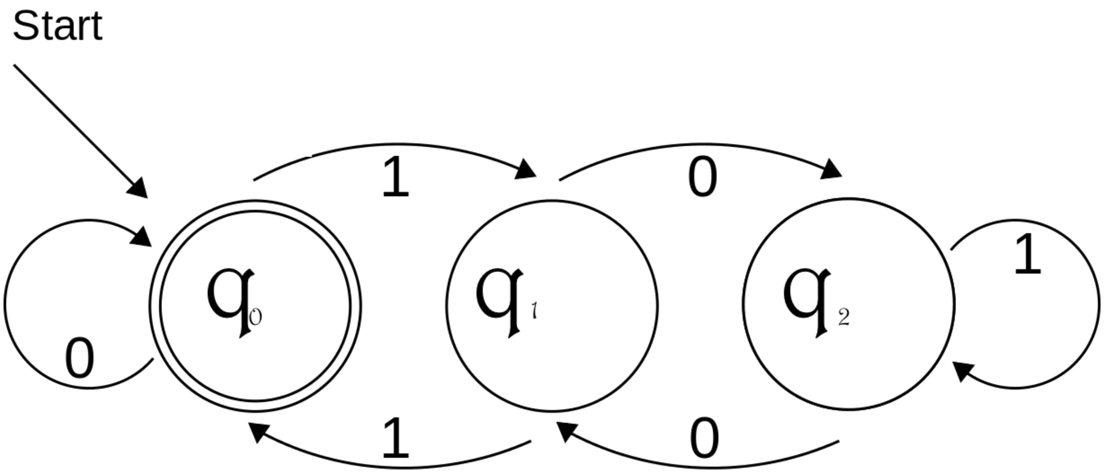
240 A10   NO OUTSIDE DISCUSSION; Consulted Wikipedia

The deterministic finite automaton A = $(Q, \Sigma, \delta, s, F)$,



is the graph below:

Lemma2: $\forall$ a, b $\in \mathbb{N}$, (3|a AND 3|b)IMPLIES3|(a + b)

(*directly from the property of exact division*)

Lemma3:

x is a binary representation of natural number, so assume x has m digits, and x =

a[1]a[2] ... a[m]. Assume x is formed of i 1s and j 0s (i + j = m).

Let x' = a[1]a[2] ... a[m]a[m + 1], where a[m+1] is a new number

that can be either 0 or 1.

If a[m+1] = 0, then x' = 2 × x .(from the property of binary system)

and x' $\equiv$ 2x(mod 3) (from the property of modulo)

Else if a[m+1] = 1, then x' = 2 × x + 1. (from the property of binary

system). and x' $\equiv$ 2x + 1(mod 3) (from the property of modulo)

Lemma4:

For deterministic finite automaton A, q_0 accepts binary representation of either empty string or a natural number $r \equiv 0 \pmod 3$. q_1 accepts binary representation of a natural number $r \equiv 1 \pmod 3$. q_2 accepts binary representation of a natural number $r \equiv 2 \pmod 3$.

Prove by induction:

Firsty, if input is an empty string, it end with q_0 , so q_0 accepts binary representation of either empty string or a natural number $r \equiv 0 \pmod 3$ is true when input is an empty string.

Let P_0 =" For deterministic finite automaton A, q_0 accepts binary representation of a natural number $r \equiv 0 \pmod 3$." P_1 =" For deterministic finite automaton A, q_1 accepts binary representation of a natural number $r \equiv 1 \pmod 3$." P_2 =" For deterministic finite automaton A, q_2 accepts binary representation of a natural number $r \equiv 2 \pmod 3$."

Base cases:

1.If x =0, the binary representation x =a[0]=0.

2. $\delta(q_0, 0) = q_0$

3. P_0 is true.

4. If x =1, the binary representation x =a[0]=1.

5. $\delta(q_0, 1) = q_1$

6. P_1 is true.

7. If x =2, the binary representation x = a[0] a[1]=10.

8. $\delta^*(q_0, x) = \delta(\delta(q_0, 1),0) = \delta(q_1,0) = q_2$

9. P_2 is true.

10. If x =3, the binary representation x = a[0] a[1]=11.

11. $\delta^*(q_0, x) = \delta(\delta(q_0, 1),1) = \delta(q_1,1) = q_0$

12. P_0 is true.

Constructor cases:

13. Assume for an arbitrary natural number x, The binary representation of x is: $x = a[1]a[2] \ldots a[m]$. Let $y = a[1]a[2] \ldots a[m-1]$. and P_0, P_1 and P_2 is true for y, which means:

$$\delta^*(q_0, y) = \begin{cases} q_0, if\ y \equiv 0 \pmod 3 \\ q_1, if\ y \equiv 1 \pmod 3 \\ q_2, if\ y \equiv 2 \pmod 3 \end{cases}$$

14. $\delta^*(q_0, x) = \delta(\delta^*(q_0, y), a[m])$

15. if a[m]=0, $x = 2\times y$ and $x \equiv 2y \pmod 3$ (from lemma 3)

16. if $\delta^*(q_0, y) = q_0$, then $y \equiv 0 \pmod 3$(from 13)

17. $x \equiv 2y \equiv 0 \pmod 3$ (from 15, 16)

18. $\delta(q_0,0) = q_0$

19. P_0 is true for this case. (from 17, 18)

20. if $\delta^*(q_0, y) = q_1$, then $y \equiv 1 \pmod 3$(from 13)

21. $x \equiv 2y \equiv 2 \pmod 3$ (from 15, 20)

22. $\delta(q_1,0) = q_2$

23. P_2 is true for this case. (from 21, 22)

24. if $\delta^*(q_0, y) = q_2$, then $y \equiv 2 \pmod 3$(from 13)

25. $x \equiv 2y \equiv 4 \equiv 1 \pmod 3$ (from 15, 24)

26. $\delta(q_2, 0) = q_1$

27. P_1 is true for this case. (from 25, 26)

28. else if a[m]=1, $x = 2\times y + 1$and $x \equiv 2y + 1 \pmod 3$ (from lemma 3)

29. if $\delta^*(q_0, y) = q_0$, then $y \equiv 0 \pmod 3$(from 13)

30. $x \equiv 2y + 1 \equiv 1 \pmod 3$

31. $\delta(q_0, 1) = q_1$

32. P_1 is true for this case.

33. if $\delta^*(q_0, y) = q_1$, then $y \equiv 1 \pmod 3$

34. $x \equiv 2y + 1 \equiv 3 \equiv 0 \pmod 3$

35. $\delta(q_1, 1) = q_0$

36. P_0 is true for this case.

37. if $\delta^*(q_0, y) = q_2$, then $y \equiv 2 \pmod 3$

38. $x \equiv 2y + 1 \equiv 5 \equiv 2 \pmod 3$

39. $\delta(q_2, 1) = q_2$

40. P_2 is true for this case.

41. So, P_0, P_1 and P_2 is true for y IMPLIES P_0, P_1 and P_2 is true for x (from 13 and (19, 23, 27, 32, 36, 40))

42. By induction, P_0, P_1 and P_2 is true for all natural numbers.

Question 1:

A is in the picture above, and since P_0 , $P_1$and $P_2$is correct from lemma4, A is correct.

Question 2:

Proof:

Consider a deterministic finite automaton B = (Q, Σ, δ, s, F) such that $\mathcal{L}(B) = L$ and |Q|<3. Then, there exist at least two possibilities that lead to the same state. Assume the two states are q_0 and q_1, q_0 is the initial state and one of q_0 ,q_1 is final state (if they are all finial states then $\mathcal{L}(B) \neq L$ because for example, 101 is not in L.).

Case 1: $\delta(q_0, 0) = q_1$, then q_1 is final state because 0 is in L. Then, $\delta(q_0, 1) = q_0$ because 1 is not in L. So, $\delta^*(q_0, 10) = q_1$, but 10 is not in L, so this form of DFA B is impossible to make $\mathcal{L}(B) = L$.

Case 2: $\delta(q_0, 0) = q_0$, then q_0 is final state because 0 is in L. Then, $\delta(q_0, 1) = q_1$ because 1 is not in L. $\delta^*(q_0, 10) = q_1$ because 10 is not in L, so we have $\delta(q_1, 0) = q_1$. What's more, $\delta^*(q_0, 11) = q_0$ because 11 is in L, so we have $\delta(q_1, 1) = q_0$. Now, we have $\delta^*(q_0, 101) = q_0$ from above analysis, but 101 is not in L, so this form of DFA B is impossible to make $\mathcal{L}(B) = L$.

Both case 1 and case 2 are impossible, which are all the possibilities exist for such B, so such B does not exist.

Question 3:

Answer: R = 0+(1((10*1) +(01*0))* 10*))

Proof, firstly, a number in binary from is either 0 or begin with 1.

When the number is 0, then it is divisible by three. So, R here is either 0 or (1((10*1) +(01*0))* 10*)). $\mathcal{L}(\mathrm{R}) =$ For the deterministic finite automaton A, $\delta(q_0,1)= q_1$. So, $\mathcal{L}(\mathrm{R}) = L$ if and only if

$\delta^*(q_1, (10^*1) + (01^*0))^* 10^*) = q_0$ and is all possible way from q_1 to q_0. (from lemma4 and the from of R). Moreover, $\delta^*(q_1, (10^*1))$ proceeds like $q_1 \rightarrow q_0 \rightarrow q_1$, $\delta^*(q_1, (01^*0))$ proceeds like $q_1 \rightarrow q_2 \rightarrow q_1$, $\delta^*(q_1, (10^*))$ proceeds like $q_1 \rightarrow q_0$, and they are all possible way from q_1 to q_0. So, $\delta^*(q_1, (10^*1) + (01^*0))^* 10^*)$ is all possible way from q_1 to q_0. So, $\mathcal{L}(\mathrm{R}) = L$.