}$$

One cycle of backstepping is now complete.

In the second cycle, dynamics of w_2 may be stabilized by Lyapunov design, following a similar procedure as for the first cycle. Let the new Lyapunov function be:

$$V_2(x_1, w_2) = V_1(x_1) + \frac{1}{2}w_2^2. \tag{17}$$

The time derivative is:

$$\dot{V}_2(x_1, w_2) = \dot{V}_1(x_1) + w_2\dot{w}_2. \tag{18}$$

Design the new pseudo-control to be:

$$\dot{w}_2 = \phi_2(w_2) = -w_2 \tag{19}$$

which yields a locally negative definite result

$$\dot{V}_2(x_1, w_2) = -x_1^2 \sin^2(x_1) - w_2^2.$$

Both variables in the second-order process have been stabilized, and there is no need to backstep the second pseudo-input $\phi_2(w_2)$. The true input u must be derived from the pseudo-inputs.

The true input signal is derived by recognizing (19) must be equivalent to (16). In other words:

$$-w_2 = \cos(x_1)u + \frac{d}{dt}(x_1 \sin(x_1)).$$

Solving algebraically for u yields the true input:

$$\begin{aligned}
u &= \frac{1}{\cos(x_1)}\left[-w_2 - \frac{d}{dt}(x_1 \sin(x_1))\right] \\
&= \frac{1}{\cos(x_1)}\left[-w_2 - \dot{x}_1\left(\sin(x_1) + x_1\cos(x_1)\right)\right] \\
&= \frac{1}{\cos(x_1)}\left[-x_2 - x_1\sin(x_1) - \sin(x_1)x_2\left(\sin(x_1) + x_1\cos(x_1)\right)\right] \qquad (20)
\end{aligned}$$

A simulated response from the initial condition $x = [1 \;\; 1]^T$ is shown in Fig. 10, for time $t = [0, 100]$. The solution asymptotically approaches the origin. Note that the backstepping solution (20) is not unique, because other pseudo-controls ϕ_i can be proposed for each cycle, thus yielding different system responses.

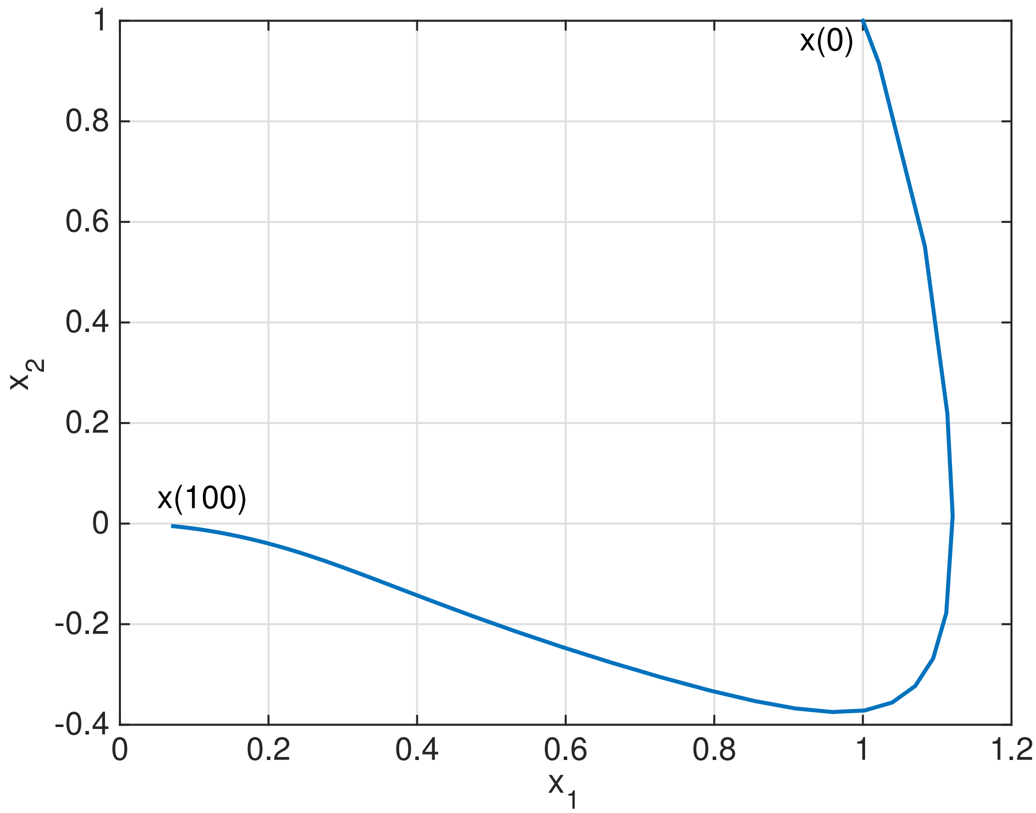

Figure 10: Simulation of backstepping control for system (10)-(11).

References

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[2] H. Khalil, *Nonlinear Systems.* Prentice-Hall, 3rd ed., 2002.