Approximate Message Passing
From Theoretical Ecology to Random Matrices

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Presentation plan

Theoretical ecology

Motivation We want to study species coexistence within large ecosystems and try to answer the following questions:

- How many species survive in a large ecosystem ?
- What is the distribution of surviving species ?

Figure: Generated with *Midjourney AI*

Lotka-Volterra system of coupled differential equations

Motivation In order to study the behavior of species coexistence within large ecosystems we will have to describe the dynamics of these interacting species and try to answer some questions:

- How many species survive at the equilibrium ?
- What are the statistical properties of these surviving species ?

Model A popular model used in this setting is given by a system of Lotka-Volterra equations:

$$\boxed{\frac{dx_i(t)}{dt} = x_i\left(r_i - x_i + \left(\frac{1}{\kappa}B\boldsymbol{x}\right)_i\right)} \qquad i \in [n]\,, \quad \boldsymbol{x} = (x_i)\,.$$

Here $(B\boldsymbol{x})_i = \sum_\ell B_{i\ell}x_\ell$.

- n is the **number of species** in a given food-web,
- $x_i = x_i(t)$ is the **abundance** (=population) of species i at time t,
- r_i is the **intrinsic growth rate** of species i,
- $B = (B_{ij})$ where B_{ij} is the **interaction** between species j and species i.
- κ an extra parameter representing the **interaction strength**.

To simplify the study we will put $r_i = 1 \quad \forall i \in [n]$**.**

Equilibrium in Lotka-Volterra system

- **Model** We model the dynamics using Lotka-Volterra's system of differential equations.

$$\frac{dx_i}{dt} = x_i \left(1 - x_i + \left(\frac{B}{\kappa}\boldsymbol{x}\right)_i\right), \quad i = 1, \cdots, n$$

- **Goal** We are interested in studying the properties of the equilibrium vector of abundances $\boldsymbol{x}^\star \in \mathbb{R}^n_+$, i.e. the vector $\boldsymbol{x}^\star$ that satisfies

$$\left.\frac{dx}{dt}\right|_{\boldsymbol{x}=\boldsymbol{x}^\star} = 0$$

$$x_i^\star \left(1 - x_i^\star + \left(\frac{B}{\kappa}\boldsymbol{x}^\star\right)_i\right) = 0$$

- **Questions**
  1. Can we describe the statistical properties of the equilibrium ?
  2. In particular, what is the proportion of surviving species at the equilibrium ?

$$\frac{\#\{i : \ x_i^\star > 0\}}{n}$$

Main assumption: a random model for the interaction matrix B

- The study of large Lotka-Volterra systems makes it **very difficult to calibrate** the model and **estimate** matrix B.
- An alternative is to consider **random matrices** [May, 1972], the statistical properties of which encode some real properties of the food-web.
- $A \sim \mathrm{Elliptic}(n, \rho)$ if

$$\begin{pmatrix} A_{ij} \\ A_{ji} \end{pmatrix} \sim \mathcal{N}_2 \left(\mathbf{0}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \right)$$

  and $A_{ii} \sim \mathcal{N}(0, 1+\rho)$.
- Interaction matrix

$$B = \frac{1}{\sqrt{n}} A.$$

- Elliptic interaction matrices are interesting in theoretical ecology as they provide more flexibility, i.e. A_{ij} and A_{ji} are not equal but only correlated.

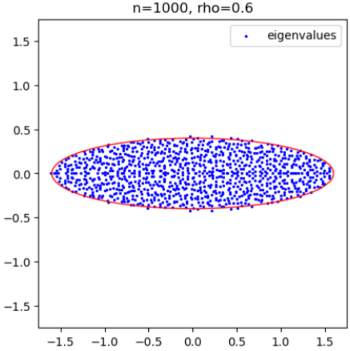


Figure: Spectrum of an elliptic matrix for $\rho = 0.6$.

From Lotka-Volterra to Linear Complementarity Problem

Given that $B = \frac{A}{\sqrt{n}}$ where $A \sim \mathrm{Elliptic}(n, \rho)$.
The equilibrium writes

$$\boxed{x_i^\star \left(1 - x_i^\star + \left(\frac{A}{\kappa\sqrt{n}} \boldsymbol{x}^\star\right)_i\right) = 0} \qquad i \in [n]\,, \quad \boldsymbol{x}^\star = (x_i^\star)\,.$$

Stable equilibrium

$$(\mathrm{LCP}) \quad \begin{cases} \boldsymbol{x}^\star & \geq \quad 0\,, \\ \left(I - \frac{A}{\kappa\sqrt{n}}\right)\boldsymbol{x}^\star - \mathbf{1} & \geq \quad 0\,, \\ x_i^\star \left(\left[\left(I - \frac{A}{\kappa\sqrt{n}}\right)\boldsymbol{x}^\star\right]_i - 1\right) & = \quad 0\,. \end{cases}$$

- **Stability condition** [Takeuchi, 1996]: Lyapunov statbility condition or non-invasibility condition.
- **Sufficient condition for the existence, uniqueness and stability of the equilibrium :** $\boxed{\kappa > \sqrt{2(1+\rho)}}$ [Clenet et al., 2022].

Statistical properties on the equilibrium

Let $A \sim \mathrm{Elliptic}(n, \rho)$ matrix and consider the LV system.

$$\frac{dx_i(t)}{dt} = x_i \left(1 - x_i + \frac{1}{\kappa}(B\boldsymbol{x})_i\right) \qquad \text{where} \qquad B = \frac{A}{\sqrt{n}}\,.$$

and assume that $\kappa > \sqrt{2(1+\rho)}$.

Theorem (G., Hachem, Najim, 2024)

- There exists a (random) **unique stable equilibrium** $\boldsymbol{x}_n^\star$: $\boxed{\boldsymbol{x}_n(t) \xrightarrow[t\to\infty]{} \boldsymbol{x}_n^\star}$ and

$$\text{(a.s.)} \quad \boxed{\mu^{\boldsymbol{x}^\star} \triangleq \frac{1}{n}\sum_{i\in[n]} \delta_{x_i^\star} \xrightarrow[n\to\infty]{weak, L^2} \mathcal{L}\left(\left(1+\rho\gamma/\delta^2\right)\left(\sigma\bar{Z}+1\right)_+\right)}$$

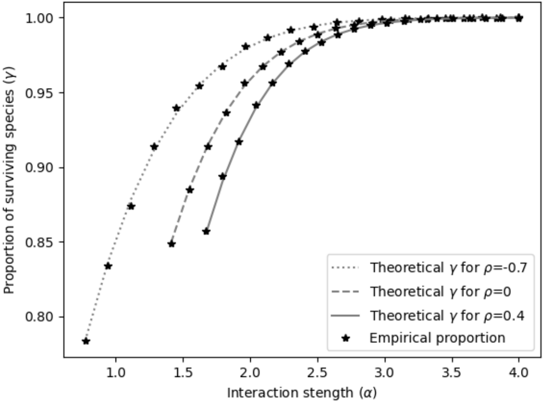
- where $(\delta, \sigma, \gamma) \in (0,\infty)^3$ is the unique triple solution of **the fixed point equations**

$$\begin{aligned}
\kappa &= \delta + \rho\frac{\gamma}{\delta}, \\
\sigma^2 &= \frac{1}{\delta^2}\mathbb{E}\left(\sigma\bar{Z}+1\right)_+^2, \qquad \bar{Z} \sim \mathcal{N}(0,1), \\
\gamma &= \mathbb{P}\Big[\sigma\bar{Z}+1>0\Big].
\end{aligned}$$

- Similar results were obtained by [Bunin, 2017] and [Galla, 2018] using statistical physics methods such as "Replica trick".

Proportion of surviving species

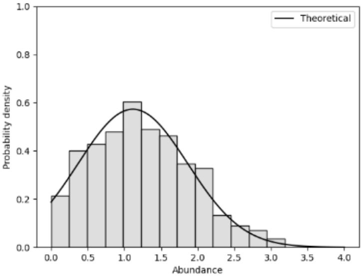
- **Theoretical values**: We solve the **fixed point equations** for (δ, σ, γ) for different values of κ. We plot $\gamma = \gamma(\kappa)$.
- **Experimental values**: (Monte-Carlo) we solve the **Equilibrium system** associated to the Lotka-Volterra equations for 500 samples of random matrices A. (using Lemke solver in python).

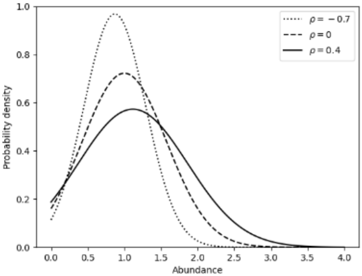


Distribution of surviving species

We fix $\kappa = 2$.

- **Theoretical values**: We plot the theoretical **truncated gaussian density function** for the adequate values of (δ, σ, γ).
- **Experimental values**: We plot the following empirical conditional distribution $\mathcal{L}\left(x_i^\star \mid x_i^\star > 0\right)$.





Outline

A fixed point equation

- We want to study the solution $\boldsymbol{x}^\star$ of the following system

$$\begin{cases} \boldsymbol{x}^\star & \geq \quad 0\,, \\ \left(I - \frac{A}{\kappa\sqrt{n}}\right)\boldsymbol{x}^\star - \mathbf{1} & \geq \quad 0\,, \\ x_i^\star\left(\left[\left(I - \frac{A}{\kappa\sqrt{n}}\right)\boldsymbol{x}^\star\right]_i - 1\right) & = \quad 0\,. \end{cases} \tag{1}$$

- To use an iterative scheme we need a fixed point equation. Consider the following equation of $\boldsymbol{z} \in \mathbb{R}^n$.

$$\boxed{\boldsymbol{z} = \frac{A}{\kappa\sqrt{n}}\boldsymbol{z}^+ + \mathbf{1}} \tag{2}$$

- If $\boldsymbol{z}$ is a solution to (2) then $\boldsymbol{z}^+$ is a solution to (1).
  - Write $\boldsymbol{z} = \boldsymbol{z}^+ - \boldsymbol{z}^-$, then

$$\boldsymbol{z}^- = \left(I - \frac{A}{\kappa\sqrt{n}}\right)\boldsymbol{z}^+ - 1$$

- **Question: We want an iterative algorithm $(\boldsymbol{z}^k)_k$ to solve (2) while tracking**

$$\boxed{\mu^{z^k} \triangleq \frac{1}{n}\sum_{i=1}^n \delta_{z_i}}$$

- An obvious algorithm $\boldsymbol{z}^{k+1} = \frac{A}{\kappa\sqrt{n}}\left(\boldsymbol{z}^k\right)^+ + 1$ does not have this property.

Elliptic Approximate Message Passing

AMP was initially introduced by [Bayati and Montanari, 2011] and later generalized by [Javanmard and Montanari, 2013], [Bayati et al., 2015], [Fan, 2021].

- **Notation** Given $f : \mathbb{R}^2 \to \mathbb{R}$ denote by $\langle f'(\boldsymbol{u}, \boldsymbol{v}) \rangle = \frac{1}{n} \sum_{i=1}^{n} \frac{\partial f}{\partial u}(u_i, v_i)$.

Assumptions

- $(f_k)_k$ Lipschitz activation functions. $\boldsymbol{a} \in \mathbb{R}^n$ is a constant parameter.
- Initialization point $\boldsymbol{u}^0 \perp\!\!\!\perp A$ and $\exists \bar{u}$ such that $\mu^{\boldsymbol{u}^0} \xrightarrow[n \to \infty]{} \mathcal{L}(\bar{u})$.

Theorem (G., Hachem, Najim, 2024)

The AMP recursion for a matrix $A \sim \mathrm{Elliptic}(n, \rho)$ is defined as

$$\begin{cases} \boldsymbol{u}^1 = \frac{1}{\sqrt{n}} A \ f_0\left(\boldsymbol{u}^0, \boldsymbol{a}\right), \\ \boldsymbol{u}^{k+1} = \frac{1}{\sqrt{n}} A f_k(\boldsymbol{u}^k, \boldsymbol{a}) - \rho \langle f_k'(\boldsymbol{u}^k, \boldsymbol{a}) \rangle f_{k-1}(\boldsymbol{u}^{k-1}, \boldsymbol{a}). \end{cases}$$

Then (a.s)

$$\boxed{\mu^{\boldsymbol{u}^k} = \frac{1}{n} \sum_{i=1}^{n} \delta_{u_i^k} \xrightarrow[n \to \infty]{weak, L^2} \mathcal{L}(Z_k) = \mathcal{N}\left(\boldsymbol{0}, \sigma_k^2\right)}$$

where $\sigma_0^2 = \mathbb{E}\left[f_0(\bar{u})^2\right]$ and $\sigma_{k+1}^2 = \mathbb{E}\left[f_k(Z_k)^2\right]$ st $Z_k \sim \mathcal{N}(0, \sigma_k^2)$.

Remark: *This result is well known in the case of symmetric matrices* ($\rho = 1$).
Tutorial: [Feng et al., 2021].

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Then (a.s) $$\boxed{\mu^{\boldsymbol{u}^1,\cdots,\boldsymbol{u}^k} = \frac{1}{n}\sum_{i=1}^{n} \delta_{(u_i^1,\cdots,u_i^k)} \xrightarrow[n\to\infty]{weak, L^2} \mathcal{L}(Z_1,\cdots,Z_k) = \mathcal{N}_k\left(\mathbf{0}, \Gamma^k\right)}$$

where $\sigma_0^2 = \mathbb{E}\left[f_0(\bar{u})^2\right]$ and $\sigma_{k+1}^2 = \mathbb{E}\left[f_k(Z_k)^2\right]$ st $Z_k \sim \mathcal{N}(0, \sigma_k^2)$.

Remark: *This result is well known in the case of symmetric matrices* ($\rho = 1$).
Tutorial: *[Feng et al., 2021].*

Design of an AMP algorithm to solve the Fixed Point equation

- We want to solve the fixed point equation $\boldsymbol{z} = \frac{A}{\kappa\sqrt{n}}\boldsymbol{z}_+ + \mathbf{1}$ using AMP.
- We need to calibrate the functions f_k and the parameter $\boldsymbol{a}$ ([Akjouj et al., 2023]) in the following recursion equation

$$\boldsymbol{u}^{k+1} = \frac{A}{\sqrt{n}} f_k(\boldsymbol{u}^k, \boldsymbol{a}) - \rho\langle f_k'(\boldsymbol{u}^k, \boldsymbol{a})\rangle f_{k-1}(\boldsymbol{u}^{k-1}, \boldsymbol{a})$$

- Given κ we determine δ and γ and we set $f_k(x,a) = \frac{(x+a)_+}{\delta}$. Parameter $\boldsymbol{a}$ will be calibrated later. Then

$$\boldsymbol{u}^{k+1} = \frac{A}{\delta\sqrt{n}}(\boldsymbol{u}^k + a)_+ - \rho\frac{\langle \mathbf{1}_{\boldsymbol{u}^k + a > 0}\rangle}{\delta^2}(\boldsymbol{u}^{k-1} + a)_+$$

- **Fact 1** $\boxed{\langle \mathbf{1}_{\boldsymbol{u}^k + a > 0}\rangle \approx \gamma}$ for k large.
- Set $\boldsymbol{\xi}^k = \boldsymbol{u}^k + \boldsymbol{a}$, then we get

$$\boldsymbol{\xi}^{k+1} = \frac{A}{\delta\sqrt{n}}\boldsymbol{\xi}_+^k - \rho\frac{\gamma}{\delta^2}\boldsymbol{\xi}_+^{k-1} + \boldsymbol{a} + \boldsymbol{\varepsilon}$$

- **Fact 2** $\boxed{\boldsymbol{\xi}^{k+1} \approx \boldsymbol{\xi}^k \approx \boldsymbol{\xi}^{k-1}}$ when k is large. Set $a = 1 + \rho\frac{\gamma}{\delta^2} = \frac{\kappa}{\delta}$. Then after massaging the previous equation.

$$\boldsymbol{\xi}_+ - \frac{\boldsymbol{\xi}_-}{1 + \rho\frac{\gamma}{\delta^2}} = \frac{A}{\kappa\sqrt{n}}\boldsymbol{\xi}_+ + \mathbf{1} + \boldsymbol{\varepsilon} \quad \Rightarrow \boxed{z \triangleq \boldsymbol{\xi}_+ - \frac{\boldsymbol{\xi}_-}{1 + \rho\frac{\gamma}{\delta^2}}}$$ satisfies the FPE.

Outline

Background

- **Theorem** Let $\boldsymbol{u}^0$ independent from A such that $\mu^{\boldsymbol{u}^0} \xrightarrow[n\to\infty]{} \mathcal{L}(\bar{u})$. The AMP recursion for a matrix $A \sim \mathrm{Elliptic}(n,\rho)$ is defined as

$$\begin{cases} \boldsymbol{u}^1 = \frac{1}{\sqrt{n}} A\ f_0\left(\boldsymbol{u}^0\right), \\ \boldsymbol{u}^{k+1} = \frac{1}{\sqrt{n}} A f_k(\boldsymbol{u}^k) - \rho\langle f_k'(\boldsymbol{u}^k)\rangle f_{k-1}(\boldsymbol{u}^{k-1}). \end{cases}$$

Then (a.s) $$\boxed{\mu^{\boldsymbol{u}^k} = \frac{1}{n}\sum_{i=1}^{n} \delta_{u_i^k} \xrightarrow[n\to\infty]{weak, L^2} \mathcal{N}\left(\mathbf{0}, \sigma_k^2\right)}$$

where $\sigma_{k+1}^2 = \mathbb{E}\left[\left(f_k(Z_k)\right)^2\right]$ and $Z_k \sim \mathcal{N}(0, \sigma_k^2)$.

- $\mu_n \xrightarrow[n\infty]{} \mu$ if $\int \varphi d\mu_n \xrightarrow[n\infty]{} \int \varphi d\mu$ for all φ.
- Let $\boldsymbol{X} \in \mathbb{R}^n$ be a random vector and $\bar{x} \in \mathbb{R}$ a random variable. We say $\mu^{\boldsymbol{X}} = \frac{1}{n}\sum_{i\in[n]} \delta_{X_i} \xrightarrow[n\infty]{} \mathcal{L}(\bar{x})$ if

$$\forall\varphi \quad \frac{1}{n}\sum_{i\in[n]} \varphi\left(X_i\right) \xrightarrow[n\infty]{\boldsymbol{c}} \mathbb{E}\left[\varphi(\bar{x})\right].$$

- Consider two random variables X and Y, we write $X \overset{\mathcal{L}}{=}_{|\mathcal{F}} Y$ if

$$\forall\varphi \quad \mathbb{E}\left[\varphi(X) \mid \mathcal{F}\right] = \mathbb{E}\left[\varphi(Y) \mid \mathcal{F}\right] \text{ (a.s.)}$$

Why the Onsager term ?

Recall the AMP recursion

$$\begin{cases} \boldsymbol{u}^1 = A\ f\left(\boldsymbol{u}^0, \boldsymbol{a}\right), \\ \boldsymbol{u}^{k+1} = Af(\boldsymbol{u}^k, \boldsymbol{a}) - \rho\langle f'(\boldsymbol{u}^k, \boldsymbol{a})\rangle f(\boldsymbol{u}^{k-1}, \boldsymbol{a}) \,. \end{cases}$$

- **Goal** We want to describe the statistical behavior of $\boldsymbol{u}$ in the high dimensional regime $n \to \infty$, where $\boldsymbol{u}$ satisfies the fixed point equation,

$$\boldsymbol{u} = Af(\boldsymbol{u}) \quad \text{and } A \text{ is random}$$

- **Idea** Iterative approximation, $\boldsymbol{u}^k \xrightarrow[n\infty]{} \boldsymbol{u}$ while tracking the $\mu^{\boldsymbol{u}^k}$.
- **Step 1** Initialize with $\boldsymbol{u}^0$ independent of A where $\mu^{\boldsymbol{u}^0}$ is known and construct

$$\boldsymbol{u}^1 = Af\left(\boldsymbol{u}^0\right)$$

  A and $\boldsymbol{u}^0$ are independent so $\mu^{\boldsymbol{u}^1}$ is easy to determine. 🙂
- **Step 2** Construct $\boldsymbol{u}^2$ such that

$$\boldsymbol{u}^2 = Af\left(\boldsymbol{u}^1\right)$$

  $\boldsymbol{u}^1$ and A are now correlated so studying $\mu^{\boldsymbol{u}^2}$ is not simple. ☹
- **Correction term** [Bayati and Montanari, 2011] subtract a correction term

$$\boxed{\boldsymbol{u}^2 = Af\left(\boldsymbol{u}^1\right) - \text{onsager term}} \qquad \left(\Leftrightarrow\ \boldsymbol{u}^2 = \widetilde{A}f\left(\boldsymbol{u}^1\right) \text{ and } \widetilde{A} \perp\!\!\!\perp \boldsymbol{u}^1\right)$$

Base case

- For $k = 1$, recall that $\boldsymbol{u}^1 = \frac{1}{\sqrt{n}} A\boldsymbol{q}^0$ where $\boldsymbol{q}^0 = f_0\left(\boldsymbol{u}^0\right)$ and $\boldsymbol{u}^0 \perp\!\!\!\perp A$. We want

$$\boxed{\mu^{\boldsymbol{u}^1} = \frac{1}{n}\sum_{i\in[n]} \delta_{u_i^1} \xrightarrow[n\to\infty]{} \mathcal{L}\left(Z_1\right)} \text{ where } Z_1 \sim \mathcal{N}\left(0, \sigma_1^2\right) \text{ and } \sigma_1^2 = \mathbb{E}\left[\left(f_0\left(\bar{u}\right)^2\right)\right].$$

Lemma

If $A \sim \mathrm{Elliptic}(n, \rho)$ and $\boldsymbol{q} \in \mathbb{R}^n$ normalized, then $A\boldsymbol{q} \sim \mathcal{N}\left(0, I + \rho\boldsymbol{q}\boldsymbol{q}^\top\right)$.
Or $A\boldsymbol{q} \overset{\mathcal{L}}{=} \boldsymbol{\xi} + \left(\sqrt{1+\rho} - 1\right)\boldsymbol{q}\boldsymbol{q}^\top\boldsymbol{\xi}$ where $\boldsymbol{\xi} \sim \mathcal{N}_n\left(0, I_n\right)$.

- Using this lemma,

$$\begin{aligned}\boldsymbol{u}^1 &= \frac{1}{\sqrt{n}} A\boldsymbol{q}^0 \\ &\overset{\mathcal{L}}{=} \boxed{\|q^0\|\boldsymbol{\xi} + c\frac{\boldsymbol{q}^0\boldsymbol{q}^{0\top}}{\sqrt{n}\|\boldsymbol{q}^0\|}\boldsymbol{\xi}} \quad \text{where } \boldsymbol{\xi} \sim \mathcal{N}_n\left(0, I_n\right).\end{aligned}$$

Proof of the previous Lemma

Lemma

If $A \sim \mathrm{Elliptic}(n, \rho)$ *and* $\boldsymbol{q} \in \mathbb{R}^n$ *normalized, then* $A\boldsymbol{q} \sim \mathcal{N}\left(0, I + \rho \boldsymbol{q}\boldsymbol{q}^\top\right)$.
Or $A\boldsymbol{q} \overset{\mathcal{L}}{=} \boldsymbol{\xi} + \left(\sqrt{1+\rho} - 1\right) \boldsymbol{q}\boldsymbol{q}^\top \boldsymbol{\xi}$ *where* $\boldsymbol{\xi} \sim \mathcal{N}_n\left(0, I_n\right)$.

Proof.

Let $\rho = 1$, we have $A \sim GOE\left(n\right)$, let $Q = [\boldsymbol{q}, *] \in \mathbb{R}^{n \times n}$ be an orthogonal matrix, using **orthogonal invariance** of GOE matrices

$$\begin{aligned} A\boldsymbol{q} &\overset{\mathcal{L}}{=} QAQ^\top \boldsymbol{q} \\ &= QAe_1 \sim \mathcal{N}_n\left(0, Q\left(I_n + \boldsymbol{e_1}\boldsymbol{e_1}^\top\right) Q^\top\right) \\ &\overset{\mathcal{L}}{=} \mathcal{N}_n\left(0, I + \boldsymbol{q}\boldsymbol{q}^\top\right). \end{aligned}$$

For a general $\rho \in [-1, 1]$ we use the following decomposition of A

$$A \overset{\mathcal{L}}{=} \sqrt{\frac{1+\rho}{2}}\, G + \sqrt{\frac{1-\rho}{2}}\, \tilde{G}$$

where $G \sim GOE(n)$ and $\tilde{G}$ (antisymmetric) are independent. □

Base case

- We have $\boldsymbol{u}^1 \overset{\mathcal{L}}{\triangleq} \frac{1}{\sqrt{n}}\|q^0\|\boldsymbol{\xi} + c\frac{\boldsymbol{q}^0\boldsymbol{q}^{0\top}}{\sqrt{n}\|\boldsymbol{q}^0\|}\boldsymbol{\xi}$ where $\boldsymbol{\xi} \sim \mathcal{N}_n\left(0, I_n\right)$ is independent of $\boldsymbol{u}^0$.
- Using the assumption $\mu^{\boldsymbol{u}^0} \xrightarrow[n\to\infty]{} \mathcal{L}(\bar{u})$ we can see that

$$\frac{1}{n}\|\boldsymbol{q}^0\|^2 = \frac{1}{n}\sum_{i\in[n]}\left(f_0\left(u_i^0\right)\right)^2 \xrightarrow[n\to\infty]{} \mathbb{E}\left[\left(f_0\left(\bar{u}\right)\right)^2\right] \triangleq \sigma_1^2$$

- The term $\frac{\boldsymbol{q}^0\boldsymbol{q}^{0\top}}{\sqrt{n}\|\boldsymbol{q}^0\|}\boldsymbol{\xi}$ can be handled by writing

$$\frac{\boldsymbol{q}^0\boldsymbol{q}^{0\top}}{\|\boldsymbol{q}^0\|^2}\boldsymbol{\xi} = \text{(rank 1 projection matrix)} \times \mathcal{N}_n\left(0, I_n\right)$$

$$\Rightarrow \frac{1}{\sqrt{n}}\left\|\frac{\boldsymbol{q}^0\boldsymbol{q}^{0\top}}{\|\boldsymbol{q}^0\|^2}\boldsymbol{\xi}\right\| \xrightarrow[n\infty]{} 0$$

- Write

$$\boldsymbol{u}^1 \overset{\mathcal{L}}{\triangleq} \sigma_1\boldsymbol{\xi} + \Delta^1$$

where $\Delta^1 = \left(\frac{1}{\sqrt{n}}\|\boldsymbol{q}^0\| - \sigma_1\right)\boldsymbol{\xi} + c\frac{\boldsymbol{q}^0\boldsymbol{q}^{0\top}}{\sqrt{n}\|\boldsymbol{q}^0\|}\boldsymbol{\xi}$ such that $\Delta^1 \approx 0$

Finally

$$\boxed{\mu^{\boldsymbol{u}^1} \approx \mu^{\sigma_1\boldsymbol{\xi}} \xrightarrow[n\infty]{} \mathcal{L}\left(Z_1\right), \quad Z_1 \sim \mathcal{N}\left(0, \sigma_1^2\right)} \text{ where } \sigma_1^2 = \mathbb{E}\left[f_0\left(\bar{u}\right)^2\right]$$

Proof for $k = 2$

- We have $\mu^{\boldsymbol{u}^1} \xrightarrow[n\infty]{} \mathcal{L}(Z_1)$ where $Z_1 \sim \mathcal{N}\left(0, \sigma_1^2\right)$ and $\sigma_1^2 = \mathbb{E}\left[f_0\left(\bar{u}\right)^2\right]$.
- We want $\mu^{\boldsymbol{u}^2} \xrightarrow[n\infty]{} \mathcal{L}(Z_2)$ where $Z_2 \sim \mathcal{N}\left(0, \sigma_2^2\right)$ and $\sigma_2^2 = \mathbb{E}\left[f_1\left(Z_1\right)^2\right]$.
- Recall the expression of $\boldsymbol{u}^2$

$$\boldsymbol{u}^2 = \frac{A}{\sqrt{n}} f_1\left(\boldsymbol{u}^1\right) - \overbrace{\rho \left\langle f_1'\left(\boldsymbol{u}^1\right)\right\rangle f_0\left(\boldsymbol{u}^0\right)}^{\text{Onsager term}} = \boxed{\frac{A}{\sqrt{n}} \boldsymbol{q}^1 - \rho d_n \boldsymbol{q}^0}$$

- Notations

$$\boldsymbol{q}^1 = f_1\left(\boldsymbol{u}^1\right) \quad \text{and} \quad d_n = \frac{1}{n} \sum_{i \in [n]} f_1'\left(\boldsymbol{u}_i^1\right)$$

$$P_0 = \boldsymbol{q}^0 \boldsymbol{q}^{0\top} / \|\boldsymbol{q}^0\|^2 \quad \text{and} \quad P_0^{\perp} = I_n - P_0$$

- Structural decomposition

$$\begin{aligned}
\boldsymbol{u}^2 &= A_n P_0 \boldsymbol{q}^1 + A_n P_0^{\perp} \boldsymbol{q}^1 - \rho d_n \boldsymbol{q}^0 \\
&= A_n P_0 \boldsymbol{q}^1 + \rho \left(A_n P_0\right)^{\top} P_0^{\perp} \boldsymbol{q}^1 - \rho d_n \boldsymbol{q}^0 + \left(A_n - \rho P_0 A_n^{\top}\right) P_0^{\perp} \boldsymbol{q}^1 \\
&= \frac{\langle \boldsymbol{q}^1, \boldsymbol{q}^0 \rangle}{\|\boldsymbol{q}^0\|^2} \boldsymbol{u}^1 + \rho \left(A_n P_0\right)^{\top} \boldsymbol{q}^1 - \rho \left(A_n P_0\right)^{\top} P_0 \boldsymbol{q}^1 - \rho d_n \boldsymbol{q}^0 + \mathcal{I}\left(A\right) \\
&= \frac{\langle \boldsymbol{q}^1, \boldsymbol{q}^0 \rangle}{\|\boldsymbol{q}^0\|^2} \boldsymbol{u}^1 + \rho \left(\langle \boldsymbol{u}^1, \boldsymbol{q}^1 \rangle - d_n \|\boldsymbol{q}^0\|^2 - \frac{\langle \boldsymbol{q}^1, \boldsymbol{q}^0 \rangle}{\|\boldsymbol{q}^0\|^2} \langle \boldsymbol{u}^1, \boldsymbol{q}^0 \rangle\right) \frac{\boldsymbol{q}^0}{\|\boldsymbol{q}^0\|^2} + \mathcal{I}\left(A\right)
\end{aligned}$$

Proof for $k=2$

$$\boldsymbol{u}^2 = \frac{\langle \boldsymbol{q}^1, \boldsymbol{q}^0 \rangle}{\|\boldsymbol{q}^0\|^2}\boldsymbol{u}^1 + \rho\left(\langle \boldsymbol{u}^1, \boldsymbol{q}^1 \rangle - d_n \|\boldsymbol{q}^0\|^2 - \frac{\langle \boldsymbol{q}^1, \boldsymbol{q}^0 \rangle}{\|\boldsymbol{q}^0\|^2}\langle \boldsymbol{u}^1, \boldsymbol{q}^0 \rangle\right)\frac{\boldsymbol{q}^0}{\|\boldsymbol{q}^0\|^2} + \mathcal{I}(A)$$

Recall that $\mu^{\boldsymbol{u}^1} \xrightarrow[n\infty]{} \mathcal{L}(Z_1)$ and $\mu^{\boldsymbol{u}^0} \xrightarrow[n\infty]{} \mathcal{L}(\bar{u})$ such that $Z_1 \sim \mathcal{N}(0, \sigma_1^2)$ and $\bar{u}$ are independent random variables where $\sigma_1^2 = \mathbb{E}\left[(f_0(\bar{u}))^2\right]$

- $\|\boldsymbol{q}^0\|^2/n = \frac{1}{n}\sum_{i\in[n]} f_0\left(u_i^0\right)^2 \xrightarrow[n\infty]{} \sigma_1^2$.
- $\langle \boldsymbol{q}^1, \boldsymbol{q}^0 \rangle / \|\boldsymbol{q}^0\|^2 = \frac{n}{\|\boldsymbol{q}^0\|^2}\frac{1}{n}\sum_{i\in[n]} f_1\left(u_i^1\right) f_0\left(u_i^0\right) \xrightarrow[n\to]{} \underbrace{\frac{1}{\sigma_1^2}\mathbb{E}\left[f_1(Z_1) f_0(\bar{u})\right]}_{\triangleq \alpha_1}$.
- $d_n = \frac{1}{n}\sum_{i\in[n]} f_1'\left(u_i^1\right) \xrightarrow[n\infty]{} \mathbb{E}\left[f_1'(Z_1)\right]$.
- $\frac{1}{n}\langle \boldsymbol{u}^1, \boldsymbol{q}^1 \rangle = \frac{1}{n}\sum_{i\in[n]} u_i^1 f_1\left(u_i^1\right) \xrightarrow[n\infty]{} \mathbb{E}\left[Z_1 f_1(Z_1)\right] \overbrace{=}^{\text{Stein}} \sigma_1^2 \mathbb{E}\left[f_1'(Z_1)\right]$.
- $\frac{1}{n}\langle \boldsymbol{u}^1, \boldsymbol{q}^0 \rangle = \frac{1}{n}\sum_{i\in[n]} u_i^1 f_0\left(u_i^0\right) \xrightarrow[n\infty]{} \mathbb{E}\left[Z_1 f_0(\bar{u})\right] = 0$.

Thus we can decompose $\boldsymbol{u}^2$ in the following way

$$\boxed{\boldsymbol{u}^2 = \alpha_1 \boldsymbol{u}^1 + \mathcal{I}(A) + \Delta}$$

where $\quad \Delta = \rho\left(\langle \boldsymbol{u}^1, \boldsymbol{q}^1 \rangle - d_n \|\boldsymbol{q}^0\|^2 - \frac{\langle \boldsymbol{q}^1, \boldsymbol{q}^0 \rangle}{\|\boldsymbol{q}^0\|^2}\langle \boldsymbol{u}^1, \boldsymbol{q}^0 \rangle\right)\frac{\boldsymbol{q}^0}{\|\boldsymbol{q}^0\|^2} + \left(\frac{\langle \boldsymbol{q}^1, \boldsymbol{q}^0 \rangle}{\|\boldsymbol{q}^0\|^2} - \alpha_1\right)$

and $\quad \Delta \approx 0$

Proof for $k = 2$

$$\boldsymbol{u}^2 = \underbrace{\alpha_1 \boldsymbol{u}^1 + \Delta}_{\mathcal{F}_1\text{-measurable}} + \mathcal{I}\left(A\right) \quad \text{where} \quad \mathcal{F}_1 = \sigma\left(\left\{\boldsymbol{u}^0, \boldsymbol{u}^1\right\}\right) \quad \text{and} \quad \Delta \approx 0.$$

- **Question** What about the term $\mathcal{I}\left(A\right) = \left(A_n - \rho P_0 A_n^\top\right) P_0^\perp \boldsymbol{q}^1$?

Lemma

Let $A \sim \mathrm{Elliptic}(n, \rho)$. Let $\boldsymbol{v} \in \mathbb{R}^n$ be a unit-norm deterministic vector, and let P be a deterministic orthogonal projection matrix on a subspace of $\mathbb{R}^n$ such that $P\boldsymbol{v} = 0$. Then,

$$(A - \rho P A^\top)\boldsymbol{v} \sim \mathcal{N}\left(0, I - \rho^2 P + \rho \boldsymbol{v}\boldsymbol{v}^\top\right).$$

$$(A - \rho P A^\top)\boldsymbol{v} \overset{\mathcal{L}}{=} \boldsymbol{\xi} + o(1) \quad \textit{where } \boldsymbol{\xi} \sim \mathcal{N}_n\left(0, I_n\right).$$

Proposition (G., Hachem, Najim, 2024)

There exists a $n \times n$ matrix $\widetilde{A}$ such that $\widetilde{A} \sim \mathrm{Elliptic}(n, \rho)$, $\widetilde{A} \perp\!\!\!\perp \mathcal{F}_1$ and

$$(A - \rho P_0 A^\top) P_0^\perp \boldsymbol{q}^1 \ \overset{\mathcal{L}}{=}_{|\mathcal{F}_1} \ (\widetilde{A} - \rho P_0 \widetilde{A}^\top) P_0^\perp \boldsymbol{q}^1 .$$

- Finally, we can write

$$\begin{aligned} \boldsymbol{u}^2 \ &\overset{\mathcal{L}}{=}_{|\mathcal{F}_1} \alpha_1 \boldsymbol{u}^1 + \mathcal{I}(\widetilde{A}) + \Delta \quad \text{where} \quad \widetilde{A} \perp\!\!\!\perp \mathcal{F}_1 \\ &\overset{\mathcal{L}}{=}_{|\mathcal{F}_1} \alpha_1 \boldsymbol{u}^1 + \|P_0^\perp \boldsymbol{q}^1\| \boldsymbol{\xi} + \Delta \end{aligned}$$

Proof for $k = 2$

$$\boldsymbol{u}^2 \overset{\mathcal{L}}{=}_{|\mathcal{F}_1} \alpha_1 \boldsymbol{u}^1 + \|P_0^\perp \boldsymbol{q}^1\| \boldsymbol{\xi} + \Delta \quad \text{where} \quad \boldsymbol{\xi} \sim \mathcal{N}_n(0, I_n) \text{ and } \Delta \approx 0$$

$$\text{and } \alpha_1 = \lim_{n\infty} \left|\langle \boldsymbol{q}^0, \boldsymbol{q}^1 \rangle\right| / \|\boldsymbol{q}^0\|^2$$

- Recall that by the induction hypothesis

$$\mu^{\boldsymbol{u}^1} \xrightarrow[n\infty]{} \mathcal{L}(Z_1) \quad \text{where} \quad Z_1 \sim \mathcal{N}(0, \sigma_1^2)$$

- $\frac{1}{n}\|P_0^\perp \boldsymbol{q}^1\|^2 = \frac{1}{n}\|\boldsymbol{q}^1\|^2 - \frac{1}{n}\left|\langle \boldsymbol{q}^0, \boldsymbol{q}^1 \rangle\right|^2 / \|\boldsymbol{q}^0\|^2 \xrightarrow[n\infty]{} \underbrace{\mathbb{E}\left[f_1(Z_1)^2\right]}_{\triangleq \sigma_2^2} - \alpha_1^2 \sigma_1^2$

- Then

$$\boldsymbol{u}^2 \overset{\mathcal{L}}{=}_{|\mathcal{F}_1} \alpha_1 \boldsymbol{u}^1 + \sqrt{\sigma_2^2 - \alpha_1^2 \sigma_1^2}\, \boldsymbol{\xi} + \Delta \Rightarrow \mu^{\boldsymbol{u}^2} \xrightarrow[n\infty]{} \mathcal{L}\left(\alpha_1 Z_1 + \sqrt{\sigma_2^2 - \alpha_1^2 \sigma_1^2}\, \xi\right)$$

$$Z_1 \sim \mathcal{N}\left(0, \sigma_1^2\right) \perp\!\!\!\perp \xi \sim \mathcal{N}(0, 1) \Rightarrow \alpha_1 Z_1 + \sqrt{\sigma_2^2 - \alpha_1^2 \sigma_1^2}\, \xi \sim \mathcal{N}\left(0, \sigma_2^2\right)$$

- Finally

$$\boxed{\mu^{\boldsymbol{u}^2} \xrightarrow[n\infty]{} \mathcal{L}(Z_2)}$$

where $Z_2 \sim \mathcal{N}(0, \sigma_2^2)$ and $\sigma_2^2 = \mathbb{E}\left[f_1(Z_1)^2\right]$

General case (Induction step)

- We prove the theorem by induction, we suppose that for certain k we have

$$\mu^{\boldsymbol{u}^k} = \frac{1}{n} \sum_{i \in [n]} \delta_{\boldsymbol{u}_i^k} \xrightarrow[n\to\infty]{} \mathcal{L}(Z_k) \quad \text{where } Z_k \sim \mathcal{N}\left(0, \sigma_k^2\right)$$

$\left(\sigma_k^2\right)_{k\in\mathbb{N}}$ is recursively defined by $\sigma_k^2 = \mathbb{E}\left[f_{k-1}(Z_{k-1})^2\right]$ and $Z_0 = \bar{u}$.

- **Step 1** Decompose $\boldsymbol{u}^{k+1}$ as follows

$$\boldsymbol{u}^{k+1} = \sum_{\ell=1}^{k} \alpha_\ell \boldsymbol{u}^\ell + \left(A - \rho P_k A^\top\right) P_k^\perp q^k + \underset{n\to\infty}{o}(1),$$

where $\boldsymbol{q}^k = f_k(\boldsymbol{u}^k)$ and P_k is the $n \times n$ projection matrix on the space $\mathrm{Span}(\boldsymbol{q}^1, \cdots, \boldsymbol{q}^k)$. And $P_k^\perp = I - P_k$.

- **Step 2** Replace with an independent matrix $\tilde{A}$

$$\left(A - \rho P_k A^\top\right) P_k^\perp \overset{\mathcal{L}}{=}\Big|_{\boldsymbol{u}^1,\cdots,\boldsymbol{u}^k} \left(\tilde{A} - \rho P_k \tilde{A}^\top\right) P_k^\perp,$$

Where $\tilde{A} \overset{\mathcal{L}}{=} A$ and $\tilde{A}$ is independent of the past $\{\boldsymbol{u}^1, \cdots, \boldsymbol{u}^k\}$.

- **Step 3** Condition on the past and apply induction hypothesis

$$\boldsymbol{u}^{k+1} \overset{\mathcal{L}}{=}\Big|_{\boldsymbol{u}^1,\cdots,\boldsymbol{u}^k} \sum_{i=1}^{k} \alpha_l \boldsymbol{u}^l + \left(\tilde{A} - \rho P_k \tilde{A}^\top\right) P_k^\perp q^k + \underset{n\to\infty}{o}(1).$$

References I

Akjouj, I., Hachem, W., Maïda, M., and Najim, J. (2023).
Equilibria of large random lotka-volterra systems with vanishing species: a mathematical approach.

Bayati, M., Lelarge, M., and Montanari, M. (2015).
Universality in polytope phase transitions and message passing algorithms.
The Annals of Applied Probability, 25(2).

Bayati, M. and Montanari, A. (2011).
The dynamics of message passing on dense graphs, with applications to compressed sensing.
IEEE Transactions on Information Theory, 57(2):764–785.

Bunin, G. (2017).
Ecological communities with lotka-volterra dynamics.
Phys. Rev. E, 95:042414.

Clenet, M., El Ferchichi, H., and Najim, J. (2022).
Equilibrium in a large lotka–volterra system with pairwise correlated interactions.
Stochastic Processes and their Applications, 153:423–444.

Fan, Z. (2021).
Approximate message passing algorithms for rotationally invariant matrices.

Feng, O. Y., Venkataramanan, R., Rush, C., and Samworth, R. J. (2021).
A unifying tutorial on approximate message passing.

References II

Galla, T. (2018).
Dynamically evolved community size and stability of random lotka-volterra ecosystems(a).
Europhysics Letters, 123(4):48004.

Javanmard, A. and Montanari, A. (2013).
State evolution for general approximate message passing algorithms, with applications to spatial coupling.
Information and Inference: A Journal of the IMA, 2(2):115–144.

May, R. M. (1972).
Will a large complex system be stable?
Nature, 238(5364):413–414.

Takeuchi, Y. (1996).
Global dynamical properties of Lotka-Volterra systems.
World Scientific.

Existence of a stable equilibrium

Theorem (Takeuchi and Adachi (1980))

If there exists a positive diagonal matrix Δ such that $\Delta M + M^\top \Delta$ is positive definite, then there exists a **unique** *and* **stable** *equilibrium of the Lotka-Volterra system.*

Lemma

Let A an elliptic matrix with coefficient ρ and $\alpha > \sqrt{2(1+\rho)}$, then $\left(I - \frac{A}{\alpha\sqrt{n}}\right) + \left(I - \frac{A}{\alpha\sqrt{n}}\right)^\top$ is a.s. positive definite.

Proof.

Let $\Delta = I$,

$$\left(I - \frac{A}{\alpha\sqrt{n}}\right) + \left(I - \frac{A}{\alpha\sqrt{n}}\right)^\top = 2I - \frac{A + A^\top}{\alpha\sqrt{n}}$$

$\square$

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A is ρ-elliptic $\Rightarrow A_{ij} + A_{ji} \sim \mathcal{N}(0, 2(1+\rho))$

$\square$

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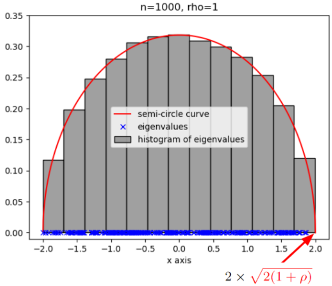
Let $\Delta = I$,

$$2I - \frac{A + A^\top}{\alpha\sqrt{n}}$$

$$\lambda_{\max}\left(\frac{A + A^\top}{\sqrt{n}}\right) \xrightarrow[n\to\infty]{} 2 \times \sqrt{2(1+\rho)}$$



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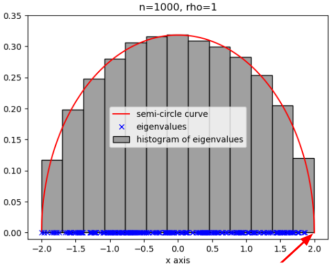
$$\lambda_{\max}\left(\frac{A + A^\top}{\sqrt{n}}\right) \xrightarrow[n\to\infty]{} 2 \times \sqrt{2(1+\rho)}$$

$$\lambda_{\min}\left(2I - \frac{A + A^\top}{\alpha\sqrt{n}}\right) = 2 - \frac{1}{\alpha}\lambda_{\max}\left(\frac{A + A^\top}{\sqrt{n}}\right)$$

$$\approx 2 - \frac{2\sqrt{2(1+\rho)}}{\alpha}$$



$2 \times \sqrt{2(1+\rho)}$

$\square$

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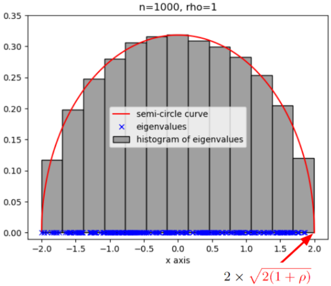
Proof.

Let $\Delta = I$,

$$\lambda_{\min}\left(2I - \frac{A + A^\top}{\alpha\sqrt{n}}\right) \approx 2 - \frac{2\sqrt{2(1+\rho)}}{\alpha} > 0$$

$$\boxed{\alpha > \sqrt{2(1+\rho)}}$$



$\square$

Proportion of surviving species

Given $\alpha > \sqrt{2(1+\rho)}$, we have $\mu^{\boldsymbol{x}^\star_n} \xrightarrow[n\to\infty]{} \pi := \mathcal{L}\left(\left(1+\rho\gamma/\delta^2\right)\left(\sigma\bar{Z}+\bar{r}\right)_+\right)$ i.e.

$$\forall\varphi \in \mathcal{C}_b \quad \frac{1}{n}\sum_{i=1}^{n}\varphi(x_i^\star) \xrightarrow[n\to\infty]{} \mathbb{E}\left[\varphi\left(\left(1+\rho\gamma/\delta^2\right)\left(\sigma\bar{Z}+\bar{r}\right)_+\right)\right]$$

In particular, the proportion of surviving species can be accessed via $\varphi(x) = \mathbf{1}_{x>0}$.

$$\frac{\#\{i \ : \ x_i^\star > 0\}}{n} \xrightarrow[n\to\infty]{} \mathbb{P}\left[\sigma\bar{Z}+r > 0\right] = \gamma.$$

Elliptic Approximate Message Passing

Theorem (G., Hachem, Najim, 2023)

The AMP recursion for a ρ-**elliptic matrix** A is defined as

$$\boldsymbol{u}^{k+1} = \frac{A}{\sqrt{n}} f_k(\boldsymbol{u}^k, \boldsymbol{a}) - \rho \langle f_k'(\boldsymbol{u}^k, \boldsymbol{a}) \rangle f_{k-1}(\boldsymbol{u}^{k-1}, \boldsymbol{a})$$

Then

$$(a.s) \qquad \mu^{\boldsymbol{u}^1, \cdots, \boldsymbol{u}^k} = \frac{1}{n} \sum_{i=1}^{n} \delta_{(u_i^1, \cdots, u_i^k)} \xrightarrow[N \to \infty]{weak, L^2} (Z_1, \cdots, Z_k) \sim \mathcal{N}_k\left(\mathbf{0}, \Gamma^k\right)$$

where Γ^k is defined recursively **(State Evolution Equations).**

State Evolution Equations

The $k \times k$ covariance matrices Γ^k are recursively defined as follows

$$\Gamma_{ij}^k = \mathbb{E}\left[f_{i-1}(Z_{i-1}) f_{j-1}(Z_{j-1})\right] \text{ where } (Z_1, \cdots, Z_k) \sim \mathcal{N}_{k-1}\left(0, \Gamma^{k-1}\right).$$

In particular

$$\mathbb{E}\left[Z_{k+1}^2\right] = \mathbb{E}\left[\left(f_k\left(Z_k\right)\right)^2\right]$$