Session 10 - 11

Probability Distribution

Prof. Jigar M. Shah

Probability Distribution

- Probability Distribution
  - Random Variable
  - Probability Distribution
  - Binomial Distribution
  - Poisson Distribution
  - Normal Distribution

Probability Distribution

Probability Distribution

- Random Variable

Common Terms	
Random Variable	A numerical description of the outcome of an experiment
Discrete Random Variable	A random variable that can assume either a finite no. of values or an infinite sequence of values such as 0, 1, 2, …
Continuous Random Variable	A random variable that may assume any numerical value in an interval or collection of intervals

Probability Distribution

Probability Distribution

- Probability Distribution

Common Terms	
Probability Distribution of a Random Variable	• Describes how probabilities are distributed over the values of the random variable
Probability Function	• Probability distribution for a discrete random variable • Denoted as f(x), it provides the probability for each value of the random variable x
Probability Density Function	• Probability distribution for a continuous random variable • Denoted as f(x), it provides the probability that the continuous random variable x assumes a value in an interval • Probability that the continuous random variable takes any particular value is 0 (since the area under the graph of f(x) at any particular point is 0)

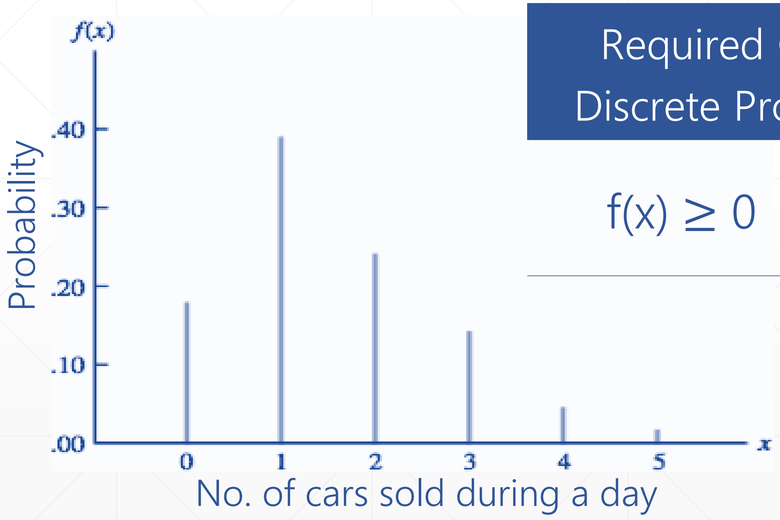
Probability Distribution

Probability Distribution

- Probability Distribution
  - Discrete Probability Function

x i.e. Number Obtained	f(x) i.e. ,Probability of x
1	1/6
2	1/6
3	1/6
4	1/6
5	1/6
6	1/6
	1

Probability Distribution for Number Obtained on One Roll of a Die



Graphical Representation of Probability Distribution of Number of Cars Sold During a Day

Required Conditions for a Discrete Probability Function

$f(x) \geq 0$ $\sum f(x) = 1$

Probability Distribution

Probability Distribution

- Binomial Distribution
  - Binomial Distribution - a discrete probability distribution associated with a multi-step experiment known as binomial experiment

Properties of a Binomial Experiment
1. The experiment consists of a sequence of n identical trials
2. Two outcomes are possible on each trial, success and failure
3. The probability of a success, denoted by p, does not change from trial to trial
4. The trials are independent

  - Random variable x is the number of successes in n trials, and f(x) is known as the Binomial Probability Distribution

Probability Distribution

Probability Distribution

- Binomial Distribution

No. of Experimental Outcomes Resulting in Exactly x Successes in n Trials		
$^{n}C_{x} = \dfrac{n!}{x!\,(n-x)!}$	where	$n! = n \times (n-1) \times (n-2) \times (n-3) \times \ldots.. 3 \times 2 \times 1$
		$0! = 1$

Probability of Particular Sequence of Trial Outcomes with x Successes in n Trials			
$p^{x}\,(1 - p)^{(n-x)}$	where	p is the probability of success on one trial	x is the number of success in n trials
		(1 – p) is the probability of failure on one trial	(n – x) is the number of failures in n trials

Probability Distribution

Probability Distribution

- Binomial Distribution

Binomial Probability Function

$$f(x) = {}^{n}C_{x}\, p^{x} (1 - p)^{(n-x)}$$

where

x is the number of success in n trials

p is the probability of success on one trial

n is the number of trials

f(x) is the probability of x successes in n trials

$${}^{n}C_{x} = \frac{n!}{x!\,(n-x)!}$$

Probability Distribution

Probability Distribution

- Binomial Distribution

Binomial Probability Function

$$\sum_{x=0}^{n} f(x) = \sum_{x=0}^{n} {}^{n}C_{x}\, p^{x} (1 - p)^{(n - x)} = 1$$

n & p characterize the Binomial Distribution, and are known as Parameters of the Binomial Distribution

If	$p = \frac{1}{2}$	the distribution is symmetrical	Mean	=	np
If	$p < \frac{1}{2}$	the distribution is skewed to the right	Variance	=	$np(1 - p) = npq$
If	$p > \frac{1}{2}$	the distribution is skewed to the left	Standard Deviation	=	$\sqrt{np(1 - p)} = \sqrt{npq}$

where $q = (1 - p)$

Probability Distribution

Probability Distribution

- Binomial Distribution
  - In a retail store, based on past experience, the manager knows that the probability that any one customer will make a purchase is 0.3. Find the probability that two of the next three customers will make a purchase?
  - According to a study, 15% of adults in USA do not use the Internet. Suppose that 10 adults in USA are selected randomly. What is the probability that:
    - None of them use Internet?
    - Three of them use Internet?
    - At least one of them uses Internet?

Probability Distribution

Probability Distribution

- Poisson Distribution
  - Poisson Distribution - a discrete probability distribution Used in estimating no. of occurrences over a specified interval of time or space. e.g. – no. of arrivals in a store in an hour, no. of repairs needed in 10 km of highway, etc.
  - The random variable, x, is the no. of occurrences, and it can be described by Poisson Probability Distribution, f(x), if following properties are satisfied:

Properties of a Poisson Experiment
1. The probability of occurrence is the same for any two intervals of equal length
2. The occurrence or non-occurrence in any interval is independent of the occurrence or non-occurrence in any other interval

Probability Distribution

Probability Distribution

- Poisson Distribution

Poisson Probability Function

$$f(x) = \frac{\mu^x e^{-\mu}}{x!}$$

where

- x is the number of occurrences in the interval
- f(x) is the probability of x occurrences in the interval
- e = 2.71828

Mean $= \mu$ Variance $= \mu$ Standard Deviation $= \sqrt{\mu}$

- Determine the probability of 5 customer arrivals at a bank teller in 15 minutes if the average no. of customer arrivals in a 15-minute period is 10.
- If major defects one month after resurfacing the highway occur at the rate of two per km, find the probability of no major defects in a particular three-km section of the highway.

Probability Distribution

Probability Distribution

- Normal Distribution
  - The most important probability distribution for describing a continuous random variable
  - Also known as Gaussian Distribution

Some Situations where Normal Distribution is Used
1. Weekly sales in a store
2. Height & weight of children at birth
3. Yield of fertilizer used in different plots of land for a crop
4. Life of items subjected to ware & tear, such as, bulbs, batteries, currency notes, etc.
5. Length & diameter of certain manufactured products like screws, pipes, etc.

Probability Distribution

Probability Distribution

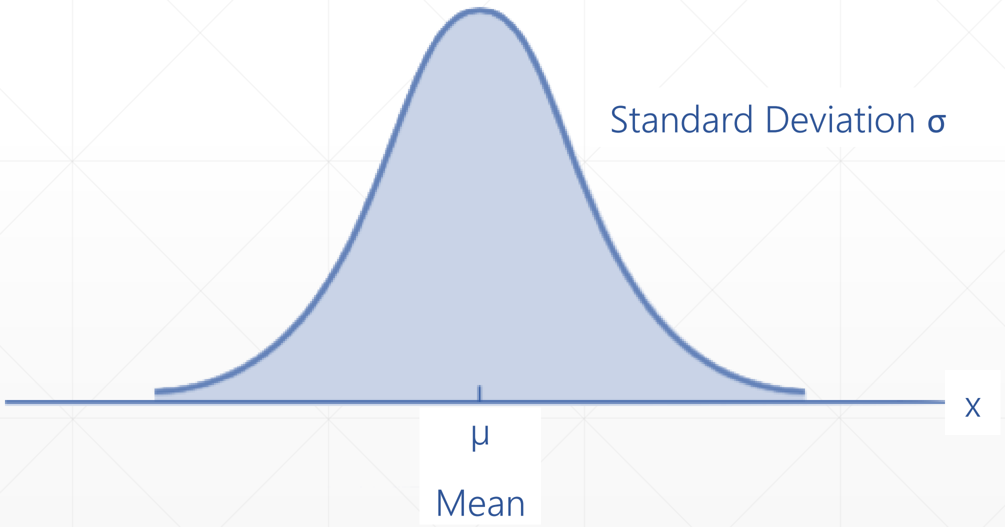
- Normal Distribution

Normal Probability Density Function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad \ldots\ldots -\infty < x < \infty$$

where

μ is the mean of random variable x	
σ is the standard deviation of the random variable x	
π = 3.14159	e = 2.71828



Bell Curve - Shape of Normal Distribution

Probability Distribution

Probability Distribution

- Normal Distribution

Normal Probability Density Function
1. It is characterized by two parameters, mean (μ) & standard deviation (σ)
2. The mean of the distribution can be any numerical value, negative, positive, 0
3. It is symmetric about the mean, and hence its skewness measure is 0
4. The tails of normal curve extend to ∞ in both the directions about the mean
5. The standard deviation determines how flat & wide the curve is
6. Probabilities for the normal random variable are given by areas under the normal curve. The total area under the normal curve is 1, with the area to the left of the mean being 0.50 and that to the right of the curve being 0.50

Probability Distribution

Probability Distribution

- Normal Distribution

Normal Probability Density Function
7. 68.3% of the values of a normal random variable are within $\pm\sigma$ from the mean
8. 95.4% of the values of a normal random variable are within $\pm 2\sigma$ from the mean
9. 99.7% of the values of a normal random variable are within $\pm 3\sigma$ from the mean
10. The measure of peaked-ness (kurtosis) of the curve is 3
11. The mean, median & mode are all equal
12. The mean being equal to the median, divides the curve in to two equal parts
13. The mean being equal to the mode, the curve has the maximum value at the mean

Probability Distribution

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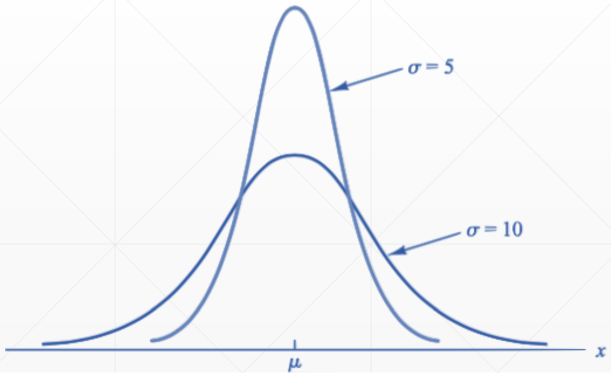
Normal Probability Density Function
14. The Mean Deviation of Normal Distribution is $\sigma\sqrt{(2/\pi)} = (4/5)\sigma = 0.8\sigma$
15. 50% of the area under the curve lies between $\mu - 0.645\sigma$ (first quartile) & $\mu + 0.645\sigma$ (third quartile)

Normal Curves with Different Means



Normal Curves with Same Means but Different Standard Deviations

Probability Distribution

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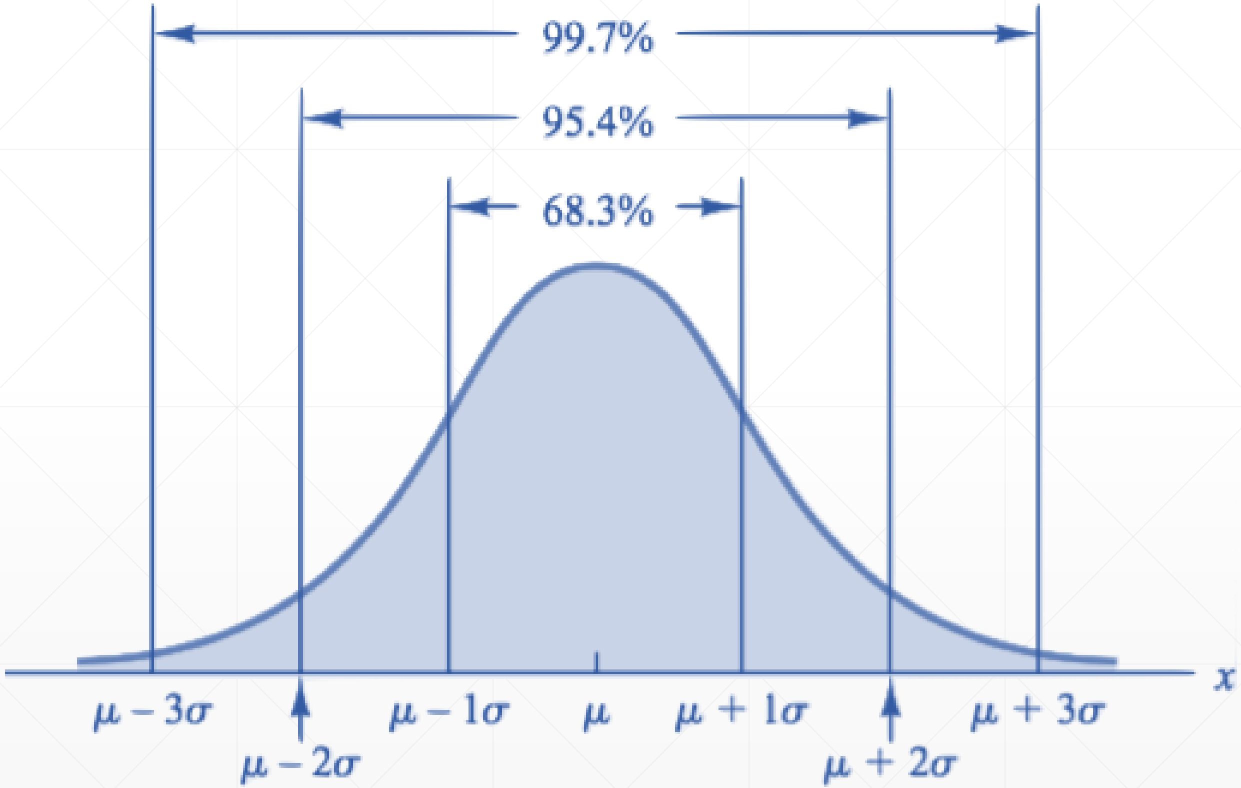
- Normal Distribution



Areas under the Curve for any Normal Distribution

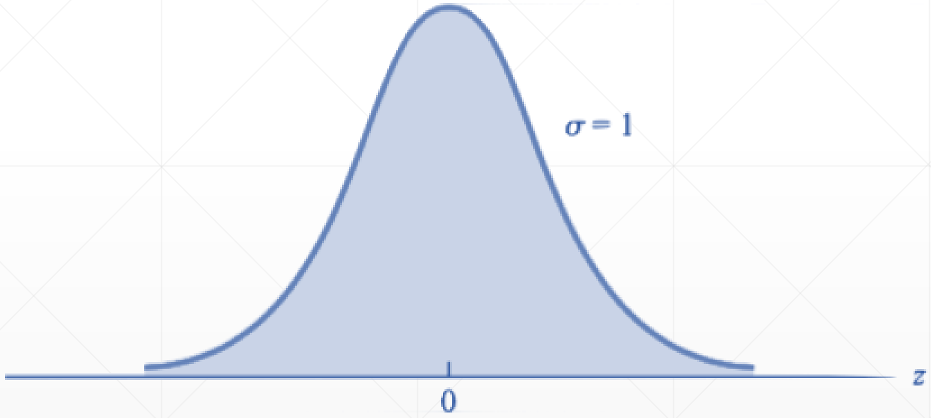
Probability Distribution

Probability Distribution

- Normal Distribution

Standard Normal Probability Distribution

- A random variable that has a normal distribution with a mean of zero and a standard deviation of one is said to have a standard normal probability distribution
- Such a random variable, that has a normal distribution with a mean of zero and a standard deviation of one, is denoted by the letter z (known as standard normal random variable)
- The standard normal probability distribution as the same general appearance as other normal distributions, but with a special properties of mean μ = 0, and standard deviation σ = 1



Standard Normal Distribution

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \quad \ldots\ldots \; -\infty < z < \infty$$

Probability Distribution

Probability Distribution

- Normal Distribution

Standard Normal Probability Distribution	
• For the standard normal distribution, areas under the normal curve are available in tables that provide cumulative probabilities for several values of z	
• Three types of probability computations	1. Probability that z will be ≤ to a given value
	2. Probability that z will be between two given values
	3. Probability that z will be ≥ to a given value

Probability Distribution

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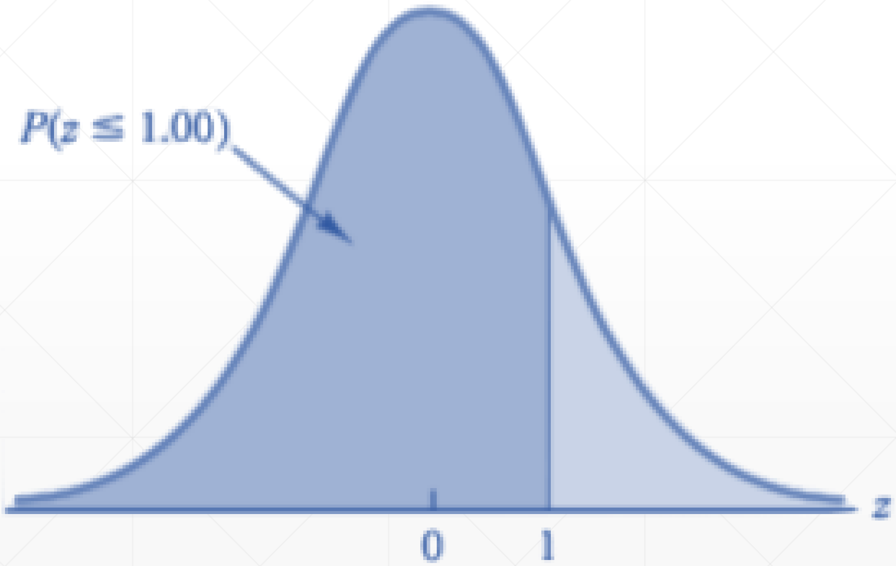
- Normal Distribution

STANDARD NORMAL DISTRIBUTION: Table Values Represent AREA to the LEFT of the Z score.

Z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
-3.9	.00005	.00005	.00004	.00004	.00004	.00004	.00004	.00004	.00003	.00003
-3.8	.00007	.00007	.00007	.00006	.00006	.00006	.00006	.00005	.00005	.00005
-3.7	.00011	.00010	.00010	.00010	.00009	.00009	.00008	.00008	.00008	.00008
-3.6	.00016	.00015	.00015	.00014	.00014	.00013	.00013	.00012	.00012	.00011
-3.5	.00023	.00022	.00022	.00021	.00020	.00019	.00019	.00018	.00017	.00017
-3.4	.00034	.00032	.00031	.00030	.00029	.00028	.00027	.00026	.00025	.00024
-3.3	.00048	.00047	.00045	.00043	.00042	.00040	.00039	.00038	.00036	.00035
-3.2	.00069	.00066	.00064	.00062	.00060	.00058	.00056	.00054	.00052	.00050
-3.1	.00097	.00094	.00090	.00087	.00084	.00082	.00079	.00076	.00074	.00071
-3.0	.00135	.00131	.00126	.00122	.00118	.00114	.00111	.00107	.00104	.00100
-2.9	.00187	.00181	.00175	.00169	.00164	.00159	.00154	.00149	.00144	.00139
-2.8	.00256	.00248	.00240	.00233	.00226	.00219	.00212	.00205	.00199	.00193
-2.7	.00347	.00336	.00326	.00317	.00307	.00298	.00289	.00280	.00272	.00264
-2.6	.00466	.00453	.00440	.00427	.00415	.00402	.00391	.00379	.00368	.00357
-2.5	.00621	.00604	.00587	.00570	.00554	.00539	.00523	.00508	.00494	.00480
-2.4	.00820	.00798	.00776	.00755	.00734	.00714	.00695	.00676	.00657	.00639
-2.3	.01072	.01044	.01017	.00990	.00964	.00939	.00914	.00889	.00866	.00842
-2.2	.01390	.01355	.01321	.01287	.01255	.01222	.01191	.01160	.01130	.01101
-2.1	.01786	.01743	.01700	.01659	.01618	.01578	.01539	.01500	.01463	.01426
-2.0	.02275	.02222	.02169	.02118	.02068	.02018	.01970	.01923	.01876	.01831
-1.9	.02872	.02807	.02743	.02680	.02619	.02559	.02500	.02442	.02385	.02330
-1.8	.03593	.03515	.03438	.03362	.03288	.03216	.03144	.03074	.03005	.02938
-1.7	.04457	.04363	.04272	.04182	.04093	.04006	.03920	.03836	.03754	.03673
-1.6	.05480	.05370	.05262	.05155	.05050	.04947	.04846	.04746	.04648	.04551
-1.5	.06681	.06552	.06426	.06301	.06178	.06057	.05938	.05821	.05705	.05592
-1.4	.08076	.07927	.07780	.07636	.07493	.07353	.07215	.07078	.06944	.06811
-1.3	.09680	.09510	.09342	.09176	.09012	.08851	.08691	.08534	.08379	.08226
-1.2	.11507	.11314	.11123	.10935	.10749	.10565	.10383	.10204	.10027	.09853
-1.1	.13567	.13350	.13136	.12924	.12714	.12507	.12302	.12100	.11900	.11702
-1.0	.15866	.15625	.15386	.15151	.14917	.14686	.14457	.14231	.14007	.13786
-0.9	.18406	.18141	.17879	.17619	.17361	.17106	.16853	.16602	.16354	.16109
-0.8	.21186	.20897	.20611	.20327	.20045	.19766	.19489	.19215	.18943	.18673
-0.7	.24196	.23885	.23576	.23270	.22965	.22663	.22363	.22065	.21770	.21476
-0.6	.27425	.27093	.26763	.26435	.26109	.25785	.25463	.25143	.24825	.24510
-0.5	.30854	.30503	.30153	.29806	.29460	.29116	.28774	.28434	.28096	.27760
-0.4	.34458	.34090	.33724	.33360	.32997	.32636	.32276	.31918	.31561	.31207
-0.3	.38209	.37828	.37448	.37070	.36693	.36317	.35942	.35569	.35197	.34827
-0.2	.42074	.41683	.41294	.40905	.40517	.40129	.39743	.39358	.38974	.38591
-0.1	.46017	.45620	.45224	.44828	.44433	.44038	.43644	.43251	.42858	.42465
-0.0	.50000	.49601	.49202	.48803	.48405	.48006	.47608	.47210	.46812	.46414

STANDARD NORMAL DISTRIBUTION: Table Values Represent AREA to the LEFT of the Z score.

Z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.50000	.50399	.50798	.51197	.51595	.51994	.52392	.52790	.53188	.53586
0.1	.53983	.54380	.54776	.55172	.55567	.55962	.56356	.56749	.57142	.57535
0.2	.57926	.58317	.58706	.59095	.59483	.59871	.60257	.60642	.61026	.61409
0.3	.61791	.62172	.62552	.62930	.63307	.63683	.64058	.64431	.64803	.65173
0.4	.65542	.65910	.66276	.66640	.67003	.67364	.67724	.68082	.68439	.68793
0.5	.69146	.69497	.69847	.70194	.70540	.70884	.71226	.71566	.71904	.72240
0.6	.72575	.72907	.73237	.73565	.73891	.74215	.74537	.74857	.75175	.75490
0.7	.75804	.76115	.76424	.76730	.77035	.77337	.77637	.77935	.78230	.78524
0.8	.78814	.79103	.79389	.79673	.79955	.80234	.80511	.80785	.81057	.81327
0.9	.81594	.81859	.82121	.82381	.82639	.82894	.83147	.83398	.83646	.83891
1.0	.84134	.84375	.84614	.84849	.85083	.85314	.85543	.85769	.85993	.86214
1.1	.86433	.86650	.86864	.87076	.87286	.87493	.87698	.87900	.88100	.88298
1.2	.88493	.88686	.88877	.89065	.89251	.89435	.89617	.89796	.89973	.90147
1.3	.90320	.90490	.90658	.90824	.90988	.91149	.91309	.91466	.91621	.91774
1.4	.91924	.92073	.92220	.92364	.92507	.92647	.92785	.92922	.93056	.93189
1.5	.93319	.93448	.93574	.93699	.93822	.93943	.94062	.94179	.94295	.94408
1.6	.94520	.94630	.94738	.94845	.94950	.95053	.95154	.95254	.95352	.95449
1.7	.95543	.95637	.95728	.95818	.95907	.95994	.96080	.96164	.96246	.96327
1.8	.96407	.96485	.96562	.96638	.96712	.96784	.96856	.96926	.96995	.97062
1.9	.97128	.97193	.97257	.97320	.97381	.97441	.97500	.97558	.97615	.97670
2.0	.97725	.97778	.97831	.97882	.97932	.97982	.98030	.98077	.98124	.98169
2.1	.98214	.98257	.98300	.98341	.98382	.98422	.98461	.98500	.98537	.98574
2.2	.98610	.98645	.98679	.98713	.98745	.98778	.98809	.98840	.98870	.98899
2.3	.98928	.98956	.98983	.99010	.99036	.99061	.99086	.99111	.99134	.99158
2.4	.99180	.99202	.99224	.99245	.99266	.99286	.99305	.99324	.99343	.99361
2.5	.99379	.99396	.99413	.99430	.99446	.99461	.99477	.99492	.99506	.99520
2.6	.99534	.99547	.99560	.99573	.99585	.99598	.99609	.99621	.99632	.99643
2.7	.99653	.99664	.99674	.99683	.99693	.99702	.99711	.99720	.99728	.99736
2.8	.99744	.99752	.99760	.99767	.99774	.99781	.99788	.99795	.99801	.99807
2.9	.99813	.99819	.99825	.99831	.99836	.99841	.99846	.99851	.99856	.99861
3.0	.99865	.99869	.99874	.99878	.99882	.99886	.99889	.99893	.99896	.99900
3.1	.99903	.99906	.99910	.99913	.99916	.99918	.99921	.99924	.99926	.99929
3.2	.99931	.99934	.99936	.99938	.99940	.99942	.99944	.99946	.99948	.99950
3.3	.99952	.99953	.99955	.99957	.99958	.99960	.99961	.99962	.99964	.99965
3.4	.99966	.99968	.99969	.99970	.99971	.99972	.99973	.99974	.99975	.99976
3.5	.99977	.99978	.99978	.99979	.99980	.99981	.99981	.99982	.99983	.99983
3.6	.99984	.99985	.99985	.99986	.99986	.99987	.99987	.99988	.99988	.99989
3.7	.99989	.99990	.99990	.99990	.99991	.99991	.99992	.99992	.99992	.99992
3.8	.99993	.99993	.99993	.99994	.99994	.99994	.99994	.99995	.99995	.99995
3.9	.99995	.99995	.99996	.99996	.99996	.99996	.99996	.99996	.99997	.99997

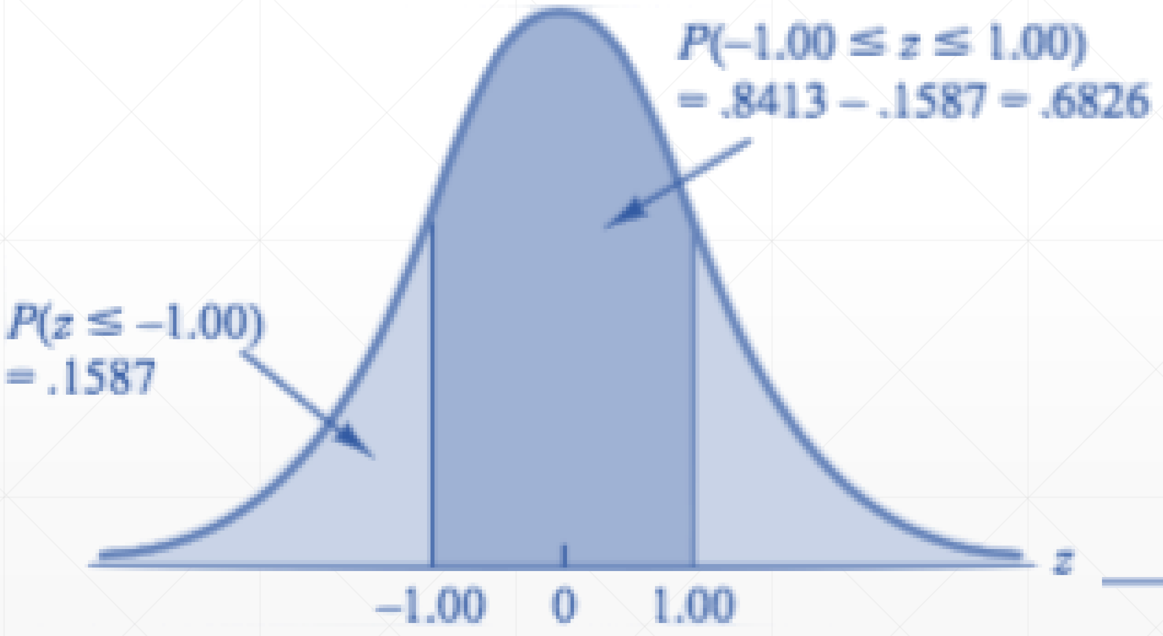


Probability that z will be ≤ to a given value

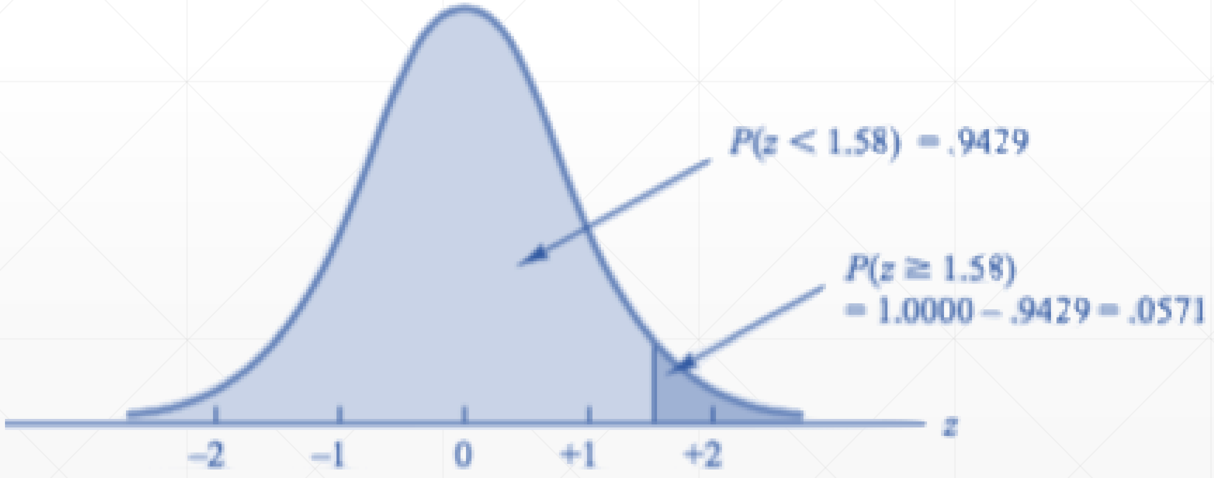
Probability Distribution

Probability Distribution

- Normal Distribution



Probability that z will be between two given values



Probability that z will be ≥ to a given value

Probability Distribution

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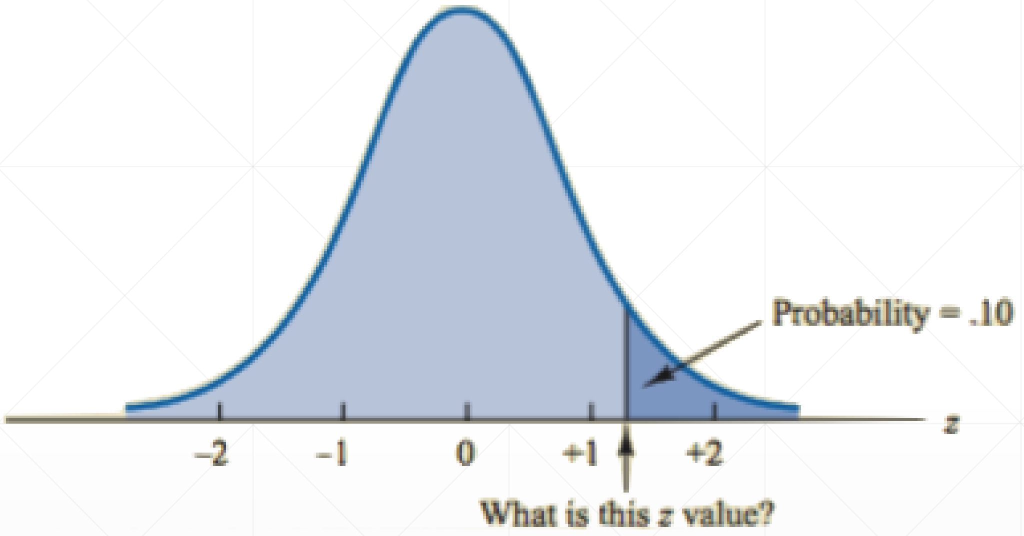
- Normal Distribution
  - Given the probabilities, the standard normal tables can be used in an inverse fashion to find the corresponding value of z



Standardizing the Normal Random Variable

- Probabilities for all normal distributions are computed using the standard normal distribution
- A normal distribution for a normal random variable x with a mean μ and a standard deviation σ can be converted into a standard normal distribution by converting the normal random variable x to the standard normal random variable z

Probability Distribution

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- Normal Distribution

Standardizing the Normal Random Variable i.e., Converting the Normal Random Variable to the Standard Normal Random Variable		
$z = \frac{x - \mu}{\sigma}$	• When $x = \mu$, $z = 0$	• When x is one standard deviation above the mean, i.e., when $x = \mu + \sigma$, $z = 1$ • When x is one standard deviation below the mean, i.e., when $x = \mu - \sigma$, $z = -1$
Thus, z can be interpreted as the no. of standard deviations that the normal random variable x is from its mean μ		

- A tire company has developed a new type of radial tire actual road tests of which have given the estimated life of the tire as 36500 km with a standard deviation of 5000 km. The data collected also indicated that the life of tire can be considered to follow a normal distribution. What percentage of tires can be expected to last more than 40000 km.

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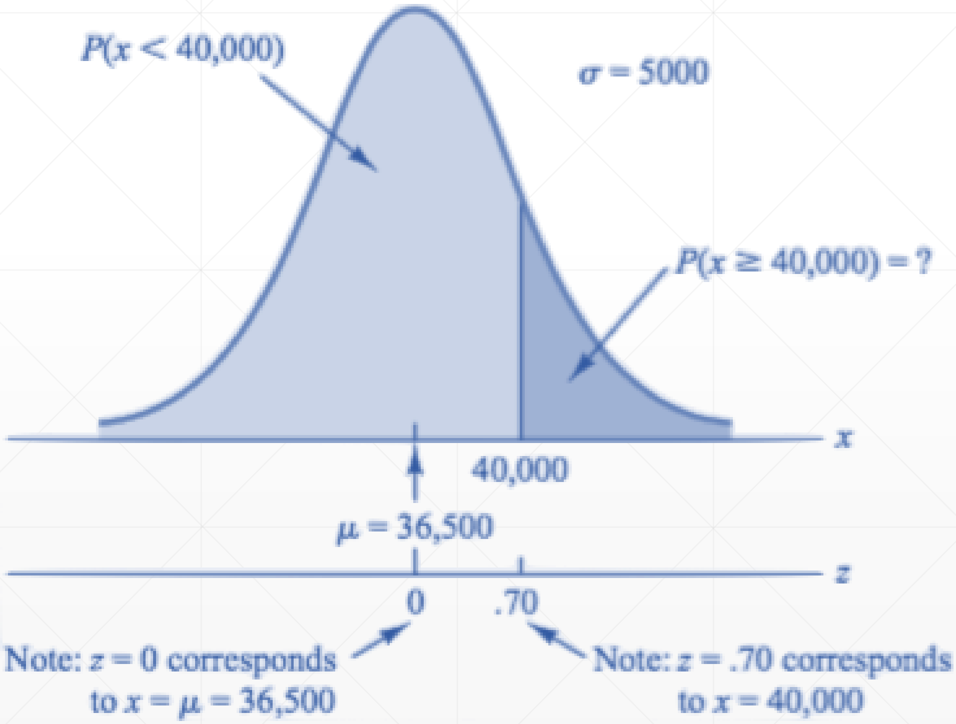

- The tire company wants to offer a guarantee that will provide a discount on replacement tires if the original tires do not last the guaranteed life. What should the guaranteed life of the tire be if the company wants no more than 10% of the tires to be eligible for the discount guarantee?

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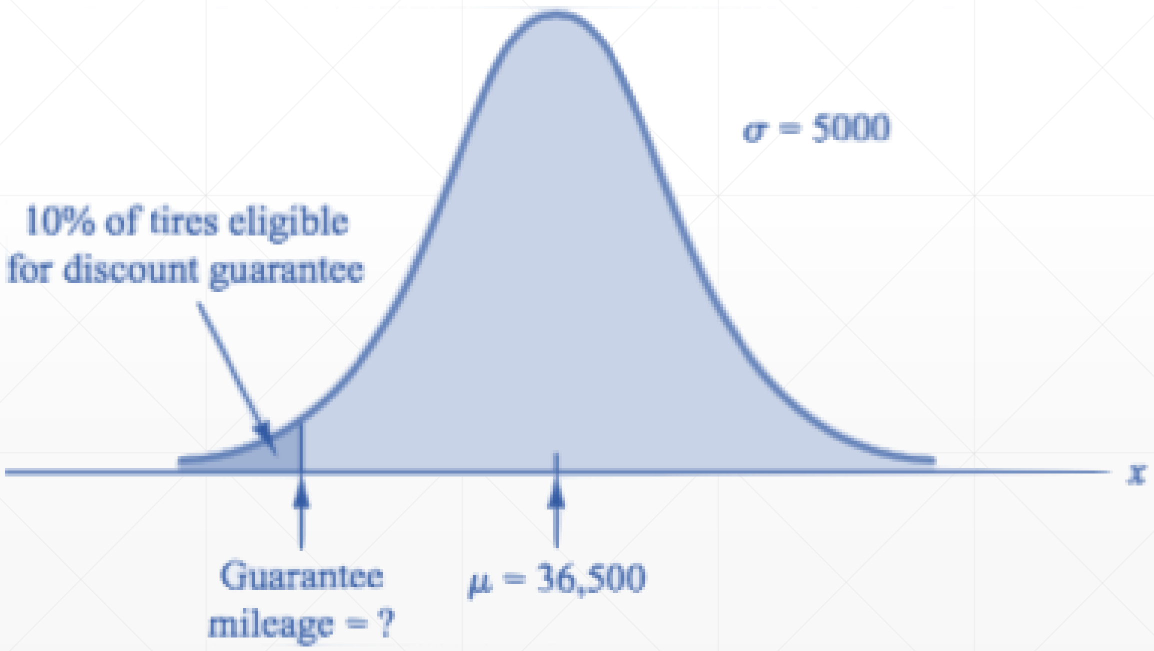
- Normal Distribution



- A person must score in the upper 2% of the population on an IQ test to qualify for membership in Mensa, the international highIQ society. If IQ scores are normally distributed with a mean of 100 and a standard deviation of 15, what score must a person have to qualify for Mensa?

Thank You

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