Permutation Statistics

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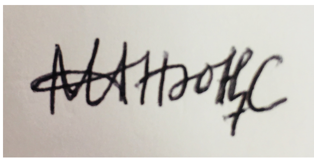
Abstract

Permutations are among the most classical objects in combinatorics and algebra. The aim of this essay is to collect different results on permutation statistics, such as the number of cycles, the number of inversions, the number of ascents and descents, and more. The study of permutation statistics has a long history. One of its motivations is the analysis of algorithms for sorting and searching. Typical results on these statistics involve the mean value, the variance, and the distribution for permutations of a large number of objects. This essay project also teaches the use of some important combinatorial techniques, such as generating functions.

Declaration

I, the undersigned, hereby declare that the work contained in this research project is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly.



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1. Introduction

1.1 Permutations

Permutations have a remarkably rich combinatorial structure. A permutation, also called an "arrangement" or "order" is a rearrangement of the elements of an ordered list into a one-to-one correspondence with itself.

1.1.1 Definition. A permutation is a linear ordering of the elements of the set $[n] = \{1, 2, 3, \cdots, n\}$.

1.1.2 Example. If $n = 3$, then $[n]$ has the six different permutations $\{123, 132, 213, 231, 312, 321\}$.

It is an easy work to count the number of possible permutations of n elements. There are n possible ways to choose the first element, for each of these we have $n - 1$ possible ways to choose the second element, $n - 2$ for the third and so forth.

1.1.3 Theorem. *The total number of permutatios of the set* $[n]$ *is given by* $n \cdot (n-1) \cdot (n-2) \cdots 1 = n!$.

1.2 Permutation Statistics

The theory of permutation statistics, which is one of the most active branches in enumerative combinatorics, dates back at least to Euler. However, MacMahon's extensive studies turned this subject to be a discipline in mathematics. In the last three decades or so, it attracted many mathematicians and much progress has been made in discovering and analyzing new statistics Clarke et al. (1996).

1.2.1 Definition. A permutation statistic is a function mapping permutations to nonnegative integers Mendes and Remmel (2008).

In the literature, five different statistics of permutations have been studied extensively. Euler started the subject and considered the number of descents. Netto and MacMahon respectively considered the number of inversions and the major index Ehrenborg and Steingrımsson (2000). Two other statistics are the excedances and the number of cycles.

Two permutation statistics are equidistributed if they have the same generating function. Foata proved the remarkable fact that the descents and the excedances are equidistributed on S_n Foata and Schützenberger (1978). MacMahon also proved the equidistribution of the number of inversions and the major index Rawlings (1981). Any permutation statistic that is equidistributed with descents is known as Eulerian and a permutation statistic that is equidistributed with inversions is known as Mahonian. Most of the permutation statistics found in the literature are either Eulerian or Mahonian.

Permutation statistics play an important role in sorting problems. For example, the analysis of bubble sort algorithms provide an interesting use of the inversion statistic. Knuth also showed that descents are connected with the theory of sorting Fulman (1998). Diaconis, McGrath, and Pitman studied a model of card shuffling in which descents play a central role Diaconis et al. (1995).

1.3 Generating Functions and Recurrence Relations

Generating functions are one of the most important and useful tools in enumerative combinatorics. They express an infinite sequence as coefficients arising from a power series in an auxiliary variable. Thus, a generating function is just a different way of writing a sequence of numbers.

1.3.1 Definition. The generating function for the sequence $(a_n)_{n\geq 0}$ of real numbers is the formal power series $A(x) = a_0 + a_1 + a_2x^2 + \cdots = \sum_{n\geq 0} a_nx^n$ or any equivalent closed form expression.

In this work, we are considering two different types of generating functions of a sequence of numbers $(a_n)_{n\geq 0}$, namely:

1. The ordinary generating function

$$A(x) = \sum_{n\geq 0} a_nx^n = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots .$$

2. The exponential generating function

$$A(x) = \sum_{n\geq 0} a_n\frac{x^n}{n!} = a_0 + a_1x + a_2\frac{x^2}{2} + a_3\frac{x^3}{6} + \cdots .$$

1.3.2 Example. The sequence $\{1, 1, 1, \cdots\}$ has the generating function $1+x+x^2+x^3+\cdots$. However, this generating function has a concise closed form expression, namely $1/(1-x)$. In working with generating functions, we shall ignore the question of convergence.

When dealing with computations of generating functions, we are particularly interested in knowing the coefficient of a certain power of x easily and whether the generating function can be written in a closed form.

In enumeration problems, it may be difficult to find the solution directly. However, we can express the solution of a problem of a given size n in terms of solutions to problems of smaller size. Thus, a recurrence relation is an expression of the n^{th} term of a sequence as a function of the preceding terms. The most general recurrence relation takes the form :

$$a_n = F_n(a_0, \cdots , a_{n-1}) \text{ for } n \geq 0.$$

For example, the famous Fibonacci sequence is defined by the recurrence relation

$$F_n = F_{n-1} + F_{n-2}$$

with initial conditions $F_0 = 0$ and $F_1 = 1$. Clearly such recurrence relations have a unique solution, and note that some initial values must be specified for the recurrence relation to define a specific sequence. To solve a recurrence relation helps us to find a formula for the term a_n of a sequence.

1.3.3 Definition. A recurrence relation of the sequence $(a_n)_{n\geq 0}$ is an equation relating the term a_n to one or more of the preceding terms a_i, $i < n$, for each $n \geq n_0$.

1.3.4 Example. Let a_n be the number of permutations of the sequence $\{1, 2, \cdots , n\}$. Then a_n satisfies the recurrence relation $a_{n+1} = (n + 1)a_n$ with initial conditions $a_1 = 1$ and $n \geq 1$. To solve this recurrence for a_n, we repeatedly apply the recurrence relation and its initial conditions and get that $a_n = na_{n-1} = n(n-1)a_{n-2} = \cdots = n!$.

1.4 Preliminary definitions

1.4.1 Definition. Let Ω be a finite set of outcomes of some sequence of trials, so that all trials of these outcomes are equally likely. A random variable is a function $X : \Omega \to \mathbb{R}$ that associates a number to the elements of the sample space.

In this work, we are considering the sample space Ω to be S_n, and the random variable to be the permutation statistic of interest.

1.4.2 Definition. Let S_n be the symmetric group on $[n]$. Let $X : S_n \to \mathbb{R}$ be a random variable. Then the expectation of X on S_n is

$$E(X) = \sum_{i \in S} iP(X = i).$$

1.4.3 Definition. Let $X : S_n \to \mathbb{R}$ be a random variable. Then the variance of X is

$$\mathrm{Var}(X) = E((X - E(X))^2) = E(X^2) - E(X)^2.$$

1.4.4 Definition. A sequence of positive real numbers $a_1, a_2, \cdots, a_n$ is called unimodal if there exists an index i such that $1 \le i \le n$ and $a_1 \le a_2 \le \cdots \le a_i \ge a_{i+1} \cdots \ge a_n$.

1.4.5 Definition. A sequence of positive real numbers $a_1, a_2, \cdots, a_n$ is called log-concave if $a_{i-1}a_{i+1} \le a_i^2$ holds for all indices i.

1.4.6 Proposition. If a sequence of positive real numbers $a_1, a_2, \cdots, a_n$ is log-concave, then it is also unimodal.

1.4.7 Definition. Bóna (2016) Let $a_1, a_2, \cdots a_n$ be a sequence of real numbers. We say the sequence has real roots if the polynomial $\sum_{i=1}^{n} a_i x^i$ has real roots only .

1.4.8 Theorem. *Bóna (2016) If the sequence $a_1, a_2, \cdots a_n$ of positive real numbers has real roots only, then it is log-concave.*

1.4.9 Definition. Let $[\mathbf{n}]$ be the polynomial $1 + q + q^2 + \cdots + q^{n-1}$, and let $[\mathbf{n}]! = [\mathbf{1}]! \cdot [\mathbf{2}]! \cdots [\mathbf{n}]!$. If we substitute $q = 1$, then $[\mathbf{i}] = i$, and therefore $[\mathbf{n}]! = n!$.

1.4.10 Definition. Let k and n be positive integers such that $k \le n$. Then the $(n, k)-$Gaussian coefficient or q-binomial coefficient is denoted by $\begin{bmatrix} \mathbf{n} \\ \mathbf{k} \end{bmatrix}$ and is given by

$$\begin{bmatrix} \mathbf{n} \\ \mathbf{k} \end{bmatrix} = \frac{[\mathbf{n}]!}{[\mathbf{k}]![\mathbf{n\text{-}k}]!}.$$

Note that $\begin{bmatrix} \mathbf{n} \\ \mathbf{k} \end{bmatrix} = \begin{bmatrix} \mathbf{n} \\ \mathbf{n\text{-}k} \end{bmatrix}$.

2. Eulerian Statistics

2.1 Descents and Ascents

2.1.1 Definition. A permutation $\pi = \pi_1\pi_2\cdots\pi_n$ has a descent at position i if $\pi_i > \pi_{i+1}$, and it has an ascent at i if $\pi_i < \pi_{i+1}$.

2.1.2 Example. Consider the permutation $\pi = 12563748$. This permutation has descents at positions 4 and 6, while positions $1, 2, 3, 5$ and 7 are ascents of π.

2.1.3 Definition. The descent set of permutation π, denoted by $\mathrm{Des}(\pi)$, is the set of all descents of π.

$$\mathrm{Des}(\pi) = \{i \mid 1 \le i \le n-1, \pi_i > \pi_{i+1}\}$$

2.1.4 Example. The permutation $\pi = 12563748$ has descent set $\mathrm{Des}(\pi) = \{4, 6\}$.

2.1.5 Remark. The total number of descents of permutation π, denoted by $\mathrm{des}(\pi)$ is the cardinality of the descent set of π.

$$\mathrm{des}(\pi) =\mid \mathrm{Des}(\pi) \mid$$

2.1.6 Theorem. *Let $S = \{s_1, s_2, \cdots, s_k\}$ be an ordered subset of $[n-1]$. The number of permutations with descent set S is*

$$B(S) = \sum_{T \subseteq S} (-1)^{|S|-|T|} \alpha(S) \tag{2.1.1}$$

where $\alpha(S)$ is the number of permutations whose descent set is contained in S, which is given by

$$\alpha(S) = \binom{n}{s_1, s_2 - s_1, s_3 - s_2, \cdots, n - s_k}. \tag{2.1.2}$$

Proof. Here we are proving the second part of the theorem, concerning the number $\alpha(S)$ of permutations whose descent set is contained in S. We can directly use the principle of inclusion and exclusion to complete the proof.
Take a permutation $\pi = \pi_1\pi_2\cdots\pi_n$. Arrange the entries of π into $k+1$ segments, so that the first i segments together have s_i entries for each i. The only positions where the permutation can have a descent is where two segments meet.
The first segment has length s_1, so we can choose its elements in $\binom{n}{s_1}$ ways. The second segment has length $s_2 - s_1$ and is disjoint from the first segment, so we can choose in $\binom{n-s_1}{s_2-s_1}$ ways. In general, segment i has length $s_i - s_{i-1}$ and can be chosen in $\binom{n-s_{i-1}}{s_i-s_{i-1}}$ ways. The last segment has only one choice and all the remaining $n - s_k$ entries have to go in the last segment. From this we get

$$\begin{aligned}
\alpha(S) &= \binom{n}{s_1}\binom{n-s_1}{s_2-s_1}\binom{n-s_2}{s_3-s_2}\cdots\binom{n-s_k}{n-s_k} \\
&= \frac{n!}{s!(n-s_1)!} \cdot \frac{(n-s_1)!}{(s_2-s_1)!(n-s_2)!} \cdots \frac{(n-s_k)!}{(n-s_k)!} \\
&= \frac{n!}{s_1! \cdot (s_2-s_1)! \cdot \cdots \cdot (n-s_k)!} \\
&= \binom{n}{s_1, s_2 - s_1, \cdots, n - s_k}.
\end{aligned}$$

□

2.1.7 Definition. Given a permutation $\pi = \pi_1\pi_2\cdots\pi_n$, a maximal sequence of elements $\pi_k, \pi_{k+1}, \cdots, \pi_{m+k-1}$ with $\pi_k < \pi_{k+1} < \cdots < \pi_{m+k-1}$ is called an ascending run of length m.

2.1.8 Example. There are three ascending runs in $\pi = 12563748$, namely $\underline{1256}$, $\underline{37}$, $\underline{48}$.

2.2 Eulerian Numbers

2.2.1 Definition. Define $A(n,k)$ to be the number of permutations of length n with $k-1$ descents.

$$A(n,k) = |\{\pi \in S_n : des(\pi) = k-1\}|. \tag{2.2.1}$$

The numbers $A(n,k)$ are called the Eulerian Numbers.

		k					
		1	2	3	4	5	6
	1	1					
	2	1	1				
n	3	1	4	1			
	4	1	11	11	1		
	5	1	26	66	26	1	
	6	1	57	302	302	57	1

Figure 2.1: The values of $A(n,k)$ for $n \leq 6$

2.2.2 Proposition. For a permutation π of length $n-1$ with k descents, there are $k+1$ possible positions to insert n without increasing the descent number.

Proof. In a permutation $\pi = \pi_1\pi_2\cdots\pi_{n-1}$ of length $n-1$ with k descents, we can insert the entry n at the end of π or between two entries that form a descent. This is not creating a new descent, so there are $k+1$ choices for the position of n. □

2.2.3 Proposition. For a permutation $\pi = \pi_1\pi_2\cdots\pi_{n-1}$ of length $n-1$ with $k-1$ descents, there are $n-k$ possible positions to insert n and increase the descent number by one.

Proof. In a permutation $\pi = \pi_1\pi_2\cdots\pi_{n-1}$ of length $n-1$ with k descents, to increase the number of descents by one, we can insert the entry n at the beginning of π or between two entries that do not make a descent. Thus there are $n-k$ choices for the position of n. □

2.2.4 Theorem. *The number of permutations of length n with k descents satisfies the recursive formula*

$$A(n,k+1) = (k+1)A(n-1,k+1) + (n-k)A(n-1,k) \tag{2.2.2}$$

with initial conditions $A(0,0) = 1$ and $A(0,k) = 0$.

Proof. Combining the propositions 2.2.2 and 2.2.3 yields the theorem. □

2.2.5 Proposition. The number of permutations with k descents equals the number of permutations with $n-k-1$ descents.

$$A(n, k+1) = A(n, n-k) \tag{2.2.3}$$

Proof. If $\pi = \pi_1\pi_2\cdots\pi_n$ has k descents, then its reverse $\pi' = \pi_n\pi_{n-1}\cdots\pi_1$ has $n-k-1$ descents. $\square$

2.2.6 Definition. The polynomial

$$A_n(x) = \sum_{k=0}^{n} A(n,k)x^k \tag{2.2.4}$$

is called the n^{th} Eulerian polynomial. When $n \geq 1$ we can write

$$A_n(x) = \sum_{k=0}^{n} A(n,k)x^k = \sum_{\pi\in S_n} x^{des(\pi)+1}. \tag{2.2.5}$$

2.2.7 Example.

$$\begin{aligned}
A_0(x) &= 1\\
A_1(x) &= x\\
A_2(x) &= x + x^2\\
A_3(x) &= x + 4x^2 + x^3\\
A_4(x) &= x + 11x^2 + 11x^3 + x^4\\
A_5(x) &= x + 26x^2 + 66x^3 + 26x^4 + x^5
\end{aligned}$$

2.2.8 Proposition. The Eulerian polynomials satisfy the following recursive formula for $n \geq 1$:

$$A_n(x) = (x - x^2)A'_{n-1}(x) + nxA_{n-1}(x) \tag{2.2.6}$$

Proof. Theorem 2.2.4 can be written as

$$A(n,k) = kA(n-1,k) + (n-(k-1))A(n-1,k-1)$$

Therefore,

$$\begin{aligned}
A_n(x) = \sum_{k=1}^{n} A(n,k)x^k &= \sum_{k=1}^{n} kA(n-1,k)x^k + \sum_{k=1}^{n}(n-(k-1))A(n-1,k-1)x^k\\
&= x\sum_{k=1}^{n} kA(n-1,k)x^{k-1} + nx\sum_{k=1}^{n} A(n-1,k-1)x^{k-1} - x^2\sum_{k=2}^{n}(k-1)A(n-1,k-1)x^{k-2}\\
&= xA'_{n-1}(x) + nxA_{n-1}(x) - x^2A'_{n-1}(x)\\
&= (x-x^2)A'_{n-1}(x) + nxA_{n-1}(x)
\end{aligned}$$

as required. $\square$

2.2.9 Theorem. *The Eulerian polynomial also can be written as*

$$A_n(x) = (1-x)^{n+1}\sum_{i=0}^{\infty} i^n x^i \quad \textit{for all} \quad n \geq 0. \tag{2.2.7}$$

2.2.10 Example. For $n = 1$,

$$A_1(x) = (1-x)^2 \sum_{i=0}^{\infty} i x^i = (1-x)^2 \frac{x}{(1-x)^2} = x.$$

For $n = 2$,

$$A_2(x) = (1-x)^3 \sum_{i=0}^{\infty} i^2 x^i = (1-x)^3 \left(\frac{2x^2}{(1-x)^3} + \frac{x}{(1-x)^2} \right) = x + x^2.$$

Here, one needs to remember the generating functions

$$\sum_{i=0}^{\infty} i x^i = \frac{x}{(1-x)^2}.$$

$$\sum_{i=0}^{\infty} i^2 x^i = \frac{2x^2}{(1-x)^3} + \frac{x}{(1-x)^2}.$$

Proof. We are proving the theorem by induction on n. When $n = 0$, equation 2.2.4 gives $A_0(x) = 1$, likewise equation 2.2.7 also shows that

$$A_0(x) = (1-x) \sum_{i=0}^{\infty} x^i = \frac{1-x}{1-x} = 1$$

Assume the theorem is true for n. Now we show that it also holds for $n + 1$. We can write equation 2.2.6 as

$$\begin{aligned}
A_{n+1}(x) &= x(1-x)A'_n(x) + (n+1)xA_n(x) \\
&= x(1-x)\left[-\left((n+1)(1-x)^n \sum_{i=0}^{\infty} i^n x^i \right) + (1-x)^{n+1} \sum_{i=1}^{\infty} i^{n+1} x^{i-1} \right] + (n+1)xA_n(x) \\
&= -\left((n+1)(1-x)^{n+1} \sum_{i=0}^{\infty} i^n x^{i+1} \right) + (1-x)^{n+2} \sum_{i=1}^{\infty} i^{n+1} x^i + (n+1)(1-x)^{n+1} \sum_{i=0}^{\infty} i^n x^{i+1} \\
&= (1-x)^{n+2} \sum_{i=1}^{\infty} i^{n+1} x^i \\
&= (1-x)^{n+2} \sum_{i=0}^{\infty} i^{n+1} x^i
\end{aligned}$$

as required. □

It is usually useful to collect all $A(n, k)$, the number of permutations of length n with $k + 1$ descents, in a generating function.

2.2.11 Theorem. *The master generating function for permutations of length n with k descents can be written as*

$$A(x,t) = \sum_{n=0}^{\infty} A_n(x) \frac{t^n}{n!} = \frac{1-x}{1-xe^{t(1-x)}} \tag{2.2.8}$$

Proof. Using equation 2.2.7 we can write

$$\begin{aligned}
A(x,t) &= \sum_{n=0}^{\infty}\left((1-x)^{n+1}\sum_{i=0}^{\infty} i^n x^i\right)\frac{t^n}{n!} = (1-x)\sum_{i=0}^{\infty}\sum_{n=0}^{\infty}\frac{((1-x)it)^n}{n!}x^i \\
&= (1-x)\sum_{i=0}^{\infty} x^i \sum_{n=0}^{\infty}\frac{((1-x)it)^n}{n!} = (1-x)\sum_{i=0}^{\infty} x^i e^{(1-x)it} \\
&= (1-x)\sum_{i=0}^{\infty}\left(xe^{(1-x)t}\right)^i \\
&= \frac{1-x}{1-xe^{t(1-x)}}
\end{aligned}$$

as required. □

2.3 Expected Value and Variance of Descents

2.3.1 Proposition. Let $des(\pi)$ be the number of descents of a permutation $\pi = \pi_1\pi_2\cdots\pi_n$. Let $X : S_n \to \mathbb{R}$ be the random variable defined by $X(\pi) = des(\pi)$. Then the average of X is

$$E(X) = \frac{n-1}{2}. \tag{2.3.1}$$

Proof. Define the indicator variable $X_i : S_n \to \mathbb{R}$ by

$$X_i(\pi) = \begin{cases} 1 & \text{if } \pi_i > \pi_{i+1} \\ 0 & \text{if not} \end{cases}$$

It follows that

$$X = \sum_{i=1}^{n-1} X_i.$$

Now, taking the expectation yields

$$E[X] = E\left[\sum_{i=1}^{n-1} X_i\right] = \sum_{i=1}^{n-1} E[X_i]$$

Since $E[X_i] = P(X_i = 1)$ and $P(X_i = 1) = P(\pi_i > \pi_{i+1}) = \frac{1}{2}$, it follows that

$$E(X) = \sum_{i=1}^{n-1}\frac{1}{2} = \frac{n-1}{2}$$

as required. □

2.3.2 Proposition. Let $des(\pi)$ be the number of descents of a permutation $\pi = \pi_1\pi_2\cdots\pi_n$. Let $X : S_n \to \mathbb{R}$ be the random variable defined by $X(\pi) = des(\pi)$. Then the variance of X is

$$\mathrm{Var}(X) = \frac{n+1}{12}. \tag{2.3.2}$$

Proof. The variance of a random variable is computed as

$$\mathrm{Var}(X) = E(X^2) - E(X)^2.$$

We computed $E(X)$ in equation 2.3.1, so to find $E(X^2)$, let us introduce the same indicator variable X_i defined for $\pi = \pi_1\pi_2\cdots\pi_n$.

Then $X = \sum_{i=1}^{n-1} X_i$ and $E(X_i) = E(X_i^2) = \frac{1}{2}$. Also, we consider the expectation $E(X_iX_j)$ of the pair (i,j) where $i < j$ in the following two cases:

- the case in which $i < j-1$ and both i and j are descents, which happens in $2\binom{n-2}{2}$ cases. Then $E(X_iX_j) = 1/4$.
- the case in which $j = i+1$ which happens in $2\binom{n-2}{1}$ cases. Then $E(X_iX_j) = 1/6$ (the permutation $\pi_{j+1}\pi_j\pi_i$ out of the six possible permutations).

Therefore

$$\begin{aligned}
E(X^2) = E\left[\left(\sum_{i=1}^{n-1} X_i\right)^2\right] &= \sum_{i=1}^{n-1} E(X_i^2) + \sum_{j=1}^{n}\sum_{i<j-1} E(X_iX_j) + \sum_{i=1}^{n-2} E(X_iX_{i+1}) \\
&= \frac{1}{2}\binom{n-1}{1} + \frac{1}{2}\binom{n-2}{2} + \frac{1}{3}\binom{n-2}{1} \\
&= \frac{n-1}{2} + \frac{(n-2)(n-3)}{4} + \frac{n-2}{3} \\
&= \frac{3n^2-5n+4}{12}.
\end{aligned}$$

Consequently

$$\begin{aligned}
Var(X) &= E(X^2) - E(X)^2 \\
&= \frac{3n^2-5n+4}{12} - \frac{(n-1)^2}{4} \\
&= \frac{n+1}{12}
\end{aligned}$$

as required. □

2.4 Alternating runs and Alternating permutations

2.4.1 Definition. A permutation $\pi = \pi_1\pi_2\cdots$ is called alternating if $\pi_1 > \pi_2 < \pi_3 > \pi_4 < \cdots$. In other words, π is an alternating permutation if $\pi_i < \pi_{i+1}$ for i even and $\pi_i > \pi_{i+1}$ for i odd.
Also, π is called reverse alternating if $\pi_1 < \pi_2 > \pi_3 < \pi_4 > \cdots$.

2.4.2 Definition. A permutation $\pi = \pi_1\pi_2\cdots\pi_n$ has $k+1$ alternating runs if there exist k indices i so that π changes direction at these positions. We say that π has a peak at i if $\pi_{i-1} < \pi_i > \pi_{i+1}$ or a valley at i if $\pi_{i-1} > \pi_i < \pi_{i+1}$.

2.4.3 Example. The permutation $\pi = 12563748$ changes positions at $i = 4, 5, 6$ and 7, so π has 5 alternating runs.

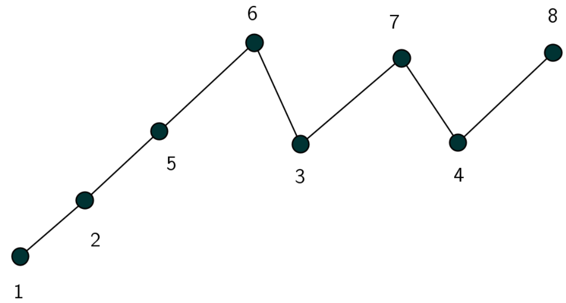

Figure 2.2

2.4.4 Definition. Let $G(n,k)$ be the number of permutations of length n with k alternating runs, and let $r(\pi)$ be the number of alternating runs of π. We denote the generating function of the numbers $G(n,k)$ by $G_n(x)$. Then

$$G_n(x) = \sum_{\pi \in S_n} x^{r(\pi)} = \sum_{k=1}^{n} G(n,k)x^k. \tag{2.4.1}$$

		k					
		1	2	3	4	5	6
	1	2					
	2	2	4				
n	3	2	12	10			
	4	2	28	58	32		
	5	2	60	236	300	122	

Figure 2.3: The values of $G(n,k)$ for $n \leq 6$

2.4.5 Proposition. The number of permutations of length n with k alternating runs satisfies the recursive formula

$$G(n,k) = kG(n-1,k) + 2G(n-1,k-1) + (n-k)G(n-1,k-2) \tag{2.4.2}$$

with initial conditions $G(1,0) = 1$ and $G(1,k) = 0$ for $k > 0$.

Proof. There are three ways to insert the entry n in a permutaion of length $n-1$.
First we can insert the entry n in a permutation π of length $n-1$ with k alternating runs without increasing the number of alternating runs. This can happen by inserting n in k different positions: right before the beginning of each descending run and right after the end of each ascending run. This gives $kG(n-1,k)$ possibilities.

Second, we can insert the entry n in a permutation π of length $n-1$ with $k-1$ alternating runs to create one more new alternating run. This can happen by inserting n in two positions:

- If the permutation π starts with an ascending run, insert n in front of π, or if π ends in an ascending run, insert n right before the last entry.
- If the permutation π starts with a descending run, insert n after the first entry of π or if π ends with a descending run, insert n at the end of π.

These two cases give $2G(n-1,k-1)$ possibilities.

Finally we can insert an entry n in a permutation π of length $n-1$ with $k-2$ alternating runs, creating two more new alternating runs. This can happen by inserting n in any of the remaining $n-(k-2)-2 = n-k$ positions. This gives $(n-k)G(n-1,k-2)$ possibilities and completes the proof. □

Using the notion of descents in permutations, we can easily define alternating and reverse alternating permutations as follows: π is alternating if $des(\pi) = \{1,3,5,\cdots\}$ and π is reverse alternating if $des(\pi) = \{2,4,6,\cdots\}$.

2.4.6 Example. The permutation $\pi = 42867153$ is an alternating permutation while its reverse $\pi' = 35176824$ is a reverse alternating permutation.

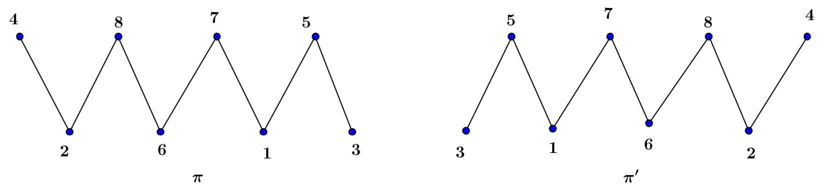

Figure 2.4

2.4.7 Theorem. *Let E_n denote the number of alternating permutations in S_n. Then E_n is called an Euler number and has the generating function*

$$\sum_{n=0}^{\infty} E_n \frac{x^n}{n!} = \sec x + \tan x \tag{2.4.3}$$

$$= 1 + x + \frac{x^2}{2!} + 2\frac{x^3}{3!} + 5\frac{x^4}{4!} + \cdots . \tag{2.4.4}$$

with initial conditions $E_0 = E_1 = 1$.

2.4.8 Example. For $n=4$, $E_4 = 5$. This means there are five permutations of length four which are alternating (reverse alternating) permutations. These are the permutations $\{2143, 3142, 3241, 4132, 4231\}$.

2.4.9 Corollary. Bóna (2016) From the fact that $\sec x$ is an even function and $\tan x$ is an odd function, it follows that

$$\sec x = \sum_{n=0}^{\infty} E_{2n} \frac{x^{2n}}{(2n)!} \tag{2.4.5}$$

and

$$\tan x = \sum_{n=0}^{\infty} E_{2n+1} \frac{x^{2n+1}}{(2n+1)!}. \tag{2.4.6}$$

So, E_{2n} is often called the secant number and E_{2n+1} is called the tangent number.

2.5 Excedences

2.5.1 Definition. In a permutation $\pi = \pi_1\pi_2\cdots\pi_n$, the position i is called an excedence if $\pi_i > i$. The case where $\pi_i \geq i$ is called a weak excedence.
Denote by $exc(\pi)$ and $wexc(\pi)$ the number of excedences and weak excedences of a permutation π, respectively.

2.5.2 Example. The permutation $\pi = 7635214$ has excedences at $1, 2$ and 4, and weak excedences at $1, 2, 3$ and 4. So $exc(\pi) = 3$ and $wexc(\pi) = 4$.

2.5.3 Proposition. The number of permutations with k weak excedences is the same as the number of permutations with $n - k$ excedences.

Proof. Let $\pi = \pi_1\pi_2\cdots\pi_n$ and $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ defined by $\sigma_i = n + 1 - \pi_{n+1-i}$. Then,

- if there is a weak excedence in π at position i, then $\pi_i \geq i$. This gives $n + 1 - i \geq n + 1 - \pi_i$ which shows that the entry σ_{n+1-i} is not an excedence in σ;
- If there is not a weak excedence in π at position i, then $\pi_i < i$. This gives $n+1-i < n+1-\pi_i$ which shows that the entry σ_{n+1-i} is an excedence in σ.

We conclude that if there are k weak excedences in π, then there are $n - k$ excedences in σ. $\square$

2.5.4 Proposition. The number of permutations of length n with $k - 1$ excedances is the same as the number of permutations of length n with k weak excedances.

Proof. Let π be a permutation of length n. Let $f(\pi)$ be the reverse complement of π defined as, the permutation of length n whose ith entry is $n + 1 - \pi_{n+1-i}$. Then there is a bijection from the set of excedances of π to the set of weak excedances of $f(\pi)$ besides n. If i is an excedance of π, then i is always greater than 1, so $n + 1 - i < n$. Also, n is always a weak excedance of any permutation, so it is a weak excedance in $f(\pi)$. The proof is complete. $\square$

2.5.5 Theorem. *The number of excedances and the number of descents of a permutations are equidistributed on S_n.*

$$\sum_{\pi\in S_n} x^{des(\pi)} = \sum_{\pi\in S_n} x^{exc(\pi)}. \tag{2.5.1}$$

Proof. We postpone the proof of this theorem until Section 4.3. $\square$

3. Mahonian Statistics

3.1 Inversions

3.1.1 Definition. Let $\pi = \pi_1\pi_2\cdots\pi_n$ be a permutation. If the indices i and j satisfy $i < j$ but $\pi_i > \pi_j$, then the pair (π_i, π_j) is called an inversion of the permutation π.

3.1.2 Example. Let $\pi = 12563748$ be a permutation. The pairs $(5,3), (5,4), (6,3), (6,4), (7,4)$ are all inversions of π.

Each inversion is a pair of elements (π_i, π_j) that is in the "wrong order". So the only permutation with no inversion is the sorted permutation where $\pi_1 < \pi_2 < \cdots < \pi_n$. The number of inversions of a permutation π is denoted by $i(\pi)$, it is a method of measuring the degree to which a permutation is violating the order. Let $I_n(k)$ denote the number of permutations of length n with k inversions.

3.1.3 Proposition. Let π be a permutation of length n. Then

$$0 \leq i(\pi) \leq \binom{n}{2} \tag{3.1.1}$$

Proof. Consider the two extreme permutations, $\pi = \pi_1\pi_2\cdots\pi_n$ and its reverse $\pi' = \pi_n\pi_{n-1}\cdots\pi_2\pi_1$. In the first permutation every $\pi_i < \pi_{i+1}$, this clearly shows that no inversions exist. In the second permutation, the last entry has n inversions, the second to last has $n-1$ inversions and so on. The total number of inversions of π' is $1 + 2 + \cdots + n$ which can be written as $\binom{n}{2}$. $\square$

3.1.4 Lemma. For a permutation π of length $n-1$, there is exactly one position to insert n that creates i new inversions.

Proof. Let $\pi = \pi_1\pi_2\cdots\pi_{n-1}$ be a permutation of length $n-1$ with inversion number k . If n is inserted into the last position of π, no inversions are created. If n is inserted in second to last position, one new inversion is created. If n is inserted in third to last position, two new inversions are created and so on. So, in general if n is inserted into π so that it precedes i entries, i new inversions are created. $\square$

3.1.5 Proposition. Any permutation $\pi = \pi_1\pi_2\cdots\pi_n$ and its inverse permutation π^{-1} have the same number of inversions.

$$i(\pi) = i(\pi^{-1})$$

Proof. The pair (i, j) create an inversion in π if and only if (π_j, π_i) is an inversion of π^{-1}. $\square$

3.1.6 Definition. A permutation π is called even if $i(\pi)$ is even, and is called odd if $i(\pi)$ is odd.

3.1.7 Proposition. Let $n \geq 2$. Then the number of even (equivalently odd) permutations of length n is $\dfrac{n!}{2}$.

Proof. Let π be any permutation of length n, and let π' be the permutation obtained from π by swapping its first two elements. Then the difference between the number of inversions of π and π' is either plus or minus one. Applying this argument to all permutations of length n, we can split into two subsets of equal size, even and odd permutations. $\square$

3.2 Generating Functions and Recursions of Inversions

3.2.1 Theorem. *For all positive integers n and k with $k < n$, the numbers $I_n(k)$ have the generating function*

$$I_n(x) = \sum_{\pi \in S_n} x^{i(\pi)} = (1+x)(1+x+x^2)\cdots(1+x+x^2+\cdots+x^{n-1}) \tag{3.2.1}$$

Proof. We prove the statement by induction on n.
For $n = 2$, there is one permutation π with no inversion and one permutation π' with one inversion, so the generating function of $I_2(x)$ is $1 + x$ as claimed.
Now assume that the statement is true for $n - 1$, and prove it for n.
Let $\pi = \pi_1\pi_2\cdots\pi_{n-1}$ be a permutation of length $n-1$. Inserting the entry n in permutation π in the i^{th} position where $i = 1, 2, \cdots, n$ creates additional $n - i$ inversions, having the generating function $1 + x + x^2 + \cdots + x^{n-1}$.
Then $I_n(x) = (1+x+x^2+\cdots+x^{n-1})I_{n-1}(x) = (1+x)(1+x+x^2)\cdots(1+x+x^2+\cdots+x^{n-1})$ □

Let $b(n, k)$ be the number of permutations of length n with k inversions. So, as we can see from 3.2.1, the numbers $b(n, k)$ are the coefficients of x^k in $I_n(x)$.

3.2.2 Proposition. For any fixed n, the sequence $b(n, 0), b(n, 1), \cdots, b(n, \binom{n}{2})$ is log-concave.

$I_n(x)$ has not real roots only. For example, if $n \geq 3$, then $I_n(x)$ is a multiple of $1 + x + x^2$ which has some complex roots.

				k								
		1	2	3	4	5	6	7	8	9	10	11
	1	1										
	2	1	1									
n	3	1	2	2	1							
	4	1	3	5	6	5	3	1				
	5	1	4	9	15	20	22	20	15	9	4	1

Figure 3.1: The values of $b(n, k)$ for $n \leq 5$

3.2.3 Lemma. Let $n \geq k$. Then the numbers $b(n, k)$ satisfy the recurrence relation

$$b(n+1, k) = b(n+1, k-1) + b(n, k). \tag{3.2.2}$$

Proof. Let $\pi = \pi_1\pi_2\cdots\pi_{n+1}$ be a permutation of length $n + 1$ with k inversions, where $k \leq n$.
To get a permutation of length n with k inversions, we can exclude the last entry of π if $\pi_{n+1} = n+1$.
To get a permutation of length $n+1$ from π with $k-1$ inversions, if an entry of permutation π is $n+1$, swap that entry and the entry immediately following it. This completes the proof. □

3.3 Major Index

3.3.1 Definition. The major index of a permutation $\pi = \pi_1\pi_2\cdots\pi_n$, sometimes called the greater index, is the sum of all descents of that permutation.

$$\mathrm{maj}(\pi) = \sum_{i \in Des(\pi)} i. \tag{3.3.1}$$

3.3.2 Example. The permutation $\pi = 12563748$ has descents 4 and 6, so $maj(\pi) = 10$.

3.3.3 Definition. Ascent spaces are defined as the spaces between a pair of entries of a permutation which are in ascending order and the space before the first entry.

3.3.4 Definition. Descent spaces are defined as the spaces between a pair of entries of a permutation which are in descending order and the space after the last entry.

3.3.5 Corollary. A permutation π with k descents has $k+1$ descent spaces and $n-k$ ascent spaces.

3.3.6 Lemma. For a permutation π of length $n-1$, there is exactly one position to insert n that increases the major index by i.

Proof. Label the descent spaces from right to left by $0, 1, 2, \cdots, k$. Then label the ascent spaces from left to right by $k+1, k+2, \cdots, n-1$.

If the entry n is inserted into a descent space, the number of descents will remain unchanged since the descent formed by n and the entry following it is replacing the descent that already existed at that position. This increases by one the label index of every descent at or to the right of the insertion space. Thus the increment in the major index is the same as the label index of the insertion space.

If the entry n is inserted into an ascent space, it creates a new descent and increases the label index of each descent to its right. The increment of the major index is the number of entries to the left of, and including, the insertion space, and the number of descents to the right of the insertion space, which is the same as the label of the ascent space. $\square$

The six permutations made by $[3]$ have the same total number of inversions and major index. So, there is a bijection between the number of inversions and major index of a permutation as the table below shows.

Permutation	Inversion	Major index
123	0	0
132	1	2
213	1	1
231	2	2
312	2	1
321	3	3

3.3.7 Theorem. *The number of inversions and the major index are equidistributed on S_n.*

$$\sum_{\pi \in S_n} x^{maj(\pi)} = \sum_{\pi \in S_n} x^{i(\pi)}. \tag{3.3.2}$$

Proof. The goal of this proof is to construct a bijection $\phi = S_n \to S_n$ such that $maj(\phi(\pi)) = i(\pi)$. Given a permutation $\pi = \pi_1\pi_2\cdots\pi_n$, label the spaces between the entries of π with the following approach:

- Label the position after π_n with a zero.
- Label the descents of π from right to left with $1, 2, \cdots, des(\pi)$.
- Label the position before π_1 with $des(\pi) + 1$.
- Finally, label the ascents of π from left to right with $des(\pi) + 2, \cdots, n$.

This completes the proof of 3.3.2 and gives a bijection that takes the inversion number of a permutation to the major index of another permutation with the same length.

3.3.8 Example. The permutation $\pi = 51342$ with major index 5 gets the labelling in the setting

$${}_{3}5_{2}1_{4}3_{5}4_{1}2_{0}$$

which gives the permutation $\pi' = 32451$ with inversion number 5.

□

3.4 Expected Value and Variance of Inversions

3.4.1 Proposition. Let $i(\pi)$ be the number of inversions of the permutation $\pi = \pi_1\pi_2\cdots\pi_n$. Let $X : S_n \to \mathbb{R}$ be the random variable defined by $X(\pi) = i(\pi)$. Then the average of X is

$$E(X) = \frac{n(n-1)}{4} \tag{3.4.1}$$

Proof. Define the indicator variable $X_{i,j} : S_n \to \mathbb{R}$ by

$$X_{i,j}(\pi) = \begin{cases} 1 & \text{if } i < j \text{ but } \pi_i > \pi_j \\ 0 & \text{if not} \end{cases}$$

It follows that

$$X = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{i,j}.$$

Taking the expectation yields

$$E[X] = E\left[\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{i,j}\right] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} E[X_{i,j}]$$

Since $E[X_{i,j}] = P(X_{i,j} = 1)$ and $P(X_{i,j} = 1) = P(\pi_i > \pi_j) = \dfrac{1}{2}$, thus

$$E(X) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{1}{2} = \frac{1}{2}\sum_{j=1}^{n} j = \frac{n(n-1)}{4}$$

as required. □

3.4.2 Proposition. Let $i(\pi)$ be the number of inversions of the permutation $\pi = \pi_1\pi_2\cdots\pi_n$. Let $X : S_n \to \mathbb{R}$ be the random variable defined by $X(\pi) = i(\pi)$. Then the variance of X is

$$\mathrm{Var}(X) = \frac{2n^3 + 3n^2 - 5n}{72}. \tag{3.4.2}$$

Proof. The variance of a random variable is computed as

$$\mathrm{Var}(X) = E(X^2) - E(X)^2.$$

In equation 3.4.1 we computed $E(X)$, so to find $E(X^2)$ let's introduce the indicator variable $X_{i,j}$ defined for $\pi = \pi_1\pi_2\cdots\pi_n$ and for all pairs of $i < j$ by

$$X_{i,j}(\pi) = \begin{cases} 1 & \text{if } i < j \text{ but } \pi_i > \pi_j \\ 0 & \text{if not} \end{cases}$$

Then $X = \sum_{i,j} X_{i,j}$ and $E(X_{i,j}) = E(X_{i,j}^2) = \frac{1}{2}$ Also, we consider the expectation $E(X_{i,j}X_{k,l})$ of the pairs $(i,j)(k,l)$ where $i < j$ and $k < l$ in the following cases:

- when the pairs $(i,j)(k,l)$ are disjoint, then $E(X_{i,j}X_{k,l}) = \frac{1}{4}$ which happens in $\binom{n}{2}\binom{n-2}{2}$ cases.
- when $i = k$ but $j \neq l$, then $E(X_{i,j}X_{k,l}) = \frac{1}{3}$ (the two permutations $\pi_l\pi_j\pi_i$ and $\pi_j\pi_l\pi_i$ out of the six possible permutations) which occurs in $2\binom{n}{3}$ cases.
- when $i \neq k$ but $j = l$, then $E(X_{i,j}X_{k,l}) = \frac{1}{3}$ (the two permutations $\pi_j\pi_k\pi_i$ and $\pi_j\pi_i\pi_k$ out of the six possible permutations) which occurs in $2\binom{n}{3}$ cases.
- when $i = k$ or $j = l$, then $E(X_{i,j}X_{k,l}) = \frac{1}{6}$ (the permutation $\pi_j\pi_i\pi_k$ or $\pi_l\pi_j\pi_i$ out of the six possible permutations), and each of these cases occurs in $\binom{n}{3}$ cases.

Therefore

$$\begin{aligned} E(X^2) &= \sum_{i<j} E(X_{i,j}^2) + \sum_{(i,j)\neq(k,l)} E(X_{i,j}X_{k,l}) \\ &= \frac{1}{2}\binom{n}{2} + \frac{1}{4}\binom{n}{2}\binom{n-2}{2} + \frac{2}{3}\binom{n}{3} + \frac{2}{3}\binom{n}{3} + \frac{1}{6}\binom{n}{3} + \frac{1}{6}\binom{n}{3} \\ &= \frac{1}{2}\binom{n}{2} + \frac{1}{4}\binom{n}{2}\binom{n-2}{2} + \frac{5}{3}\binom{n}{3} = \frac{9n^4 - 14n^3 + 15n^2 - 10n}{144} \end{aligned}$$

and consequently

$$\begin{aligned} \mathrm{Var}(X) &= E(X^2) - E(X)^2 \\ &= \frac{9n^4 - 14n^3 + 15n^2 - 10n}{144} - \frac{n^2(n-1)^2}{16} \\ &= \frac{2n^3 + 3n^2 - 5n}{72} \end{aligned}$$

as required. $\square$

3.5 Inversions of Permutations of Multisets

3.5.1 Definition. A multiset is a generalisation of a set that allows elements to appear a number of times, rather than being simply present or absent. For example, $M = \{1, 1, 2, 5, 4, 4, 2\}$ is a multiset.

3.5.2 Definition. An inversion of a permutation $\pi = \pi_1\pi_2\cdots\pi_n$ of a multiset M is a pair of indices (i, j) such that $i < j$ but $\pi_i > \pi_j$.

3.5.3 Example. The multiset permutation $M = \{1, 4, 2, 2, 2, 3, 1\}$ has inversions $(2, 3), (2, 4), (2, 5), (2, 6),$ $(2, 7), (3, 7), (4, 7), (5, 7)$ and $(6, 7)$.

3.5.4 Definition. Let $a_1, a_2, \cdots, a_k$ be positive integers so that $\sum_{i=1}^{k} a_i = n$. Then the Gaussian coefficient, or q-multinomial coefficient is denoted by $\begin{bmatrix} n \\ a_1, a_2, \cdots, a_k \end{bmatrix}$ and is given by

$$\begin{bmatrix} n \\ a_1, a_2, \cdots, a_k \end{bmatrix} = \frac{[n]!}{[a_1]![a_2]!\cdots[a_k]!} \tag{3.5.1}$$

The q-multinomial coefficients satisfy the identity

$$\begin{bmatrix} n \\ a_1, a_2, \cdots, a_k \end{bmatrix} = \begin{bmatrix} n \\ a_1 \end{bmatrix} \begin{bmatrix} n - a_1 \\ a_2 \end{bmatrix} \begin{bmatrix} n - a_1 - a_2 \\ a_3 \end{bmatrix} \cdots \begin{bmatrix} a_k \\ a_k \end{bmatrix} \tag{3.5.2}$$

and also satisfy the recurrence relation for $a \leq n$

$$\begin{bmatrix} n \\ a \end{bmatrix} = q^{n-a} \begin{bmatrix} n-1 \\ a-1 \end{bmatrix} + \begin{bmatrix} n-1 \\ a \end{bmatrix} \tag{3.5.3}$$

with initial conditions $\begin{bmatrix} n \\ 0 \end{bmatrix} = \begin{bmatrix} n \\ n \end{bmatrix} = 1.$

3.5.5 Theorem. *Let $K = \{1^{a_1}, 2^{a_2}, \cdots, k^{a_k}\}$ be a multiset so that $\sum_{i=1}^{k} a_i = n$, and let S_K denote the set of all permutations of K. Then*

$$\sum_{\pi \in S_K} q^{i(\pi)} = \begin{bmatrix} n \\ a_1, a_2, \cdots, a_k \end{bmatrix} \tag{3.5.4}$$

Proof. Consider the special case where $K = 2$. In this case, the multiset K contains a_1 copies of 1 and a_2 copies of 2, so that $a_1 + a_2 = n$. An inversion occurs if 2 is left of 1.
In this special case, we can show that

$$\sum_{\pi \in S_K} q^{i(\pi)} = \begin{bmatrix} n \\ a_1 \end{bmatrix} \tag{3.5.5}$$

This statement can be proved by induction. For $n = 1$, the statement becomes

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1$$

which is trivially true. Assume that the statement is true for $n-1$, and prove it for n.
If the multiset permutation of k ends in a 2, then its last entry does not form any inversion, and if it ends in a 1, then its last entry forms $a_2 = n-1$ inversions. This can be written as

$$\sum_{\pi \in S_K} q^{i(\pi)} = q^{n-a_1} \begin{bmatrix} n-1 \\ a_1 - 1 \end{bmatrix} + \begin{bmatrix} n-1 \\ a_1 \end{bmatrix}$$

which is equivalent to

$$\sum_{\pi \in S_K} q^{i(\pi)} = \begin{bmatrix} n \\ a_1 \end{bmatrix}$$

This completes the induction for $k = 2$. Now, we are proving the theorem in its general form by induction on K. For the case $K = 1$ it is trivially true and the case $K = 2$ is also true as we proved.
Assume that the statement of the theorem is true for $K = \{1^{a_1}, 2^{a_2}, \cdots, k^{a_k}\}$ and prove it for $K^+ = \{1^{a_1}, 2^{a_2}, \cdots, k^{a_k}, (k+1)^{a_{k+1}}\}$.
A permutation π of any K^+ can be decomposed into two independent multiset permutations, π' and π''. The multiset permutation π' is the multiset permutation obtained from π by replacing all entries less than $(k+1)$ by 1, and the permutation π'' is the multiset permutation obtained from π by removing all copies of $(k+1)$. Then we can say that

$$i(\pi) = i(\pi') + i(\pi'').$$

The permutation π' consists of $n - a_{k+1}$ copies of 1 and a_{k+1} copies of $k+1$. Then $\sum_{\pi'} q^{i(\pi')}$ is equivalent to the previous special case. Therefore

$$\sum_{\pi} q^{i(\pi')} = \begin{bmatrix} n \\ a_{k+1} \end{bmatrix}$$

To find $\sum_{\pi''} q^{i(\pi'')}$, we can use the induction hypothesis and get

$$\sum_{\pi''} q^{i(\pi'')} = \begin{bmatrix} n - a_{k+1} \\ a_1 \end{bmatrix} \cdot \begin{bmatrix} n - a_k - a_1 \\ a_2 \end{bmatrix} \cdots \begin{bmatrix} a_k \\ a_k \end{bmatrix}$$

Finally,

$$\begin{aligned}
\sum_{\pi \in S_k} q^{i(\pi)} &= \sum_{\pi \in S_k} q^{i(\pi') + i(\pi'')} \\
\sum_{\pi \in S_k} q^{i(\pi)} &= \sum_{\pi'} q^{i(\pi')} \cdot \sum_{\pi''} q^{i(\pi'')} \\
&= \begin{bmatrix} n \\ a_{k+1} \end{bmatrix} \cdot \begin{bmatrix} n - a_{k+1} \\ a_1 \end{bmatrix} \cdot \begin{bmatrix} n - a_k - a_1 \\ a_2 \end{bmatrix} \cdots \begin{bmatrix} a_k \\ a_k \end{bmatrix} \\
&= \begin{bmatrix} n \\ a_1, a_2, \cdots, a_{k+1} \end{bmatrix}
\end{aligned}$$

This completes the induction proof of the theorem. □

3.5.6 Theorem. *Let $K = \{1^{a_1}, 2^{a_2}, \cdots, k^{a_k}\}$ be a multiset so that $\sum_{i=1}^{k} a_i = n$. Then the statistics inversion and major index are equidistributed on the set S_K of all permutations of K.*

$$\sum_{\pi \in S_K} q^{i(\pi)} = \sum_{\pi \in S_K} q^{maj(\pi)} = \begin{bmatrix} n \\ a_1, a_2, \cdots, a_k \end{bmatrix} \tag{3.5.6}$$

Proof. As we did in Theorem 3.3.2, also the goal of this proof is to build a bijection $\phi = S_K \to S_K$ such that $maj(\phi(\pi)) = i(\pi)$.
Given a multiset permutation $\pi = \pi_1\pi_2\cdots\pi_n$, label the spaces between the entries of π with the following approach:

- Label the position after π_n with a zero.
- Label the descents of π from right to left with $1, 2, \cdots, des(\pi)$.
- Label the position before π_1 with $des(\pi) + 1$.
- Label the non-descents of π from left to right with $des(\pi) + 2, \cdots, K$.

This completes the proof of Theorem 3.5.6 and gives a bijection that takes the inversion number of a multiset permutation to the major index of another multiset permutation with the same length. □

4. Cycles

4.1 Cycle type permutations

A permutation $\pi_1\pi_2\ldots\pi_n$ can also be regarded as a bijection $\pi : [n] \to [n]$, where $\pi(j) = \pi_j$.

4.1.1 Example. The permutation 4123 can be viewed as a bijective function $\pi = [4] \to [4]$ defined by $\pi(1) = 4$, $\pi(2) = 1$, $\pi(3) = 2$ and $\pi(4) = 3$.

In the rest of this chapter, we are introducing a new way of writing permutations, as functions. The permutation $\pi = 461532$ can be written as

$$f = \begin{matrix} 1\ 2\ 3\ 4\ 5\ 6 \\ 4\ 6\ 1\ 5\ 3\ 2 \end{matrix}$$

expressing that f maps 1 to 4, 2 to 6, 3 to 1, 4 to 5, 5 to 3, and 6 to 2.
This notation is called the two-line notation. The advantage of writing permutations in this approach rather than the one-line notation which involves writing $f(1)f(2)\cdots f(n)$ in one line, is that we can define the product of two permutations on $[n]$ by simply taking their composition as a composition of functions.

4.1.2 Example. Consider the two permutations $f = 4123$ and $g = 4231$. Then $(f \cdot g)(1) = g(f(1)) = g(4) = 1$, $(f \cdot g)(2) = g(f(2)) = g(1) = 4$, $(f \cdot g)(3) = g(f(3)) = g(2) = 2$, and $(f \cdot g)(4) = g(f(4)) = g(3) = 3$. Then $gf = 1423$.

A closer look at our running example, the permutation $f = 461532$ reveals that f permutes the elements $1, 4, 5$ and 3 among themselves, and the elements 2 to 6 among themselves.
In other words, f cyclically permutes $1, 4, 5$ and 3, and f cyclically permutes 2 and 6 . We can write $f = (1453)(26)$. This notation is known as cycle notation.

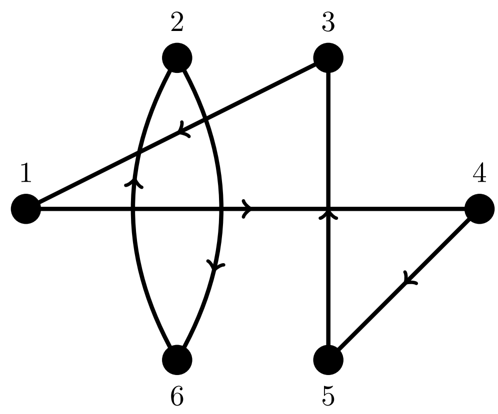

Figure 4.1: The cycles of the permutation 461532.

There is a unique way of writing permutations using the cycle notation. The permutation π is written as a list of cycles in which each starts with its largest entry, and the cycle are written in increasing order of their first entries. This is known as canonical cycle notation.

4.1.3 Example. The permutation $\pi = (6235)(81)(947)$ is in canonical cycle notation.

4.1.4 Definition. Let π be a permutation of length n with a_i cycles of length i, for all positive integers $i \in [n]$. Then the array $(a_1, a_2, \cdots, a_n)$ is called the type of π.

4.1.5 Example. Let $\pi = (3)(412)(79)(856)$. Then the type of π is $(1, 1, 2, 0, 0, 0, 0, 0, 0)$.

4.1.6 Theorem. *Let $a = (a_1, a_2, \cdots, a_n)$ be an n-tuple of nonnegative integers so that the equality $\sum_{i=1}^{n} ia_i = n$ holds. Then the number of permutations of length n with a_i cycles of length i is*

$$\frac{n!}{1^{a_1}a_1!2^{a_2}a_2!\cdots n^{a_n}a_n!}. \tag{4.1.1}$$

Proof. Start by writing down all the elements of $[n]$ in a linear order. Then place parentheses going from left to right between the entries to form the required cycle lengths. Insert the first a_1 pairs of parenthesis around the first a_1 elements creating a_1 1-cycles, insert following a_2 pairs of parentheses around pairs of entries creating a_2 2-cycles, and so on. The permutations obtained will all be of type $a = (a_1, a_2, \cdots, a_n)$ and there are $n!$ ways to do this.
Each cycle permutation is obtained in $a_1!a_2!\cdots a_n!$ ways and the entries of each cycle can be written down in $1^{a_1} \cdot 2^{a_2} \cdots n^{a_n}$ different ways. This shows that each permutation of the desired type $(a_1, a_2, \cdots, a_n)$ is obtained in $a_1!a_2!\cdots a_n!1^{a_1} \cdot 2^{a_2} \cdots n^{a_n}$ different ways, and completes the proof. □

4.1.7 Example. The number of permutations of length 8 of type $(1, 2, 1, 0, 0, 0, 0, 0)$ is

$$\frac{8!}{1^1 \cdot 1! \cdot 2^2 \cdot 2! \cdot 3^1 \cdot 1!} = 1680$$

4.1.8 Example. The number of permutations of length n of type $(0, 0, \cdots, 1)$, in other words, the number of permutations of length n having only one cycle is equal to $(n-1)!$.

4.2 Generating Functions of Stirling Numbers

4.2.1 Definition. Let $c(n, k)$ denote the number of permutations of length n with k cycles. Then $c(n, k)$ is called the signless Stirling number of the first kind.

4.2.2 Proposition. The numbers $c(n, k)$ satisfy the recurrence relation

$$c(n, k) = c(n-1, k-1) + (n-1)c(n-1, k). \tag{4.2.1}$$

with initial conditions $c(n, 0) = 0$ if $n \geq 0$ and $c(0, 0) = 1$

Proof. The right hand side counts the numbers of permutations of length n with k cycles. In such permutations there are two possibilities for the position of the entry n. Either the entry n forms its own 1-cycle and the remaining $n-1$ entries form $k-1$ cycles. This occurs in $c(n-1, k-1)$ ways. Or the entry n does not form a cycle itself and the remaining $n-1$ entries form k cycles. Then the entry n is in one of these cycles and can be inserted in any of the $n-1$ positions immediately following some entry. This gives $(n-1)c(n-1, k)$ possibilities. □

4.2.3 Theorem. *For all positive integers n,*

$$\sum_{k=0}^{n} c(n, k)x^k = x(x+1)\cdots(x+n-1). \tag{4.2.2}$$

Proof. We are proving that the coefficients of x^k on the right hand side of equation 4.2.2 satisfy the recursive formula 4.2.1.
Let

$$F_n(x) = \sum_{k=0}^{n} b(n,k)x^k = x(x+1)\cdots(x+n-1). \tag{4.2.3}$$

The initial values are $b(0,0) = 1$ and $b(n,0) = b(0,k) = 0$.

Then we can write 4.2.3 in this form:

$$\begin{aligned} F_n(x) &= (x+n-1)\sum_{k=0}^{n-1} b(n-1,k)x^k \\ &= \sum_{k=0}^{n-1} b(n-1,k)x^{k+1} + (n-1)\sum_{k=0}^{n-1} b(n-1,k)x^k \\ &= \sum_{k=1}^{n} b(n-1,k-1)x^k + (n-1)\sum_{k=0}^{n-1} b(n-1,k)x^k. \end{aligned}$$

Therefore $b(n,k)$ satisfies the recurrence relation 4.2.1,

$$b(n,k) = b(n-1,k-1) + (n-1)b(n-1,k)$$

Hence, $b(n,k)$ satisfies the same recurrence and initial conditions as $c(n,k)$, so they agree. $\square$

The number $c(n,k)$ is the coefficient of x^k in $x(x+1)\cdots(x+n-1)$. Figure 4.2 shows the values of $c(n,k)$ for $0 \le n \le 5$.

		k						
		1	2	3	4	5	6	7
	1	1						
	2	0	1					
n	3	0	1	1				
	4	0	2	3	1			
	5	0	6	11	6	1		
	6	0	24	50	35	10	1	

Figure 4.2: The values of $c(n,k)$ for $n \le 6$

4.2.4 Proposition. Bóna (2016) For any fixed positive integer n, the sequence $\{c(n,k)\}_{0\le k\le n}$ has real zeros only. In particular, this sequence is log-concave, and therefore, unimodal.

4.2.5 Definition. Let $s(n,k)$ be the Stirling numbers of the first kind defined by

$$s(n,k) = (-1)^{n-k}c(n,k). \tag{4.2.4}$$

The number $s(n,k)$ is the coefficient of x^k in $x(x-1)\cdots(x-n+1)$. The values of $s(n,k)$ for $n \ge 5$ are shown in Figure 4.3.

		k						
		1	2	3	4	5	6	7
	1	1						
	2	0	1					
n	3	0	-1	1				
	4	0	2	-3	1			
	5	0	-6	11	-6	1		
	6	0	24	-50	35	-10	1	

Figure 4.3: The values of $s(n, k)$ for $n \leq 5$

4.2.6 Proposition. For all positive integers n,

$$\sum_{k=0}^{n} s(n,k)x^k = x(x-1)\cdots(x-n+1). \tag{4.2.5}$$

Proof. Substitute $x \to -x$ in equation 4.2.2.

$$\begin{aligned}\sum_{k=0}^{n} c(n,k)(-1)^k x^k &= -x(-x+1)\cdots(-x+n-1)\\ &= (-1)^n x(x-1)\cdots(x-n+1)\end{aligned}$$

Multiply both sides by $(-1)^n$.

$$\sum_{k=0}^{n} c(n,k)(-1)^{n-k} x^k = x(x-1)\cdots(x-n+1)$$

Set $s(n,k) = c(n,k)(-1)^{n-k}$. Thus,

$$\sum_{k=0}^{n} s(n,k)x^k = x(x-1)\cdots(x-n+1)$$

as required. $\square$

4.2.7 Definition. Define the double generating function of Stirling numbers of the first kind as

$$f(x,u) = \sum_{n\leq 0}\sum_{k=0}^{n} s(n,k)x^k \frac{u^n}{n!} \tag{4.2.6}$$

4.2.8 Proposition. Bóna (2016) The generating function 4.2.6 can be written as

$$f(x,u) = (1+u)^x \tag{4.2.7}$$

4.3 Permutations with restricted cycle structure

4.3.1 Lemma (Transition Lemma)**.** Let π be a permutation of length n written in canonical cycle notation. Let $f(\pi)$ be the permutation obtained from π by omitting all the parentheses and reading the entries in one line notation. Then the map $f : S_n \to S_n$ is a bijection.

Proof. We have to show that for every permutation π there is exactly one permutation $\pi^\star \in S_n$ such that $f(\pi^\star) = \pi$.
Let $\pi = \pi_1\pi_2\cdots\pi_n$ be any permutation of length n written in one line notation. It is always true that π_1 starts the first cycle, so the first left parentheses is inserted to the left of π_1. To end the first cycle, the right parentheses must be inserted to the left of the smallest index i so that $\pi_i > \pi_1$. Thus the second cycle of π must start with π_i, so the second left parentheses is inserted to the left of π_i. Again to end the second cycle, the second right parentheses must be inserted to the left of the smallest index j so that $\pi_j > \pi_i$. Repeating this procedure gives the third, fourth, and so on. The procedure will stop when a cycle starts with the entry n.
In general, we have to start a new cycle at π_k if and only if π_k is greater than the leading entries of all previous cycles, which means that π_k is larger than all entries on its left. This guarantees the existence of a unique preimage of π. $\square$

4.3.2 Example. Let the permutation $\pi = (512)(6)(734)(8)$. Then $f(\pi) = f((512)(6)(734)(8))$ is $\pi^\star = 51267348$.

4.3.3 Definition. Let $\pi = \pi_1\pi_2\cdots\pi_n$ be a permutation. The element π_i is called a left-to-right maximum if, for all $k < i$, we have $\pi_k < \pi_i$.

4.3.4 Example. The permutation $\pi = 52176384$ has three left-to-right maxima namely $5, 7$ and 8.

4.3.5 Remark. If π has k left-to-right maxima, then $f^{-1}(\pi)$ has k cycles.

4.3.6 Proposition. The number of permutations of length n with exactly k left-to-right maxima is $c(n, k)$.

Proof. The bijection f of the transition lemma maps the set of permutations of length n with exactly k cycles onto the set of permutations of length n with exactly k left-to-right maxima. $\square$

Proof. (of Theorem 2.5.5) We show that the bijection f of the transition lemma maps the permutations of length n with k weak excedances to the permutations of length n with $k-1$ ascents.
Let π be a permutation written in canonical cycle notation such that $\pi = (p_1 \cdots p_{i_1})(p_{i_1+1} \cdots p_{i_2}) \cdots (p_{i_j-1+1} \cdots p_{ij})$. We apply f to π and count the ascents $f(\pi)$. We see that index i is an ascent of $f(\pi)$ if and only if $i \neq n$ and $p_i \leq \pi(p_i)$.
Unless i is not at the end of a cycle, we have $\pi(p_i) = p_{i+1}$, since each element of a cycle is mapped to the one on its right, except for the last entry of a cycle. Therefore $p_i < p_{i+1}$ implies $p_i < \pi(p_i)$ and proves the "only if part".
When i is at the end of a cycle, we have $\pi(p_i) = p_{i+1}$ and therefore $p_i < p_{i+1}$. When i is at the end of a cycle, we have $\pi(p_i) = p_i$ if p_i is contained in a $1-$cycle, and $p_i < p_{i+1}$. $p_i \leq \pi(p_i) < p_{i+1}$ otherwise. Notice that p_i is less than the first entry of its cycle, which means also less than the first element of the next cycle by the canonical cycle property.
We have shown that the bijection f of the transition lemma takes the weak excedances of π other than n and maps them to the ascents of $f(\pi)$. The proposition 2.5.4 completes the proof which shows that the

number of permutations of length n with $k-1$ excedances is the same as the number of permutations of length n with k weak excedances. $\square$

4.3.7 Example. Let $\pi = (61)(5423)$. Then writing π in one-line notation, we have $\pi = 635241$. This one-line notation has three weak excedances, namely $1, 2$ and 3. On the other hand, $f(\pi) = 615423$ has two ascents, namely 2 and 5. This is where π has its weak excedances besides n.

4.4 Expected Value of Cycles

4.4.1 Proposition. Let $c(\pi)$ be the number of cycles of the permutation $\pi = \pi_1\pi_2\cdots\pi_n$ in canonical cycle form. Let $X(n) : S_n \to \mathbb{R}$ be the random variable defined by $X(n) = c(\pi)$. Then the average of $X(n)$ is

$$X(n) = \sum_{i=1}^{n} \frac{1}{i}. \tag{4.4.1}$$

Proof. We prove the statement by induction on n. When $n = 1$, we have a permutation of only one entry, there exists exactly one cycle and the statement is true.
Assume the statement is true for n. Now we show that it is also true for $n+1$. Take a permutation π of length n. We can insert the entry $n+1$ into any of the $n+1$ gaps of π. A new cycle is created if the entry $n+1$ is inserted into the last gap position of π, that is, in $1/(n+1)$ of all cases, and no new cycles are created if the entry $n+1$ is inserted into one of the other n gaps, that is, in $n/(n+1)$ cases. Thus the average number of cycles in a randomly selected permutation of length $n+1$ is

$$\begin{aligned}
X(n)\frac{n}{n+1} + (X(n)+1)\frac{1}{n+1} &= X(n)\frac{n}{n+1} + X(n)\frac{1}{n+1} + \frac{1}{n+1} \\
&= X(n) + \frac{1}{n+1} \\
&= \sum_{i=1}^{n} \frac{1}{i} + \frac{1}{n+1} \\
&= \sum_{i=1}^{n+1} \frac{1}{i}
\end{aligned}$$

as required. $\square$

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