Welcome

To

MKS TUTORIALS

By

Manoj Sir

Topics covered under playlist of Partial Differential Equation: Formation of Partial Differential Equation, Solution of Partial Differential Equation by Direct Integration Method, Lagrange's Linear Equation of First Order, Homogeneous Linear Equation with Constant Coefficients, Non-Homogeneous Linear Equations, Method of Separation of Variables.

Complete playlist of Partial Differential Equations (in hindi) -
https://www.youtube.com/playlist?list=PLhSp9OSVmeyJoNnAqghUK-Lit3qBgfa6o

Complete playlist of Partial Differential Equations (in english) -
https://www.youtube.com/playlist?list=PL0d0PH4hlFOe-SWbDg41sfPXsnF7qezFu

Please Subscribe to our Hindi YouTube Channel – MKS TUTORIALS:
https://www.youtube.com/channel/UCbDs7CHAWVtyu81-6WIqZXg

Please Subscribe to our English YouTube Channel – Manoj Sir TUTES:
https://www.youtube.com/channel/UCj_4NZ5kRbWGbCn0VtSEmlQ

PARTIAL DIFFERENTIAL EQUATIONS

Basic Points

1. A differential eq.ⁿ containing one or more partial derivatives is called a PDE.

2. Partial differentiation occurs only when there are atleast two independents variables.

$$z = f(x, y)$$

Eg. $z = x^2 y$ $\quad \frac{\partial z}{\partial x} = 2xy \quad \frac{\partial z}{\partial y} = x^2$

3. The order of a PDE is the order of the highest order partial derivative present in the equation.

Order 1: $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0$, $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right) = 0$

Order 2: $\frac{\partial^2 z}{\partial t^2} = K^2\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right)$

4. The degree of a PDE is the power of the highest order partial derivative present in the eq.ⁿ.

5. Partial derivatives $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x \partial y}, \frac{\partial^2 z}{\partial y^2}$ are denoted by p, q, r, s, t respectively.

FORMATION of PDE

The PDE can be formed using the following methods:

(I) By elimination of arbitrary constants.

(II) By elimination of arbitrary functions.

Problems:

1. $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

2. $z = a(x+y) + b(x-y) + abt + c$

3. $(x-a)^2 + (y-b)^2 + z^2 = c^2$.

4. $z = f(y/x)$

5. $z = f(x+it) + g(x-it)$

6. $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$

7. $z = x^2 f(y) + y^2 g(x)$

Que ① Form PDE by eliminating the arbitrary constants: $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ —— ①

Sol.ⁿ Differentiating partially ① w.r.t x,

$$\cancel{2}\frac{\partial z}{\partial x} = \frac{\cancel{2}x}{a^2} \Rightarrow \frac{\partial z}{\partial x} = \frac{x}{a^2} \Rightarrow \frac{1}{a^2} = \frac{1}{x}\frac{\partial z}{\partial x} = \frac{p}{x}$$

Again differentiating partially ① w.r.t y,

$$\cancel{2}\frac{\partial z}{\partial y} = \frac{\cancel{2}y}{b^2} \Rightarrow \frac{\partial z}{\partial y} = \frac{y}{b^2} \Rightarrow \frac{1}{b^2} = \frac{1}{y}\frac{\partial z}{\partial y} = \frac{q}{y}$$

eq.ⁿ ① becomes,

$$2z = x^{\cancel{2}}\left(\frac{p}{\cancel{x}}\right) + y^{\cancel{2}}\left(\frac{q}{\cancel{y}}\right)$$

$$2z = xp + qy$$

which is the reqd. PDE.

Que ② Form PDE by eliminating the arbitrary constants :

(i) $z = a(x+y) + b(x-y) + abt + c$

Solⁿ $z = ax + ay + bx - by + abt + c$ —— ①

Differentiating partially ① w.r.t x, y and t,

$\frac{\partial z}{\partial x} = a+b = p,$

$\frac{\partial z}{\partial y} = a-b = q \quad , \quad \frac{\partial z}{\partial t} = ab$

∵ we know,

$(a+b)^2 - (a-b)^2 = 4ab$

$\Rightarrow \quad p^2 - q^2 = 4\frac{\partial z}{\partial t}$

which is the reqd. PDE.

Ans

(ii) $(x-a)^2+(y-b)^2+z^2=c^2$ ——— (1)

Sol^n Differentiating partially (1) w.r.t x,

$$2(x-a)+2z\frac{\partial z}{\partial x}=0$$

$$\Rightarrow 2(x-a)=-2zp \Rightarrow x-a=-zp$$

Differentiating partially (1) w.r.t. y,

$$2(y-b)+2z\frac{\partial z}{\partial y}=0$$

$$\Rightarrow 2(y-b)=-2zq \Rightarrow y-b=-zq.$$

eq.^{n} (1) becomes,

$$(-zp)^2+(-zq)^2+z^2=c^2$$

$$\Rightarrow z^2p^2+z^2q^2+z^2=c^2$$

$$\Rightarrow z^2(p^2+q^2+1)=c^2.$$

which is the reqd. PDE.

Ans

Que (3) Form PDE by eliminating the arbitrary functions: (i) $z = f(y/x)$ — (1)

Sol.^{n} Differentiating partially (1) w.r.t x and y,

$$\frac{\partial z}{\partial x} = f'(y/x) \cdot y \cdot \left(-\frac{1}{x^2}\right)$$

$$\Rightarrow p = f'(y/x)\left(-\frac{y}{x^2}\right) \quad \text{— (ii)}$$

Also, $$\frac{\partial z}{\partial y} = f'(y/x)\left(\frac{1}{x}\right)(1)$$

$$\Rightarrow q = f'(y/x)\left(\frac{1}{x}\right) \quad \text{— (iii)}$$

(ii) ÷ (iii) gives,

$$\frac{p}{q} = \frac{f'(y/x)(-y/x^2)}{f'(y/x)(1/x)}$$

$$\Rightarrow \frac{p}{q} = -\frac{y}{x}$$

$$\Rightarrow xp = -yq$$

$$\Rightarrow xp + yq = 0.$$

which is the reqd. PDE.

(ii) $z = f(x+it) + g(x-it)$ — (i)

Solⁿ Differentiating partially (i) w.r.t x,

$$\frac{\partial z}{\partial x} = f'(x+it) + g'(x-it)$$

Again diff partially w.r.t x,

$$\frac{\partial^2 z}{\partial x^2} = f''(x+it) + g''(x-it) \quad \text{— (ii)}$$

Diff. partially (i) w.r.t t,

$$\frac{\partial z}{\partial t} = i f'(x+it) + (-i) g'(x-it)$$

Again diff partially w.r.t t,

$$\frac{\partial^2 z}{\partial t^2} = i^2 f''(x+it) + (-i)^2 g''(x-it)$$

$$= i^2 \left[f''(x+it) + g''(x-it) \right] \quad \text{— (iii)}$$

Comparing (ii) and (iii),

$$\frac{\partial^2 z}{\partial t^2} = i^2 \frac{\partial^2 z}{\partial x^2}$$

which is the reqd. PDE.

Ans

Que (4) Form PDE by eliminating the arbitrary function f : (i) $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$ —— (1)

Sol.^{n} Differentiating partially (1) w.r.t x,

$$p = \frac{\partial z}{\partial x} = 0 + 2f'\left(\frac{1}{x} + \log y\right)\left(-\frac{1}{x^2}\right)$$

$$\Rightarrow \quad x^2 p = -2f'\left(\frac{1}{x} + \log y\right) \text{ —— (ii)}$$

Again, diff partially (1) w.r.t y,

$$q = \frac{\partial z}{\partial y} = 2y + 2f'\left(\frac{1}{x} + \log y\right)\cdot\frac{1}{y}$$

$$\Rightarrow \quad qy = 2y^2 + 2f'\left(\frac{1}{x} + \log y\right) \text{ —— (iii)}$$

from (ii) and (iii),

$$qy = 2y^2 - x^2 p$$

$$\Rightarrow \quad x^2 p + qy = 2y^2$$

which is the reqd. PDE.

(iii) $z = x^2 f(y) + y^2 g(x)$ — (1)

Solⁿ Differentiating partially (1) w.r.t x,

$$p = \frac{\partial z}{\partial x} = 2x f(y) + y^2 g'(x) \quad \text{— (A)}$$

Multiplying x on both sides,

$$px = 2x^2 f(y) + xy^2 g'(x) \quad \text{— (ii)}$$

Again diff. partially (1) w.r.t. y,

$$q = \frac{\partial z}{\partial y} = x^2 f'(y) + 2y\, g(x)$$

Multiplying y on both sides,

$$qy = x^2 y f'(y) + 2y^2 g(x) \quad \text{— (iii)}$$

Adding (ii) and (iii),

$$px + qy = 2[x^2 f(y) + y^2 g(x)] + xy[y g'(x) + x f'(y)] \quad \text{— (iv)}$$

Diff. partially (A) w.r.t y,

$$s = \frac{\partial^2 z}{\partial y \partial x} = 2x f'(y) + 2y\, g'(x)$$

$$= 2[x f'(y) + y g'(x)] \quad \text{— (v)}$$

From (iv) and (v),

$$px + qy = 2z + xy \cdot \frac{s}{2}$$

$$\Rightarrow 2px + 2qy - xys = 4z$$

which is the reqd. PDE. Ans.

SOLUTION of PDE by Direct Integration Method

Que ① Solve by direct integration method:

$$\frac{\partial^3 z}{\partial x^2 \partial y} + 18xy^2 + \sin(2x-y) = 0$$

Sol.ⁿ Given PDE is $\frac{\partial^3 z}{\partial x^2 \partial y} + 18xy^2 + \sin(2x-y) = 0$

Integrating w.r.t. x, $\frac{\partial^2 z}{\partial x \partial y} + 18 \cdot \frac{x^2}{2} \cdot y^2 - \frac{1}{2}\cos(2x-y) = f(y)$

$$\Rightarrow \frac{\partial^2 z}{\partial x \partial y} + 9x^2y^2 - \frac{1}{2}\cos(2x-y) = f(y)$$

Integrating w.r.t x, $\frac{\partial z}{\partial y} + 9\frac{x^3}{3} \cdot y^2 - \frac{1}{2} \cdot \frac{1}{2}\sin(2x-y)$
$= xf(y) + g(y)$

$$\Rightarrow \frac{\partial z}{\partial y} + 3x^3y^2 - \frac{1}{4}\sin(2x-y) = xf(y) + g(y)$$

Integrating w.r.t y, $z + 3x^3\frac{y^3}{3} + \frac{1}{4}\cos(2x-y)(-1)$
$= x\int f(y)dy + \int g(y)dy + h(x)$

$$\Rightarrow z + x^3y^3 - \frac{1}{4}\cos(2x-y) = x\int f(y)dy + \int g(y)dy + h(x)$$

$$\Rightarrow z = \frac{1}{4}\cos(2x-y) - x^3y^3 + x\,u(y) + v(y) + h(x)$$

where, $\int f(y)dy = u(y)$

and $\int g(y)dy = v(y)$ Ans

which is the reqd. solution.

Que ② Solve by direct integration method:

$$\frac{\partial^3 z}{\partial x^2 \partial y} - \cos(2x+3y) = 0$$

Sol^n Given PDE is $\dfrac{\partial^3 z}{\partial x^2 \partial y} - \cos(2x+3y) = 0$

$$\Rightarrow \frac{\partial^3 z}{\partial x^2 \partial y} = \cos(2x+3y)$$

Integrating w.r.t. x,

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{1}{2}\sin(2x+3y) + f(y)$$

Integrating w.r.t. x,

$$\frac{\partial z}{\partial y} = \frac{1}{2}(-1)\cos(2x+3y)\cdot\frac{1}{2} + xf(y) + g(y)$$

$$= -\frac{1}{4}\cos(2x+3y) + xf(y) + g(y)$$

Integrating w.r.t. y,

$$z = -\frac{1}{4}\sin(2x+3y)\cdot\frac{1}{3} + x\int f(y)dy + \int g(y)dy + h(x)$$

$$\Rightarrow z = -\frac{1}{12}\sin(2x+3y) + xu(y) + v(y) + h(x)$$

which is the reqd. PDE.

where, $\int f(y)\,dy = u(y)$

and $\int g(y)\,dy = v(y)$

Lagrange's Linear PDE

Equation of the type

$$Pp + Qq = R \quad \text{--- (1)}$$

is called Lagrange's Linear PDE.

where P, Q, R are the functions of x, y, z.

The subsidiary / auxiliary eq.^{ns} are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}.$$

The solution of eq.^{n} (1) is given by

$$f(u, v) = 0, \quad u = f(v) \quad \text{or} \quad v = f(u)$$

where u and v are functions of x, y, z.

Solve :

(1) $yzp + zxq = xy$

(2) $x^2(y-z)p + y^2(z-x)q = z^2(x-y)$

(3) $(mz - ny)\frac{\partial z}{\partial x} + (nx - lz)\frac{\partial z}{\partial y} = ly - mx$

(4) $y^2p - xyq = x(z-2y)$

(5) $(z^2 - 2yz - y^2)p + (xy + zx)q = xy - zx$

(6) $(x^2 - yz)p + (y^2 - zx)q = (z^2 - xy)$

Que ① Solve: $yzp + zxq = xy$

Sol.ⁿ Given PDE is

$$yzp + zxq = xy \quad \text{——} ①$$

which is of the form $Pp + Qq = R$

$\therefore$ The eq.ⁿ ① is Lagrange's Linear PDE.

Here, the auxiliary eq.ⁿˢ are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \Rightarrow \frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{xy}$$

Taking first two fractions,

$$\frac{dx}{yz} = \frac{dy}{zx} \Rightarrow x\,dx = y\,dy \quad \text{(By SOV)}$$

Integrating both sides, $\frac{x^2}{2} = \frac{y^2}{2} + c_1 \Rightarrow \frac{x^2}{2} - \frac{y^2}{2} = c_1$

$$\Rightarrow x^2 - y^2 = 2c_1 = c_2$$

Taking last two fractions, $\frac{dy}{zx} = \frac{dz}{xy} \Rightarrow y\,dy = z\,dz$ (By SOV)

On integrating, $\frac{y^2}{2} = \frac{z^2}{2} + c_3 \Rightarrow \frac{y^2}{2} - \frac{z^2}{2} = c_3$

$$\Rightarrow y^2 - z^2 = 2c_3 = c_4.$$

$\therefore$ The reqd. solution is

$$f(x^2 - y^2,\ y^2 - z^2) = 0 \quad \text{Ans}$$

$\hookrightarrow$ Arbitrary function.

Que ② Solve : $x^2(y-z)p + y^2(z-x)q = z^2(x-y)$

Sol.^{n} Given PDE is

$$x^2(y-z)p + y^2(z-x)q = z^2(x-y) \quad \text{—— ①}$$

which is of the form $Pp + Qq = R$

$\therefore$ eq.^{n} ① is Lagrange's Linear PDE.

The AE are $\dfrac{dx}{x^2(y-z)} = \dfrac{dy}{y^2(z-x)} = \dfrac{dz}{z^2(x-y)}$

Using $1/x, 1/y, 1/z$ as multipliers,

$$\frac{dx/x}{x(y-z)} = \frac{dy/y}{y(z-x)} = \frac{dz/z}{z(x-y)} = \frac{dx/x + dy/y + dz/z}{0}$$

$$\Rightarrow \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$$

On integrating, $\log x + \log y + \log z = \log c_1 \Rightarrow xyz = c_1$

Again using $1/x^2, 1/y^2, 1/z^2$ as multipliers,

Each fraction $= \dfrac{dx/x^2 + dy/y^2 + dz/z^2}{0}$

$$\Rightarrow \frac{dx}{x^2} + \frac{dy}{y^2} + \frac{dz}{z^2} = 0$$

On integrating, $-\dfrac{1}{x} - \dfrac{1}{y} - \dfrac{1}{z} = c_2 \Rightarrow \dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = -c_2 = c_3$

$\therefore$ The reqd solution is

$$f\left(xyz, \frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = 0$$

Ans.

Que ③ Solve: $(mz - ny)\frac{\partial z}{\partial x} + (nx - lz)\frac{\partial z}{\partial y} = ly - mx$.

Solⁿ Given Lagrange's Linear PDE is

$$(mz - ny)p + (nx - lz)q = ly - mx.$$

The AE are $\frac{dx}{mz - ny} = \frac{dy}{nx - lz} = \frac{dz}{ly - mx}$

Using x, y, z as multipliers,

Each fraction $= \frac{x\,dx + y\,dy + z\,dz}{0}$

$\Rightarrow x\,dx + y\,dy + z\,dz = 0$

On integrating, $\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c_1 \Rightarrow x^2 + y^2 + z^2 = 2c_1 = c_2$

Again using l, m, n as multipliers,

Each fraction $= \frac{l\,dx + m\,dy + n\,dz}{0}$

$\Rightarrow l\,dx + m\,dy + n\,dz = 0$

On integrating, $lx + my + nz = c_3$

$\therefore$ The reqd solution is

$$f(x^2 + y^2 + z^2,\ lx + my + nz) = 0$$

Ans.

Que (4) Solve: $y^2p - xyq = x(z-2y)$

Sol.n Given Lagrange's linear PDE is

$$y^2p - xyq = x(z-2y)$$

The AE are $\dfrac{dx}{y^2} = \dfrac{dy}{-xy} = \dfrac{dz}{x(z-2y)}$

Taking first two fractions,

$$\frac{dx}{y^2} = \frac{dy}{-xy} \Rightarrow xdx = -ydy \quad \text{(By SOV)}$$

On integrating, $\dfrac{x^2}{2} = -\dfrac{y^2}{2} + c_1 \Rightarrow \dfrac{x^2}{2} + \dfrac{y^2}{2} = c_1$

$$\Rightarrow x^2 + y^2 = 2c_1 = c_2$$

Taking last two fractions,

$$\frac{dy}{-xy} = \frac{dz}{x(z-2y)} \Rightarrow \frac{dy}{-y} = \frac{dz}{z-2y} \Rightarrow (z-2y)dy = -ydz$$

$$\Rightarrow zdy - 2ydy = -ydz \Rightarrow zdy + ydz = 2ydy$$

$$\Rightarrow d(zy) = 2y\,dy \qquad (\because d(zy) = zdy + ydz)$$

On integrating,

$$zy = 2\cdot\frac{y^2}{2} + c_3 \Rightarrow zy - y^2 = c_3$$

$\therefore$ The reqd. solution is

$$f(x^2+y^2,\ zy - y^2) = 0 \quad \text{Ans.}$$

Que 5 Solve : $(z^2-2yz-y^2)p+(xy+zx)q = xy-zx$

Sol^n Given Lagrange's Linear PDE is

$(z^2-2yz-y^2)p+(xy+zx)q = xy-zx$

The AE are $\dfrac{dx}{(z^2-2yz-y^2)} = \dfrac{dy}{(xy+zx)} = \dfrac{dz}{(xy-zx)}$

Taking last two fractions, $\dfrac{dy}{x(y+z)} = \dfrac{dz}{x(y-z)}$

$\Rightarrow (y-z)\,dy = (y+z)\,dz$

$\Rightarrow y\,dy - z\,dy = y\,dz + z\,dz$

$\Rightarrow y\,dy - z\,dy - y\,dz - z\,dz = 0$

$\Rightarrow y\,dy - d(zy) - z\,dz = 0 \quad \{\because d(zy) = z\,dy + y\,dz\}$

On integrating,

$\dfrac{y^2}{2} - zy - \dfrac{z^2}{2} = c_1 \Rightarrow y^2 - 2zy - z^2 = 2c_1 = c_2.$

Using x, y, z as multipliers,

Each fractions $= \dfrac{x\,dx + y\,dy + z\,dz}{0}$

$\Rightarrow x\,dx + y\,dy + z\,dz = 0$

On integrating, $\dfrac{x^2}{2} + \dfrac{y^2}{2} + \dfrac{z^2}{2} = c_3 \Rightarrow x^2+y^2+z^2 = 2c_3 = c_4.$

$\therefore$ The reqd solution is

$f(y^2-2zy-z^2,\ x^2+y^2+z^2) = 0$

Ans

Que (6) Solve: $(x^2-yz)p+(y^2-zx)q=(z^2-xy)$

Sol^n Given Lagrange's Linear PDE is

$$(x^2-yz)p+(y^2-zx)q=z^2-xy$$

The AE are $$\frac{dx}{x^2-yz}=\frac{dy}{y^2-zx}=\frac{dz}{z^2-xy}$$

$$\Rightarrow \frac{dx-dy}{x^2-yz-y^2+zx}=\frac{dy-dz}{y^2-zx-z^2+xy}=\frac{dz-dx}{z^2-xy-x^2+yz}$$

$$\Rightarrow \frac{dx-dy}{(x^2-y^2)+z(x-y)}=\frac{dy-dz}{(y^2-z^2)+x(y-z)}=\frac{dz-dx}{(z^2-x^2)+y(z-x)}$$

$$\Rightarrow \frac{dx-dy}{(x-y)(x+y+z)}=\frac{dy-dz}{(y-z)(y+z+x)}=\frac{dz-dx}{(z-x)(z+x+y)}$$

$$\Rightarrow \frac{dx-dy}{x-y}=\frac{dy-dz}{y-z}=\frac{dz-dx}{z-x}$$

$$\Rightarrow \frac{dx}{x-y}-\frac{dy}{x-y}=\frac{dy}{y-z}-\frac{dz}{y-z}=\frac{dz}{z-x}-\frac{dx}{z-x}$$

Taking first two fractions,

$$\frac{dx}{x-y} - \frac{dy}{x-y} = \frac{dy}{y-z} - \frac{dz}{y-z}$$

On integrating,

$$\log(x-y) - \log(x-y)(-1) = \log(y-z) - \log(y-z)(-1) + \log c_1$$

$$\Rightarrow 2\log(x-y) = 2\log(y-z) + \log c_1$$

$$\Rightarrow \log(x-y)^2 - \log(y-z)^2 = \log c_1$$

$$\Rightarrow \log\left(\frac{x-y}{y-z}\right)^2 = \log c_1 \Rightarrow \left(\frac{x-y}{y-z}\right)^2 = c_1$$

$$\Rightarrow \frac{x-y}{y-z} = \sqrt{c_1} = c_2$$

Taking last two fractions,

$$\frac{dy}{y-z} - \frac{dz}{y-z} = \frac{dz}{z-x} - \frac{dx}{z-x}$$

On integrating,

$$\log(y-z) - \log(y-z)(-1) = \log(z-x) - \log(z-x)(-1) + \log c_3$$

$$\Rightarrow 2\log(y-z) = 2\log(z-x) + \log c_3$$

$$\Rightarrow \log(y-z)^2 - \log(z-x)^2 = \log c_3 \Rightarrow \log\left(\frac{y-z}{z-x}\right)^2 = \log c_3$$

$$\Rightarrow \left(\frac{y-z}{z-x}\right)^2 = c_3 \Rightarrow \frac{y-z}{z-x} = \sqrt{c_3} = c_4$$

$\therefore$ The reqd. solution is

$$f\left(\frac{x-y}{y-z}, \frac{y-z}{z-x}\right) = 0 \quad \underline{Ans.}$$

CHARPIT'S METHOD

Que ① Solve : $(p^2+q^2)y = qz$

Sol.^{n} Given PDE is $(p^2+q^2)y = qz$

let $f(x,y,z,p,q) = (p^2+q^2)y - qz = 0$ — ①

Here, Charpit's AE are

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}$$

from eq.^{n} ①,

$$\frac{\partial f}{\partial x} = 0,\ \frac{\partial f}{\partial y} = p^2+q^2,\ \frac{\partial f}{\partial z} = -q,\ \frac{\partial f}{\partial p} = 2py,\ \frac{\partial f}{\partial q} = 2qy - z$$

∴ Charpit's AE are

$$\frac{dp}{-pq} = \frac{dq}{p^2+q^2-q^2} = \frac{dz}{-2p^2y-2q^2y+qz} = \frac{dx}{-2py} = \frac{dy}{z-2qy}$$

Taking first two fractions,

$$\frac{dp}{-pq} = \frac{dq}{p^2} \Rightarrow p\,dp = -q\,dq \quad (\text{By SOV})$$

On integration, $\frac{p^2}{2} = -\frac{q^2}{2} + c_1 \Rightarrow \frac{p^2}{2} + \frac{q^2}{2} = c_1$.

$\Rightarrow p^2+q^2 = 2c_1 = c_2$ — ⑪

from ① and ⑪,

$c_2 \cdot y - qz = 0 \Rightarrow c_2 y = qz$

$\Rightarrow q = \frac{c_2 y}{z}$

eq.ⁿ (11) becomes,

$$p^2 + \frac{c_2^2 y^2}{z^2} = c_2 \Rightarrow p^2 = c_2 - \frac{c_2^2}{z^2} y^2$$

$$\Rightarrow p = \sqrt{\frac{c_2 z^2 - c_2^2 y^2}{z^2}} = \frac{\sqrt{c_2}}{z} \cdot \sqrt{z^2 - c_2 y^2}$$

$\because$ z depends on x and y,

$$dz = \frac{\partial z}{\partial x} \cdot dx + \frac{\partial z}{\partial y} \cdot dy = p\,dx + q\,dy.$$

$$\Rightarrow dz = \frac{\sqrt{c_2}}{z} \sqrt{z^2 - c_2 y^2}\, dx + \frac{c_2 y}{z} dy$$

$$\Rightarrow z\,dz = \sqrt{c_2} \cdot \sqrt{z^2 - c_2 y^2}\, dx + c_2 y\, dy$$

$$\Rightarrow z\,dz - c_2 y\,dy = \sqrt{c_2} \cdot \sqrt{z^2 - c_2 y^2}\, dx$$

let $\sqrt{c_2} = c \Rightarrow c_2 = c^2$

$$\Rightarrow z\,dz - c^2 y\,dy = c\sqrt{z^2 - c^2 y^2}\, dx$$

$$\Rightarrow \frac{z\,dz - c^2 y\,dy}{\sqrt{z^2 - c^2 y^2}} = c\,dx$$

$$\Rightarrow \frac{d(z^2/2) - d(c^2 y^2/2)}{\sqrt{z^2 - c^2 y^2}} = c\,dx \Rightarrow \frac{1}{2} \frac{d(z^2 - c^2 y^2)}{\sqrt{z^2 - c^2 y^2}} = c\,dx$$

$\because \frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$

$d\sqrt{x} = \frac{1}{2} \frac{dx}{\sqrt{x}}$

On integration,

$$\sqrt{z^2 - c^2 y^2} = cx + c_3$$

$$\Rightarrow z^2 - c^2 y^2 = (cx + c_3)^2$$

$$\Rightarrow z^2 = c^2 y^2 + (cx + c_3)^2$$

which is the reqd. solution.

Ans

Que (2) Solve: $z^2 = pqxy$

Soln Given PDE is $z^2 = pqxy$

Let $f(x,y,z,p,q) = z^2 - pqxy = 0$ — (i)

The Charpit's AE are

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}$$

from eqn (i),

$$\frac{\partial f}{\partial x} = -pqy,\ \frac{\partial f}{\partial y} = -pqx,\ \frac{\partial f}{\partial z} = 2z,\ \frac{\partial f}{\partial p} = -qxy,\ \frac{\partial f}{\partial q} = -pxy$$

∴ Charpit's AE are

$$\frac{dp}{-pqy + 2pz} = \frac{dq}{-pqx + 2qz} = \frac{dz}{pqxy + pqxy} = \frac{dx}{qxy} = \frac{dy}{pxy}$$

Taking x and p as multipliers in first and fourth fractions. Again taking y and q as multipliers in second and last fractions, then equating both terms,

$$\frac{x\,dp + p\,dx}{-pqxy + 2pxz + pqxy} = \frac{y\,dq + q\,dy}{-pqxy + 2qyz + pqxy}$$

$$\Rightarrow \frac{x\,dp + p\,dx}{2pxz} = \frac{y\,dq + q\,dy}{2qyz} \Rightarrow \frac{x\,dp + p\,dx}{px} = \frac{y\,dq + q\,dy}{qy}$$

$$\Rightarrow \frac{d(xp)}{xp} = \frac{d(yq)}{yq}$$ On integration,

$$\log xp = \log yq + \log c$$

$$\Rightarrow \log xp - \log yq = \log c \Rightarrow \log \frac{xp}{yq} = \log c$$

$$\Rightarrow \frac{xp}{yq} = c \Rightarrow p = \frac{cyq}{x} \quad \text{— (ii)}$$

eq.^{n} (i) becomes, $z^2 = pqxy = cy^2q^2 \Rightarrow q = \frac{z}{y\sqrt{c}}$

from (ii), $p = \frac{cy}{x} \cdot \frac{z}{y\sqrt{c}} = \frac{z\sqrt{c}}{x} = \frac{zc_1}{x}$ $\boxed{\text{Let } \sqrt{c} = c_1}$

$\because$ z depends on x and y,

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = p\,dx + q\,dy$$

$$\Rightarrow dz = \frac{zc_1}{x}dx + \frac{z}{yc_1}dy$$

$$\Rightarrow \frac{dz}{z} = c_1 \cdot \frac{dx}{x} + \frac{1}{c_1}\frac{dy}{y}$$

On integration, $\log z = c_1 \log x + \frac{1}{c_1}\log y + \log c_2$

$$\Rightarrow \log z = \log x^{c_1} + \log y^{1/c_1} + \log c_2$$

$$\Rightarrow \log z = \log(x^{c_1} \cdot y^{1/c_1} \cdot c_2)$$

$$\Rightarrow z = c_2 x^{c_1} \cdot y^{1/c_1}$$

which is the reqd. solution. Ans

Homogeneous Linear Equation with constant co-efficients:

Eq.^{n} of the form

$$\frac{\partial^n z}{\partial x^n} + K_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + \cdots + K_n \frac{\partial^n z}{\partial y^n} = F(x,y)$$

called a Homogeneous linear PDE of n^{th} order with constant co-efficients.

Note: All the terms contains derivatives of order 'n'.

Its symbolic form is given by

$$\left(D^n + K_1 D^{n-1} D' + \cdots + K_n D'^n\right) z = F(x,y) \qquad \left\{ \frac{\partial}{\partial x} = D, \quad \frac{\partial}{\partial y} = D' \right.$$

$$\Rightarrow \quad f(D, D') z = F(x,y)$$

To find CF :

Write the auxiliary eq.^{n} (Replace D by m and D' by 1)

i.e., $m^n + K_1 m^{n-1} + \cdots + K_n = 0$

and get values/roots of m.

1. If roots are distinct ($m_1, m_2, m_3, \cdots$)

$$CF = f_1(y+m_1x) + f_2(y+m_2x) + f_3(y+m_3x) + \cdots$$

2. If any two roots are repeated ($m_1, m_1, m_3, \cdots$)

$$CF = f_1(y+m_1x) + x f_2(y+m_1x) + f_3(y+m_3x) + \cdots$$

3. If all the roots are repeated ($m_1, m_1, m_1, \cdots$)

$$CF = f_1(y+m_1x) + x f_2(y+m_1x) + x^2 f_3(y+m_1x) + \cdots$$

To find PI

Symbolic form, $PI = \frac{1}{f(D,D')} \cdot F(x,y)$

$$F(x,y) \rightarrow \begin{cases} e^{ax+by} \\ \sin(mx+ny) \text{ or } \cos(mx+ny) \\ x^m y^n \\ \text{Combination of functions.} \end{cases}$$

(1) When $F(x,y) = e^{ax+by}$

$$PI = \frac{1}{f(D,D')} \cdot e^{ax+by}$$

Put $D = a$, $D' = b$.

(2) When $F(x,y) = \sin(mx+ny)$ or $\cos(mx+ny)$

$$PI = \frac{1}{f(D^2, DD', D'^2)} \sin(mx+ny) \text{ or } \cos(mx+ny)$$

Put $D^2 = -m^2$, $DD' = -mn$, $D'^2 = -n^2$

(3) When $F(x,y) = x^m y^n$

$$PI = \frac{1}{f(D,D')} \cdot x^m y^n = f(D,D')^{-1} \cdot x^m y^n$$

Expand $[f(D,D')]^{-1}$ in ascending power of D or D' and operate on $x^m y^n$ term by term.

(4) When $F(x,y)$ is any function of x and y.

$$PI = \frac{1}{f(D,D')} \cdot F(x,y)$$

Resolve $\frac{1}{f(D,D')}$ into partial fractions considering $f(D,D')$ as a function of D alone and operate each partial fraction on $F(x,y)$ resembling that

$$\frac{1}{D-mD'} \cdot F(x,y) = \int F(x, c-mx)\, dx$$

where c is replaced by $y+mx$ after integration.

Que ① Solve: $\dfrac{\partial^2 z}{\partial x^2} - \dfrac{\partial^2 z}{\partial x \partial y} = \cos x \cos 2y.$

Sol^n Given PDE in symbolic form is

$$(D^2 - DD')z = \cos x \cos 2y$$

Its AE is $\quad m^2 - m = 0 \quad \Rightarrow m(m-1) = 0$

$\Rightarrow m = 0, 1 \quad$ (distinct roots)

$$\therefore \; C.F = f_1(y + 0.x) + f_2(y + 1.x)$$
$$= f_1(y) + f_2(y+x)$$

$$PI = \frac{1}{D^2 - DD'} \cdot \cos x \cos 2y = \frac{1}{D^2 - DD'} \cdot \frac{1}{2}\left[\cos(x+2y) + \cos(x-2y)\right]$$

$$\cos A \cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$$

$$= \frac{1}{2}\left[\frac{1}{D^2 - DD'}\cos(x+2y) + \frac{1}{D^2 - DD'}\cos(x-2y)\right]$$

For 1st term, Put $D^2 = -1^2 = -1$, $DD' = -(1\times 2) = -2$

For 2nd term, Put $D^2 = -1^2 = -1$, $DD' = -\{1 \times (-2)\} = 2.$

$$PI = \frac{1}{2}\left[\frac{1}{-1-(-2)}\cos(x+2y) + \frac{1}{-1-2}\cos(x-2y)\right]$$

$$= \frac{1}{2}\left[\cos(x+2y) - \frac{1}{3}\cos(x-2y)\right] = \frac{1}{2}\cos(x+2y) - \frac{1}{6}\cos(x-2y)$$

Hence, the complete solution is

$$z = CF + PI$$
$$= f_1(y) + f_2(y+x) + \frac{1}{2}\cos(x+2y) - \frac{1}{6}\cos(x-2y)$$

Ans.

Que ② Solve $(D^2+4DD'-5D'^2)z = \sin(2x+3y)$

Sol.ⁿ Given PDE is $(D^2+4DD'-5D'^2)z = \sin(2x+3y)$ which is in symbolic form.

Its AE is $m^2+4m-5=0$

$\Rightarrow m^2+5m-m-5=0$

$\Rightarrow m(m+5)-1(m+5)=0$

$\Rightarrow (m+5)(m-1)=0 \quad \Rightarrow m=1,-5$ (distinct roots)

$\therefore CF = f_1(y+1.x) + f_2(y+(-5)x)$

$= f_1(y+x) + f_2(y-5x)$

$$PI = \frac{1}{D^2+4DD'-5D'^2}\cdot \sin(2x+3y)$$

Put $D^2=-2^2=-4$, $DD'=-(2\times3)=-6$, $D'^2=-3^2=-9$

$$PI = \frac{1}{-4+4(-6)-5(-9)}\cdot \sin(2x+3y)$$

$$= \frac{1}{-4-24+45}\sin(2x+3y) = \frac{1}{17}\sin(2x+3y)$$

$\therefore$ The complete solution is

$z = CF + PI$

$= f_1(y+x) + f_2(y-5x) + \frac{1}{17}\sin(2x+3y)$

Ans.

Que (3) Solve: $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = \cos(2x+y)$

Sol.^{n} Given PDE in symbolic form is

$(D^2 + DD' - 6D'^2)z = \cos(2x+y)$

Its AE is $m^2 + m - 6 = 0 \Rightarrow m^2 + 3m - 2m - 6 = 0$

$\Rightarrow m(m+3) - 2(m+3) = 0 \quad \Rightarrow (m+3)(m-2) = 0$

$\Rightarrow m = -3, 2$ (distinct roots)

$\therefore$ CF $= f_1(y-3x) + f_2(y+2x)$

$$PI = \frac{1}{D^2 + DD' - 6D'^2} \cdot \cos(2x+y)$$

But $D^2 + DD' - 6D'^2 = -2^2 + (-2\times 1) - 6(-1^2) = -4 - 2 + 6 = 0$

$\therefore$ we have to apply the general method.

$$PI = \frac{1}{(D+3D')(D-2D')} \cdot \cos(2x+y) = \frac{1}{D+3D'}\left[\frac{1}{D-2D'} \cdot \cos(2x+y)\right]$$

$$= \frac{1}{D+3D'} \int \cos(2x + c - 2x)\,dx \qquad \left\{ \begin{array}{l} D - mD' \\ y = c - mx \end{array} \right.$$

$$= \frac{1}{D+3D'} \int \cos c\, dx = \frac{1}{D+3D'} x \cdot \cos c$$

$$= \frac{1}{D+3D'} x \cdot \cos(y+2x) \quad = \int x \cos(c + 3x + 2x)\,dx$$

$$= \int x \cos(5x + c)\,dx = x\frac{\sin(5x+c)}{5} - (1)\left\{-\frac{\cos(5x+c)}{25}\right\}$$

$$= \frac{x}{5}\sin(5x+c) + \frac{1}{25}\cos(5x+c) = \frac{x}{5}\sin(5x+y-3x) + \frac{1}{25}\cos(5x+y-3x)$$

$$= \frac{x}{5}\sin(2x+y) + \frac{1}{25}\cos(2x+y)$$

$\therefore$ The complete solution is

$$z = CF + PI$$

$$= f_1(y-3x) + f_2(y+2x) + \frac{x}{5}\sin(2x+y) + \frac{1}{25}\cos(2x+y)$$

Que (4) Solve: $r - 4s + 4t = e^{2x+y}$

Sol.^{n} Given PDE is

$$\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial x \partial y} + 4\frac{\partial^2 z}{\partial y^2} = e^{2x+y}$$

In symbolic form

$$(D^2 - 4DD' + 4D'^2)z = e^{2x+y}$$

Its AE is $m^2 - 4m + 4 = 0 \Rightarrow (m-2)^2 = 0$

$\Rightarrow m = 2, 2.$ (repeated roots)

$\therefore CF = f_1(y+2x) + x f_2(y+2x)$

$$PI = \frac{1}{D^2 - 4DD' + 4D'^2} \cdot e^{2x+y}$$

$$= x \cdot \frac{1}{2D - 4D'} e^{2x+y}$$

$$= x^2 \cdot \frac{1}{2} e^{2x+y} = \frac{x^2}{2} \cdot e^{2x+y}$$

$\therefore$ The complete solution is

$$z = CF + PI$$

$$= f_1(y+2x) + x f_2(y+2x) + \frac{x^2}{2} e^{2x+y}.$$

Ans

Que ⑤ Solve: $\dfrac{\partial^3 z}{\partial x^3} - 2\dfrac{\partial^3 z}{\partial x^2 \partial y} = 2e^{2x} + 3x^2y$

Solⁿ Given PDE in symbolic form

$$(D^3 - 2D^2D')z = 2e^{2x} + 3x^2y$$

Its AE is $m^3 - 2m^2 = 0 \Rightarrow m^2(m-2) = 0$

$\Rightarrow m = 0, 0, 2.$

$\therefore$ CF $= f_1(y) + x f_2(y) + f_3(y+2x)$

$$PI = \frac{1}{D^3 - 2D^2D'}(2e^{2x} + 3x^2y)$$

$$= 2\cdot\frac{1}{D^3 - 2D^2D'}\cdot e^{2x} + 3\,\frac{1}{D^3\left(1 - 2\frac{D'}{D}\right)}\cdot x^2y$$

$$= 2\cdot\frac{1}{2^3 - 2\cdot 2^2\cdot 0}\cdot e^{2x} + \frac{3}{D^3}\left(1 - 2\frac{D'}{D}\right)^{-1} x^2y$$

$$= \frac{1}{4}e^{2x} + \frac{3}{D^3}\left(1 + 2\frac{D'}{D} + \cdots\right)x^2y$$

$$= \frac{1}{4}e^{2x} + \frac{3}{D^3}\left(x^2y + \frac{2}{D}\cdot x^2\right) = \frac{1}{4}e^{2x} + \frac{3}{D^3}\left(x^2y + 2\cdot\frac{x^3}{3}\right)$$

$$= \frac{1}{4}e^{2x} + \frac{3}{D^3}\left(x^2y + \frac{2}{3}x^3\right) = \frac{1}{4}e^{2x} + \frac{x^5}{20}y + \frac{x^6}{60}$$

$\therefore$ The complete solution is

$$z = CF + PI.$$

$$= f_1(y) + x f_2(y) + f_3(y+2x) + \frac{1}{4}e^{2x} + \frac{x^5}{20}y + \frac{x^6}{60}$$

Ans

Que 6) Solve: $4\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 16\log(x+2y)$

Sol^n Given PDE in symbolic form

$$(4D^2 - 4DD' + D'^2)z = 16\log(x+2y)$$

Its AE is $4m^2 - 4m + 1 = 0 \Rightarrow (2m-1)^2 = 0$

$\Rightarrow m = 1/2, 1/2$ (repeated roots)

$\therefore CF = f_1(y + \frac{1}{2}x) + x f_2(y + \frac{1}{2}x)$

$$PI = \frac{1}{(2D-D')^2} \cdot 16\log(x+2y) = \frac{1}{4(D-\frac{D'}{2})^2} \cdot 16\log(x+2y)$$

$$= 4\frac{1}{(D-\frac{1}{2}D')}\left[\frac{1}{D-\frac{1}{2}D'} \cdot \log(x+2y)\right] \quad \left\{ \begin{array}{l} D - mD' \\ y = c - mx \end{array} \right.$$

$$= 4\frac{1}{(D-\frac{1}{2}D')}\int \log\{x + 2(c - \tfrac{1}{2}x)\}dx$$

$$= 4\frac{1}{D-\frac{1}{2}D'}\int \log(x + 2c - x)dx = 4\frac{1}{D-\frac{1}{2}D'}\int \log(2c)\,dx$$

$$= 4 \cdot \frac{1}{D-\frac{1}{2}D'} x\log 2c = 4 \cdot \frac{1}{D-\frac{1}{2}D'} \cdot x\log(x+2y)$$

$$= 4\int x\log\{x + 2(c - \tfrac{1}{2}x)\}dx = 4\int x\log(x+2c-x)dx$$

$$= 4\int x\log 2c\,dx = 4\log 2c\int x\,dx = 4\log 2c \cdot \frac{x^2}{2}$$

$$= 2x^2\log(x+2y)$$

$\therefore$ The complete solution is

$$z = CF + PI$$

$$= f_1(y + x/2) + x f_2(y + x/2) + 2x^2\log(x+2y)$$

Ans

Que(7) Solve: $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = y\cos x$

Solⁿ Given PDE in symbolic form is

$$(D^2 + DD' - 6D'^2)z = y\cos x$$

The AE is $m^2 + m - 6 = 0 \Rightarrow m^2 + 3m - 2m - 6 = 0$

$\Rightarrow m(m+3) - 2(m+3) = 0 \Rightarrow (m+3)(m-2) = 0$

$\Rightarrow m = -3, 2$. (distinct roots)

$\therefore CF = f_1(y-3x) + f_2(y+2x)$

$$PI = \frac{1}{D^2 + DD' - 6D'^2} y\cos x = \frac{1}{(D-2D')(D+3D')} \cdot y\cos x$$

$$= \frac{1}{D-2D'}\left[\frac{1}{D+3D'} \cdot y\cos x\right]$$

$$\left\{ \begin{array}{l} D - mD' \\ y = c - mx \end{array} \right.$$

$$= \frac{1}{D-2D'} \int (c+3x)\cos x \, dx$$

$$= \frac{1}{D-2D'}\left[(c+3x)\sin x - 3(-\cos x)\right] = \frac{1}{D-2D'}\left[y\sin x + 3\cos x\right]$$

$$= \int \left[(c-2x)\sin x + 3\cos x\right] dx$$

$$= (c-2x)(-\cos x) - (-2)(-\sin x) + 3\sin x$$

$$= -y\cos x - 2\sin x + 3\sin x = \sin x - y\cos x.$$

$\therefore$ The complete solution is

$$z = CF + PI$$

$$= f_1(y-3x) + f_2(y+2x) + \sin x - y\cos x.$$

Ans

Que (8) Solve: $(D^2 - DD' - 2D'^2)z = (y-1)e^x$

Sol^n Given PDE is $(D^2 - DD' - 2D'^2)z = (y-1)e^x$

Its AE is $m^2 - m - 2 = 0 \Rightarrow m^2 - 2m + m - 2 = 0$

$\Rightarrow m(m-2) + 1(m-2) = 0 \Rightarrow (m-2)(m+1) = 0$

$\Rightarrow m = 2, -1$ (distinct roots)

$\therefore$ CF $= f_1(y+2x) + f_2(y-x)$

$$PI = \frac{1}{D^2 - DD' - 2D'^2} \cdot (y-1)e^x = \frac{1}{(D-2D')(D+D')} \cdot (y-1)e^x$$

$$= \frac{1}{D-2D'}\left[\frac{1}{D+D'}(y-1)e^x\right] \quad \left\{ \begin{array}{l} D - mD' \\ y = c - mx \end{array} \right.$$

$$= \frac{1}{D-2D'}\left[\int (c+x-1)e^x\,dx\right]$$

$$= \frac{1}{D-2D'}\left[(c+x-1)e^x - (1)e^x\right] = \frac{1}{D-2D'}\left[(y-1)e^x - e^x\right]$$

$$= \int\left[(c-2x-1)e^x - e^x\right]dx$$

$$= (c-2x-1)e^x - (-2)e^x - e^x$$

$$= (y-1)e^x + 2e^x - e^x = (y-1)e^x + e^x.$$

$$= ye^x - e^x + e^x = ye^x.$$

$\therefore$ The complete solution is

$$z = CF + PI$$

$$= f_1(y+2x) + f_2(y-x) + ye^x.$$

Ans

Non-Homogeneous Linear Equations

27 Que ① Solve : $(D^2 + 2DD' + D'^2 - 2D - 2D')z = \sin(x+2y)$

78 Que ② Solve : $(D + D' - 1)(D + 2D' - 3)z = 4 + 3x + 6y$

To find CF

↳ Factorize $f(D, D')$ in the factors of the form $D - mD' - c$.

↳ The solution of $(D - mD' - c)z = 0$ is given by $z = e^{cx}\,\phi(y + mx)$.

↳ The solution corresponding to various factors are added up to give the CF.

Que ① Solve: $(D^2+2DD'+D'^2-2D-2D')z = \sin(x+2y)$

Solⁿ Given linear eqⁿ is non-homogeneous.

Here, $f(D,D') = D^2+2DD'+D'^2-2D-2D'$

$= (D+D')^2 - 2(D+D') = (D+D')(D+D'-2)$

$\because$ the solution corresponding to the factor $(D-mD'-c)$ is given by

$$z = e^{cx}\,\phi(y+mx)$$

$\therefore$ CF $= \phi_1(y-x) + e^{2x}\phi_2(y-x)$

$$PI = \frac{1}{D^2+2DD'+D'^2-2D-2D'}\cdot \sin(x+2y)$$

Put $D^2 = -1^2 = -1$, $DD' = -(1\times 2) = -2$, $D'^2 = -2^2 = -4$

$$PI = \frac{1}{-1-4-4-2(D+D')}\cdot\sin(x+2y) = -\frac{1}{2(D+D')+9}\cdot\sin(x+2y)$$

$$= -\frac{[2(D+D')-9]}{[2(D+D')]^2-9^2}\cdot\sin(x+2y) = \frac{[9-2(D+D')]}{4(D^2+2DD'+D'^2)-81}\cdot\sin(x+2y)$$

$$= \frac{9\sin(x+2y) - 2\cos(x+2y) - 2\cos(x+2y)\cdot 2}{4[-1-4-4]-81}$$

$$= -\frac{1}{117}\left[9\sin(x+2y) - 6\cos(x+2y)\right]$$

$$= \frac{3}{117}\left[2\cos(x+2y) - 3\sin(x+2y)\right]$$

$$= \frac{1}{39}\left[2\cos(x+2y) - 3\sin(x+2y)\right]$$

$\therefore$ The complete solution is

$$z = \phi_1(y-x) + e^{2x}\phi_2(y-x) + \frac{1}{39}\left[2\cos(x+2y) - 3\sin(x+2y)\right]$$

Ans.

Que ② Solve: $(D+D'-1)(D+2D'-3)z = 4+3x+6y$

Sol^n Given linear eq^n is non-homogeneous.

Here, $f(D,D') = (D+D'-1)(D+2D'-3)$

$\because$ the solution corresponding to $(D-mD'-c)$

is $\quad z = e^{cx}\,\phi(y+mx)$

$\therefore\ CF = e^{x}\,\phi_1(y-x) + e^{3x}\,\phi_2(y-2x)$

$$PI = \frac{1}{(D+D'-1)(D+2D'-3)}\cdot(4+3x+6y) = \frac{1}{(1-D-D')(3-D-2D')}\cdot(4+3x+6y)$$

$$= \frac{1}{[1-(D+D')]\,3\left[1-\left(\frac{D}{3}+\frac{2}{3}D'\right)\right]}\cdot(4+3x+6y)$$

$$= \frac{1}{3}\left[1-(D+D')\right]^{-1}\left[1-\left(\frac{D}{3}+\frac{2}{3}D'\right)\right]^{-1}(4+3x+6y)$$

$$= \frac{1}{3}\left[1+(D+D')+\cdots\right]\left[1+\left(\frac{D}{3}+\frac{2}{3}D'\right)+\cdots\right](4+3x+6y)$$

$$= \frac{1}{3}\left[1+D+D'+\cdots\right]\left[4+3x+6y+\frac{1}{3}(3)+\frac{2}{3}(6)\right]$$

$$= \frac{1}{3}\left[1+D+D'+\cdots\right](9+3x+6y)$$

$$= \frac{1}{3}(9+3x+6y+3+6) = \frac{1}{3}(3x+6y+18)$$

$$= x+2y+6$$

$\therefore$ The complete solution is

$$z = CF + PI$$

$$= e^{x}\,\phi_1(y-x) + e^{3x}\,\phi_2(y-2x) + x+2y+6$$

Ans.

Method of Separation of Variables

Steps to solve:

1. Assume the trial solution
$$u(x,y) = X(x)\cdot Y(y) \quad \text{or} \quad u = XY.$$
Put the values of trial solution in given PDE.
2. Separate the variables and assume it equal to same constant k.
3. Take the separated variables individually and solve it. (find X and Y)
4. Put the values of X and Y in the trial solution to give reqd. solution of given PDE.

Que Using method of SOV, solve

29 1. $\dfrac{\partial u}{\partial x} = 2\dfrac{\partial u}{\partial t} + u$, $u(x,0) = 6e^{-3x}$

30 2. $\dfrac{\partial u}{\partial x} = 4\dfrac{\partial u}{\partial y}$, $u(0,y) = 8e^{-3y}$

31 3. $4\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} = 3u$, $u(0,y) = 3e^{-y} - e^{-5y}$

32 4. $3\dfrac{\partial u}{\partial x} + 2\dfrac{\partial u}{\partial y} = 0$, $u(x,0) = 4e^{-x}$

33 5. $\dfrac{\partial^2 z}{\partial x^2} - 2\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y} = 0$.

Que ① Using method of separation of variables, solve: $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$, where $u(x,0) = 6e^{-3x}$.

Sol^n Given $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$ —①

Let the solution of eq.① be

$$u(x,t) = X(x)\cdot T(t)$$

Eq.^{n} ① becomes, $X'T = 2XT' + XT \Rightarrow (X' - X)T = 2XT'$

$$\Rightarrow \frac{X'-X}{X} = 2\frac{T'}{T} = K.$$

Solving $\frac{X'-X}{X} = K \Rightarrow \frac{X'}{X} - 1 = K \Rightarrow \frac{X'}{X} = K+1$

On integration, $\log X = (K+1)x + \log C_1 = (K+1)x.\log e + \log C_1$

$$\Rightarrow \log X = \log e^{(K+1)x} + \log C_1 = \log C_1\cdot e^{(K+1)x}$$

$$\Rightarrow X = C_1 e^{(K+1)x}$$

Solving $\frac{T'}{T} = \frac{K}{2}$ On integration, $\log T = \frac{K}{2}t + \log C_2$

$$\Rightarrow \log T = \frac{K}{2}t.\log e + \log C_2 = \log e^{Kt/2} + \log C_2 = \log C_2\cdot e^{Kt/2}$$

$$\Rightarrow T = C_2 e^{Kt/2}.$$

$$\therefore u(x,t) = C_1 e^{(K+1)x}\cdot C_2 e^{Kt/2} = C_1C_2\, e^{(K+1)x + Kt/2}.$$

At $t=0$, $u(x,0) = C_1C_2\, e^{(K+1)x}$

Also, $u(x,0) = 6e^{-3x}$ (given)

On comparing both eq.^{ns}, $C_1C_2 = 6$, $K+1 = -3 \Rightarrow K = -4$

$\therefore$ The reqd. solution is

$$u(x,t) = 6\, e^{(-4+1)x + (-4)t/2} = 6e^{-3x-2t}$$

$$= 6\, e^{-(3x+2t)}$$

Ans

Que ⑧ Using method of separation of variables, solve $\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y}$; given $u(0,y) = 8e^{-3y}$.

Sol.ⁿ Given: $\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y}$ —— ①

Let the solution of eq.ⁿ ① be

$$u(x,y) = X(x)\cdot Y(y)$$

$\therefore$ Eq.ⁿ ① becomes, $X'Y = 4XY' \Rightarrow \frac{X'}{X} = \frac{4Y'}{Y} = K.$

Solving $\frac{X'}{X} = K$ On integration, $\log X = Kx + \log c_1$

$\Rightarrow \log X = Kx \cdot \log e + \log c_1 = \log e^{Kx} + \log c_1 = \log c_1 e^{Kx}$

$\Rightarrow X = c_1 e^{Kx}$.

Solving $\frac{Y'}{Y} = \frac{K}{4}$ On integration, $\log Y = \frac{K}{4}y + \log c_2$

$\Rightarrow \log Y = \frac{K}{4}y \log e + \log c_2 = \log e^{Ky/4} + \log c_2 = \log c_2 e^{Ky/4}$

$\Rightarrow Y = c_2 e^{Ky/4}$.

$$\therefore u(x,y) = c_1 e^{Kx} \cdot c_2 e^{Ky/4} = c_1 c_2 e^{K(x+y/4)}$$

At $x=0$, $u(0,y) = c_1 c_2 e^{Ky/4}$

Also, $u(0,y) = 8e^{-3y}$ (given)

On comparing both eq.ⁿˢ, $c_1 c_2 = 8$, $\frac{K}{4} = -3 \Rightarrow K = -12$

$\therefore$ The reqd. solution is

$$u(x,y) = 8e^{-12(x+y/4)} = 8e^{-12\left(\frac{4x+y}{4}\right)}$$

$$= 8e^{-3(4x+y)}$$

Ans

Que (3) Using method of separation of variables, solve : $4\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=3u$; given $u=3e^{-y}-e^{-5y}$ when $x=0$.

Sol.^{n} Given: $4\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=3u$. —— (1)

let the solution of eq.^{n} (1) be

$$u(x,y)=X(x)\cdot Y(y)$$

Eq.^{n} (1) becomes, $4X'Y+XY'=3XY$

$\Rightarrow 4X'Y=X(3Y-Y') \Rightarrow 4\frac{X'}{X}=\frac{3Y-Y'}{Y}=K.$

Solving $\frac{X'}{X}=\frac{K}{4}$, On integration, $\log X=\frac{K}{4}x+\log c_1$

$\Rightarrow \log X=\frac{Kx}{4}\log e+\log c_1=\log e^{Kx/4}+\log c_1=\log c_1 e^{Kx/4}.$

$\Rightarrow X=c_1 e^{Kx/4}.$

Solving $\frac{Y'}{Y}=3-K$, On integration, $\log Y=(3-K)y+\log c_2$

$\Rightarrow \log Y=(3-K)y.\log e+\log c_2=\log e^{(3-K)y}+\log c_2=\log c_2 e^{(3-K)y}$

$\Rightarrow Y=c_2 e^{(3-K)y}.$

$\therefore u(x,y)=c_1 e^{Kx/4}\cdot c_2 e^{(3-K)y}=c_1c_2 e^{[Kx/4+(3-K)y]}$

At $x=0$, $u(0,y)=c_1c_2 e^{(3-K)y}$.

Also, $u(0,y)=3e^{-y}-e^{-5y}$ (given)

On comparing $3e^{-y}=c_1c_2 e^{(3-K)y} \Rightarrow c_1c_2=3,\ 3-K=-1$
$\Rightarrow K=4$

$\therefore u(x,y)=3e^{(x-y)}$ —— (2)

On comparing $-e^{-5y}=c_1c_2 e^{(3-K)y} \Rightarrow c_1c_2=-1,\ 3-K=-5$
$\Rightarrow K=8$

$\therefore u(x,y)=-e^{2x-5y}$ —— (3)

from (2) and (3),

$$u(x,y)=3e^{x-y}-e^{2x-5y}$$

Ans

Que (4) Using method of separation of variables,
solve: $3\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0$; given $u(x,0) = 4e^{-x}$

Solⁿ Given: $3\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0$ —— (1)

Let the solution of eqⁿ (1) be
$u(x,y) = X(x) \cdot Y(y)$

Eqⁿ (1) becomes, $3X'Y + 2XY' = 0 \Rightarrow 3X'Y = -2XY'$
$\Rightarrow 3\frac{X'}{X} = -\frac{2Y'}{Y} = k$.

Solving $\frac{X'}{X} = \frac{k}{3}$ On integration, $\log X = \frac{k}{3}x + \log c_1$
$\Rightarrow \log X = \frac{k}{3}x \log e + \log c_1 = \log e^{kx/3} + \log c_1 = \log c_1 e^{kx/3}$
$\Rightarrow X = c_1 e^{kx/3}$

Solving $\frac{Y'}{Y} = -\frac{k}{2}$ On integration, $\log Y = -\frac{k}{2}y + \log c_2$
$\Rightarrow \log Y = -\frac{k}{2}y \log e + \log c_2 = \log e^{-ky/2} + \log c_2 = \log c_2 e^{-ky/2}$
$\Rightarrow Y = c_2 e^{-ky/2}$

$\therefore u(x,y) = c_1 e^{kx/3} \cdot c_2 e^{-ky/2} = c_1 c_2 e^{k(x/3 - y/2)}$

At $y=0$, $u(x,0) = c_1 c_2 e^{kx/3}$.

Also, $u(x,0) = 4e^{-x}$ (given)

On comparing, $c_1 c_2 = 4$, $k/3 = -1 \Rightarrow k = -3$.

$\therefore$ The reqd. solution is
$$u(x,y) = 4\, e^{-3(x/3 - y/2)}$$
$$= 4 \cdot e^{-x + 3y/2}$$

Ans

Que(5) Using method of separation of variables,
Solve: $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0.$

Sol.ⁿ Given: $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$ —— (1)

let the solution of eq.ⁿ (1) be
$$Z(x,y) = X(x)\cdot Y(y)$$

Eq.ⁿ (1) becomes, $X''Y - 2X'Y + XY' = 0$
$\Rightarrow (X'' - 2X')Y = -XY' \Rightarrow \frac{X'' - 2X'}{X} = -\frac{Y'}{Y} = K.$

Solving $\frac{X'' - 2X'}{X} = K \Rightarrow X'' - 2X' = KX.$
$\Rightarrow X'' - 2X' - KX = 0$ —— (ii)

which is a second order linear differential eq.ⁿ.
Its AE is $m^2 - 2m - K = 0$
$$m = \frac{+2 \pm \sqrt{4 + 4K}}{2} = 1 \pm \sqrt{1+K}$$

$\therefore$ The solution of eq.ⁿ (ii) is
$$X = C_1 e^{(1+\sqrt{1+K})x} + C_2 e^{(1-\sqrt{1+K})x}$$

Also, solving $\frac{Y'}{Y} = -K$ On integration,
$$\log Y = -Ky + \log C_3.$$
$\Rightarrow \log Y = -Ky \log e + \log C_3 = \log e^{-Ky} + \log C_3 = \log C_3 e^{-Ky}$
$\Rightarrow Y = C_3 e^{-Ky}.$

$\therefore$ The required solution is
$$Z(x,y) = \left[C_1 e^{(1+\sqrt{1+K})x} + C_2 e^{(1-\sqrt{1+K})x}\right] \cdot C_3 e^{-Ky}$$

Ans.

THANK YOU SO MUCH