High dimensional Lotka-Volterra with random interactions

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A Motivation From theoretical ecology

- **Interactions in ecological systems**: The interactions between species in an ecological environment can be modeled using Lotka Volterra differential equations, i.e. $\forall i \in [n]$

$$\begin{cases} \dfrac{d}{dt}\boldsymbol{x}_i(t) = \boldsymbol{x}_i(t)\left(\boldsymbol{r}_i - \boldsymbol{x}_i(t) + \displaystyle\sum_j A_{ij}\boldsymbol{x}_i(t)\right) \\ \boldsymbol{x}_i(t) \geq 0, \quad \forall t \end{cases}$$

$\rightarrow$ A_{ij} is the effect of species j on species i.

$\rightarrow$ x_i and r_i are natural abundance and intrinsic growth of species i respectively.

- **Model**: The interaction matrix A is a **large random** matrix such that $\mathrm{Var}(A_{ij}) = v_{ij}$ and $\mathrm{Corr}(A_{ij}, A_{ji}) = \tau_{ij}$. $V = (v_{ij})$ and $T = (\tau_{ij})$ are variance and correlation profiles.

- **Problem**: We study the existence, uniqueness as well as the statistical properties of the equilibrium point $\boldsymbol{x}^\star$ (ex: proportion of surviving species),

$$\boldsymbol{x}_i^\star\left(\boldsymbol{r}_i - \boldsymbol{x}_i^\star + [A\boldsymbol{x}^\star]_i\right) = 0,\ \forall i \in [n] \quad (1)$$

Existence of equilibrium

A non spectral property of random matrices : In addition to equation (1), the equilibrium $\boldsymbol{x}^\star$ is taken to be Lyapunov stable (non-invadavility condition). Hence:

$$\begin{cases} \boldsymbol{x}_i^\star \geq 0, \\ \boldsymbol{x}_i^\star\left(\boldsymbol{r}_i - [(I_n - A)\,\boldsymbol{x}^\star]_i\right) = 0, \\ \boldsymbol{r}_i - (I_n - A)\,\boldsymbol{x}_i^\star \leq 0, \end{cases}$$

for all $i \in [n]$.

Consequence: $\boldsymbol{x}^\star$ is a solution to LCP $(I_n - A, -\boldsymbol{r})$ "Linear Complementarity Problem" a classical non-linear optimization problem generalizing LP and QP.

Contribution

- **Goal**: Study of the asymptotic behavior of the empirical distribution $\boxed{\mu^{\boldsymbol{x}^\star} = \dfrac{1}{n}\displaystyle\sum_{k=1}^{n}\delta_{x_k^\star}}$ when n is large.
- **Tool**: The iterates of Approximate Message Passing (AMP) $\boldsymbol{x}^{\boldsymbol{k}}$ converge to the equilibrium $\boldsymbol{x}^\star$. This algorithm provides rich information about $\lim\limits_{n\to\infty}\mu^{\boldsymbol{x}^{\boldsymbol{k}}}$.
- **Contribution**: We proved the existence and uniqueness of $\boldsymbol{x}^\star$. We also generalized AMP for sparse and correlated matrices.

Strategy of proof [2, 3]

- We prove AMP for $f_k = p_k$ polynomial.
- Having $V_{ii} = 0$, we use combinatorial techniques to study the moments of $\boldsymbol{x}^{\boldsymbol{k}}$.
- We use density arguments to capture general functions

$$f_k \approx p_k^{(\varepsilon)} \quad \text{with error } \varepsilon.$$

- Finally, we use a perturbation argument to prove AMP of a general (nonzero diagonal) variance profile.

$$V = \Delta^{\text{diag}} + V^{\text{zero diag}}.$$

AMP for elliptic matrices

For a matrix A with uniform variance profile and a fixed correlation ρ, AMP gets the following form

$$\boldsymbol{x}^{\boldsymbol{k+1}} = Af_k\left(\boldsymbol{x}^{\boldsymbol{k}}\right) - \rho\langle f_k'(x^k)\rangle_n f_{k-1}\left(\boldsymbol{x}^{\boldsymbol{k-1}}\right),$$

The "Onsager term" is modified. We show that

$$\mu^{(\boldsymbol{x}^{\boldsymbol{1}},\cdots,\boldsymbol{x}^{\boldsymbol{k}})} \xrightarrow[n\to\infty]{w,L^2} \mathcal{L}(Z^1,\cdots,Z^k) \text{ a.s.}$$

where $(Z^1,\cdots,Z^k) \sim \mathrm{DE}\,(f,k)$.

Recursive definition of DE: $(Z^1,\cdots,Z^{k+1}) \sim \mathcal{N}_k(0, R^{k+1})$ such that

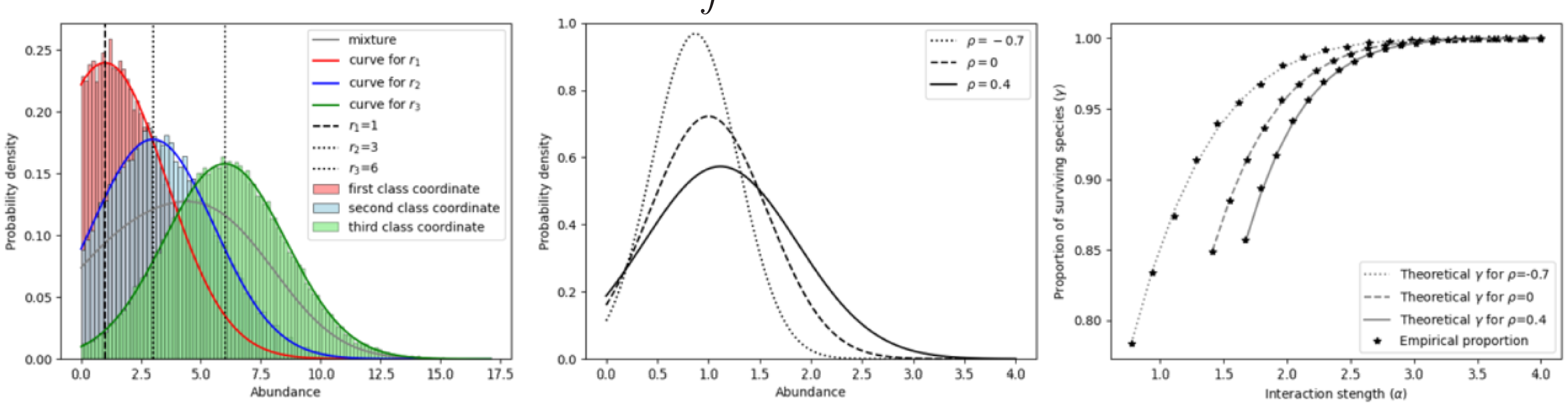
$$R^{k+1} = \alpha^{-1}\mathbb{E}\begin{bmatrix} f_0(x^0) \\ f_1(Z^1) \\ \cdots \\ f_k(Z^k) \end{bmatrix}^{\otimes 2},$$

where $(Z^1,\cdots,Z^k) \sim \mathcal{N}_k(0, R^k)$.

A mixture of truncated gaussians !

We rigorously show that $\mu^{\boldsymbol{x}^*}$ **is a mixture of truncated gaussians** as n goes to ∞.

Proportion of surviving species is $\gamma = \displaystyle\int 1_{\mathbb{R}_+^*}\,d\mu^{\boldsymbol{x}^*}$.



(a) (b) (c)

- **Figure (a)** Mixture of 3 truncated gaussians for $\boldsymbol{r} = (1,\cdots 1,\ 3,\cdots 3,\ 6,\cdots 6)^\top$.
- **Figure (b)** Effect of the correlation on the surviving species' distribution.
- **Figure (c)** Proportion of surviving species for correlations $\rho = 0.7, 0, 0.4$.

Statistics of equilibrium

Consider $A = (\alpha\sqrt{n})^{-1}G$, where $G \sim$ Elliptic(ρ). Suppose $\mu^{\boldsymbol{r}} \to \mathcal{L}(\bar{r})$. The system of unknown $(\delta, \sigma^2, \gamma)$ has a unique solution if $\alpha > \sqrt{2(1+\rho)}$.

$$\begin{cases} \alpha &= \delta + \rho\dfrac{\gamma}{\delta}, \\ \sigma^2 &= \dfrac{1}{\delta^2}\mathbb{E}\left[(\sigma\xi + \bar{r})_+^2\right], \\ \gamma &= \mathbb{P}\left[\sigma\xi + \bar{r} > 0\right], \end{cases}$$

where $\xi \sim \mathcal{N}(0,1)$.

Result - Asymptotic behavior

$$\mu^{\boldsymbol{x}^*} \xrightarrow[n\to\infty]{w,L^2} \mathcal{L}\left(\left(1 + \rho\frac{\gamma}{\delta^2}\right)(\sigma\xi + \bar{r})_+\right) \text{ a.s.}$$

Remark. For k large enough, $\boldsymbol{x}^{\boldsymbol{k}} \approx \boldsymbol{x}^*$.

Conclusion

- We proved the existence and uniqueness of the equilibrium in a general setting.
- We analyzed the statistical properties of the equilibrium $\boldsymbol{x}^*$ by tracking $\mu^{\boldsymbol{x}^{\boldsymbol{k}}_{\text{AMP}}}$.
- We developed of a new AMP algorithm adapted to a general class of random matrices.

References

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