Approximate Message Passing for elliptic random matrices

M-Y. Gueddari[1], W. Hachem[1], J. Najim[1]

[1] *CNRS and Université Gustave Eiffel, France*

5 Bd Descartes, 77420 Champs-sur-Marne

Motivation : From theoretical ecology

- **Model**: The interactions between species in an ecological environment can be modeled using Lotka Volterra differential equations.

$$\begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{x}(t) \odot \left(\boldsymbol{r} - \boldsymbol{x}(t) + \dfrac{1}{\alpha\sqrt{n}} A\boldsymbol{x}(t) \right) \\ \boldsymbol{x}(t) \succcurlyeq 0, \quad \forall t \end{cases}$$

→ $A_{i,j}$ is the effect of species i on species j.

- **Goal**: We are interested in the equilibrium point $\boldsymbol{x}^*$ and its statistical properties (distribution, number of positive components, etc).

$$\boldsymbol{x}^* \odot \left(\boldsymbol{r} - \left(I_n - \frac{A}{\alpha\sqrt{n}} \right) \boldsymbol{x}^* \right) = 0 \quad (1)$$

- **Specs**: The interaction matrix A is a large random matrix such that the interactions $A_{i,j}$ and $A_{j,i}$ are correlated ($\rho \in [-,1,1]$). This model is very popular in theoretical ecology.

LCP

In addition to equation 1, the equilibrium $\boldsymbol{x}^*$ is taken to be Lyapunov stable which adds another constraint for $\boldsymbol{x}^*$. Hence:

$$\begin{cases} \boldsymbol{x}^* \succcurlyeq 0, \\ \boldsymbol{x}^* \odot \left(\boldsymbol{r} - \left(I_n - \dfrac{A}{\alpha\sqrt{n}} \right) \boldsymbol{x}^* \right) = 0, \\ \boldsymbol{r} - \left(I_n - \dfrac{A}{\alpha\sqrt{n}} \right) \boldsymbol{x}^* \preccurlyeq 0 \end{cases}$$

Consequence: $\boldsymbol{x}^*$ is a solution to $LCP\left(I_n - \dfrac{A}{\alpha\sqrt{n}}, -\boldsymbol{r} \right)$ which is a classical non-linear optimization problem.

Objectives

- **Goal**: The analysis of the asymptotic behavior of the empirical distribution $\boxed{\mu^{\boldsymbol{x}^*} = \frac{1}{n}\sum_{k=1}^{n} \delta_{x_k^*}}$ when n is large.

- **Tool**: Approximate Message Passing (AMP) can solve this problem by approximating of the measure $\mu^{\boldsymbol{x}^*}$ and by providing rich information of its properties.

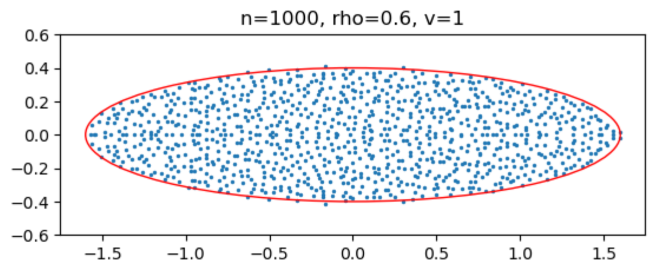
- **Procedure**: For a well-chosen activation function, the iterates $\boldsymbol{x^k}$ of AMP converge to a solution of LCP.

- **Our contribution**: AMP is known for GOE matrices. We generalize AMP convergence results for a more general type of matrices: elliptic matrices.

Elliptic matrices

$A \sim Ellip(n, \rho)$ if $A_{i,j} \sim \mathcal{N}(0,1)$, $A_{i,i} \sim \mathcal{N}(0, 1+\rho)$ and $cov(A_{i,j}, A_{j,i}) = \rho$.

The asymptotic spectrum of A is the uniform law over the ellipse of semi-major axis $= 1+\rho$ and semi-minor axis$= 1-\rho$.



AMP for GOE matrices

Approximate Message Passing algorithms for GOE matrices:

$$\boldsymbol{x^{k+1}} = \frac{1}{\sqrt{n}} A f_k\left(\boldsymbol{x^k}\right) - \overline{f_t'(\boldsymbol{x^k})}^{\,n} f_{k-1}\left(\boldsymbol{x^{k-1}}\right)$$

Result: Under some mild initial conditions, we have

$$\mu^{(\boldsymbol{x^1}, \cdots, \boldsymbol{x^p})} \xrightarrow[n\to\infty]{W_2} \mathcal{L}(Z^1, \cdots, Z^p) \text{ a.s.}$$

Where $(Z^1, \cdots, Z^p)$ is a centered Gaussian vector whose co-variance matrix is given by Density Evolution equations.

Elements of proof

Bolthausen's conditioning technique

Let $\boldsymbol{q^k} = f_k(\boldsymbol{x^k})$, P_k is the projection matrix over $span\left(\boldsymbol{q^0}, \cdots, \boldsymbol{q^{k-1}}\right)$ and $\mathcal{F}_k = \sigma\left(\boldsymbol{x^0}, \cdots, \boldsymbol{x^k}\right)$. Then

$$\boldsymbol{x^{k+1}} = \frac{1}{\sqrt{n}} A f_k\left(\boldsymbol{x^k}\right) - \rho b_k \boldsymbol{q^{k-1}}$$

$$\overset{\mathcal{L}}{=} |_{\mathcal{F}_k} \frac{1}{\sqrt{n}} \sum_{l=1}^{k} \alpha_l \boldsymbol{x^l} + o(1) + \left(\tilde{A} - \rho P_k \tilde{A}^\top\right) P_k^\perp$$

Where green is $\mathcal{F}_k$ measurable and red is independant from $\mathcal{F}_k$.

Fixed point equations

Let $\rho \in [-1,1]$ and $Z \sim \mathcal{N}(0,1)$, the system of unknown $(\delta, \sigma^2, \gamma)$ has a unique solution if $\alpha > 2$.

$$\begin{cases} \alpha &= \delta + \rho\dfrac{\gamma}{\delta} \\ \sigma^2 &= \dfrac{1}{\delta^2}\mathbb{E}\left(\sigma Z + r\right)_+^2 \\ \gamma &= \mathbb{P}\left[\sigma Z + r > 0\right] \end{cases}$$

Key ideas:

- Using the AMP scheme with specific activation functions $f_k(x) = \dfrac{(x+r_k)_+}{\delta}$, the iterates $\boldsymbol{x^k}$ and $\boldsymbol{x^{k+1}}$ become more correlated when k grows.
- Density evolution equations explicitly describe the variances which lead to a fixed point equation for σ^2.

Result:

$$\mu^{\boldsymbol{x}^*} \xrightarrow[n\to\infty]{W_2} \mathcal{L}\left(\left(1 + \rho\frac{\gamma}{\delta^2}\right)(\sigma Z + r)_+\right) \text{ a.s.}$$

Remark. For k large enough, $\boldsymbol{x^k} \approx \boldsymbol{x}^*$.

AMP for elliptic matrices

For the elliptic matrix model: AMP iterations will be defined as:

$$\boldsymbol{x^{k+1}} = \frac{1}{\sqrt{n}} A f_k\left(\boldsymbol{x^k}\right) - \rho \overline{f_k'(\boldsymbol{x^k})}^{\,n} f_{k-1}\left(\boldsymbol{x^{k-1}}\right)$$

The "Onsager term" $\rho \overline{f_k'(\boldsymbol{x^k})}^{\,n} f_{k-1}\left(\boldsymbol{x^{k-1}}\right)$ is an interpolation between the GOE case : $\rho = 1$ and the i.i.d. case : $\rho = 0$ (where the Onsager term disappear).

Perspectives

- Prove a universality result of the convergence of AMP (elliptic model), i.e. when the entries are not necessarily gaussian.
- Generalize the AMP scheme to matrices with variance profile, i.e. models of the form $M = V \odot A$, where $A \sim Ellip(n, \rho)$ and V is a deterministic matrix.

Conclusion

In this work,

- we analysed the statistical properties of the equilibrium $\boldsymbol{x}^*$.
- we developed of a new AMP algorithm.

References

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[3] Akjouj, I. and Hachem, W. and Maïda, M. and Najim, J. Equilibria of large random Lotka-Volterra systems with vanishing species : a mathematical approach, 2023.