1 Quick Notes on Recurrences and Characteristic Polynomials

An addendum by Serguei Makarov to Tutorial 7.

There were several questions in Tutorial 7 about the characteristic polynomial method for solving recurrences. These included questions about why certain numbers appear where they do in the characteristic equation, why the homogeneous and inhomogeneous parts are solved the way they are, and so forth. To make sure these questions get resolved, I've assembled the following notes which unpack the quiz solution and one of the tutorial problems in more detail.

Note: The *Mathematics of Computer Science* textbook (available at `http://courses.csail.mit.edu/6.042/spring16/mcs.pdf`) covers recurrences in Chapter 22, with the general techniques discussed in section 22.3. (There is also material on recurrences in Chapter 3 of the CSCB36/236/240 course notes at `http://www.cs.toronto.edu/~vassos/b36-notes/notes.pdf`, but not so much on the characteristic polynomial.)

1.1 Characteristic Polynomial of a Homogeneous Recurrence

The precise reasons for why the characteristic polynomial works the way it does come from linear algebra and can be studied in excruciating detail from textbooks and online resources such as (`http://www.artofproblemsolving.com/wiki/index.php/Characteristic_polynomial#Linear_recurrences`). However, it would be useful to give an intuition for the method that doesn't require digging into linear algebra proofs.

Example Recurrence. For example, consider the recurrence from the (corrected) quiz:

$$f(n) = \begin{cases} 7 & \text{if } n = 0 \\ -1 & \text{if } n = 1 \\ 3f(n-2) - 2f(n-1) & \text{if } n \geq 2 \end{cases}$$

First of all, this is a *homogeneous linear recurrence*. That means the inductive part of the definition is actually linear in the previous values of the function, $f(n-1)$ and $f(n-2)$. That is, if we replace the previous values by variables x_{n-1} and x_{n-2} we get a *linear function* that looks like this:

$$x_n = g(x_{n-1}, x_{n-2}) = 3x_{n-2} - 2x_{n-1}$$

A linear function satisfies properties like

$$\begin{aligned} g(2x_{n-1}, 2x_{n-2}) &= 2g(x_{n-1}, x_{n-2}) \\ g(x_{n-1} + y_{n-1}, x_{n-2} + y_{n-2}) &= g(x_{n-1}, x_{n-2}) + g(y_{n-1}, y_{n-2}) \end{aligned}$$

and these properties are what the characteristic polynomial method is based on.

(Recurrences that involve terms like $[f(n-1)]^2$ are not linear recurrences, do not satisfy these properties, and the characteristic polynomial method does not work on them. Similarly, if f is defined by an inhomogeneous recurrence these properties would not apply.)

Homogeneous also means that there is no extra term in the equation to worry about. This would not be true if the third line of the definition were something like $3f(n-2) - 2f(n-1) + n^2$. (How to solve this kind of *inhomogeneous* case is described in the second section of these notes.)

The Characteristic Polynomial. To get the characteristic polynomial of the recurrence, you're instructed to take function terms like $f(n-1)$ and replace it with a corresponding exponent x^{n-1}. Doing this for the entire equation gives:

$$x^n = 3x^{n-2} - 2x^{n-1}$$

which you simplify (divide both sides by x^{n-2}) to get $x^2 = 3 - 2x$, or

$$x^2 + 2x - 3 = 0.$$

This is the *characteristic polynomial* of your recurrence. You can solve it with the quadratic formula or factor it as $x^2 + 2x - 3 = (x-1)(x+3)$ to get the roots 1 and -3. But why are the roots of this polynomial interesting?

Consider what happens when we define the operation T:

$$T(x_{n-1}, x_{n-2}) = (x_n, x_{n-1}) = (3x_{n-2} - 2x_{n-1}, x_{n-1})$$

This is a linear operation that takes two values x_{n-1} and x_{n-2} and returns two values x_n and x_{n-1}. Basically, given the values needed to compute $f(n)$ using the inductive step, it returns the values needed to compute $f(n+1)$.

Observe that the roots of the characteristic polynomial show up when we do the following:

$$\begin{aligned} T(1,1) &= (3-2,1) = 1 \cdot (1,1) \\ T(-3,1) &= (3-2(-3),-3) = (3+6,-3) = (9,-3) = -3 \cdot (-3,1) \end{aligned}$$

and when we apply T repeatedly to these initial values, we get:

$$\begin{aligned} T^2(1,1) &= T(T(1,1)) = (1,1) \\ T^3(1,1) &= T(T(T(1,1))) = (1,1) \\ &\dots \\ T^2(-3,1) &= T(T(-3,1)) = -3 \cdot (-3) \cdot (-3,1) \\ T^3(-3,1) &= T(T(T(-3,1))) = (-3)^3 \cdot (-3,1) \\ &\dots \end{aligned}$$

so in general $T^n(1,1) = (1,1)$ and $T^n(-3,1) = (-3)^n \cdot (-3,1)$.

The values $1, -3$ which satisfy this property are called *characteristic values* or *eigenvalues* of T, and the input values $(1,1)$ and $(-3,1)$ could be called *eigenvectors* by analogy to linear algebra.

(For those who have taken a linear algebra course, recall how you obtain the eigenvalues of a matrix by solving its characteristic polynomial: `https://en.wikipedia.org/wiki/Eigenvalues_and_eigenvectors#Characteristic_polynomial`. You could write the operation T as a matrix:

$$\begin{bmatrix} -2 & 3 \\ 1 & 0 \end{bmatrix}$$

and get the eigenvalues of this matrix – which are, again, 1 and -3.)

Using the characteristic polynomial to solve the recurrence. The values of the recurrence $f(n)$ can be obtained by repeatedly applying T to the values from the base case:

$$\begin{aligned}
(f(0), f(1)) &= (7, -1) \\
(f(2), f(1)) &= T(f(0), f(1)) = T(7, -1) \\
(f(3), f(2)) &= T(f(2), f(1)) = T(T(7, -1)) \\
(f(4), f(3)) &= T(f(3), f(2)) = T(T(T(7, -1))) \\
&\dots \\
(f(n), f(n-1)) &= T^{n-1}(7, 1)
\end{aligned}$$

And because T is a linear operator, you can consider the pair of values given to it as a vector, and apply T to a sum of two vectors $a \cdot (1, 1)$ and $b \cdot (-3, 1)$, where a and b are constants:

$$\begin{aligned}
T[a \cdot (1, 1) + b \cdot (-3, 1)] &= T(a, a) + T(-3b, b) = (a, a) + -3(-3b, b) \\
T[T[a \cdot (1, 1) + b \cdot (-3, 1)]] &= T[(a, a) + -3 \cdot (-3b, b)] = T(a, a) + -3 \cdot T(-3b, b) = (a, a) + (-3)^2(-3b, b) \\
&\dots \\
T^n[a \cdot (1, 1) + b \cdot (-3, 1)] &= (a, a) + (-3)^n(-3b, b)
\end{aligned}$$

We've thus derived a simple formula for $T^n[a \cdot (1, 1) + b \cdot (-3, 1)]$.

Now assume you could pick a and b such that $a \cdot (1, 1) + b \cdot (-3, 1) = (7, -1)$. Then we use the formula we just derived to get $(f(n+1), f(n)) = T^n(7, 1) = (a + b(-3)^{n+1}, a + b(-3)^n)$.

If you consider only the second values in each pair of numbers, we get $f(n) = a + b(-3)^n$. From this, we know what our solution looks like.

What's left is to solve for a and b. The formula we derived for $f(n)$ must also apply to the base cases (also known as *boundary conditions*) $f(0)$ and $f(1)$, for which we are given the values. This results in the system of equations

$$\begin{cases} a + b(-3)^0 & = f(0) = 7 \\ a + b(-3)^1 & = f(1) = -1 \end{cases}$$

solving which gives us the solution $a = 5$ and $b = 2$.

Thus $f(n) = 5 + 2(-3)^n$.

Summary: for a homogeneous linear recurrence you should:

- Obtain its characteristic polynomial (replacing terms $f(n-k)$ with x^{n-k}).
- Solve the characteristic polynomial to obtain its roots $p, q, \dots$. The solution to the recurrence will have the form $f(n) = a \cdot p^n + b \cdot q^n + \dots$.
- Now solve for any base cases you are given. For example, if $f(0) = X$, $f(1) = Y$, you should solve the system of equations consisting of $X = a \cdot p^0 + b \cdot q^0 + \dots$ and $Y = a \cdot p^1 + b \cdot q^1 + \dots$.

1.2 Inhomogeneous Recurrences

Example Inhomogeneous Recurrence. Now consider a recurrence with a constant term in the inductive step, such as the Towers of Hanoi recurrence discussed in tutorial:

$$H(n) = \begin{cases} 0 & \text{if } n = 0 \\ 2H(n-1) + 1 & \text{if } n \geq 1 \end{cases}$$

This recurrence is *inhomogeneous* since the inductive part of the definition has an extra term. In general, the extra term in an inhomogeneous recurrence can be any function based on n (e.g. $2n^2 + n + 1$), but for this problem things are simpler because the extra term is constant.

We can consider the inductive step from this recurrence as a single-parameter function, $T(x) = 2x + 1$. (Since the inductive step for this recurrence only depends on one previous value, T takes only one parameter and returns only one value.)

The reasoning from the previous section does not apply because $T(n + m) = 2(n + m) + 1 \neq T(n) + T(m)$.

Instead we solve the recurrence by splitting the into the *inhomogeneous part* and the *homogeneous part*.

Homogeneous Part. The *homogeneous part* of the recurrence is what we get if we remove the extra term, getting a similar recurrence $G(n) = 2G(n-1)$. This recurrence has characteristic polynomial $x - 2 = 0$, with root 2, and therefore it has a solution of the form $G(n) = a2^n$. All the reasoning from the previous section applies here.

Inhomogeneous Part. The *inhomogeneous part* of the recurrence is what we get if we add back the constant term, getting $J(n) = 2J(n-1) + 1$, and try to solve without worrying about the base case. J has the same recurrence relation as H, but I'm denoting it with a different letter to draw your attention to the fact that we'll be deriving a different solution for it.

What we have to do is try to find any *particular solution* for J which satisfies the recurrence, but does not need to satisfy the base cases which H has to satisfy. There is no single strategy for doing this part; you have to guess and then check that plugging the particular solution into the inhomogeneous formula will satisfy it. In general the particular solution should be as simple as possible.

For example, if we make our guess complicated and assume that we are looking for a solution of the form $J(n) = bn$ for some constant b, we can plug this into the recurrence to verify if it makes sense:

$$\begin{aligned} J(n) &= 2J(n-1) + 1 \\ (bn) &= 2(bn - b) + 1 \\ bn &= 2bn - 2b + 1 \\ 0 &= bn - b + 1 \end{aligned}$$

... we see that this *does not* work as a solution no matter what value of b we choose. The recurrence should be valid for *every* value of n, but replacing $J(n)$ with bn yields a formula which is only true when $n = (1 - b)/b$.

Of course, for the Towers of Hanoi, the form of the solution happens to be much simpler. Let's guess that $J(n) = b$ for some constant b. Then plugging this guess into the recurrence gives:

$$\begin{aligned} J(n) &= 2J(n-1) + 1 \\ b &= 2(b) + 1 \\ 0 &= b + 1 \end{aligned}$$

... which happens to be true for *every* value of n when $b = -1$.

So for guessing these particular solutions you should start as simple as you can and make it more complicated if the simple solution doesn't work.

Combining the Two Parts. What you're told to do next is to combine the solutions you got for the homogeneous part and the inhomogeneous part, getting $H(n) = G(n) + J(n) = a2^n - 1$, which is a valid solution for $H(n)$. This may seem a bit strange, but it will start to make sense when you think about the properties of G and J. We know that

$$\begin{aligned} G(n) &= 2G(n-1) \\ J(n) &= 2J(n-1) + 1 \end{aligned}$$

so, when we sum the two together to get $H(n)$, we realize that:

$$H(n) = G(n) + J(n) = 2G(n-1) + 2J(n-1) + 1 = 2H(n-1) + 1$$

and so $H(n)$ is a valid solution to the inhomogeneous recurrence.

In general, this is true for all inhomogeneous recurrences: if you have a valid solution to the entire recurrence (the inhomogeneous part) and you sum it with the solution for just the homogeneous part, you get another valid solution to the entire recurrence. You use this fact to take a simple particular solution (in this case, -1) and add enough free variables to it (in this case, a) that you can try to choose values for those variables to satisfy the base case.

So we know that $H(n) = a2^n - 1$ satisfies the inductive step of the recurrence relation, and we know it has enough unknown variables a for which we can pick values to satisfy the base cases.

When we plug in values the base case $0 = H(0) = a2^0 - 1$, we get $a = 1$. So we know that $H(n) = 2^n - 1$.

Summary: for an inhomogeneous linear recurrence you should:

- Get the homogeneous part of the recurrence by tossing out the term that doesn't refer to a previous value of the recurrence. This might be a constant term, or some term that only depends on n.
- Find the solution for the homogeneous part. It should have the form $f(n) = a \cdot p^n + b \cdot q^n + \ldots$ where $p, q, \ldots$ are the roots of the characteristic polynomial.
- Solve the inhomogeneous part of the recurrence (by guessing a solution). If the inductive step of the recurrence is given by $g(n) = ag(n-1) + bg(n-2) + \cdots + X(n)$, then you should take your guess for $g(n)$ and plug it into the inductive step to make sure it is true for any value of n.
- Combine the two parts to get another valid solution $f(n) = g(n) + a \cdot p^n + b \cdot q^n + \ldots$ and solve for $C, a, b, \ldots$ by plugging in the base cases for $f(n)$.