csc240 Assignment 7
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NO OUTSIDE DISCUSSION
NO EXTRA MATERIAL CONSULTED
* basic assumption:

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def pt(n):
    for k←1 to n do:
        print
```

the function pt(n) above print n times.

(a)

M(n) = 0 if n = 1

M(n) = n + n*M(n-1) if n > 1

explanation: det(B, n) performs no operation when n = 1, so M(n) = 0 if n = 1. When n > 1, det(B, n) goes into a while loop that loops n times. Every loop ends up with a conditional sentence that perform multiplication once no matter what, and no inner loop performs multiplications. Then, it perform det(C, n-1), goes into another determinant calculation. So, M(n) = (1+M(n-1))n = n + n*M(n-1)

A(n) = 0 if n = 1

A(n) = $2n^3 - 3n^2 + 3n + 1 + nA(n-1)$ if n > 1

explanation: det(B, n) performs no operation when n = 1, so A(n) = 0 if n = 1. When n > 1, d ← 0 is the first assignment, then det(B, n) goes into a while loop that loops n times. Every loop ends up with a conditional sentence that perform assignment once and and calculates det(C, n-1), goes into another determinant calculation. The inner loop for i ← 1 to n – 1 performs to separate inner loop, which assign j from 1 to n-1 and assigns an element in matrix C to an element in matrix B, which takes 2(n-1) steps. Moreover, every loop, no matter it is inner loop or outer loop, it assigns the number of loop more steps. So, A(n) = 1 + n(1+(n-1)(2(n-1)+1) + 1 + A(n-1)) = $2n^3 - 3n^2 + 3n + 1 + nA(n-1)$

(b)

lemma: $\forall n \in \mathbb{Z}^+,\ (n > 1) IMPLIES (n^2 - n - 1 > 0\ )$

proof :

$$(\mathrm{n} > 1)\mathrm{IMPLIES}\left(n - \frac{1}{2}\right)^2 \geq \left(2 - \frac{1}{2}\right)^2$$

$$\left(2 - \frac{1}{2}\right)^2 = \frac{9}{4} > \frac{5}{4}$$

$so, \ \forall n \in \mathbb{Z}^{+}, \ (n > 1) IMPLIES (n^2 - n - 1 > 0\ )$

base case:

1. when n = 1, n! -1 = 0, 2n! – n = 1, $\ \mathrm{n}! - 1 \ \leq \mathrm{M(n)} \leq 2n! - n$

   when n = 2, n! -1=1, 2n! – n = 2, M(n)=2, $\mathrm{n}! - 1 \ \leq \mathrm{M(n)} \leq 2n! - n$

constructor case:

2. let $n \ \geq 2 \in \mathbb{Z}^{+}$ be arbitrary

   3. Assume $\mathrm{n}! - 1 \ \leq \mathrm{M(n)} \leq 2\mathrm{n}! - \mathrm{n}$

   4. $\mathrm{n(n! - 1)} \leq \mathrm{nM(n)}$ (direct prove from 3)

   5. $\mathrm{n} * \mathrm{n}! \ \leq \mathrm{nM(n)} + \mathrm{n}$ (direct prove from 4)

   6. (n+1)M(n) $\leq 2(\mathrm{n}+1)\mathrm{n}! - (\mathrm{n}+1)\mathrm{n}$ (direct prove from 3)

   7. (n+1)M(n)+(n+1) $\leq 2(\mathrm{n}+1)\mathrm{n}! - (1-\mathrm{n})(1+\mathrm{n})$ (direct prove from 6)

   8. $(\mathrm{n}+1)\mathrm{M(n)} + (\mathrm{n}+1) \leq 2(\mathrm{n}+1)\mathrm{n}! - (\mathrm{n}+1)$ (direct prove from 2, 6 and lemma)

   9. For k = n+1, $\mathrm{M(k)} = \mathrm{k} + \mathrm{k\,M(n)}$

   10. $\mathrm{k}! - 1 = \mathrm{n}! - 1 + \mathrm{n} * \mathrm{n}! \leq \mathrm{M(n)} + \mathrm{n} * \mathrm{n}! \leq \mathrm{M(n)} + \ \mathrm{nM(n)} + \mathrm{n} =$

       $\mathrm{kM(n)} + \mathrm{n} < \ \mathrm{k} + \mathrm{k\,M(n)} = \mathrm{M(k)}$(direct prove from 5, 9)

   11. $\mathrm{M(k)} = \mathrm{k} + \mathrm{k\,M(n)} \leq 2\mathrm{k}! - \mathrm{k}$ when k (direct prove from 2, 8, 9)

   12. $\mathrm{k}! - 1 \leq \mathrm{M(k)} \leq 2\mathrm{k}! - \mathrm{k}$ when $\mathrm{k} > 2$(direct prove from 11, 12)

13. $\mathrm{n}! - 1 \ \leq \mathrm{M(n)} \leq 2n! - n \, for \ all \ n \ \in \mathbb{Z}^{+},$

$(weak\ induction\ from\ 1, 2, 12)$

(c)

lemma : $2n^3 - 3n^2 + 3n + 1 > n \ \ for\ all\ n \ \in \mathbb{Z}^{+}$

proof:

$$for\ all\ n \ \in \mathbb{Z}^{+}, 2n^3 - 3n^2 + 3n + 1 - n = 2n^3 - 3n^2 + 2n + 1$$
$$= \mathrm{n(2n - 1)(n - 1)} + 2\mathrm{n} + 1 > 1 > 0$$

base case:

1.when n = 1, $\mathrm{M(n)} = A(n) = 0, So\ \mathrm{M(n)} \le A(n)$

constructor case:

2. let $n \in \mathbb{Z}^+$ be arbitrary

3. Assume $\mathrm{M(n)} \le A(n)$

4. $\mathrm{kM(n)} \le kA(n)$ (direct prove from 3)

5. $\mathrm{kM(n)} + k \le kA(n)$+$2k^3 - 3k^2 + 3k + 1$(direct prove from 4 and lemma)

6. for k = n+1, M(k)= $\mathrm{kM(n)} + k \le kA(n)$+$2k^3 - 3k^2 + 3k + 1 = A(k)$
(direct prove from 5)

7. $\mathrm{M(n)} \le A(n)\ for\ all\ n \in \mathbb{Z}^+ (weak\ induction\ from\ 1, 2, 6)$

(d) let $\mathrm{u} = 100 \in \mathbb{Z}^+, h(n) = n^4$, $un! - h(n) = 100n!$- n^4

lemma: $for\ every\ n \in \mathbb{Z}^+ (n \ge 5)\ IMPLIES\ (n(\mathrm{n}-1)^4 - n^4 - (2n^3 - 3n^2 + 3n + 1) \ge 0)$

proof:

$$n(\mathrm{n}-1)^4 - n^4 - (2n^3 - 3n^2 + 3n + 1) = n^5 - 5n^4 + 4n^3 - n^2 - 2n + 1$$

$$when\ n = 5, n^5 - 5n^4 + 4n^3 - 2n + 1 = 464 > 0$$

$$\frac{d(n^5 - 5n^4 + 4n^3 - n^2 - 2n + 1)}{dn} = 5n^4 - 20n^3 + 12n^2 - 2n - 2$$
$$= 5(n-4)n^3 + 2(3n+1)(2n-1)$$

$$(n \ge 5) IMPLIES ((n-4) > 0\ AND\ (3n+1) > 0\ AND (2n-1) > 0$$

$$so, (n \ge 5) IMPLIES (5n^4 - 20n^3 + 12n^2 - 2n - 2 > 0)$$

$$so, (n^5 - 5n^4 + 4n^3 - 2n + 1) is\ increasing\ when\ n \ge 5$$

$$so, for\ every\ n \in \mathbb{Z}^+ (n \ge 5)\ IMPLIES\ (n(\mathrm{n}-1)^4 - n^4 - (2n^3 - 3n^2 + 3n + 1) \ge 0)$$

proof:

base case:

1.when n=1, $\mathrm{A(n)} = 0, un! - h(n) = u - 1 = 99, \mathrm{A(n)} \leq un! - h(n)$

when n=2, $\mathrm{A(n)} = 7, un! - h(n) = 2u - 16 = 184, \mathrm{A(n)} \leq un! - h(n)$

when n=3, $\mathrm{A(n)} = 35, un! - h(n) = 2u - 16 = 519, \mathrm{A(n)} \leq un! - h(n)$

when n=4, $\mathrm{A(n)} = 112, un! - h(n) = 2u - 16 = 2144, \mathrm{A(n)} \leq un! - h(n)$

when n=5, $\mathrm{A(n)} = 278, un! - h(n) = 2u - 16 = 11375, \mathrm{A(n)} \leq un! - h(n)$

constructor case:

2. let $n \geq 5 \ \in \mathbb{Z}^+$ be arbitrary

3. Assume $\mathrm{A(n)} \leq un! - h(n)$

4.for k=n+1, $\mathrm{A(k)} = 2k^3 - 3k^2 + 3k + 1 + kA(n) \leq k(\mathrm{k} - 1)^4 - k^4 + kA(n)$ (direct prove from 2 and lemma)

5. $k(\mathrm{k} - 1)^4 - k^4 + kA(n) \leq k(\mathrm{k} - 1)^4 - k^4 + k(100n!\text{-}\,(k - 1)^4) = k100n! - k^4$ (direct prove from 4)

6. $\mathrm{A(k)} \leq k100n! - k^4 \ when\ k > 5$ (direct prove from 2, 5)

7. $\mathrm{A(n)} \leq un! - h(n) for\ all\ n \ \in \mathbb{Z}^+(\ (weak\ induction\ from\ 1, 2, 6)$

So, there exist a constant u= 100$\in \mathbb{Z}^+ \ and\ a\ polynomial\ h(n) = n^4\ such\ that\ \mathrm{A(n)} \leq un! - h(n)\ for\ every\ n \ \in \mathbb{Z}^+$