Welcome

To

MKS TUTORIALS

By

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Topics covered in playlist of Fourier Series: Introduction (Fourier Series), Euler's Formulae, Conditions for a Fourier Expansion- Dirichlet's Conditions, Functions having Points of Discontinuity, Change of Intervals, Odd and Even Functions- Expansions of Odd or Even Periodic Functions, Half Range Series, Practical harmonic Analysis, Parseval's Identities or Parseval's Formula for Fourier Series, Complex Form of Fourier Series.

Complete playlist link for Fourier Series (in Hindi):
https://www.youtube.com/playlist?list=PLhSp9OSVmeyLke5_cby8i8ZhK8FHpw3qs

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FOURIER SERIES

If $f(x)$ is defined and periodic in the interval $[a, b]$, then its fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2n\pi x}{b-a}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2n\pi x}{b-a}\right)$$

By Euler's formulae,

$$a_0 = \frac{2}{b-a} \int_a^b f(x)\,dx$$

$$a_n = \frac{2}{b-a} \int_a^b f(x) \cdot \cos\left(\frac{2n\pi x}{b-a}\right) dx$$

$$b_n = \frac{2}{b-a} \int_a^b f(x) \cdot \sin\left(\frac{2n\pi x}{b-a}\right) dx$$

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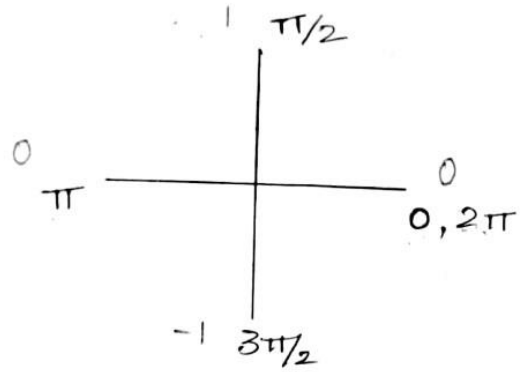
for $b - a = 2\pi$

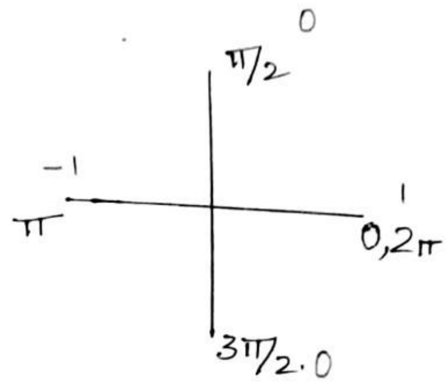
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cdot \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where, $a_0 = \frac{1}{\pi} \int_a^b f(x)\,dx$

$$a_n = \frac{1}{\pi} \int_a^b f(x) \cdot \cos nx\,dx$$

$$b_n = \frac{1}{\pi} \int_a^b f(x) \cdot \sin nx\,dx$$





$\sin n\pi = 0$

$\sin 2n\pi = 0$

$\cos n\pi = (-1)^n$

$\cos 2n\pi = 1$

$\cos (2n+1)\pi = -1$

$\cos (2n-1)\pi = -1$

Problems to be solved:

1. $f(x) = x^2, \ [-\pi, \pi]$
2. $f(x) = e^{-x}, \ [0, 2\pi]$
3. $f(x) = x - x^2, \ [-\pi, \pi]$
4. $f(x) = \sqrt{(1-\cos x)}, \ [0, 2\pi]$
5. $f(x) = x \sin x, \ [0, 2\pi]$

Que① Prove that

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} (-1)^n \frac{\cos nx}{n^2}, \quad (-\pi < x < \pi)$$

Also, show that

(i) $\sum_1 \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$

(ii) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$.

(iii) $\sum \frac{1}{(2n-1)^2} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$

Solⁿ Here, $f(x) = x^2$ and $b - a = 2\pi$

$\therefore$ The fourier series for $f(x) = x^2$ is

$$x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{—— ①}$$

Now, $a_0 = \frac{1}{\pi}\int_a^b f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx$

$$= \frac{1}{\pi}\left|\frac{x^3}{3}\right|_{-\pi}^{\pi} = \frac{1}{3\pi}\left[\pi^3 - (-\pi)^3\right]$$

$$= \frac{2\pi^3}{3\pi} = \frac{2\pi^2}{3} \qquad \boxed{a_0 = \frac{2\pi^2}{3}}$$

$$a_n = \frac{1}{\pi}\int_a^b f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx\,dx$$

$$= \frac{1}{\pi}\left| x^2\left(\frac{\sin nx}{n}\right) - (2x)\left(-\frac{\cos nx}{n^2}\right) + (2)\left(-\frac{\sin nx}{n^3}\right)\right|_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left| \frac{x^2 \sin nx}{n} + \frac{2x\cos nx}{n^2} - \frac{2\sin nx}{n^3}\right|_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\pi^2 \sin n\pi}{n} - \frac{(-\pi)^2 \sin n(-\pi)}{n} + \frac{2\pi\cos n\pi}{n^2} - \frac{2(-\pi)\cos n(-\pi)}{n^2} - \frac{2\sin n\pi}{n^3} + \frac{2\sin n(-\pi)}{n^3}\right]$$

$$= \frac{1}{\pi}\left[\frac{2\pi\cos n\pi}{n^2} + \frac{2\pi\cos n\pi}{n^2}\right]$$

$$= \frac{1}{\pi}\,\frac{4\pi\cos n\pi}{n^2} = \frac{4(-1)^n}{n^2}$$

$$\left\{\begin{array}{l}\sin n\pi = 0\\ \cos n\pi = (-1)^n\end{array}\right.$$

$$\boxed{a_n = \frac{4(-1)^n}{n^2}}$$

$$b_n = \frac{1}{\pi}\int_a^b f(x)\,\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \sin nx\,dx$$

$$= \frac{1}{\pi}\left| x^2\left(-\frac{\cos nx}{n}\right) - (2x)\left(-\frac{\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right|_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left| -\frac{x^2\cos nx}{n} + \frac{2x\sin nx}{n^2} + \frac{2\cos nx}{n^3}\right|_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{\pi^2\cos n\pi}{n} + \frac{(-\pi)^2\cos n(-\pi)}{n} + \frac{2\pi\sin n\pi}{n^2} \right.$$

$$\left. - \frac{2(-\pi)\sin n(-\pi)}{n^2} + \frac{2\cos n\pi}{n^3} - \frac{2\cos n(-\pi)}{n^3}\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi^2\cos n\pi}{n} + \frac{\pi^2\cos n\pi}{n} + \frac{2\cos n\pi}{n^3} - \frac{2\cos n\pi}{n^3}\right]$$

$$\boxed{b_n = 0}$$

$\therefore$ eq.^{n} (1) becomes,

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n\frac{\cos nx}{n^2}$$

$$x^2 = \frac{\pi^2}{3} + 4\left[-\frac{1}{1^2}\cos x + \frac{1}{2^2}\cos 2x - \frac{1}{3^2}\cos 3x + \cdots\right] \quad \text{(11)}$$

(i) Put $x = \pi$ in eq^n (ii),

$$\pi^2 = \frac{\pi^2}{3} + 4\left[-\frac{1}{1^2}\cos\pi + \frac{1}{2^2}\cos 2\pi - \frac{1}{3^2}\cos 3\pi + \cdots\right.$$

$$\pi^2 - \frac{\pi^2}{3} = 4\left[-\frac{1}{1^2}(-1) + \frac{1}{2^2}(1) - \frac{1}{3^2}(-1) + \cdots\right]$$

$$\frac{2\pi^2}{3} = 4\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right]$$

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \sum \frac{1}{n^2} \quad \text{Proved}$$

(iii)

(ii) Put $x = 0$ in eq. (ii),

$$0 = \frac{\pi^2}{3} + 4\left[-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \cdots\right]$$

$$-\frac{\pi^2}{3} = 4\left[-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \cdots\right]$$

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12} \quad \text{Proved}$$

(iv)

(iii) Adding (iii) and (iv),

$$\frac{\pi^2}{6} + \frac{\pi^2}{12} = 2\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right)$$

$$\frac{\pi^2}{4} = 2\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right)$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \sum \frac{1}{(2n-1)^2}$$

Proved

Que(2) Obtain the fourier series for $f(x) = e^{-x}$ in the interval $0 < x < 2\pi$.

Sol^n Here, $f(x) = e^{-x}$ and $b - a = 2\pi$

$\therefore$ The fourier series for $f(x) = e^{-x}$ is

$$e^{-x} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{--- (1)}$$

Now, $a_0 = \frac{1}{\pi}\int_a^b f(x)dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}dx$

$$= \frac{1}{\pi}\left|\frac{e^{-x}}{-1}\right|_0^{2\pi} = \frac{-1}{\pi}\left[e^{-2\pi} - e^0\right] = \boxed{\frac{1 - e^{-2\pi}}{\pi} = a_0}$$

$$a_n = \frac{1}{\pi}\int_a^b f(x)\cdot \cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\cos nx\,dx$$

Use formula:

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$$

$$= \frac{1}{\pi}\left|\frac{e^{-x}}{(1+n^2)}(-\cos nx + n\sin nx)\right|_0^{2\pi}$$

$$= \frac{1}{\pi(1+n^2)}\left|n e^{-x}\sin nx - e^{-x}\cos nx\right|_0^{2\pi}$$

$$= \frac{1}{\pi(1+n^2)}\left[(0-0) - (e^{-2\pi} - 1)\right] = \frac{1 - e^{-2\pi}}{\pi(1+n^2)}$$

$$\therefore \boxed{a_n = \left(\frac{1}{n^2+1}\right)\left(\frac{1 - e^{-2\pi}}{\pi}\right)}$$

$$b_n = \frac{1}{\pi}\int_a^b f(x)\, \sin nx\, dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x} \cdot \sin nx\, dx$$

Use formula:

$$\int e^{ax} \cdot \sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b\cos bx)$$

$$b_n = \frac{1}{\pi}\left| \frac{e^{-x}}{(1+n^2)}(-\sin nx - n\cos nx)\right|_0^{2\pi}$$

$$= \frac{-1}{\pi(1+n^2)}\left| e^{-x}\sin nx + n e^{-x}\cos nx \right|_0^{2\pi}$$

$$= \frac{-1}{\pi(1+n^2)}\left[(0-0) + (n e^{-2\pi} - n)\right]$$

$$= \frac{n(1-e^{-2\pi})}{\pi(n^2+1)} = \boxed{\left(\frac{n}{n^2+1}\right)\left(\frac{1-e^{-2\pi}}{\pi}\right) = b_n}$$

eq.ⁿ(1) becomes,

$$e^{-x} = \left(\frac{1-e^{-2\pi}}{2\pi}\right) + \sum_{n=1}^{\infty}\left(\frac{1-e^{-2\pi}}{\pi}\right)\left(\frac{1}{n^2+1}\right)\cos nx + \sum_{n=1}^{\infty}\left(\frac{1-e^{-2\pi}}{\pi}\right)\left(\frac{n}{n^2+1}\right)\sin nx$$

$$= \left(\frac{1-e^{-2\pi}}{\pi}\right)\left\{\frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{1}{n^2+1}\right)\cos nx + \sum_{n=1}^{\infty}\left(\frac{n}{n^2+1}\right)\sin nx\right\}$$

$$= \left(\frac{1-e^{-2\pi}}{\pi}\right)\left\{\frac{1}{2} + \left(\frac{1}{2}\cos x + \frac{1}{5}\cos 2x + \frac{1}{10}\cos 3x + \cdots\right) + \left(\frac{1}{2}\sin x + \frac{2}{5}\sin 2x + \frac{3}{10}\sin 3x + \cdots\right)\right\}$$

Ans

Que (3) Find a fourier series to represent $(x-x^2)$ from $-\pi < x < \pi$.

Sol^n Here, $f(x) = x - x^2$ and $b-a = 2\pi$

The fourier series for $f(x) = x - x^2$ is

$$x - x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{—— (1)}$$

$$a_0 = \frac{1}{\pi}\int_a^b f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\,dx$$

$$= \frac{1}{\pi}\left| \frac{x^2}{2} - \frac{x^3}{3} \right|_{-\pi}^{\pi} = \frac{1}{\pi}\left[\frac{\pi^2}{2} - \frac{(-\pi)^2}{2} - \frac{\pi^3}{3} + \frac{(-\pi)^3}{3}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} - \frac{\pi^2}{2} - \frac{\pi^3}{3} - \frac{\pi^3}{3}\right] = -\frac{2\pi^3}{3\pi} = \boxed{-\frac{2\pi^2}{3} = a_0}$$

$$a_n = \frac{1}{\pi}\int_a^b f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[(x-x^2)\frac{\sin nx}{n} - (1-2x)\left(-\frac{\cos nx}{n^2}\right) + (-2)\left(-\frac{\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(x-x^2)\frac{\sin nx}{n} + (1-2x)\frac{\cos nx}{n^2} + 2\frac{\sin nx}{n^3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(\pi-\pi^2)\frac{\sin n\pi}{n} - (-\pi-\pi^2)\frac{\sin n(-\pi)}{n} + (1-2\pi)\frac{\cos n\pi}{n^2} - \frac{(1+2\pi)\cos n(-\pi)}{n^2} + 2\frac{\sin n\pi}{n^3} - 2\frac{\sin n(-\pi)}{n^3}\right]$$

$$= \frac{1}{\pi}\left[(1-2\pi)\frac{(-1)^n}{n^2} - \frac{(1+2\pi)(-1)^n}{n^2}\right]$$

$$= \frac{1}{n^2\pi}\left[(-1)^n - 2\pi(-1)^n - (-1)^n - 2\pi(-1)^n\right] = \boxed{-\frac{4(-1)^n}{n^2} = a_n}$$

$$b_n = \frac{1}{\pi}\int_a^b f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[(x-x^2)\left(-\frac{\cos nx}{n}\right)-(1-2x)\left(-\frac{\sin nx}{n^2}\right)+(-2)\frac{\cos nx}{n^3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(x^2-x)\frac{\cos nx}{n}+(1-2x)\frac{\sin nx}{n^2}-2\frac{\cos nx}{n^3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(\pi^2-\pi)\frac{\cos n\pi}{n}-(\pi^2+\pi)\frac{\cos n(-\pi)}{n}+(1-2\pi)\frac{\sin n\pi}{n^2}\right.$$
$$\left.-(1+2\pi)\frac{\sin n(-\pi)}{n^2}-2\frac{\cos n\pi}{n^3}+2\frac{\cos n(-\pi)}{n^3}\right]$$

$$= \frac{1}{\pi}\left[(\pi^2-\pi)\frac{(-1)^n}{n}-(\pi^2+\pi)\frac{(-1)^n}{n}\right]$$

$$= \frac{(-1)^n}{n\pi}\left[\pi^2-\pi-\pi^2-\pi\right] = \frac{(-1)^n}{n\pi}(-2\pi)$$

$$\therefore \boxed{b_n = -\frac{2(-1)^n}{n}}$$

from (1),

$$x-x^2 = \left(-\frac{2\pi^2}{3}\right)\cdot\frac{1}{2}+\sum_{n=1}^{\infty}\frac{(-4)(-1)^n}{n^2}\cos nx$$
$$+\sum_{n=1}^{\infty}\frac{(-2)(-1)^n}{n}\sin nx$$

$$= -\frac{\pi^2}{3}-4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx-2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx$$

$$= -\frac{\pi^2}{3}+4\left(\frac{\cos x}{1^2}-\frac{\cos 2x}{2^2}+\frac{\cos 3x}{3^2}-\cdots\right)$$
$$+2\left(\frac{\sin x}{1}-\frac{\sin 2x}{2}+\frac{\sin 3x}{3}-\cdots\right)$$

Ans.

Que(4) Expand $f(x) = \sqrt{(1-\cos x)}$; $0 < x < 2\pi$ in a fourier series. Hence, evaluate $\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots$

Sol^n Here, $f(x) = \sqrt{(1-\cos x)}$

$$= \sqrt{2\sin^2 x/2} = \sqrt{2}\cdot \sin x/2$$

$$\left\{\begin{aligned} \cos 2x &= 2\cos^2 x - 1 \\ &= 1 - 2\sin^2 x \\ 2\sin^2 x &= 1 - \cos 2x \\ 2\sin^2 x/2 &= 1 - \cos x \end{aligned}\right\}$$

Also, $b - a = 2\pi$.

$\therefore$ The fourier series is

$$\sqrt{(1-\cos x)} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{—— (1)}$$

$$a_0 = \frac{1}{\pi}\int_a^b f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} \sqrt{2}\cdot \sin\frac{x}{2}\,dx = \frac{\sqrt{2}}{\pi}\left| -\frac{\cos x/2}{1/2}\right|_0^{2\pi}$$

$$= -\frac{\sqrt{2}}{\pi}\times 2\left[\cos\pi - \cos 0\right] = -\frac{2\sqrt{2}}{\pi}(-1-1) = \boxed{\frac{4\sqrt{2}}{\pi} = a_0}$$

$$a_n = \frac{1}{\pi}\int_a^b f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi}\sqrt{2}\cdot\sin\frac{x}{2}\cdot\cos nx\,dx$$

$$= \frac{\sqrt{2}}{\pi}\times\frac{1}{2}\int_0^{2\pi} 2\cdot\cos nx\cdot\sin\frac{x}{2}\,dx \qquad \left\{\begin{aligned} &2\cos A\cdot\sin B \\ &= \sin(A+B) - \sin(A-B)\end{aligned}\right\}$$

$$= \frac{1}{\pi\sqrt{2}}\int_0^{2\pi}\left\{\sin\left(n+\frac{1}{2}\right)x - \sin\left(n-\frac{1}{2}\right)x\right\}dx$$

$$= \frac{1}{\pi\sqrt{2}}\int_0^{2\pi}\left\{\sin\left(\frac{2n+1}{2}\right)x - \sin\left(\frac{2n-1}{2}\right)x\right\}dx$$

$$= \frac{1}{\pi\sqrt{2}}\left| -\frac{\cos\left(\frac{2n+1}{2}\right)x}{\left(\frac{2n+1}{2}\right)} + \frac{\cos\left(\frac{2n-1}{2}\right)x}{\left(\frac{2n-1}{2}\right)}\right|_0^{2\pi}$$

$$= \frac{1}{\pi\sqrt{2}}\left|\left(\frac{-2}{2n+1}\right)\cos\left(\frac{2n+1}{2}\right)x + \left(\frac{2}{2n-1}\right)\cos\left(\frac{2n-1}{2}\right)x\right|_0^{2\pi}$$

$$= \frac{2}{\pi\sqrt{2}}\left[\left(\frac{-1}{2n+1}\right)\left\{\cos(2n+1)\pi - \cos 0\right\} + \left(\frac{1}{2n-1}\right)\left\{\cos(2n-1)\pi - \cos 0\right\}\right]$$

$$= \frac{\sqrt{2}}{\pi}\left[\left(\frac{-1}{2n+1}\right)(-1-1) + \left(\frac{1}{2n-1}\right)(-1-1)\right]$$

$$= \frac{\sqrt{2}}{\pi}\left(\frac{2}{2n+1} - \frac{2}{2n-1}\right) = \frac{2\sqrt{2}}{\pi}\left(\frac{2n-1-2n-1}{4n^2-1}\right)$$

$$= \frac{2\sqrt{2}}{\pi}\frac{(-2)}{(4n^2-1)} = \boxed{\frac{-4\sqrt{2}}{\pi(4n^2-1)} = a_n}$$

$$b_n = \frac{1}{\pi}\int_a^b f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi}\sqrt{2}\cdot\sin\frac{x}{2}\cdot\sin nx\,dx$$

$$= \frac{\sqrt{2}}{\pi}\times\frac{1}{2}\int_0^{2\pi} 2\sin nx\cdot\sin\frac{x}{2}\,dx \quad \left\{\begin{array}{l}2\sin A\cdot\sin B\\ = \cos(A-B) - \cos(A+B)\end{array}\right\}$$

$$= \frac{1}{\pi\sqrt{2}}\int_0^{2\pi}\left\{\cos\left(n-\frac{1}{2}\right)x - \cos\left(n+\frac{1}{2}\right)x\right\}dx$$

$$= \frac{1}{\pi\sqrt{2}}\int_0^{2\pi}\left\{\cos\left(\frac{2n-1}{2}\right)x - \cos\left(\frac{2n+1}{2}\right)x\right\}dx$$

$$= \frac{1}{\pi\sqrt{2}}\left[\frac{\sin\left(\frac{2n-1}{2}\right)x}{\left(\frac{2n-1}{2}\right)} - \frac{\sin\left(\frac{2n+1}{2}\right)x}{\left(\frac{2n+1}{2}\right)}\right]_0^{2\pi}$$

$$= \frac{1}{\pi\sqrt{2}}\left[\left(\frac{2}{2n-1}\right)\sin\left(\frac{2n-1}{2}\right)x - \left(\frac{2}{2n+1}\right)\sin\left(\frac{2n+1}{2}\right)x\right]_0^{2\pi}$$

$$= \frac{1}{\pi\sqrt{2}}\left[\left(\frac{2}{2n-1}\right)\sin(2n-1)\pi - \left(\frac{2}{2n-1}\right)\sin 0 - \left(\frac{2}{2n+1}\right)\sin(2n+1)\pi + \left(\frac{2}{2n+1}\right)\sin 0\right]$$

$$= 0.$$

from eq.^{n} (1),

$$\sqrt{(1-\cos x)} = \left(\frac{4\sqrt{2}}{\pi}\right)\frac{1}{2} + \sum_{n=1}^{\infty}\left[\frac{-4\sqrt{2}}{\pi(4n^2-1)}\right]\cos nx$$

$$= \frac{2\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\left(\frac{\cos nx}{4n^2-1}\right)$$

Put $x=0$,

$$\sqrt{1-\cos 0} = \frac{2\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\left(\frac{1}{4n^2-1}\right)$$

$$\frac{2\sqrt{2}}{\pi} = \frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\left(\frac{1}{4n^2-1}\right)$$

$$\frac{1}{2} = \frac{1}{3} + \frac{1}{15} + \frac{1}{35} + \cdots$$

$$\Rightarrow \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots = \frac{1}{2}$$

Ans

Que(5) Expand $f(x) = x\sin x$ as a fourier series in the interval $[0, 2\pi]$.

Sol^n Here, $f(x) = x\sin x$ and $b-a = 2\pi$

$\therefore$ The fourier series is given by

$$x\sin x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{—(1)}$$

$$a_0 = \frac{1}{\pi}\int_a^b f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} x\sin x\,dx = \frac{1}{\pi}\Big[x(-\cos x) - 1(-\sin x)\Big]_0^{2\pi}$$

$$= \frac{1}{\pi}\Big[-x\cos x + \sin x\Big]_0^{2\pi} = \frac{1}{\pi}\Big[-2\pi\cos 2\pi + 0 + \sin 2\pi - \sin 0\Big]$$

$$= -\frac{2\pi}{\pi} = -2.$$

$$a_n = \frac{1}{\pi}\int_a^b f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x\sin x\cos nx\,dx$$

$$= \frac{1}{\pi}\times\frac{1}{2}\int_0^{2\pi} x\,(2\cos nx\cdot\sin x)\,dx \qquad \boxed{\begin{aligned}&2\cos A\sin B\\ &= \sin(A+B) - \sin(A-B)\end{aligned}}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\left\{\sin(n+1)x - \sin(n-1)x\right\}dx$$

$$= \frac{1}{2\pi}\left[x\left\{-\frac{\cos(n+1)x}{(n+1)} + \frac{\cos(n-1)x}{(n-1)}\right\} - (1)\left\{-\frac{\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2}\right\}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[2\pi\left\{-\frac{\cos(n+1)2\pi}{(n+1)} + \frac{\cos(n-1)2\pi}{(n-1)}\right\} - 0 - \left\{-\frac{\sin(n+1)2\pi}{(n+1)^2} + \frac{\sin 0}{(n+1)^2} + \frac{\sin(n-1)2\pi}{(n-1)^2} - \frac{\sin 0}{(n-1)^2}\right\}\right]$$

$$= \frac{1}{2\pi}\times 2\pi\left\{\frac{-1}{n+1} + \frac{1}{n-1}\right\} = \frac{-n+1+n+1}{n^2-1} = \frac{2}{n^2-1} \quad (n\neq 1)$$

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} x \sin x \cos x \, dx = \frac{1}{\pi}\times\frac{1}{2}\int_0^{2\pi} x(2\sin x\cos x)\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x \sin 2x \, dx = \frac{1}{2\pi}\left[x\left(-\frac{\cos 2x}{2}\right) - 1\left(-\frac{\sin 2x}{4}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[-\frac{x\cos 2x}{2} + \frac{\sin 2x}{4}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[-\frac{2\pi.\cos 4\pi}{2} + 0 + \frac{\sin 4\pi}{4} - \frac{\sin 0}{4}\right]$$

$$= -\frac{2\pi}{4\pi} = -\frac{1}{2} = a_1$$

$$b_n = \frac{1}{\pi}\int_a^b f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x \sin x \cdot \sin nx \, dx$$

$$= \frac{1}{\pi}\times\frac{1}{2}\int_0^{2\pi} x(2\sin nx\cdot\sin x)\,dx \qquad \left\{\begin{aligned} &2\sin A.\sin B \\ &= \cos(A-B) - \cos(A+B)\end{aligned}\right\}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\{\cos(n-1)x - \cos(n+1)x\}\,dx$$

$$= \frac{1}{2\pi}\left[x\left\{\frac{\sin(n-1)x}{(n-1)} - \frac{\sin(n+1)x}{(n+1)}\right\} - (1)\left\{-\frac{\cos(n-1)x}{(n-1)^2} + \frac{\cos(n+1)x}{(n+1)^2}\right\}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[2\pi\left\{\frac{\sin(n-1)2\pi}{(n-1)} - \frac{\sin(n+1)2\pi}{(n+1)}\right\} - 0 - \left\{-\frac{\cos(n-1)2\pi}{(n-1)^2} + \frac{\cos(n+1)2\pi}{(n+1)^2} + \frac{\cos 0}{(n-1)^2} - \frac{\cos 0}{(n+1)^2}\right\}\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{(n-1)^2} - \frac{1}{(n+1)^2} - \frac{1}{(n-1)^2} + \frac{1}{(n+1)^2}\right]$$

$$\boxed{b_n = 0} \qquad (n \neq 1)$$

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} x\sin^2 x\,dx = \frac{1}{\pi}\int_0^{2\pi} x\left(\frac{1-\cos 2x}{2}\right)dx$$

$$= \frac{1}{2\pi}\left[x\left(x - \frac{\sin 2x}{2}\right) - 1\left(\frac{x^2}{2} + \frac{\cos 2x}{4}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[x^2 - \frac{x\sin 2x}{2} - \frac{x^2}{2} - \frac{\cos 2x}{4}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{x^2}{2} - \frac{x\sin 2x}{2} - \frac{\cos 2x}{4}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{4\pi^2}{2} - 0 - \frac{2\pi\sin 4\pi}{2} + 0 - \frac{\cos 4\pi}{4} + \frac{\cos 0}{4}\right]$$

$$= \frac{1}{2\pi}\left[\frac{4\pi^2}{2}\right]$$

$$b_1 = \pi$$

$\therefore$ eq.^{n} (1) becomes,

$$x\sin x = -1 - \frac{1}{2}\cos x + \pi\sin x + \frac{2}{2^2-1}\cos 2x + \frac{2}{3^2-1}\cos 3x + \cdots$$

Ans

FUNCTIONS HAVING POINTS OF DISCONTINUITY

Que (1) Find the fourier series expansion for $f(x)$, if $f(x) = \begin{cases} -\pi, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$

Sol.ⁿ Here, $b - a = 2\pi$

$\therefore$ The fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{— (1)}$$

$$a_0 = \frac{1}{\pi}\int_a^b f(x)dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)dx + \int_0^{\pi} x\,dx\right]$$

$$= \frac{1}{\pi}\left[(-\pi)\left|x\right|_{-\pi}^{0} + \left|\frac{x^2}{2}\right|_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[(-\pi)(0+\pi) + \left(\frac{\pi^2}{2} - 0\right)\right] = \frac{1}{\pi}\left(-\pi^2 + \frac{\pi^2}{2}\right)$$

$$\boxed{a_0 = -\frac{\pi}{2}}$$

$$a_n = \frac{1}{\pi}\int_a^b f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\cos nx\,dx + \int_0^{\pi} x\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[(-\pi)\left|\frac{\sin nx}{n}\right|_{-\pi}^{0} + \left|x\frac{\sin nx}{n} - (1)\left(-\frac{\cos nx}{n^2}\right)\right|_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\left(-\frac{\pi}{n}\right)\left\{\sin 0 - \sin n(-\pi)\right\} + \pi\frac{\sin n\pi}{n} - 0 + \frac{\cos n\pi}{n^2} - \frac{\cos 0}{n^2}\right]$$

$$= \frac{1}{\pi}\left\{\frac{(-1)^n - 1}{n^2}\right\} = \boxed{\frac{(-1)^n - 1}{n^2\pi} = a_n}$$

$$b_n = \frac{1}{\pi}\int_a^b f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\sin nx\,dx + \int_0^{\pi} x\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[(-\pi)\left(-\frac{\cos nx}{n}\right)_{-\pi}^{0} + \left| x\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right|_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\pi\left\{\frac{\cos 0}{n} - \frac{\cos n(-\pi)}{n}\right\} - \pi\frac{\cos n\pi}{n} + 0 + \frac{\sin n\pi}{n^2} - \frac{\sin 0}{n^2}\right]$$

$$= \frac{1}{\pi}\left[\pi\left(\frac{1}{n} - \frac{\cos n\pi}{n}\right) - \pi\frac{\cos n\pi}{n}\right]$$

$$= \frac{1}{\pi}\cdot\frac{\pi}{n}\left[1 - 2\cos n\pi\right] = \boxed{\frac{1}{n}\left[1 - 2(-1)^n\right] = b_n}$$

$\therefore$ eq.^{n} (1) becomes,

$$f(x) = -\frac{\pi}{4} + \sum_{n=1}^{\infty}\left\{\frac{(-1)^n - 1}{n^2\pi}\right\}\cos nx + \sum_{n=1}^{\infty}\left\{\frac{1 - 2(-1)^n}{n}\right\}\sin nx$$

Ans

Que ② If $f(x) = \begin{cases} 0, & -\pi \le x \le 0 \\ \sin x, & 0 \le x \le \pi \end{cases}$; prove that

$$f(x) = \frac{1}{\pi} + \frac{\sin x}{2} - \frac{2}{\pi}\sum_{n=2}^{\infty} \frac{\cos nx}{n^2-1}$$ and hence

show that $\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \cdots \infty = \frac{1}{4}(\pi - 2)$

Sol.^{n} Here, $b-a = 2\pi$

∴ The fourier series for function $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{—— ①}$$

$$a_0 = \frac{1}{\pi}\int_a^b f(x)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\cdot dx + \int_0^{\pi} \sin x\,dx\right]$$

$$= \frac{1}{\pi}\left[\left|-\cos x\right|_0^{\pi}\right] = \frac{-1}{\pi}(\cos\pi - \cos 0) = \frac{-1}{\pi}(-1-1)$$

$$\boxed{a_0 = \frac{2}{\pi}}$$

$$a_n = \frac{1}{\pi}\int_a^b f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\cdot\cos nx\,dx + \int_0^{\pi} \sin x\cdot\cos nx\,dx\right]$$

$$= \frac{1}{2\pi}\int_0^{\pi} 2\cos nx\cdot\sin x\,dx = \frac{1}{2\pi}\int_0^{\pi}\{\sin(n+1)x - \sin(n-1)x\}\,dx$$

$$= \frac{1}{2\pi}\left[-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]_0^{\pi}$$

$2\cos A\cdot\sin B = \sin(A+B) - \sin(A-B)$

$$= \frac{1}{2\pi}\left[\frac{\cos(n-1)\pi}{n-1} - \frac{\cos 0}{n-1} - \frac{\cos(n+1)\pi}{n+1} + \frac{\cos 0}{n+1}\right]$$

$$= \frac{1}{2\pi}\left[\frac{(-1)^{n-1}-1}{n-1} + \frac{1-(-1)^{n+1}}{n+1}\right] \quad (n \ne 1)$$

when n is odd, $a_n = 0$

when n is even, $a_n = \frac{1}{2\pi}\left[\frac{-2}{n-1} + \frac{2}{n+1}\right]$

$$= \frac{1}{2\pi}\left[\frac{-2n-2+2n-2}{n^2-1}\right] = \frac{-4}{2\pi(n^2-1)} = \frac{-2}{(n^2-1)\pi} = a_n \quad (n \ne 1)$$

$$a_1 = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\cdot \cos x\,dx + \int_0^{\pi} \sin x \cdot \cos x\,dx\right] = \frac{1}{2\pi}\int_0^{\pi} \sin 2x\,dx$$

$$= \frac{1}{2\pi}\left|-\frac{\cos 2x}{2}\right|_0^{\pi} = \frac{-1}{4\pi}\left(\cos 2\pi - \cos 0\right) = -\frac{1}{4\pi}(1-1) = 0$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\cdot \sin nx\,dx + \int_0^{\pi} \sin x \cdot \sin nx\,dx\right] \quad \begin{pmatrix} 2\sin A \sin B \\ = \cos(A-B) - \\ \cos(A+B) \end{pmatrix}$$

$$= \frac{1}{2\pi}\int_0^{\pi} 2\sin nx \cdot \sin x\,dx = \frac{1}{2\pi}\int_0^{\pi}\left\{\cos(n-1)x - \cos(n+1)x\right\}dx$$

$$= \frac{1}{2\pi}\left[\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right]_0^{\pi} = 0 \quad (n \neq 1)$$

$$b_1 = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\cdot \sin x\,dx + \int_0^{\pi} \sin^2 x\,dx\right] \quad \begin{pmatrix} \cos 2x = 1 - 2\sin^2 x \\ \sin^2 x = \left(\frac{1-\cos 2x}{2}\right) \end{pmatrix}$$

$$= \frac{1}{\pi}\int_0^{\pi}\left(\frac{1-\cos 2x}{2}\right)dx = \frac{1}{2\pi}\left[\int_0^{\pi} dx - \int_0^{\pi}\cos 2x\,dx\right]$$

$$= \frac{1}{2\pi}\left[|x|_0^{\pi} - \left|\frac{\sin 2x}{2}\right|_0^{\pi}\right] = \frac{1}{2\pi}\left[\pi - \frac{1}{2}\left(\sin 2\pi - \sin 0\right)\right]$$

$$\Rightarrow b_1 = \frac{1}{2}$$

$$\therefore f(x) = \frac{1}{\pi} - \frac{2}{\pi}\sum_{n=2}^{\infty}\frac{\cos nx}{n^2-1} + \frac{1}{2}\sin x.$$

$$= \frac{1}{\pi} - \frac{2}{\pi}\left(\frac{\cos 2x}{2^2-1} + \frac{\cos 4x}{4^2-1} + \frac{\cos 6x}{6^2-1} + \cdots\right) + \frac{1}{2}\sin x$$

Put $x = \pi/2$

$$\sin \pi/2 = \frac{1}{\pi} - \frac{2}{\pi}\left(\frac{\cos \pi}{3} + \frac{\cos 2\pi}{15} + \frac{\cos 3\pi}{35} + \cdots\right) + \frac{1}{2}$$

$$1 - \frac{1}{2} = \frac{1}{\pi} - \frac{2}{\pi}\left(-\frac{1}{3} + \frac{1}{15} - \frac{1}{35} + \cdots\right)$$

$$\frac{1}{2} - \frac{1}{\pi} = \frac{2}{\pi}\left(\frac{1}{3} - \frac{1}{15} + \frac{1}{35} - \cdots\right) = \frac{\pi - 2}{2\pi}$$

$$\Rightarrow \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \cdots = \frac{1}{4}(\pi - 2)$$

Hence proved

Que (3) Find the fourier series for $f(x)$ in the interval $(-\pi, \pi)$ where

$$f(x) = \begin{cases} -x+1, & -\pi < x < 0 \\ x+1, & 0 < x < \pi \end{cases}$$

Sol.ⁿ Here, $b-a = 2\pi$

$\therefore$ The fourier series for $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{——} (1)$$

$$a_0 = \frac{1}{\pi}\int_a^b f(x)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-x+1)\,dx + \int_0^{\pi}(x+1)\,dx\right]$$

$$= \frac{1}{\pi}\left(\left|-\frac{x^2}{2} + x\right|_{-\pi}^{0} + \left|\frac{x^2}{2} + x\right|_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[-0+0+\frac{(-\pi)^2}{2} - (-\pi) + \frac{\pi^2}{2} + \pi - 0 - 0\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} + \pi + \frac{\pi^2}{2} + \pi\right] = \frac{1}{\pi}\left(\pi^2 + 2\pi\right) = \boxed{\pi + 2 = a_0}$$

$$a_n = \frac{1}{\pi}\int_a^b f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-x+1)\cos nx\,dx + \int_0^{\pi}(x+1)\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left|(-x+1)\frac{\sin nx}{n} - (-1)\left(-\frac{\cos nx}{n^2}\right)\right|_{-\pi}^{0} + \left|(x+1)\frac{\sin nx}{n} - (1)\left(-\frac{\cos nx}{n^2}\right)\right|_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\left|(1-x)\frac{\sin nx}{n} - \frac{\cos nx}{n^2}\right|_{-\pi}^{0} + \left|(x+1)\frac{\sin nx}{n} + \frac{\cos nx}{n^2}\right|_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[(1)\frac{\sin 0}{n} - (1+\pi)\frac{\sin n(-\pi)}{n} - \frac{\cos 0}{n^2} + \frac{\cos n(-\pi)}{n^2} + (\pi+1)\frac{\sin n\pi}{n} - (1)\frac{\sin 0}{n} + \frac{\cos n\pi}{n^2} - \frac{\cos 0}{n^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{-1+(-1)^n+(-1)^n-1}{n^2}\right] = \boxed{\frac{2}{n^2\pi}\left[(-1)^n-1\right] = a_n}$$

$$b_n = \frac{1}{\pi}\int_a^b f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-x+1)\sin nx\,dx + \int_0^{\pi}(x+1)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left|(-x+1)\left(-\frac{\cos nx}{n}\right) - (-1)\left(-\frac{\sin nx}{n^2}\right)\right|_{-\pi}^{0} + \left|(x+1)\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right|_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\left|(x-1)\frac{\cos nx}{n} - \frac{\sin nx}{n^2}\right|_{-\pi}^{0} + \left|-(x+1)\frac{\cos nx}{n} + \frac{\sin nx}{n^2}\right|_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[(-1)\frac{\cos 0}{n} - (-\pi-1)\frac{\cos n(-\pi)}{n} - \frac{\sin 0}{n^2} + \frac{\sin n(-\pi)}{n^2} - (\pi+1)\frac{\cos n\pi}{n} + (1)\frac{\cos 0}{n} + \frac{\sin n\pi}{n^2} - \frac{\sin 0}{n^2}\right]$$

$$= \frac{1}{\pi}\left[-\frac{1}{n} + (\pi+1)\cdot\frac{\cos n\pi}{n} - (\pi+1)\frac{\cos n\pi}{n} + \frac{1}{n}\right]$$

$$\boxed{b_n = 0}$$

$\therefore$ from (i),
$$f(x) = \frac{\pi+2}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\left[(-1)^n - 1\right]}{n^2}\cos nx.$$

which is the reqd solution.

Que(4) Find the fourier series for

$$f(t) = \begin{cases} -1, & -\pi < t < -\pi/2 \\ 0, & -\pi/2 < t < \pi/2 \\ 1, & \pi/2 < t < \pi \end{cases}$$

Solⁿ Here, $b - a = 2\pi$

∴ The fourier series is given by

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin nt \quad \text{——} (1)$$

$$a_0 = \frac{1}{\pi}\int_a^b f(t)\,dt = \frac{1}{\pi}\left[\int_{-\pi}^{-\pi/2}(-1)dt + \int_{-\pi/2}^{\pi/2} 0\,dt + \int_{\pi/2}^{\pi} 1\cdot dt\right]$$

$$= \frac{1}{\pi}\left[(-1)\left|t\right|_{-\pi}^{-\pi/2} + \left|t\right|_{\pi/2}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[(-1)\left(-\frac{\pi}{2}+\pi\right) + \left(\pi - \frac{\pi}{2}\right)\right] = \frac{1}{\pi}\left[\frac{\pi}{2} - \pi + \pi - \frac{\pi}{2}\right]$$

$$\boxed{a_0 = 0}$$

$$a_n = \frac{1}{\pi}\int_a^b f(t)\cos nt\,dt$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{-\pi/2}(-1)\cos nt\,dt + \int_{-\pi/2}^{\pi/2} 0\cdot\cos nt\,dt + \int_{\pi/2}^{\pi} 1\cdot\cos nt\,dt\right]$$

$$= \frac{1}{\pi}\left[-\left|\frac{\sin nt}{n}\right|_{-\pi}^{-\pi/2} + \left|\frac{\sin nt}{n}\right|_{\pi/2}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[-\frac{1}{n}\left\{\sin n(-\pi/2) - \sin n(-\pi)\right\} + \frac{1}{n}\left\{\sin n\pi - \sin n\frac{\pi}{2}\right\}\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{n}\sin n\frac{\pi}{2} - \frac{1}{n}\sin n\frac{\pi}{2}\right] = \boxed{0 = a_n}$$

$$b_n = \frac{1}{\pi} \int_a^b f(t) \sin nt \, dt$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^{-\pi/2} (-1) \sin nt \, dt + \int_{-\pi/2}^{\pi/2} 0 \cdot \sin nt \, dt + \int_{\pi/2}^{\pi} 1 \cdot \sin nt \, dt \right]$$

$$= \frac{1}{\pi} \left[- \left| -\frac{\cos nt}{n} \right|_{-\pi}^{-\pi/2} + \left| -\frac{\cos nt}{n} \right|_{\pi/2}^{\pi} \right]$$

$$= \frac{1}{\pi} \left[\frac{\cos n(-\pi/2)}{n} - \frac{\cos n(-\pi)}{n} - \frac{\cos n\pi}{n} + \frac{\cos n\pi/2}{n} \right]$$

$$= \frac{1}{n\pi} \left[2\cos n\pi/2 - 2(-1)^n \right]$$

$$\boxed{b_n = \frac{2}{n\pi} \left[\cos n\pi/2 - (-1)^n \right]}$$

$\therefore$ from eq. (i),

$$f(t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left[\cos n\frac{\pi}{2} - (-1)^n \right] \sin nt$$

which is the reqd. solution.

CHANGE OF INTERVALS

Que ① Find the fourier series expansion of $f(x) = 2x - x^2$ in the interval $(0,3)$.

Sol.ⁿ Here, $f(x) = 2x - x^2$ and $b-a = 3$

$\therefore$ The fourier series of $f(x)$ is given by

$$f(x) = 2x - x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi x}{3} + \sum_{n=1}^{\infty} b_n \sin\frac{2n\pi x}{3} \quad \text{①}$$

$$a_0 = \frac{2}{3}\int_0^3 (2x - x^2)\,dx = \frac{2}{3}\left| 2\cdot\frac{x^2}{2} - \frac{x^3}{3}\right|_0^3$$

$$= \frac{2}{3}\left| x^2 - \frac{x^3}{3}\right|_0^3 = \frac{2}{3}\left(3^2 - \frac{3^3}{3}\right) = \boxed{0 = a_0}$$

$$a_n = \frac{2}{3}\int_0^3 (2x - x^2)\cdot\cos\left(\frac{2n\pi}{3}x\right)dx$$

$$= \frac{2}{3}\left| (2x - x^2)\frac{\sin(2n\pi x/3)}{2n\pi/3} - (2-2x)\left\{-\frac{\cos(2n\pi x/3)}{4n^2\pi^2/9}\right\} + (0-2)\left\{-\frac{\sin(2n\pi x/3)}{8n^3\pi^3/27}\right\}\right|_0^3$$

$$= \frac{2}{3}\left| \frac{3(2x - x^2)}{2n\pi}\cdot\sin\left(\frac{2n\pi}{3}x\right) + \frac{9(2-2x)}{4n^2\pi^2}\cos\left(\frac{2n\pi x}{3}\right) + \frac{54}{8n^3\pi^3}\cdot\sin\left(\frac{2n\pi x}{3}\right)\right|_0^3$$

$$= \frac{2}{3}\left[\frac{3(-3)}{2n\pi}\cdot\sin 2n\pi - 0 + \frac{(-36)}{4n^2\pi^2}\cdot\cos 2n\pi - \frac{18}{4n^2\pi^2}\cos 0 + \frac{54}{8n^3\pi^3}\sin 2n\pi - 0\right]$$

$$a_n = \frac{2}{3}\left[\frac{-9}{n^2\pi^2}(1) - \frac{9}{2n^2\pi^2}\right] = -\frac{2}{3} \times \frac{9}{n^2\pi^2}\left(1+\frac{1}{2}\right)$$

$$= -\frac{2}{3} \times \frac{9}{n^2\pi^2}\left(\frac{3}{2}\right) = \boxed{\frac{-9}{n^2\pi^2} = a_n}$$

$$b_n = \frac{2}{3}\int_0^3 (2x-x^2)\sin\left(\frac{2n\pi x}{3}\right)dx$$

$$= \frac{2}{3}\left| (2x-x^2)\left\{-\frac{\cos(2n\pi x/3)}{2n\pi/3}\right\} - (2-2x)\left\{-\frac{\sin(2n\pi x/3)}{4n^2\pi^2/9}\right\} + (-2)\frac{\cos(2n\pi x/3)}{8n^3\pi^3/27}\right|_0^3$$

$$= \frac{2}{3}\left| \frac{3(x^2-2x)}{2n\pi}\cos(2n\pi x/3) + \frac{9(2-2x)}{4n^2\pi^2}\sin(2n\pi x/3) - \frac{54}{8n^3\pi^3}\cos(2n\pi x/3)\right|_0^3$$

$$= \frac{2}{3}\left[\frac{9}{2n\pi}\cos 2n\pi - 0 + \frac{(-36)}{4n^2\pi^2}\sin 2n\pi - \frac{18}{4n^2\pi^2}\cdot\sin 0 - \frac{27}{4n^3\pi^3}\cdot\cos 2n\pi + \frac{27}{4n^3\pi^3}\cdot\cos 0\right]$$

$$= \frac{2}{3}\left[\frac{9}{2n\pi}\right] = \boxed{\frac{3}{n\pi} = b_n}$$

$\therefore$ from (1),

$$2x - x^2 = -\frac{9}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\left(\frac{2n\pi x}{3}\right) + \frac{3}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{2n\pi x}{3}\right)$$

which is the reqd. fourier series.

Que(5) Obtain a fourier series for the function

$$f(x)=\begin{cases}\pi x, & 0\le x\le 1\\ \pi(2-x), & 1\le x\le 2.\end{cases}$$

Deduce that $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots=\frac{\pi^2}{8}$

Sol.ⁿ Here, $b-a=2$.

$\therefore$ The fourier series of $f(x)$ is given by

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos n\pi x+\sum_{n=1}^{\infty}b_n\sin n\pi x \quad \text{— (1)}$$

$$a_0=1\int_0^2 f(x)\,dx=\int_0^1\pi x\,dx+\int_1^2\pi(2-x)\,dx$$

$$=\pi\left|\frac{x^2}{2}\right|_0^1+2\pi\int_1^2 dx-\pi\int_1^2 x\,dx$$

$$=\pi\left(\tfrac{1}{2}\right)+2\pi\left|x\right|_1^2-\pi\left|\frac{x^2}{2}\right|_1^2$$

$$=\frac{\pi}{2}+2\pi(2-1)-\frac{\pi}{2}(4-1)=\boxed{\pi=a_0}$$

$$a_n=1\int_0^2 f(x)\cos n\pi x\,dx=\int_0^1\pi x\cos n\pi x\,dx+\int_1^2\pi(2-x)\cos n\pi x\,dx$$

$$=\pi\left|x\cdot\frac{\sin n\pi x}{n\pi}-(1)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right|_0^1+\pi\left|(2-x)\frac{\sin n\pi x}{n\pi}-(-1)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right|_1^2$$

$$=\pi\left|\frac{x\sin n\pi x}{n\pi}+\frac{\cos n\pi x}{n^2\pi^2}\right|_0^1+\pi\left|(2-x)\frac{\sin n\pi x}{n\pi}-\frac{\cos n\pi x}{n^2\pi^2}\right|_1^2$$

$$=\pi\left[\frac{1\cdot\sin n\pi}{n\pi}-0+\frac{\cos n\pi}{n^2\pi^2}-\frac{\cos 0}{n^2\pi^2}+(0)\frac{\sin n\pi 2}{n\pi}-(1)\frac{\sin n\pi}{n\pi}-\frac{\cos 2n\pi}{n^2\pi^2}+\frac{\cos n\pi}{n^2\pi^2}\right]$$

$$a_n = \frac{\pi}{n^2\pi^2}\left[2(-1)^n - 2\right] = \frac{2}{n^2\pi}\left[(-1)^n - 1\right]$$

when n is even, $a_n = 0$

when n is odd, $a_n = \frac{2}{n^2\pi}(-1-1) = -\frac{4}{n^2\pi}$

$$b_n = \frac{1}{1}\int_0^2 f(x)\sin n\pi x\,dx = \int_0^1 \pi x \sin n\pi x\,dx + \int_1^2 \pi(2-x)\sin n\pi x\,dx$$

$$= \pi\left|x\left(-\frac{\cos n\pi x}{n\pi}\right) - (1)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right|_0^1 + \pi\left|(2-x)\left(-\frac{\cos n\pi x}{n\pi}\right) - (-1)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right|_1^2$$

$$= \pi\left|-\frac{x\cos n\pi x}{n\pi} + \frac{\sin n\pi x}{n^2\pi^2}\right|_0^1 + \pi\left|(x-2)\frac{\cos n\pi x}{n\pi} - \frac{\sin n\pi x}{n^2\pi^2}\right|_1^2$$

$$= \pi\left[-\frac{\cos n\pi}{n\pi} + 0 + \frac{\sin n\pi}{n^2\pi^2} - \frac{\sin 0}{n^2\pi^2} + 0 - (-1)\frac{\cos n\pi}{n\pi} - \frac{\sin 2n\pi}{n^2\pi^2} + \frac{\sin n\pi}{n^2\pi^2}\right]$$

$$= \pi\left[-\frac{\cos n\pi}{n\pi} + \frac{\cos n\pi}{n\pi}\right] = \boxed{0 = b_n}$$

$\therefore$ from (1), $f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos n\pi x}{n^2} \quad (n = 1, 3, 5, \cdots)$

Put $x = 2$,

$$\pi(2-2) = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos 2\pi}{1^2} + \frac{\cos 6\pi}{3^2} + \frac{\cos 10\pi}{5^2} + \cdots\right)$$

$$\frac{\pi}{2} = \frac{4}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right)$$

or, $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$

Hence proved

EVEN OR ODD FUNCTION

Interval $(-a, a)$

eg: $(-\pi, \pi)$ or $(-1, 1)$ or $(-l, l)$

Then check for even or odd functions

For Even function

Put $x = -x$ in $f(x)$

If $f(-x) = f(x)$

then function is even. $(b_n = 0)$

The fourier series of an even function is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2n\pi x}{b-a}\right)$$

For odd function

Put $x = -x$ in $f(x)$

If $f(-x) = -f(x)$

then function is odd. $(a_0 = 0, a_n = 0)$

The fourier series of an odd function is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{2n\pi x}{b-a}\right)$$

To solve a_0, a_n and b_n, we adopt this methodology,

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

NUMERICALS

1. $f(x) = e^{-|x|}$, $[-\pi, \pi]$

2. $f(x) = |\cos x|$, $[-\pi, \pi]$

3. $f(x) = \begin{cases} 1 + \dfrac{2x}{\pi}, & -\pi \le x \le 0 \\ 1 - \dfrac{2x}{\pi}, & 0 \le x \le \pi \end{cases}$ (Triangular wave form)

Que ① Find the fourier series of $f(x) = e^{-|x|}$ in $(-\pi, \pi)$.

Sol^{n} Here, $f(x) = e^{-|x|}$

Put $x = -x$; $f(-x) = e^{-|-x|} = e^{-|x|} = f(x)$

Hence, $f(x)$ is an even function.

$\Rightarrow b_n = 0$

Now,

$$f(x) = e^{-(-x)} = e^{x} \; ; \quad -\pi < x < 0$$
$$= e^{-(x)} = e^{-x} \; ; \quad 0 < x < \pi$$

$\therefore$ The fourier series of an even function is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{2n\pi x}{b-a}$$

Here, $b - a = 2\pi$

$$\therefore e^{-|x|} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{——} ①$$

$$a_0 = \frac{2}{b-a} \int_a^b f(x)\,dx = \frac{2}{2\pi} \int_{-\pi}^{\pi} e^{-|x|}\,dx$$

$$= \frac{1}{\pi} \cdot 2 \int_0^{\pi} e^{-x}\,dx = \frac{2}{\pi} \left| \frac{e^{-x}}{-1} \right|_0^{\pi} = -\frac{2}{\pi}\left(e^{-\pi} - e^{0}\right)$$

$$= \boxed{\frac{2}{\pi}\left(1 - e^{-\pi}\right) = a_0}$$

$$a_n = \frac{2}{b-a}\int_a^b f(x)\cos\left(\frac{2n\pi x}{b-a}\right)dx = \frac{2}{2\pi}\int_{-\pi}^{\pi} e^{-|x|}\cos nx\,dx$$

$$= \frac{1}{\pi}\, 2\int_0^{\pi} e^{-x}\cos nx\,dx$$

Use formula: $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$

$$= \frac{2}{\pi}\left|\frac{e^{-x}}{(1+n^2)}(-\cos nx + n\sin nx)\right|_0^{\pi}$$

$$= \frac{2}{(1+n^2)\pi}\Big[e^{-\pi}(-\cos n\pi + n\sin n\pi) - e^{0}(-\cos 0 + n\sin 0)\Big]$$

$$= \frac{2}{(1+n^2)\pi}\left[-(-1)^n e^{-\pi} + 1\right]$$

$$a_n = \frac{2}{(1+n^2)\pi}\left[1 - (-1)^n e^{-\pi}\right]$$

$\therefore$ from eq^n (1),

$$e^{-|x|} = \left(\frac{1-e^{-\pi}}{\pi}\right) + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{(1+n^2)}\left[1-(-1)^n e^{-\pi}\right]\cos nx$$

which is the reqd. fourier series.

Que (2) If $f(x) = |\cos x|$, expand $f(x)$ as a fourier series in the interval $(-\pi, \pi)$.

Sol.^{n} Put $x = -x$

$$\Rightarrow f(-x) = |\cos(-x)| = |\cos x| = f(x)$$

$\therefore f(x)$ is an even function $(b_n = 0)$

The fourier series of an even function is

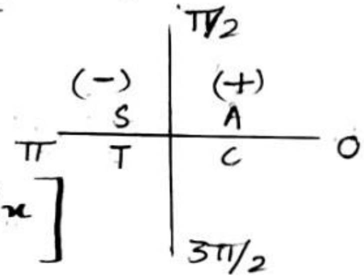
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2n\pi x}{b-a}\right)$$

Here, $b-a = 2\pi$

$$\therefore |\cos x| = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{— (1)}$$

$$a_0 = \frac{2}{b-a}\int_a^b f(x)\,dx = \frac{2}{2\pi}\int_{-\pi}^{\pi} |\cos x|\,dx$$

$$= \frac{1}{\pi} \times 2\int_0^{\pi} |\cos x|\,dx$$



$$= \frac{1}{\pi} \times 2\left[\int_0^{\pi/2} \cos x\,dx + \int_{\pi/2}^{\pi} (-\cos x)\,dx\right]$$

$$= \frac{2}{\pi}\left[\left|\sin x\right|_0^{\pi/2} - \left|\sin x\right|_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\sin \pi/2 - \sin 0 - \sin \pi + \sin \pi/2\right]$$

$$= \frac{2}{\pi}\left[1 + 1\right] = \boxed{\frac{4}{\pi} = a_0}$$

$$a_n = \frac{2}{b-a}\int_a^b f(x)\cos\left(\frac{2n\pi x}{b-a}\right)dx = \frac{2}{2\pi}\int_{-\pi}^{\pi} |\cos x| \cos nx\,dx$$

$$= \frac{1}{\pi} \times 2\int_0^{\pi} |\cos x| \cos nx\,dx$$

$$a_n = \frac{2}{\pi}\left[\int_0^{\pi/2} \cos nx \cos x\, dx + \int_{\pi/2}^{\pi} \cos nx(-\cos x)\, dx\right]$$

$$= \frac{2}{\pi}\left[\frac{1}{2}\int_0^{\pi/2} 2\cos nx \cos x\, dx - \frac{1}{2}\int_{\pi/2}^{\pi} 2\cos nx \cdot \cos x\, dx\right]$$

$$= \frac{2}{\pi}\left[\frac{1}{2}\int_0^{\pi/2}\{\cos(n+1)x + \cos(n-1)x\}dx - \frac{1}{2}\int_{\pi/2}^{\pi}\{\cos(n+1)x + \cos(n-1)x\}dx\right]$$

Use formula:

$$2\cos A \cdot \cos B = \cos(A+B) + \cos(A-B)$$

$$= \frac{1}{\pi}\left[\left|\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right|_0^{\pi/2} - \left|\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right|_{\pi/2}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\sin(n+1)\pi/2}{n+1} - \frac{\sin 0}{n+1} + \frac{\sin(n-1)\pi/2}{n-1} - \frac{\sin 0}{n-1}\right.$$

$$\left. - \frac{\sin(n+1)\pi}{n+1} + \frac{\sin(n+1)\pi/2}{n+1} - \frac{\sin(n-1)\pi}{n-1} + \frac{\sin(n-1)\pi/2}{n-1}\right]$$

$$= \frac{2}{\pi}\left[\frac{\sin(n+1)\pi/2}{n+1} + \frac{\sin(n-1)\pi/2}{n-1}\right]$$

$$= \frac{2}{\pi}\left[\frac{\sin(n\pi/2 + \pi/2)}{n+1} + \frac{\sin(n\pi/2 - \pi/2)}{n-1}\right]$$

$$= \frac{2}{\pi}\left[\frac{\sin(\pi/2 + n\pi/2)}{n+1} - \frac{\sin(\pi/2 - n\pi/2)}{n-1}\right]$$

$$= \frac{2}{\pi}\left[\frac{\cos n\pi/2}{n+1} - \frac{\cos n\pi/2}{n-1}\right]$$

$$= \frac{2}{\pi} \cdot \cos n\pi/2 \left(\frac{1}{n+1} - \frac{1}{n-1}\right) = \frac{2\cos n\pi/2}{\pi}\left(\frac{n-1-n-1}{n^2-1}\right)$$

$$\boxed{a_n = -\frac{4\cos n\pi/2}{\pi(n^2-1)}} \quad (\text{when } n \neq 1)$$

$$a_1 = \frac{2}{b-a}\int_a^b |\cos x| \cos x \, dx = \frac{2}{2\pi}\int_{-\pi}^{\pi} |\cos x| \cos x \, dx$$

$$= \frac{1}{\pi} \times 2 \int_0^{\pi} |\cos x| \cos x \, dx$$

$$\cos 2x = 2\cos^2 x - 1$$
$$\cos^2 x = \frac{1+\cos 2x}{2}$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} \cos^2 x \, dx - \int_{\pi/2}^{\pi} \cos^2 x \, dx\right]$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2}\left(\frac{\cos 2x + 1}{2}\right) dx - \int_{\pi/2}^{\pi}\left(\frac{\cos 2x + 1}{2}\right) dx\right]$$

$$= \frac{1}{\pi}\left[\left|\frac{\sin 2x}{2} + x\right|_0^{\pi/2} - \left|\frac{\sin 2x}{2} + x\right|_{\pi/2}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\sin \pi}{2} + \frac{\pi}{2} - \frac{\sin 0}{2} - 0 - \frac{\sin 2\pi}{2} - \pi + \frac{\sin \pi}{2} + \frac{\pi}{2}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2} - \pi + \frac{\pi}{2}\right] = \boxed{0 = a_1}$$

from eq.ⁿ (1),

$$|\cos x| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=2}^{\infty} \frac{\cos n\pi/2}{n^2-1} \cdot \cos nx$$

which is the reqd. fourier series.

Que(3) Obtain a fourier series for the function

$$f(x) = \begin{cases} 1 + \frac{2x}{\pi}, & -\pi \le x \le 0, \\ 1 - \frac{2x}{\pi}, & 0 \le x \le \pi \end{cases}$$

Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$

Solⁿ Put $x = -x$

$$f(-x) = 1 - \frac{2x}{\pi} \text{ in } (-\pi, 0) = f(x) \text{ in } (0, \pi).$$

Also, $f(-x) = 1 + \frac{2x}{\pi}$ in $(0, \pi) = f(x)$ in $(-\pi, 0)$.

$\therefore f(x)$ is an even function in $(-\pi, \pi)$

The fourier series of an even function is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2n\pi x}{b-a}\right)$$

Here, $b - a = 2\pi$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{—— (1)}$$

$$a_0 = \frac{2}{b-a} \int_a^b f(x)\,dx = \frac{2}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi} \times 2 \int_0^{\pi} f(x)\,dx = \frac{2}{\pi} \int_0^{\pi} \left(1 - \frac{2x}{\pi}\right) dx$$

$$= \frac{2}{\pi} \left| x - \frac{2}{\pi} \cdot \frac{x^2}{2} \right|_0^{\pi} = \frac{2}{\pi} \left| x - \frac{x^2}{\pi} \right|_0^{\pi}$$

$$= \frac{2}{\pi} \left(\pi - \frac{\pi^2}{\pi} \right) = \boxed{0 = a_0}$$

$$a_n = \frac{2}{b-a}\int_a^b f(x)\cos\left(\frac{2n\pi x}{b-a}\right)dx = \frac{2}{2\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\times 2\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi}\left(1-\frac{2x}{\pi}\right)\cos nx\,dx$$

$$= \frac{2}{\pi}\left|\left(1-\frac{2x}{\pi}\right)\frac{\sin nx}{n} - \left(-\frac{2}{\pi}\right)\left(-\frac{\cos nx}{n^2}\right)\right|_0^{\pi}$$

$$= \frac{2}{\pi}\left|\left(1-\frac{2x}{\pi}\right)\frac{\sin nx}{n} - \frac{2}{\pi n^2}(\cos nx)\right|_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(1-\frac{2\pi}{\pi}\right)\frac{\sin n\pi}{n} - (1)\frac{\sin 0}{n} - \frac{2}{\pi n^2}\cos n\pi + \frac{2}{\pi n^2}\cos 0\right]$$

$$a_n = \frac{2}{\pi}\left[\frac{-2}{\pi n^2}(-1)^n + \frac{2}{\pi n^2}\right] = \frac{4}{n^2\pi^2}\left[1-(-1)^n\right]$$

$a_n = \frac{8}{n^2\pi^2}$, when 'n' is odd.

$= 0$, when 'n' is even.

from eq.^{n} (1),

$$f(x) = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos nx}{n^2} \quad \underline{\text{Ans}}$$

Put $x=0$,

$$1 = \frac{8}{\pi^2}\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right)$$

or, $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots = \frac{\pi^2}{8}$

Hence proved

HALF - RANGE SERIES

Cosine Series
or
Half Range Cosine Series

The fourier half range cosine series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{b-a}\right)$$

where,

$$a_0 = \frac{2}{b-a} \int_a^b f(x)\, dx$$

$$a_n = \frac{2}{b-a} \int_a^b f(x) \cos\left(\frac{n\pi x}{b-a}\right) dx$$

Sine Series
or
Half Range Sine Series

The fourier half range sine series is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{b-a}\right)$$

where,

$$b_n = \frac{2}{b-a}\int_a^b f(x) \sin\left(\frac{n\pi x}{b-a}\right) dx$$

Numericals

1. $f(t) = \begin{cases} 2t & ; 0 < t < 1 \\ 2(2-t) & ; 1 < t < 2 \end{cases}$ (Cosine series)

2. $f(x) = x \sin x \; ; \; (0, \pi)$ (Cosine series)

3. $f(x) = \begin{cases} \frac{1}{4} - x \; ; & 0 < x < 1/2 \\ x - \frac{3}{4} \; ; & 1/2 < x < 1 \end{cases}$ (Sine series)

4. $f(x) = \begin{cases} K(x) \; ; & 0 \le x \le l/2 \\ K(l-x) \; ; & l/2 \le x \le l. \end{cases}$ (Cosine series)

Que (1) Find the fourier half range cosine series of the function

$$f(t) = \begin{cases} 2t & ; \ 0 < t < 1 \\ 2(2-t) & ; \ 1 < t < 2 \end{cases}$$

Sol.ⁿ The reqd. fourier half range cosine series is given by

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi t}{b-a}\right)$$

Here, $b-a = 2-0 = 2$.

$$\therefore f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi t}{2}\right) \quad \text{—— (1)}$$

$$a_0 = \frac{2}{b-a}\int_a^b f(t)\,dt = \frac{2}{2}\left[\int_0^1 2t\,dt + \int_1^2 2(2-t)\,dt\right]$$

$$= 2\int_0^1 t\,dt + 4\int_1^2 dt - 2\int_1^2 t\,dt$$

$$= 2\left|\frac{t^2}{2}\right|_0^1 + 4\left|t\right|_1^2 - 2\left|\frac{t^2}{2}\right|_1^2$$

$$= (1-0) + 4(2-1) - (4-1) = \boxed{2 = a_0}$$

$$a_n = \frac{2}{b-a}\int_a^b f(t)\cos\left(\frac{n\pi t}{b-a}\right)dt$$

$$= \frac{2}{2}\left[\int_0^1 2t\cos\left(\frac{n\pi t}{2}\right)dt + \int_1^2 (4-2t)\cos\left(\frac{n\pi t}{2}\right)dt\right]$$

$$a_n = 2\left| t\cdot \frac{\sin(n\pi t/2)}{n\pi/2} - (1)\left\{-\frac{\cos(n\pi t/2)}{n^2\pi^2/4}\right\}\right|_0^1$$

$$+ \left|(4-2t)\frac{\sin(n\pi t/2)}{n\pi/2} - (-2)\left\{-\frac{\cos(n\pi t/2)}{n^2\pi^2/4}\right\}\right|_1^2$$

$$= 2\left|\frac{2t}{n\pi}\cdot \sin\left(\frac{n\pi t}{2}\right) + \frac{4}{n^2\pi^2}\cos\left(\frac{n\pi t}{2}\right)\right|_0^1$$

$$+ \left|\frac{2(4-2t)}{n\pi}\cdot \sin\left(\frac{n\pi t}{2}\right) - \frac{8}{n^2\pi^2}\cdot\cos\left(\frac{n\pi t}{2}\right)\right|_1^2$$

$$= 2\left[\frac{2}{n\pi}\cdot\sin\frac{n\pi}{2} - 0 + \frac{4}{n^2\pi^2}\cdot\cos\frac{n\pi}{2} - \frac{4}{n^2\pi^2}\cos 0\right]$$

$$+ 0 - \frac{4}{n\pi}\sin\frac{n\pi}{2} - \frac{8}{n^2\pi^2}\cos n\pi + \frac{8}{n^2\pi^2}\cos\frac{n\pi}{2}$$

$$= \frac{4}{n\pi}\cdot\sin\frac{n\pi}{2} + \frac{8}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{8}{n^2\pi^2} - \frac{4}{n\pi}\sin\frac{n\pi}{2}$$

$$- \frac{8}{n^2\pi^2}\cos n\pi + \frac{8}{n^2\pi^2}\cos\frac{n\pi}{2}.$$

$$a_n = \frac{8}{n^2\pi^2}\left[2\cos\frac{n\pi}{2} - 1 - (-1)^n\right]$$

from eq.^{n} (1),

$$f(t) = 1 + \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[2\cos\frac{n\pi}{2} - 1 - (-1)^n\right]\cos\frac{n\pi t}{2}$$

Ans

Que ② Obtain the fourier expansion of $x \sin x$ as a cosine series in $(0, \pi)$.

Sol.ⁿ The fourier half range cosine series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{b-a}\right)$$

Here, $f(x) = x \sin x$, $b-a = \pi$

$$\therefore x \sin x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{—— ①}$$

$$a_0 = \frac{2}{b-a} \int_a^b f(x)\,dx = \frac{2}{\pi} \int_0^{\pi} x \sin x\,dx$$

$$= \frac{2}{\pi} \left| x(-\cos x) - (1)(-\sin x) \right|_0^{\pi}$$

$$= \frac{2}{\pi} \left| -x\cos x + \sin x \right|_0^{\pi}$$

$$= \frac{2}{\pi} \left[-\pi \cos\pi + 0 + \sin\pi - \sin 0 \right]$$

$$= \frac{2}{\pi} \times \pi = \boxed{2 = a_0}$$

$$a_n = \frac{2}{b-a} \int_a^b f(x) \cos\left(\frac{n\pi x}{b-a}\right) dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x \sin x \cdot \cos nx\,dx$$

$$= \frac{2}{\pi} \times \frac{1}{2} \int_0^{\pi} x(2\cos nx \cdot \sin x)\,dx$$

$$\boxed{\text{Use formula: } 2\cos A \cdot \sin B = \sin(A+B) - \sin(A-B)}$$

$$a_n = \frac{1}{\pi} \int_0^{\pi} x \left\{ \sin(n+1)x - \sin(n-1)x \right\} dx$$

$$= \frac{1}{\pi} \left[\int_0^{\pi} x \sin(n+1)x\, dx - \int_0^{\pi} x \sin(n-1)x\, dx \right]$$

$$= \frac{1}{\pi} \left[x \left\{ -\frac{\cos(n+1)x}{n+1} \right\} - (1) \left\{ -\frac{\sin(n+1)x}{(n+1)^2} \right\} - x \left\{ \frac{-\cos(n-1)x}{n-1} \right\} + (1) \left\{ -\frac{\sin(n-1)x}{(n-1)^2} \right\} \right]_0^{\pi}$$

$$= \frac{1}{\pi} \left[-\pi \frac{\cos(n+1)\pi}{n+1} + 0 + \pi \frac{\cos(n-1)\pi}{n-1} - 0 \right]$$

$$= \frac{\pi}{\pi} \left[\frac{\cos(n-1)\pi}{n-1} - \frac{\cos(n+1)\pi}{n+1} \right]$$

$$\boxed{a_n = \frac{\cos(n-1)\pi}{n-1} - \frac{\cos(n+1)\pi}{n+1}} \quad \text{when } (n \neq 1)$$

$$a_1 = \frac{2}{\pi} \int_0^{\pi} x \sin x \cos x\, dx = \frac{2}{\pi} \times \frac{1}{2} \int_0^{\pi} x\, 2\sin x \cos x\, dx$$

$$= \frac{1}{\pi} \int_0^{\pi} x \sin 2x\, dx = \frac{1}{\pi} \left| x\left(-\frac{\cos 2x}{2}\right) - (1)\left(-\frac{\sin 2x}{4}\right) \right|_0^{\pi}$$

$$= \frac{1}{\pi} \left| -\frac{x}{2}\cos 2x + \frac{\sin 2x}{4} \right|_0^{\pi} \Rightarrow a_1 = -\frac{1}{2}$$

from ①, $$x \sin x = 1 + \left(-\frac{1}{2}\cos x\right) + \sum_{n=2}^{\infty} \left\{ \frac{\cos(n-1)\pi}{n-1} - \frac{\cos(n+1)\pi}{n+1} \right\} \times \cos nx$$

$$\Rightarrow x \sin x = 1 - \frac{1}{2}\cos x + \sum_{n=2}^{\infty} \left\{ \frac{\cos(n-1)\pi}{n-1} - \frac{\cos(n+1)\pi}{n+1} \right\} \cos nx$$

Ans

Que (3) Expand $f(x) = \begin{cases} \frac{1}{4} - x \; ; & 0 < x < 1/2 \\ x - \frac{3}{4} \; ; & 1/2 < x < 1 \end{cases}$

as a fourier series of sine terms.

Sol.ⁿ The fourier half range sine series is $f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{b-a}\right)$

Given: $b - a = 1 - 0 = 1$

$$\therefore f(x) = \sum_{n=1}^{\infty} b_n \sin n\pi x \quad \text{——— (1)}$$

$$b_n = \frac{2}{b-a} \int_a^b f(x) \sin\left(\frac{n\pi x}{b-a}\right) dx = 2\int_0^1 f(x) \sin n\pi x \, dx$$

$$= 2\left[\int_0^{1/2} \left(\frac{1}{4} - x\right) \sin n\pi x \, dx + \int_{1/2}^{1} \left(x - \frac{3}{4}\right) \sin n\pi x \, dx\right]$$

$$= 2\left[\left|\left(\frac{1}{4} - x\right)\left(-\frac{\cos n\pi x}{n\pi}\right) - (-1)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right|_0^{1/2} + \left|\left(x - \frac{3}{4}\right)\left(-\frac{\cos n\pi x}{n\pi}\right) - \left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right|_{1/2}^{1}\right]$$

$$= 2\left[\left|\left(x - \frac{1}{4}\right)\frac{\cos n\pi x}{n\pi} - \frac{\sin n\pi x}{n^2\pi^2}\right|_0^{1/2} + \left|\left(\frac{3}{4} - x\right)\frac{\cos n\pi x}{n\pi} + \frac{\sin n\pi x}{n^2\pi^2}\right|_{1/2}^{1}\right]$$

$$b_n = 2\left[\left(\frac{1}{2}-\frac{1}{4}\right)\frac{\cos n\pi/2}{n\pi} - \left(-\frac{1}{4}\right)\frac{\cos 0}{n\pi} - \frac{\sin n\pi/2}{n^2\pi^2}\right.$$

$$+\frac{\sin 0}{n^2\pi^2} + \left(\frac{3}{4}-1\right)\frac{\cos n\pi}{n\pi} - \left(\frac{3}{4}-\frac{1}{2}\right)\frac{\cos n\pi/2}{n\pi}$$

$$\left. + \frac{\sin n\pi}{n^2\pi^2} - \frac{\sin n\pi/2}{n^2\pi^2}\right]$$

$$= 2\left[\frac{1}{4}\frac{\cos n\pi/2}{n\pi} + \frac{1}{4n\pi} - \frac{\sin n\pi/2}{n^2\pi^2} - \frac{1}{4}\frac{\cos n\pi}{n\pi}\right.$$

$$\left. - \frac{1}{4}\frac{\cos n\pi/2}{n\pi} - \frac{\sin n\pi/2}{n^2\pi^2}\right]$$

$$= \frac{1}{2n\pi} - 4\frac{\sin n\pi/2}{n^2\pi^2} - \frac{(-1)^n}{2n\pi}$$

$$b_n = \frac{1}{2n\pi}\left[1-(-1)^n\right] - 4\frac{\sin n\pi/2}{n^2\pi^2}$$

$\therefore$ from (1),

$$f(x) = \sum_{n=1}^{\infty}\left\{\frac{1}{2n\pi}\left[1-(-1)^n\right] - \frac{4\sin n\pi/2}{n^2\pi^2}\right\}\sin n\pi x$$

Ans.

Que(4) Obtain the half range cosine series for $f(x) = \begin{cases} Kx & , 0 \le x \le l/2 \\ K(l-x) & , l/2 \le x \le l. \end{cases}$

Sol^n The fourier half range cosine series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{b-a}\right)$$

Here, $b-a = l$

$$\therefore \quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) \quad \text{——(1)}$$

$$a_0 = \frac{2}{b-a}\int_a^b f(x)\,dx = \frac{2}{l}\left[\int_0^{l/2} Kx\,dx + \int_{l/2}^{l}(Kl - Kx)\,dx\right]$$

$$= \frac{2}{l}\left[K\left|\frac{x^2}{2}\right|_0^{l/2} + \left|Klx - \frac{Kx^2}{2}\right|_{l/2}^{l}\right]$$

$$= \frac{2}{l}\left[\frac{K}{2}\cdot\frac{l^2}{4} + Kl\left(l - \frac{l}{2}\right) - \frac{K}{2}\left(l^2 - \frac{l^2}{4}\right)\right]$$

$$= \frac{2}{l}\left[\frac{Kl^2}{8} + \frac{Kl^2}{2} - \frac{3Kl^2}{8}\right] = \frac{2}{l}\left(\frac{Kl^2}{4}\right) = \frac{Kl}{2} = a_0$$

$$a_n = \frac{2}{b-a}\int_a^b f(x)\cos\left(\frac{n\pi x}{b-a}\right)dx$$

$$= \frac{2}{l}\left[\int_0^{l/2} Kx\cdot\cos\left(\frac{n\pi x}{l}\right)dx + \int_{l/2}^{l} K(l-x)\cos\left(\frac{n\pi x}{l}\right)dx\right]$$

$$= \frac{2}{l}\left[\left| Kx\,\frac{\sin n\pi x/l}{n\pi/l} - K\left(-\frac{\cos n\pi x/l}{n^2\pi^2/l^2}\right)\right|_0^{l/2} + \right.$$

$$\left.\left| K(l-x)\frac{\sin n\pi x/l}{n\pi/l} - K(-1)\left(-\frac{\cos n\pi x/l}{n^2\pi^2/l^2}\right)\right|_{l/2}^{l}\right]$$

$$= \frac{2}{l}\left[\left|\frac{lKx}{n\pi}\cdot \sin\left(\frac{n\pi x}{l}\right) + \frac{l^2K}{n^2\pi^2}\cos\left(\frac{n\pi x}{l}\right)\right|_0^{l/2} + \right.$$

$$\left.\left|\frac{lK(l-x)}{n\pi}\sin\left(\frac{n\pi x}{l}\right) - \frac{l^2K}{n^2\pi^2}\cos\left(\frac{n\pi x}{l}\right)\right|_{l/2}^{l}\right]$$

$$= \frac{2}{l}\left[\frac{lK}{n\pi}\cdot\frac{l}{2}\cdot\sin\left(\frac{n\pi}{l}\cdot\frac{l}{2}\right) - 0 + \frac{l^2K}{n^2\pi^2}\cos\left(\frac{n\pi}{l}\frac{l}{2}\right) - \frac{l^2K}{n^2\pi^2}\cdot\cos 0\right.$$

$$\left. + 0 - \frac{lK}{n\pi}\cdot\frac{l}{2}\sin\frac{n\pi}{l}\frac{l}{2} - \frac{l^2K}{n^2\pi^2}\cos n\pi + \frac{l^2K}{n^2\pi^2}\cos\left(\frac{n\pi}{l}\frac{l}{2}\right)\right]$$

$$= \frac{2}{l}\left[\frac{l^2K}{2n\pi}\sin\frac{n\pi}{2} + \frac{l^2K}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{l^2K}{n^2\pi^2} - \frac{l^2K}{2n\pi}\sin\frac{n\pi}{2}\right.$$

$$\left. - \frac{l^2K}{n^2\pi^2}(-1)^n + \frac{l^2K}{n^2\pi^2}\cos\frac{n\pi}{2}\right]$$

$$= \frac{2}{l}\left[\frac{l^2K}{n^2\pi^2}\left\{2\cos\frac{n\pi}{2} - 1 - (-1)^n\right\}\right]$$

$$a_n = \frac{2lK}{n^2\pi^2}\left\{2\cos\frac{n\pi}{2} - 1 - (-1)^n\right\}$$

from eq. (1),

$$f(x) = \frac{Kl}{4} + \frac{2lK}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[2\cos\left(\frac{n\pi}{2}\right) - 1 - (-1)^n\right]\cdot\cos\left(\frac{n\pi x}{l}\right)$$

which is the reqd. solution.

EULER'S FORMULAE for FOURIER SERIES.

The fourier series for the function $f(x)$ in the interval $[a, b]$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2n\pi x}{b-a}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2n\pi x}{b-a}\right)$$

where,

$$\left.\begin{aligned} a_0 &= \frac{2}{b-a}\int_a^b f(x)\,dx \\ a_n &= \frac{2}{b-a}\int_a^b f(x)\cdot\cos\left(\frac{2n\pi x}{b-a}\right)dx \\ b_n &= \frac{2}{b-a}\int_a^b f(x)\cdot\sin\left(\frac{2n\pi x}{b-a}\right)dx \end{aligned}\right\} \text{Euler's formulae}$$

For $b-a = 2\pi$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

where,

$$\left.\begin{aligned} a_0 &= \frac{1}{\pi}\int_a^b f(x)\,dx \\ a_n &= \frac{1}{\pi}\int_a^b f(x)\cos nx\,dx \\ b_n &= \frac{1}{\pi}\int_a^b f(x)\sin nx\,dx. \end{aligned}\right\} \text{Euler's formulae}$$

DIRICHLET'S CONDITIONS

(Conditions of a fourier expansion)

Any function $f(x)$ can be formed as a fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where, a_0, a_n and b_n are constants.

provided $f(x)$:

(i) is periodic, single-valued and finite.

(ii) has at the most a finite no. of maxima and minima.

(iii) has a finite no. of dis-continuities in any one period.

PRACTICAL HARMONIC ANALYSIS

Que (1) The following table gives the variations of periodic current over a period :-

t (sec) :	0	T/6	T/3	T/2	2T/3	5T/6	T
A (amp) :	1.98	1.30	1.05	1.30	−0.88	−0.25	1.98

Show that there is a direct current part of 0.75 A in the variable current and obtain the amplitude of the first harmonic.

Sol.ⁿ ∵ The fourier series of function is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2n\pi x}{b-a}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2n\pi x}{b-a}\right)$$

Here, $b-a = T-0 = T$.

$$A(t) = \frac{a_0}{2} + a_1 \cos\frac{2\pi t}{T} + a_2 \cos\frac{4\pi t}{T} + \dots$$
$$+ b_1 \sin\frac{2\pi t}{T} + b_2 \sin\frac{4\pi t}{T} + \dots$$

$$A = \frac{a_0}{2} + a_1 \cos\frac{2\pi t}{T} + b_1 \sin\frac{2\pi t}{T} + \dots$$

where, $a_0 = 2 \times \frac{\Sigma A}{N}$, $a_1 = 2 \times \frac{\Sigma A \cos\frac{2\pi t}{T}}{N}$, $b_1 = 2 \times \frac{\Sigma A \sin 2\pi t/T}{N}$.

Here, $N = 6$.

$$\boxed{\begin{aligned} a_0 &= 2 \times [\text{Mean value of } f(x) \text{ in } (0, 2\pi)] \\ a_1 &= 2 \times [\text{Mean value of } f(x)\cos x \text{ in } (0, 2\pi)] \\ b_1 &= 2 \times [\text{Mean value of } f(x)\sin x \text{ in } (0, 2\pi)] \end{aligned}}$$

t	$2\pi t/T$	$\cos\frac{2\pi t}{T}$	$\sin\frac{2\pi t}{T}$	A	$A\cos\frac{2\pi t}{T}$	$A\cdot\sin\frac{2\pi t}{T}$
0	0	1	0	1.98	1.98	0
$T/6$	$\pi/3$	0.5	0.866	1.30	0.65	1.126
$T/3$	$2\pi/3$	−0.5	0.866	1.05	−0.525	0.909
$T/2$	π	−1	0	1.30	−1.3	0
$2T/3$	$4\pi/3$	−0.5	−0.866	−0.88	0.44	0.762
$5T/6$	$5\pi/3$	0.5	−0.866	−0.25	−0.125	0.217
			$\Sigma =$	4.5	1.12	3.014

$$a_0 = 2 \times \frac{\Sigma A}{N} = 2 \times \frac{4.5}{6} = 1.5$$

$$a_1 = 2 \times \frac{\Sigma A \cdot \cos 2\pi t/T}{N} = 2 \times \frac{1.12}{6} = 0.373.$$

$$b_1 = 2 \times \frac{\Sigma A \cdot \sin 2\pi t/T}{N} = 2 \times \frac{3.014}{6} = 1.005$$

Hence, the direct current part in the variable current $= \frac{a_0}{2} = \frac{1.5}{2} = 0.75\ A.$

Also, the amplitude of first harmonic is

$$= \sqrt{a_1^2 + b_1^2} = \sqrt{(0.373)^2 + (1.005)^2}$$

$$= 1.072$$

Ans.

Que② Obtain the first three co-efficients in the fourier cosine series for y, where y is given in the following table:

x:	0	1	2	3	4	5
y:	4	8	15	7	6	2.

2π

$N=6$

Sol.ⁿ Let the interval size be 60°.

∴ θ:	0°	60°	120°	180°	240°	300°
$y(\theta)$:	4	8	15	7	6	2.

The fourier cosine series be

$$f(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2n\pi\theta}{b-a}\right)$$

Here, $b-a = 2\pi$

$$\therefore\ y(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\theta$$

$$= \frac{a_0}{2} + a_1\cos\theta + a_2\cos 2\theta + a_3\cos 3\theta + \cdots$$

θ	$\cos\theta$	$\cos 2\theta$	$\cos 3\theta$	y	$y\cos\theta$	$y\cos 2\theta$	$y\cos 3\theta$
$0°$	1	1	1	4	4	4	4
$60°$	$1/2$	$-1/2$	-1	8	4	-4	-8
$120°$	$-1/2$	$-1/2$	1	15	-7.5	-7.5	15
$180°$	-1	1	-1	7	-7	7	-7
$240°$	$-1/2$	$-1/2$	1	6	-3	-3	6
$300°$	$1/2$	$-1/2$	-1	2	1	-1	-2
			$\Sigma =$	42	-8.5	-4.5	8

$$a_0 = 2 \times \frac{\Sigma y}{N} = 2 \times \frac{42}{6} = 14$$

$$\left.\begin{aligned} a_1 &= 2 \times \frac{\Sigma y \cos\theta}{N} = 2 \times \frac{(-8.5)}{6} = -2.8 \\ a_2 &= 2 \times \frac{\Sigma y \cos 2\theta}{N} = 2 \times \frac{(-4.5)}{6} = -1.5 \\ \text{Also, } a_3 &= 2 \times \frac{\Sigma y \cos 3\theta}{N} = 2 \times \frac{8}{6} = 2.7 \end{aligned}\right\} \text{Ans}$$

Que (3) The turning moment T is given for a series of values of the crank angle:

θ :	0°	30°	60°	90°	120°	150°	180°
T :	0	5224	8097	7850	5499	2626	0

Obtain the first four terms in a series of sine to represent T and calculate T for $\theta = 75°$.

Sol.ⁿ The fourier sine series is given by

$$f(\theta) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi\theta}{b-a}$$

Here, $b-a = \pi$

$$\therefore \; T(\theta) = \sum_{n=1}^{\infty} b_n \sin n\theta$$

$$= b_1 \sin\theta + b_2 \sin 2\theta + b_3 \sin 3\theta + b_4 \sin 4\theta + \cdots$$

θ	$\sin\theta$	$\sin 2\theta$	$\sin 3\theta$	$\sin 4\theta$	T	$T\sin\theta$	$T\sin 2\theta$	$T\sin 3\theta$	$T\sin 4\theta$
0°	0	0	0	0	0	0	0	0	0
30°	0.5	0.866	1	0.866	5224	2612	4524	5224	4524
60°	0.866	0.866	0	-0.866	8097	7012	7012	0	-7012
90°	1	0	-1	0	7850	7850	0	-7850	0
120°	0.866	-0.866	0	0.866	5499	4762	-4762	0	4762
150°	0.5	-0.866	1	-0.866	2626	1313	-2274	2626	-2274
					$\sum =$	23549	4500	0	0

$$b_1 = 2 \times \frac{\sum T \sin\theta}{N} = 2 \times \frac{23549}{6} = 7850$$

$$b_2 = 2 \times \frac{\sum T \sin 2\theta}{N} = 2 \times \frac{4500}{6} = 1500$$

$$b_3 = 2 \times \frac{\sum T \sin 3\theta}{N} = 2 \times \frac{0}{6} = 0$$

$$b_4 = 2 \times \frac{\sum T \sin 4\theta}{N} = 2 \times \frac{0}{6} = 0$$

Ans

$$\therefore \quad T(\theta) = 7850 \sin\theta + 1500 \sin 2\theta$$

Now, at $\theta = 75^\circ$,

$$T(75^\circ) = 7850 \sin 75^\circ + 1500 \sin 150^\circ$$

$$= 8332$$ Ans

Que(4) Obtain the constant term and the coefficients of first cosine and sine terms in the fourier expansion of y as given in the following table:

x:	0	1	2	3	4	5
y:	9	18	24	28	26	20

Sol.ⁿ Taking interval size = 60°

θ:	0	60	120	180	240	300
$y(\theta)$:	9	18	24	28	26	20

The fourier series of function is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2n\pi x}{b-a}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2n\pi x}{b-a}\right)$$

Here, $b-a = 2\pi$

$$y(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2n\pi\theta}{2\pi}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2n\pi\theta}{2\pi}\right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\theta + \sum_{n=1}^{\infty} b_n \sin n\theta.$$

$$y(\theta) = \frac{a_0}{2} + a_1 \cos\theta + b_1 \sin\theta + \cdots$$

θ	$\cos\theta$	$\sin\theta$	y	$y\cos\theta$	$y\sin\theta$
0°	1	0	9	9	0
60°	$1/2$	0.866	18	9	15.588
120°	$-1/2$	0.866	24	−12	20.784
180°	−1	0	28	−28	0
240°	$-1/2$	−0.866	26	−13	−22.516
300°	$1/2$	−0.866	20	10	−17.32
		$\Sigma =$	125	−25	−3.464.

$$\left.\begin{aligned} a_0 &= 2\times\frac{\Sigma y}{N} = 2\times\frac{125}{6} = 41.67 \\ a_1 &= 2\times\frac{\Sigma y\cos\theta}{N} = 2\times\left(-\frac{25}{6}\right) = -8.34 \\ b_1 &= 2\times\frac{\Sigma y\sin\theta}{N} = 2\times\left(-\frac{3.464}{6}\right) = -1.155 \end{aligned}\right\} \text{Ans.}$$

$a_0 \rightarrow$ Constant term.

$a_1 \rightarrow$ Co-efficient of first cosine term.

$b_1 \rightarrow$ Co-efficient of first sine term.

PARSEVAL'S IDENTITY/FORMULA for FOURIER SERIES:

The fourier series of $f(x)$ in interval $[a,b]$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2n\pi x}{b-a}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2n\pi x}{b-a}\right) \quad \text{--- (1)}$$

where,

$$\left.\begin{aligned} a_0 &= \frac{2}{b-a}\int_a^b f(x)\,dx \\ a_n &= \frac{2}{b-a}\int_a^b f(x)\cdot\cos\left(\frac{2n\pi x}{b-a}\right)dx \\ b_n &= \frac{2}{b-a}\int_a^b f(x)\sin\left(\frac{2n\pi x}{b-a}\right)dx. \end{aligned}\right\} \text{Euler's formulae}$$

According to Parseval's identity/formula,

$$\int_a^b [f(x)]^2\,dx = \left(\frac{b-a}{2}\right)\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 + \sum_{n=1}^{\infty} b_n^2\right]$$

provided that the fourier series for $f(x)$ converges uniformly in the interval $[a,b]$.

Multiplying eq.^{n} (1) by $f(x)$ both sides and integrating from $x=a$ to $x=b$ w.r.t. x,

$$\int_a^b [f(x)]^2 dx = \frac{a_0}{2}\int_a^b f(x)\,dx + \sum_{n=1}^{\infty} a_n \int_a^b f(x)\cos\left(\frac{2n\pi x}{b-a}\right)dx + \sum_{n=1}^{\infty} b_n \int_a^b f(x)\cdot \sin\left(\frac{2n\pi x}{b-a}\right)dx$$

$$= \frac{a_0}{2}\cdot a_0\left(\frac{b-a}{2}\right) + \sum_{n=1}^{\infty} a_n\cdot a_n\cdot\left(\frac{b-a}{2}\right) + \sum_{n=1}^{\infty} b_n\cdot b_n\left(\frac{b-a}{2}\right)$$

$$\int_a^b [f(x)]^2 dx = \left(\frac{b-a}{2}\right)\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 + \sum_{n=1}^{\infty} b_n^2\right]$$

Hence proved.

If the interval is $[-l, l]$ i.e., $b-a=2l$.

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

then, $$\int_{-l}^{l} [f(x)]^2 dx = l\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 + \sum_{n=1}^{\infty} b_n^2\right]$$

If the interval is $[0, 2l]$ i.e., $b-a=2l$.

then, $$\int_0^{2l} [f(x)]^2 dx = l\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 + \sum_{n=1}^{\infty} b_n^2\right].$$

If the interval in the half range cosine series is $(0, l)$ i.e, $b-a = l$.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

then, $$\int_0^l [f(x)]^2 dx = \frac{l}{2}\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2\right]$$

If the interval in the half range sine series in $(0, l)$ i.e, $b-a = l$.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

then, $$\int_0^l [f(x)]^2 dx = \frac{l}{2}\left[\sum_{n=1}^{\infty} b_n^2\right]$$

Que ① Obtain the fourier series for $y = x^2$ in $-\pi < x < \pi$. Also, deduce that

$$\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots = \frac{\pi^4}{90}.$$

Que ② Prove that in $0 < x < l$,

$$x = \frac{l}{2} - \frac{4l}{\pi^2}\left(\frac{1}{1^2}\cos\frac{\pi x}{l} + \frac{1}{3^2}\cos\frac{3\pi x}{l} + \frac{1}{5^2}\cos\frac{5\pi x}{l} + \cdots\right)$$

Also, deduce that

$$\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots = \frac{\pi^4}{96}.$$

Que① Obtain the fourier series for $y = x^2$ in $-\pi < x < \pi$.
Also, show that $\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots = \frac{\pi^4}{90}$.

Sol.^{n}

We have,

$$a_0 = \frac{2\pi^2}{3}, \quad a_n = \frac{4(-1)^n}{n^2}, \quad b_n = 0$$

Applying parseval's identity for $f(x)$ in $(-\pi, \pi)$,

$$\int_{-\pi}^{\pi} [f(x)]^2 dx = \pi \left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 + \sum_{n=1}^{\infty} b_n^2 \right]$$

$$\int_{-\pi}^{\pi} (x^2)^2 dx = \pi \left[\frac{1}{2}\left(\frac{2\pi^2}{3}\right)^2 + \sum_{n=1}^{\infty} \left\{ \frac{4(-1)^n}{n^2} \right\}^2 + 0 \right]$$

$$\left| \frac{x^5}{5} \right|_{-\pi}^{\pi} = \frac{1}{5}[\pi^5 - (-\pi)^5] = \frac{2}{5}\pi^5 = \pi \left[\frac{2}{9}\pi^4 + \sum_{n=1}^{\infty} \frac{16(-1)^{2n}}{n^4} \right]$$

$$\Rightarrow \frac{2}{5}\pi^4 = \left[\frac{2}{9}\pi^4 + \sum_{n=1}^{\infty} \frac{16}{n^4}\{(-1)^2\}^n \right] = \frac{2}{9}\pi^4 + 16 \sum_{n=1}^{\infty} \frac{1}{n^4}$$

$$\Rightarrow \left(\frac{2}{5} - \frac{2}{9} \right) \pi^4 = 16 \sum_{n=1}^{\infty} \frac{1}{n^4}$$

$$\Rightarrow \left(\frac{18 - 10}{45} \right) \pi^4 = \frac{8}{45}\pi^4 = 16 \sum_{n=1}^{\infty} \frac{1}{n^4}$$

$$\Rightarrow \frac{\pi^4}{90} = \sum_{n=1}^{\infty} \frac{1}{n^4}$$

$$\Rightarrow \boxed{\frac{\pi^4}{90} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots}$$

Hence proved.

Que ② Prove that in $0 < x < l$,

$$x = \frac{l}{2} - \frac{4l}{\pi^2}\left(\frac{1}{1^2}\cos\frac{\pi x}{l} + \frac{1}{3^2}\cos\frac{3\pi x}{l} + \frac{1}{5^2}\cos\frac{5\pi x}{l} + \cdots\right)$$

Also, deduce that

$$\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots = \frac{\pi^4}{96}.$$

Sol.ⁿ Here, $f(x) = x$ and $b - a = l$.

The fourier half range cosine series for $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{b-a}$$

$$x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} \quad \text{——— (1)}$$

$$a_0 = \frac{2}{b-a}\int_a^b f(x)\,dx = \frac{2}{l}\int_0^l x\,dx = \frac{2}{l}\left|\frac{x^2}{2}\right|_0^l$$

$$= \frac{l^2}{l} = l.$$

Also, $a_n = \frac{2}{b-a}\int_a^b f(x)\cdot\cos\left(\frac{n\pi x}{b-a}\right)dx = \frac{2}{l}\int_0^l x\cdot\cos\frac{n\pi x}{l}\,dx$

$$= \frac{2}{l}\left|\, x\cdot\frac{\sin n\pi x/l}{n\pi/l} - (1)\left(-\frac{\cos n\pi x/l}{n^2\pi^2/l^2}\right)\right|_0^l$$

$$= \frac{2}{l}\left|\frac{xl}{n\pi}\cdot\sin\frac{n\pi x}{l} + \frac{l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}\right|_0^l$$

$$= \frac{2}{l}\left[0 - 0 + \frac{l^2}{n^2\pi^2}\left(\cos n\pi - \cos 0\right)\right] = \frac{2}{l}\cdot\frac{l^2}{n^2\pi^2}\left[(-1)^n - 1\right]$$

$$a_n = \frac{2l}{n^2\pi^2}\left[(-1)^n - 1\right]$$

Now, $a_n = 0$, when n is even

$a_n = -\frac{4l}{n^2\pi^2}$, when n is odd.

from (i),

$$x = \frac{l}{2} + \sum_{n=1}^{\infty}\left(-\frac{4l}{n^2\pi^2}\right)\cos\frac{n\pi x}{l} = \frac{l}{2} - \frac{4l}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cdot\cos\frac{n\pi x}{l}$$

$$\boxed{x = \frac{l}{2} - \frac{4l}{\pi^2}\left(\frac{1}{1^2}\cos\frac{\pi x}{l} + \frac{1}{3^2}\cos\frac{3\pi x}{l} + \frac{1}{5^2}\cos\frac{5\pi x}{l} + \cdots\right)}$$

Proved

Now, applying Parseval's identity, in the interval $(0, l)$ for half range cosine series,

$$\int_0^l [f(x)]^2 dx = \frac{l}{2}\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2\right]$$

$$\int_0^l x^2 dx = \left|\frac{x^3}{3}\right|_0^l = \frac{l^3}{3} = \frac{l}{2}\left[\frac{l^2}{2} + \sum_{n=1}^{\infty}\left(-\frac{4l}{n^2\pi^2}\right)^2\right]$$

$$\frac{2l^2}{3} = \frac{l^2}{2} + \sum_{n=1}^{\infty}\frac{16l^2}{n^4\pi^4} \Rightarrow \frac{2l^2}{3} - \frac{l^2}{2} = \sum_{n=1}^{\infty}\frac{16l^2}{n^4\pi^4}$$

$$\Rightarrow \frac{4l^2 - 3l^2}{6} = \frac{16\cdot l^2}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4} \Rightarrow \frac{1}{6} = \frac{16}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\Rightarrow \frac{\pi^4}{96} = \sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\Rightarrow \boxed{\frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots}$$

Hence Proved.

COMPLEX FORM of FOURIER SERIES

The fourier series of periodic function $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2n\pi x}{b-a}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2n\pi x}{b-a}\right)$$

where, $a_0 = \frac{2}{b-a}\int_a^b f(x) \cdot dx$

$$a_n = \frac{2}{b-a}\int_a^b f(x) \cdot \cos\left(\frac{2n\pi x}{b-a}\right) dx$$

$$b_n = \frac{2}{b-a}\int_a^b f(x) \cdot \sin\left(\frac{2n\pi x}{b-a}\right) dx$$

Let the interval be $[-l, l]$, so $b-a = 2l$.

then, $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$.

where, $a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\, dx$; $a_n = \frac{1}{l}\int_{-l}^{l} f(x) \cdot \cos\frac{n\pi x}{l} dx$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x) \cdot \sin\frac{n\pi x}{l} dx.$$

$\because$ We know, $e^{i\theta} = \cos\theta + i\sin\theta$ and $e^{-i\theta} = \cos\theta - i\sin\theta$

$\Rightarrow \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$ and $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$

$\Rightarrow \cos\frac{n\pi x}{l} = \frac{e^{in\pi x/l} + e^{-in\pi x/l}}{2}$ and $\sin\frac{n\pi x}{l} = \frac{e^{in\pi x/l} - e^{-in\pi x/l}}{2i}$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\left(\frac{e^{in\pi x/l} + e^{-in\pi x/l}}{2}\right) + \sum_{n=1}^{\infty} b_n\left(\frac{e^{in\pi x/l} - e^{-in\pi x/l}}{2i}\right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[\frac{a_n \cdot e^{in\pi x/l}}{2} + \frac{a_n e^{-in\pi x/l}}{2} + \frac{b_n e^{in\pi x/l}}{2i} - \frac{b_n e^{-in\pi x/l}}{2i}\right]$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[\frac{a_n \cdot e^{in\pi x/l}}{2} + \frac{a_n e^{-in\pi x/l}}{2} - \frac{i^2 \cdot b_n e^{in\pi x/l}}{2i} + \frac{i^2 b_n e^{-in\pi x/l}}{2i}\right]$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[\frac{a_n \cdot e^{in\pi x/l}}{2} + \frac{a_n e^{-in\pi x/l}}{2} - \frac{i \cdot b_n e^{in\pi x/l}}{2} + \frac{i b_n e^{-in\pi x/l}}{2}\right]$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[\left(\frac{a_n - ib_n}{2}\right) \cdot e^{in\pi x/l} + \left(\frac{a_n + ib_n}{2}\right) e^{-in\pi x/l}\right]$$

$$f(x) = C_0 + \sum_{n=1}^{\infty}\left[C_n \cdot e^{in\pi x/l} + C_{-n}\, e^{-in\pi x/l}\right]$$

where, $C_0 = \frac{a_0}{2}$, $C_n = \frac{a_n - ib_n}{2}$, $C_{-n} = \frac{a_n + ib_n}{2}$

$$C_n = \frac{1}{2}\left[a_n - ib_n\right] = \frac{1}{2}\left[\frac{1}{l}\int_{-l}^{l} f(x) \cos\frac{n\pi x}{l}\, dx - i \cdot \frac{1}{l}\int_{-l}^{l} f(x) \cdot \sin\frac{n\pi x}{l}\, dx\right]$$

$$= \frac{1}{2l}\left[\int_{-l}^{l} f(x)\left(\cos\frac{n\pi x}{l} - i \sin\frac{n\pi x}{l}\right) dx\right]$$

$$\Rightarrow C_n = \frac{1}{2l}\int_{-l}^{l} f(x) \cdot e^{-in\pi x/l}\, dx.$$

$$C_{-n} = \frac{1}{2}\left[a_n + ib_n\right] = \frac{1}{2}\left[\frac{1}{l}\int_{-l}^{l} f(x) \cos\frac{n\pi x}{l}\, dx + i \cdot \frac{1}{l}\int_{-l}^{l} f(x) \cdot \sin\frac{n\pi x}{l}\, dx\right]$$

$$= \frac{1}{2l}\left[\int_{-l}^{l} f(x) \cdot \left(\cos\frac{n\pi x}{l} + i \sin\frac{n\pi x}{l}\right) dx.\right.$$

$$\Rightarrow C_{-n} = \frac{1}{2l}\int_{-l}^{l} f(x) \cdot e^{in\pi x/l}\, dx$$

and $f(x)$ can be compactly written as

$$f(x) = \sum_{n=-\infty}^{\infty} C_n \cdot e^{in\pi x/l}$$

l = half length of the interval size.

Que (1) Find the complex form of fourier series of $f(x) = e^{-x}$ in $-1 \le x \le 1$.

Sol.^{n} ∵ The fourier series in complex form is given by

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \cdot e^{in\pi x/l}$$

$l \to$ half length of interval size.

Here, $b-a = 2 = 2l \Rightarrow l = 1$.

$$\therefore \boxed{f(x) = e^{-x} = \sum_{n=-\infty}^{\infty} c_n \cdot e^{in\pi x}} \quad \text{--- (1)}$$

Now, $c_n = \dfrac{1}{2l} \displaystyle\int_{-l}^{l} f(x) \cdot e^{-in\pi x/l} \, dx$

$$= \frac{1}{2} \int_{-1}^{1} e^{-x} \cdot e^{-in\pi x} \, dx.$$

$$= \frac{1}{2} \int_{-1}^{1} e^{-(1+in\pi)x} \, dx$$

$$= \frac{1}{2} \left| \frac{e^{-(1+in\pi)x}}{-(1+in\pi)} \right|_{-1}^{1}$$

$$= \frac{-1}{2(1+in\pi)} \left[e^{-(1+in\pi)} - e^{(1+in\pi)} \right]$$

$$= \frac{e^{(1+in\pi)} - e^{-(1+in\pi)}}{2(1+in\pi)}$$

$$= \frac{e^{1} \cdot e^{in\pi} - e^{-1} \cdot e^{-in\pi}}{2(1+in\pi)}$$

$$\left\{ \begin{aligned} e^{ix} &= \cos x + i \sin x \\ e^{-ix} &= \cos x - i \sin x \end{aligned} \right.$$

$$C_n = \frac{e\left(\cos n\pi + i\sin n\pi\right) - e^{-1}\left(\cos n\pi - i\sin n\pi\right)}{2(1+in\pi)}$$

$$= \frac{e(-1)^n - e^{-1}(-1)^n}{2(1+in\pi)}$$

$$= \left(\frac{e - e^{-1}}{2}\right)(-1)^n \left(\frac{1}{1+in\pi} \times \frac{1-in\pi}{1-in\pi}\right)$$

$$\left\{\begin{array}{l} \sinh x = \dfrac{e^{x}-e^{-x}}{2} \\ \cosh x = \dfrac{e^{x}+e^{-x}}{2} \end{array}\right.$$

$$= \frac{(-1)^n(\sinh 1)(1-in\pi)}{1-i^2n^2\pi^2}$$

$$\boxed{C_n = \frac{(-1)^n(\sinh 1)(1-in\pi)}{1+n^2\pi^2}}$$

from ①, $\boxed{e^{-x} = \sum_{n=-\infty}^{\infty} (-1)^n(\sinh 1)\left(\frac{1-in\pi}{1+n^2\pi^2}\right) \cdot e^{in\pi x}}$

which is the reqd.
complex form of fourier Series

Ans.

THANK YOU SO MUCH