CHAPTER 2

VERTEX ANTIMAGIC TOTAL LABELING OF GENERALIZED PETERSEN GRAPHS

2.1 Introduction

Generalized Petersen graphs were first defined by Watkins in 1969. In general, a graph in which no two vertices has the same weight is called an vertex antimagic graph. In this chapter, we study the constructions of vertex antimagic labeling of Generalized Petersen graphs.

Basic Definitions

Let G = (V,E) be a simple graph with v vertices and e edges.

Definition 2.1.1

For every vertex $v \in V$, its weight $w(v) = f(v) + \sum_{u \epsilon N(v)} f(uv)$ where $N(v)$ is the neighborhood of v.

Definition 2.1.2

A *vertex antimagic total labeling or valuation* **(VATL)** of G is a bijection

$f : V \cup E \rightarrow \{1,2,\ldots\ v+e\}$ so that $w(v)$ is distinct for all $v \in V$. A *vertex antimagic total labeling f* is called *super vertex antimagic total labeling* if

$f(V) = \{1,2,\ldots,v\}$. A graph is called *vertex antimagic total* if it admits an *vertex antimagic total labeling*.

Definition 2.1.3

An (a,d)-*vertex antimagic total labeling*(**VATL**)of G is a bijection $f : V \cup E \to \{1,2,\ldots v+e\}$ so that the set of vertex weights of all edges in G is $\{a, a+d, a+2d, \ldots a+(e-1)d\}$ where a, d are two fixed positive integers and G is called *vertex antimagic total* **(VAT).** An (a,d)-*vertex antimagic total labeling* f is called *super* (a,d)-*vertex antimagic total labeling* if $f(V) = \{1,2,\ldots,v\}$.

Definition 2.1.4

A generalized Petersen graph $P(n,m)$, $n \geq 3$, $1 \leq m < \frac{n}{2}$ is a 3-regular graph with $2n$ vertices $u_0, u_1, \ldots u_{n-1}, v_0, v_1, \ldots v_{n-1}$ and edges (u_i, v_i), (u_i, u_{i+1}), (v_i, v_{i+m}) for all $i \in \{0,1,2,\ldots n-1\}$, where the subscripts are taken modulo n. $P(5,2)$ is the standard Petersen graph which is shown in Fig 2.1.1.

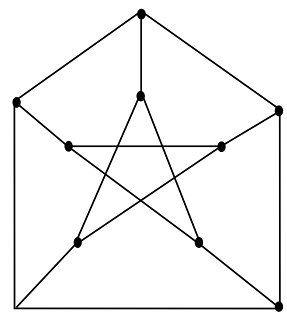


Fig 2.1.1 - P(5,2)

We now present some results from literature.

Proposition 2.1.5 [8]

Let G be a regular graph of degree r with v vertices and e edges. Then G has an (a,d)-*vertex antimagic total labeling* if and only if G has an (a',d)-*vertex antimagic total labeling* where $a' = (r+1)(v+e+1) - a - (v-1)d$.

Proposition 2.1.6 [9]

For n odd, $n \geq 3$, the prism D_n has a (a,d)-*vertex antimagic total labeling* for

$(a, d) \in \left\{\left(\frac{15n+5}{2}, 1\right), \left(\frac{11n+7}{2}, 3\right), \left(\frac{21n+5}{2}, 1\right), \left(\frac{17n+7}{2}, 3\right)\right\}$.

Proposition 2.1.7 [9]

Every prism D_n with even cycles admits a (a,d)-*vertex antimagic total labeling* for $(a, d) \in \left\{\left(\frac{13n+6}{2}, 2\right), \left(\frac{9n+8}{2}, 4\right), \left(\frac{19n+6}{2}, 2\right), \left(\frac{15n+8}{2}, 4\right)\right\}$.

2.2 Vertex Antimagic Total Labeling of P(*n*, *m*)

In this section, we present some vertex antimagic total labelings of Generalized Petersen graphs $P(n, m)$.

Theorem 2.2.1

For n odd, $n \geq 5$, the generalized Petersen graph $P(n,2)$ has a $\left(\frac{15n+5}{2}, 1\right)$- vertex antimagic total labeling.

Proof

Consider $G = P(n,2)$ with $v = 2n$ vertices and $e = 3n$ edges where n is odd, $n \geq 5$.

Define $f: V \cup E \rightarrow \{1,2,\ldots,5n\}$ as follows:

$$f(u_i) = \begin{cases} \frac{1}{2}(8n-i), & for \ \ i \equiv 0 \pmod 2, \\ \frac{1}{2}(7n-i), & for \ \ i \equiv 1 \pmod 2. \end{cases}$$

$$f(v_i) = \begin{cases} \frac{1}{2}(10n-i), & for \ \ i \equiv 0 \pmod 2, \\ \frac{1}{2}(9n-i), & for \ \ i \equiv 1 \pmod 2. \end{cases}$$

$$f(u_i v_i) = \begin{cases} \frac{1}{2}(2+i), & \textit{for } i \equiv 0 \pmod 2, \\ \frac{1}{2}(n+2+i), & \textit{for } i \equiv 1 \pmod 2. \end{cases}$$

$$f(u_i u_{i+1}) = \begin{cases} 2n+1, & \textit{for } i = 0, \\ \frac{1}{2}(6n+2-i), & \textit{for } i \equiv 0 \pmod 2, i \neq 0, \\ \frac{1}{2}(5n+2-i), & \textit{for } i \equiv 1 \pmod 2. \end{cases}$$

Case $n \equiv 1 \pmod 4$:

$$f(v_i v_{i+2}) = \begin{cases} n+1, & \textit{for } i = 0, \\ \frac{1}{4}(5n+4-i), & \textit{for } i \equiv 1 \pmod 4, \\ \frac{1}{4}(6n+4-i), & \textit{for } i \equiv 2 \pmod 4, \\ \frac{1}{4}(7n+4-i), & \textit{for } i \equiv 3 \pmod 4, \\ \frac{1}{4}(8n+4-i), & \textit{for } i \equiv 0 \pmod 4, i \neq 0. \end{cases}$$

Case $n \equiv 3 \pmod 4$:

$$f(v_i v_{i+2}) = \begin{cases} n+1, & \textit{for } i = 0, \\ \frac{1}{4}(7n+4-i), & \textit{for } i \equiv 1 \pmod 4, \\ \frac{1}{4}(6n+4-i), & \textit{for } i \equiv 2 \pmod 4, \\ \frac{1}{4}(5n+4-i), & \textit{for } i \equiv 3 \pmod 4, \\ \frac{1}{4}(8n+4-i), & \textit{for } i \equiv 0 \pmod 4, i \neq 0. \end{cases}$$

Now, let us evaluate the weights of u_i ‘s and v_i ‘s .

For any vertex $v \in V$, its weight $w(v) = f(v) + \sum_{u \in N(v)} f(uv)$

Thus $w(u_i) = f(u_i) + f(u_i u_{i+1}) + f(u_{i-1} u_i) + f(u_i v_i)$

and $w(v_i) = f(v_i) + f(u_i v_i) + f(v_i v_{i+2}) + f(v_{n-2+i} v_i)$

which are derived and observed as follows:

$$w(u_i) = \begin{cases} \frac{1}{2}(17n+3)+(2-i), & for\ \ i = 0,1, \\ \frac{1}{2}(19n+3)+\frac{1}{2}(4-2i), & for\ \ i = 2,3,4,\ldots n-1. \end{cases}$$

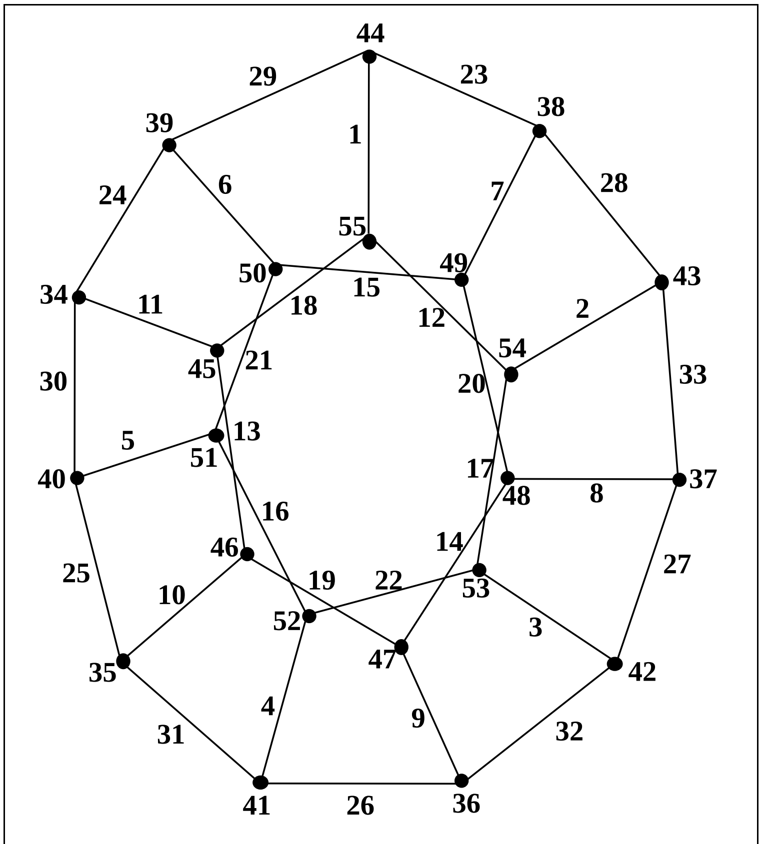
$$w(v_i) = \begin{cases} \frac{1}{2}(15n+5)+\frac{1}{2}(2-i), & for\ \ i = 0,2, \\ \frac{1}{2}(16n+4)+\frac{1}{2}(3-i), & for\ \ i = 1,3,5,\ldots n-2, \\ \frac{1}{2}(17n+3)+\frac{1}{2}(4-i), & for\ \ i = 4,6,8,\ldots n-1. \end{cases}$$

Hence the set of vertex-weights is $\left\{ \frac{1}{2}(15n+5), \frac{1}{2}(15n+7), \ldots, \frac{1}{2}(19n+3) \right\}$.

Thus f is a super $\left(\frac{15n+5}{2},\ 1\right)$ - vertex antimagic total labeling.

Therefore P(n,2) is a super $\left(\frac{15n+5}{2},\ 1\right)$ - vertex antimagic total graph.

Example: This theorem is illustrated in Fig 2.2.1.



$w(u_i)$ = {97,96,106,105,104,103,102,101,100,99,98}

$w(v_i)$ = {86,91,85,90,95,89,94,88,93,87,92}

$w(u_i) \cup w(v_i)$ = {85,86,87,…106}

Fig 2.2.1 - $\left(\frac{15n+5}{2},\ 1\right)$ **- VATL of P(11,2)**

Theorem 2.2.2

For $n \geq 3$, $1 \leq m \leq \left\lfloor \frac{n-1}{2} \right\rfloor$, every generalized Petersen graph P(n,m) has a ($8n + 3$, 2) - vertex antimagic total labeling.

Proof

Consider G = $P(n,m)$ with $v = 2n$ vertices and $e = 3n$ edges where $n \geq 3$, $1 \leq m \leq \left\lfloor \frac{n-1}{2} \right\rfloor$ .Define $f: V \cup E \rightarrow \{ 1,2,\ldots,5n \}$ as follows:

$$f(u_i) = 3n + 1 + i, \ for\ 0 \leq i \leq n-1$$

$$f(v_i) = \begin{cases} n+m+i, & for\ 0 \leq i \leq n-m, \\ m+i, & for\ n-m+1 \leq i \leq n-1. \end{cases}$$

$$f(u_i v_i) = \begin{cases} 4n+1, & for\ i=0, \\ 5n+1-i, & for\ 1 \leq i \leq n-1. \end{cases}$$

$$f(u_i u_{i+1}) = 1 + i, \ for\ 0 \leq i \leq n-1.$$

$$f(v_i v_{i+m}) = \begin{cases} 2n+m+i, & for\ 0 \leq i \leq n-m, \\ n+m+i, & for\ n-m+1 \leq i \leq n-1. \end{cases}$$

Now, let us evaluate the weights of u_i ‘s and v_i ‘s .

For any vertex $v \in V$, its weight $w(v) = f(v) + \sum_{u \in N(v)} f(uv)$

Thus $w(u_i) = f(u_i) + f(u_i u_{i+1}) + f(u_{i-1} u_i) + f(u_i v_i)$

and $w(v_i) = f(v_i) + f(u_i v_i) + f(v_i v_{i+m}) + f(v_{n-m+i} v_i)$

which are derived and observed as follows:

$w(u_i) = 8n + 3 + 2i$, *for* $0 \le i \le n - 1$.

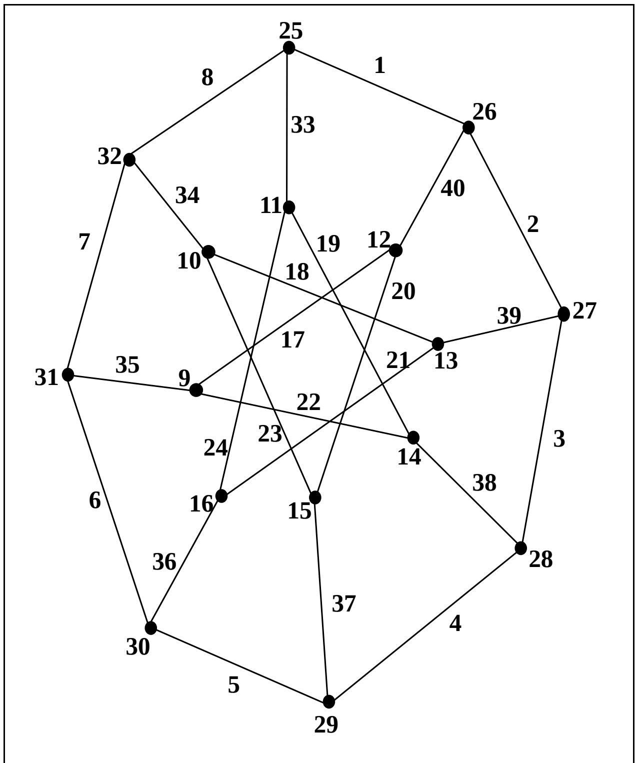
$w(v_i) = 10n + 2m + 1 + 2i$, *for* $0 \le i \le n - m$,

$\quad = 8n + 2m + 1 + 2i$, *for* $n - m + 1 \le i \le n - 1$.

Thus the vertex weights are given by $8n + 3, 8n + 5, \ldots, 8n + 3+(2n-1)d$.

Thus the set of all vertex-weights of $P(n,m)$ is $\{ a, a+d, a+2d, \ldots a+(2n-1)d \}$ where $a = 8n + 3$ and $d = 2$.

Example: This theorem is illustrated in Fig 2.2.2.



$w(u_i) = \{67,69,71,73,75,77,79,81\}$
$w(v_i) = \{87,89,91,93,95,97,83,85\}$
$w(u_i) \cup w(v_i) = \{67,69,71,\ldots,97\}$

Fig 2.2.2 - (8*n* + 3, 2) – VATL of P(8,3)

2.3 Vertex Antimagic Total Labeling of *r*P(*n, m*)

In this section, we study some vertex antimagic total labelings of *r* copies of Generalized Petersen graphs.

Theorem 2.3.1

For *n* odd, $n \geq 5$, r P(n,2) is a vertex antimagic graph.

Proof

Consider the labeling $f^t : V(rP(n,2)) \cup E(rP(n,2)) \rightarrow \{1,2,\ldots 5nr\}$ defined as follows:

For $i = nt, nt+1, nt+2, \ldots, nt+(n-1)$; where $t = 0,1,2,\ldots r-1$,

$$f(u_i) = \begin{cases} \frac{1}{2}(8n - i + 11nt), & for\ \ i - nt \equiv 0 \pmod{2}, \\ \frac{1}{2}(7n - i + 11nt), & for\ \ i - nt \equiv 1 \pmod{2}. \end{cases}$$

$$f(v_i) = \begin{cases} \frac{1}{2}(10n - i + 11nt), & for\ \ i - nt \equiv 0 \pmod{2}, \\ \frac{1}{2}(9n - i + 11nt), & for\ \ i - nt \equiv 1 \pmod{2}. \end{cases}$$

$$f(u_i v_i) = \begin{cases} \frac{1}{2}(2 + i + 9nt), & for\ \ i - nt \equiv 0 \pmod{2}, \\ \frac{1}{2}(n + 2 + i + 9nt), & for\ \ i - nt \equiv 1 \pmod{2}. \end{cases}$$

$$f(u_i u_{i+1}) = \begin{cases} 2n + 5nt + 1, & for\ \ i - nt = 0, \\ \frac{1}{2}(6n - i + 11nt + 2), & for\ \ i - nt \equiv 0 \pmod{2}, i - nt \neq 0, \\ \frac{1}{2}(5n - i + 11nt + 2), & for\ \ i - nt \equiv 1 \pmod{2}. \end{cases}$$

Consider the following two subcases to label the edges $v_i v_{i+2}$:

Case (i) : $n \equiv 1 \pmod 4$

$$f(v_i v_{i+2}) = \begin{cases} n + 5nt + 1, & \text{for } i - nt = 0\,, \\ \frac{1}{4}(5n - i + 21nt + 4), & \text{for } i - nt \equiv 1 \pmod 4, \\ \frac{1}{4}(6n - i + 21nt + 4), & \text{for } i - nt \equiv 2 \pmod 4, \\ \frac{1}{4}(7n - i + 21nt + 4), & \text{for } i - nt \equiv 3 \pmod 4, \\ \frac{1}{4}(8n - i + 21nt + 4), & \text{for } i - nt \equiv 0 \pmod 4,\ i - nt \neq 0. \end{cases}$$

Case (ii) : $n \equiv 3 \pmod 4$

$$f(v_i v_{i+2}) = \begin{cases} n + 5nt + 1, & \text{for } i - nt = 0\,, \\ \frac{1}{4}(7n - i + 21nt + 4), & \text{for } i - nt \equiv 1 \pmod 4, \\ \frac{1}{4}(6n - i + 21nt + 4), & \text{for } i - nt \equiv 2 \pmod 4, \\ \frac{1}{4}(5n - i + 21nt + 4), & \text{for } i - nt \equiv 3 \pmod 4, \\ \frac{1}{4}(8n - i + 21nt + 4), & \text{for } i - nt \equiv 0 \pmod 4,\ i - nt \neq 0. \end{cases}$$

Now, let us evaluate the weights of u_i ‘s and v_i ‘s .

For any vertex $v \in V$, its weight $w(v) = f(v) + \sum_{u \in N(v)} f(uv)$

Thus $w(u_i) = f(u_i) + f(u_i u_{i+1}) + f(u_{i-1} u_i) + f(u_i v_i)$

and $w(v_i) = f(v_i) + f(u_i v_i) + f(v_i v_{i+2}) + f(v_{n-2+i} v_i)$

which are derived for all possible cases and observed as follows:

For $i = nt, nt+1, nt+2, \ldots, nt+(n-1)$; where $t = 0,1,2,\ldots r-1$,

$$w(u_i) = \begin{cases} \frac{1}{2}(17n+40nt+7), & for\ \ i-nt=0\,, \\ \frac{1}{2}(17n+40nt+5), & for\ \ i-nt=1\,, \\ \frac{1}{2}(19n-2i+42nt+7), & for\ \ i-nt\neq 0,\ \ i-nt\neq 1\,. \end{cases}$$

$$w(v_i) = \begin{cases} \frac{1}{4}(30n+80nt+14), & for\ \ i-nt=0\,, \\ \frac{1}{4}(30n-i+81nt+12), & for\ \ i-nt=2, \\ \frac{1}{4}(32n-2i+82nt+14), & for\ \ i-nt\equiv 1\,(\text{mod }4)\,\&\,3\,(\text{mod }4), \\ \frac{1}{4}(30n-2i+82nt+14), & for\ \ i-nt\equiv 2\,(\text{mod }4)\,;\ i-nt<2, \\ \frac{1}{4}(34n-2i+82nt+14), & for\ \ i-nt\equiv 2\,(\text{mod }4)\,;\ i-nt>2, \\ \frac{1}{4}(34n-2i+82nt+14), & for\ \ i-nt\equiv 0\,(\text{mod }4),\ i-nt\neq 0. \end{cases}$$

We can prove $w(u_i) \neq w(u_j)$; $w(v_i) \neq w(v_j)$ and $w(u_i) \neq w(v_j)$ for all i, j .

For instance consider $i\text{-}nt \neq 0$ and $\neq 1$.

Then $w(u_i) = \frac{1}{2}\ (19n - 2i + 42nt + 7) \neq \frac{1}{2}\ (19n - 2j + 42nt + 7) = w(u_j)$.

Similarly, consider $i\text{-}nt \equiv 2$ (mod4).

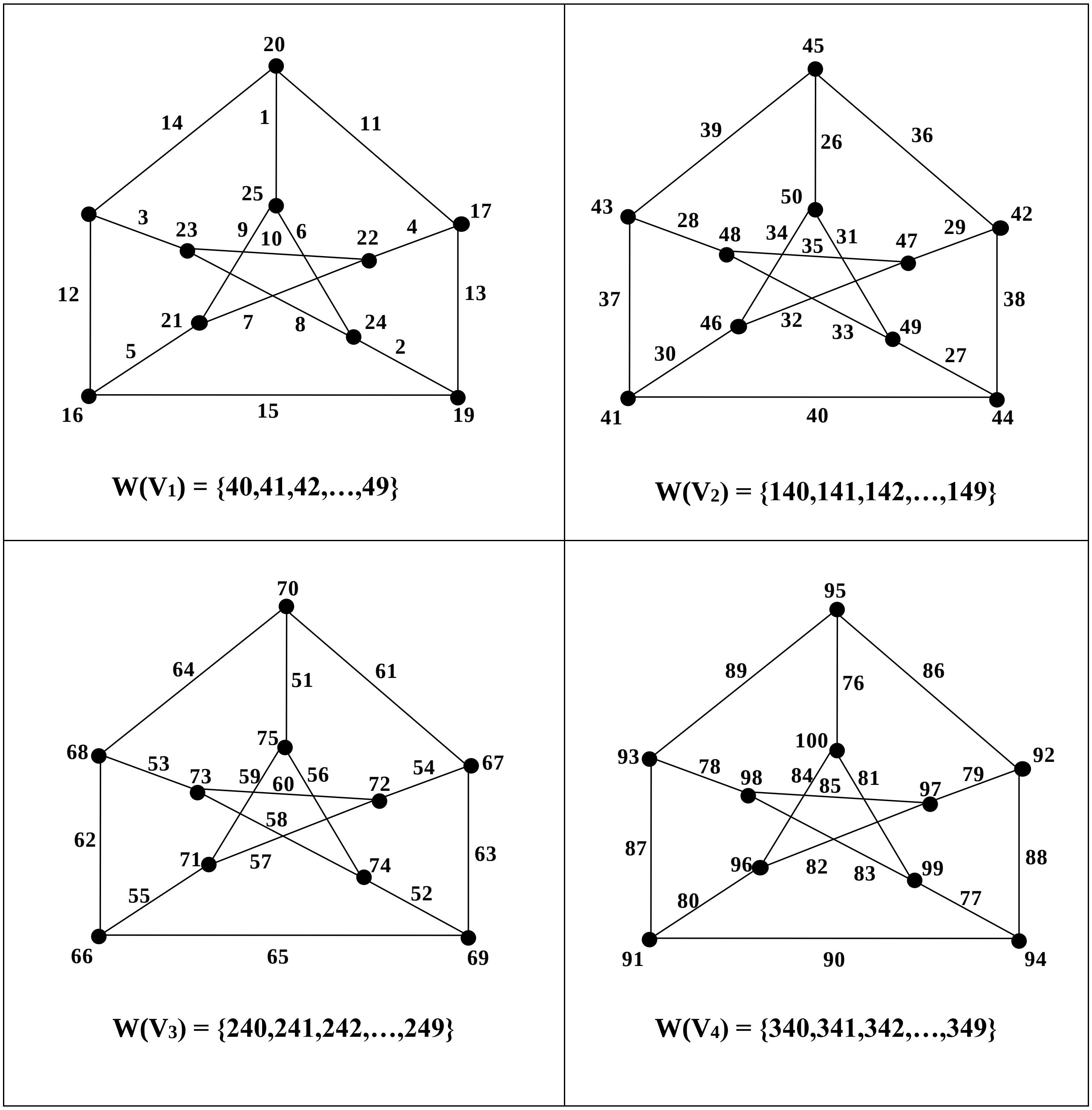
Then $w(v_i) = \frac{1}{2}\ (30n - 2i + 82nt + 14) \neq \frac{1}{2}(30n - 2j + 82nt + 14) = w(v_j)$.

Thus $w(u_i) \neq w(u_j)$; $w(v_i) \neq w(v_j)$ and $w(u_i) \neq w(v_j)$

for all i , $j = nt, nt+1, nt+2, \ldots, nt+(n-1)$; where $t = 0,1,2,\ldots r-1$.

Hence r P(n,2) is a vertex antimagic graph.

Example: This theorem is illustrated in Fig 2.3.1.



We can observe that $w(u_i) \neq w(u_j)$; $w(v_i) \neq w(v_j)$ and $w(u_i) \neq w(v_j)$ for any i,j

Fig 2.3.1 - VATL of 4 P(5,2)

Theorem 2.3.2

For n odd, $n \geq 7$, $r\,P(n,3)$ is a vertex antimagic graph.

Proof

Consider the labeling f : $V(rP(n,3)) \cup E(rP(n,3)) \rightarrow \{1,2,\ldots 5nr\}$ defined as follows:

We consider three possible cases.

Case 1: If $n \equiv 1 \pmod 6$

For $i = nt, nt+1, nt+2, \ldots, nt+(n-1)$; where $t = 0,1,2,\ldots r-1$,

$$f(u_i) = \begin{cases} \frac{1}{3}(12n - i + 16nt), & \text{for } i - nt \equiv 0 \pmod 3, \\ \frac{1}{3}(10n - i + 16nt), & \text{for } i - nt \equiv 1 \pmod 3, \\ \frac{1}{3}(11n - i + 16nt), & \text{for } i - nt \equiv 2 \pmod 3. \end{cases}$$

$$f(v_i) = \begin{cases} \frac{1}{3}(15n - i + 16nt), & \text{for } i - nt \equiv 0 \pmod 3, \\ \frac{1}{3}(13n - i + 16nt), & \text{for } i - nt \equiv 1 \pmod 3, \\ \frac{1}{3}(14n - i + 16nt), & \text{for } i - nt \equiv 2 \pmod 3. \end{cases}$$

$$f(u_i v_i) = \begin{cases} \frac{1}{3}(i + 14nt + 3), & \text{for } i - nt \equiv 0 \pmod 3, \\ \frac{1}{3}(2n + i + 14nt + 3), & \text{for } i - nt \equiv 1 \pmod 3, \\ \frac{1}{3}(n + i + 14nt + 3), & \text{for } i - nt \equiv 2 \pmod 3. \end{cases}$$

$$f(u_i u_{i+1}) = \begin{cases} 2n+5nt+1, & \text{for } i-nt=0\,, \\ \frac{1}{2}(6n-i+11nt+2), & \text{for } i-nt\equiv 0 \pmod 2, i-nt\neq 0, \\ \frac{1}{2}(5n-i+11nt+2), & \text{for } i-nt\equiv 1 \pmod 2. \end{cases}$$

$$f(v_i v_{i+3}) = \begin{cases} n+5nt+1, & \text{for } i-nt=0\,, \\ \frac{1}{6}(7n-i+31nt+6), & \text{for } i-nt\equiv 1 \pmod 6, \\ \frac{1}{6}(8n-i+31nt+6), & \text{for } i-nt\equiv 2 \pmod 6, \\ \frac{1}{6}(9n-i+31nt+6), & \text{for } i-nt\equiv 3 \pmod 6, \\ \frac{1}{6}(10n-i+31nt+6), & \text{for } i-nt\equiv 4 \pmod 6, \\ \frac{1}{6}(11n-i+31nt+6), & \text{for } i-nt\equiv 5 \pmod 6, \\ \frac{1}{6}(12n-i+31nt+6), & \text{for } i-nt\equiv 0 \pmod 6, i-nt\neq 0. \end{cases}$$

Now, let us evaluate the weights of u_i ‘s and v_i ‘s .

For any vertex $v \in V$, its weight $w(v) = f(v) + \sum_{u \in N(v)} f(uv)$

Thus $w(u_i) = f(u_i) + f(u_i u_{i+1}) + f(u_{i-1} u_i) + f(u_i v_i)$

and $w(v_i) = f(v_i) + f(u_i v_i) + f(v_i v_{i+3}) + f(v_{n-3+i} v_i)$

which are derived for all possible cases and observed as follows:

For $i = nt, nt+1, nt+2, \ldots, nt+(n-1)$; where $t = 0,1,2,\ldots r-1$,

$$w(u_i) = \begin{cases} \frac{1}{6}(51n + 120nt + 21), & for\ \ i - nt = 0\ , \\ \frac{1}{6}(51n + 120nt + 15), & for\ \ i - nt = 1\ , \\ \frac{1}{6}(57n - 6i + 126nt + 21), & for\ \ i - nt \neq 0,\ \ i - nt \neq 1\ . \end{cases}$$

$$w(v_i) = \begin{cases} \frac{1}{6}(45n + 120nt + 21), & for\ \ i - nt = 0\ , \\ \frac{1}{6}(45n + 120nt + 15), & for\ \ i - nt = 3, \\ \frac{1}{6}(51n - 2i + 122nt + 21), & for\ \ i - nt \equiv 0 \pmod 3), i - nt \neq 0, i - nt \neq 3, \\ \frac{1}{6}(47n - 2i + 122nt + 21), & for\ \ i - nt \equiv 1 \pmod 3)\ , \\ \frac{1}{6}(49n - 2i + 122nt + 21), & for\ \ i - nt \equiv 2 \pmod 3)\ , \end{cases}$$

We can prove $w(u_i) \neq w(u_j)$; $w(v_i) \neq w(v_j)$ and $w(u_i) \neq w(v_j)$ for all i, j .

For instance consider i-$nt \neq 0$ and $\neq 1$.

Then $w(u_i) = \frac{1}{6}\ (57n - 6i + 126nt + 21) \neq \frac{1}{6}\ (57n - 6j + 126nt + 21) = w(u_j)$.

Similarly, consider i-$nt \equiv 1$ (mod3).

Then $w(v_i) = \frac{1}{6}\ (47n - 2i + 122nt + 21) \neq \frac{1}{6}\ (47n - 2j + 122nt + 21) = w(v_j)$.

Thus $w(u_i) \neq w(u_j)$; $w(v_i) \neq w(v_j)$ and $w(u_i) \neq w(v_j)$

for all i , $j = nt, nt+1, nt+2, \ldots, nt+(n-1)$; where $t = 0,1,2,\ldots r-1$.

Hence r P(n,3) is a vertex antimagic graph.

Case 2: If $n \equiv 3 \pmod 6$

For $i = nt, nt+1, nt+2, \ldots, nt+(n-1)$; where $t = 0,1,2,\ldots r-1$,

$$f(u_i) = \begin{cases} \frac{1}{3}(12n - i + 16nt), & for\ \ i - nt \equiv 0 \pmod 3, \\ \frac{1}{3}(11n - i + 16nt + 1), & for\ \ i - nt \equiv 1 \pmod 3, \\ \frac{1}{3}(10n - i + 16nt + 2), & for\ \ i - nt \equiv 2 \pmod 3. \end{cases}$$

$$f(v_i) = \begin{cases} \frac{1}{3}(15n - i + 16nt), & for\ \ i - nt \equiv 0 \pmod 3, \\ \frac{1}{3}(14n - i + 16nt + 1), & for\ \ i - nt \equiv 1 \pmod 3, \\ \frac{1}{3}(13n - i + 16nt + 2), & for\ \ i - nt \equiv 2 \pmod 3. \end{cases}$$

$$f(u_i v_i) = \begin{cases} \frac{1}{3}(i + 14nt + 3), & for\ \ i - nt \equiv 0 \pmod 3, \\ \frac{1}{3}(n + i + 14nt + 2), & for\ \ i - nt \equiv 1 \pmod 3, \\ \frac{1}{3}(2n + i + 14nt + 1), & for\ \ i - nt \equiv 2 \pmod 3. \end{cases}$$

$$f(u_i u_{i+1}) = \begin{cases} 2n + 5nt + 1, & for\ \ i - nt = 0, \\ \frac{1}{2}(6n - i + 11nt + 2), & for\ \ i - nt \equiv 0 \pmod 2, i - nt \neq 0, \\ \frac{1}{2}(5n - i + 11nt + 2), & for\ \ i - nt \equiv 1 \pmod 2. \end{cases}$$

$$
f(v_i v_{i+3}) = \begin{cases} n+5nt+1, & for\ \ i-nt=0\,, \\ \frac{1}{6}(8n-i+31nt+7), & for\ \ i-nt\equiv 1\ (\mathrm{mod}\ 6), \\ \frac{1}{6}(10n-i+31nt+8), & for\ \ i-nt\equiv 2\ (\mathrm{mod}\ 6), \\ \frac{1}{6}(9n-i+31nt+6), & for\ \ i-nt\equiv 3\ (\mathrm{mod}\ 6), \\ \frac{1}{6}(11n-i+31nt+7), & for\ \ i-nt\equiv 4\ (\mathrm{mod}\ 6), \\ \frac{1}{6}(7n-i+31nt+8), & for\ \ i-nt\equiv 5\ (\mathrm{mod}\ 6), \\ \frac{1}{6}(12n-i+31nt+6), & for\ \ i-nt\equiv 0\ (\mathrm{mod}\ 6), i-nt\neq 0. \end{cases}
$$

Now, let us evaluate the weights of u_i 's and v_i 's .

For any vertex $v \in V$, its weight $w(v) = f(v) + \sum_{u\epsilon N(v)} f(uv)$

Thus $w(u_i) = f(u_i) + f(u_i u_{i+1}) + f(u_{i-1} u_i) + f(u_i v_i)$

and $w(v_i) = f(v_i) + f(u_i v_i) + f(v_i v_{i+3}) + f(v_{n-3+i} v_i)$ which are derived for all possible cases and observed as follows:

For $i = nt, nt+1, nt+2, \ldots, nt+(n-1)$; where $t = 0,1,2,\ldots r-1$,

$$
w(u_i) = \begin{cases} \frac{1}{6}(51n+120nt+21), & for\ \ i-nt=0\,, \\ \frac{1}{6}(51n+120nt+15), & for\ \ i-nt=1\,, \\ \frac{1}{6}(57n-6i+126nt+21), & for\ \ i-nt\neq 0,\ \ i-nt\neq 1\,. \end{cases}
$$

$$
w(v_i) = \begin{cases} \frac{1}{6}(47n + 120nt + 21), & for\ \ i - nt = 0\ , \\ \frac{1}{6}(45n + 120nt + 21), & for\ \ i - nt = 1\ , \\ \frac{1}{6}(49n + 120nt + 21), & for\ \ i - nt = 2\ , \\ \frac{1}{6}(45n + 120nt + 15), & for\ \ i - nt = 3, \\ \frac{1}{6}(51n - 2i + 122nt + 21), & for\ \ i - nt \equiv 0\ (\text{mod } 3), i - nt \neq 0, i - nt \neq 3, \\ \frac{1}{6}(49n - 2i + 122nt + 23), & for\ \ i - nt \equiv 1\ (\text{mod } 3)\ , i - nt \neq 1, \\ \frac{1}{6}(47n - 2i + 122nt + 25), & for\ \ i - nt \equiv 2\ (\text{mod } 3)\ , i - nt \neq 2. \end{cases}
$$

We can prove $w(u_i) \neq w(u_j)$; $w(v_i) \neq w(v_j)$ and $w(u_i) \neq w(v_j)$ for all i, j .

For instance consider i-$nt \neq 0$ and $\neq 1$

Then $w(u_i) = \frac{1}{6}\ (57n - 6i + 126nt + 21) \neq \frac{1}{6}\ (57n - 6j + 126nt + 21) = w(u_j)$.

Similarly, consider i-$nt \equiv 2$ (mod3) , i-$nt \neq 2$

Then $w(v_i) = \frac{1}{6}\ (47n - 2i + 122nt + 25) \neq \frac{1}{6}\ (47n - 2j + 122nt + 25) = w(v_j)$.

Thus $w(u_i) \neq w(u_j)$; $w(v_i) \neq w(v_j)$ and $w(u_i) \neq w(v_j)$
for all i , $j = nt, nt+1, nt+2, \ldots, nt+(n-1)$; where $t = 0,1,2,\ldots r-1$.

Hence r P(n,3) is a vertex antimagic graph.

Case 3: If $n \equiv 5 \pmod 6$

For $i = nt, nt+1, nt+2, \ldots, nt+(n-1)$; where $t = 0,1,2,\ldots r-1$,

$$f(u_i) = \begin{cases} \frac{1}{3}(12n - i + 16nt), & for\ \ i - nt \equiv 0 \ (\mathrm{mod}\ 3), \\ \frac{1}{3}(11n - i + 16nt), & for\ \ i - nt \equiv 1 \ (\mathrm{mod}\ 3), \\ \frac{1}{3}(10n - i + 16nt), & for\ \ i - nt \equiv 2 \ (\mathrm{mod}\ 3). \end{cases}$$

$$f(v_i) = \begin{cases} \frac{1}{3}(15n - i + 16nt), & for\ \ i - nt \equiv 0 \ (\mathrm{mod}\ 3), \\ \frac{1}{3}(14n - i + 16nt), & for\ \ i - nt \equiv 1 \ (\mathrm{mod}\ 3), \\ \frac{1}{3}(13n - i + 16nt), & for\ \ i - nt \equiv 2 \ (\mathrm{mod}\ 3). \end{cases}$$

$$f(u_i v_i) = \begin{cases} \frac{1}{3}(i + 14nt + 3), & for\ \ i - nt \equiv 0 \ (\mathrm{mod}\ 3), \\ \frac{1}{3}(n + i + 14nt + 3), & for\ \ i - nt \equiv 1 \ (\mathrm{mod}\ 3), \\ \frac{1}{3}(2n + i + 14nt + 3), & for\ \ i - nt \equiv 2 \ (\mathrm{mod}\ 3). \end{cases}$$

$$f(u_i u_{i+1}) = \begin{cases} 2n + 5nt + 1, & for\ \ i - nt = 0, \\ \frac{1}{2}(6n - i + 11nt + 2), & for\ \ i - nt \equiv 0 \ (\mathrm{mod}\ 2), i - nt \neq 0, \\ \frac{1}{2}(5n - i + 11nt + 2), & for\ \ i - nt \equiv 1 \ (\mathrm{mod}\ 2). \end{cases}$$

$$f(v_i v_{i+3}) = \begin{cases} n+5nt+1, & for \;\; i-nt=0\,, \\ \frac{1}{6}(11n-i+31nt+6), & for \;\; i-nt \equiv 1 \pmod 6, \\ \frac{1}{6}(10n-i+31nt+6), & for \;\; i-nt \equiv 2 \pmod 6, \\ \frac{1}{6}(9n-i+31nt+6), & for \;\; i-nt \equiv 3 \pmod 6, \\ \frac{1}{6}(8n-i+31nt+6), & for \;\; i-nt \equiv 4 \pmod 6, \\ \frac{1}{6}(7n-i+31nt+6), & for \;\; i-nt \equiv 5 \pmod 6, \\ \frac{1}{6}(12n-i+31nt+6), & for \;\; i-nt \equiv 0 \pmod 6, i-nt \neq 0. \end{cases}$$

Now, let us evaluate the weights of u_i ‘s and v_i ‘s .

For any vertex $v \in V$, its weight $w(v) = f(v) + \sum_{u \epsilon N(v)} f(uv)$

Thus $w(u_i) = f(u_i) + f(u_i u_{i+1}) + f(u_{i-1} u_i) + f(u_i v_i)$

and $w(v_i) = f(v_i) + f(u_i v_i) + f(v_i v_{i+3}) + f(v_{n-3+i} v_i)$ which are derived for all possible cases and observed as follows:

For $i = nt, nt+1, nt+2, \ldots, nt+(n-1)$; where $t = 0,1,2,\ldots r-1$,

$$w(u_i) = \begin{cases} \frac{1}{6}(51n+120nt+21), & for \;\; i-nt=0\,, \\ \frac{1}{6}(51n+120nt+15), & for \;\; i-nt=1\,, \\ \frac{1}{6}(57n-6i+126nt+21), & for \;\; i-nt \neq 0, \; i-nt \neq 1\,. \end{cases}$$

$$w(v_i) = \begin{cases} \frac{1}{6}(45n + 120nt + 21), & for \;\; i - nt = 0\,, \\ \frac{1}{6}(45n + 120nt + 15), & for \;\; i - nt = 3\,, \\ \frac{1}{6}(51n - 2i + 122nt + 21), & for \;\; i - nt \equiv 0 \pmod 3, i - nt \neq 0, i - nt \neq 3, \\ \frac{1}{6}(49n - 2i + 122nt + 21), & for \;\; i - nt \equiv 1 \pmod 3\,, \\ \frac{1}{6}(47n - 2i + 122nt + 21), & for \;\; i - nt \equiv 2 \pmod 3\,. \end{cases}$$

We can prove $w(u_i) \neq w(u_j)$; $w(v_i) \neq w(v_j)$ and $w(u_i) \neq w(v_j)$ for all i, j .

For instance consider $i\text{-}nt \neq 0$ and $\neq 1$.

Then $w(u_i) = \frac{1}{6}\,(57n - 6i + 126nt + 21) \neq \frac{1}{6}\,(57n - 6j + 126nt + 21) = w(u_j)$.

Similarly, consider $i\text{-}nt \equiv 1$ (mod3) .

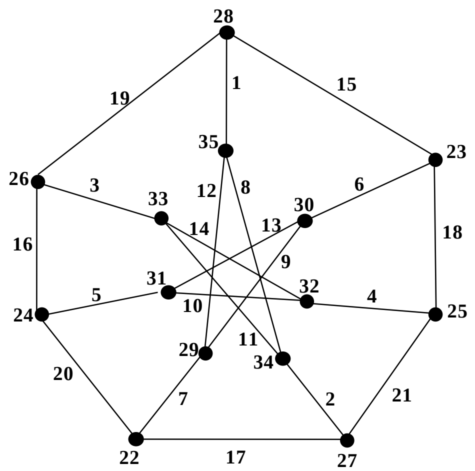
Then $w(v_i) = \frac{1}{6}\,(49n - 2i + 122nt + 21) \neq \frac{1}{6}\,(49n - 2j + 122nt + 21) = w(v_j)$.

Thus $w(u_i) \neq w(u_j)$; $w(v_i) \neq w(v_j)$ and $w(u_i) \neq w(v_j)$

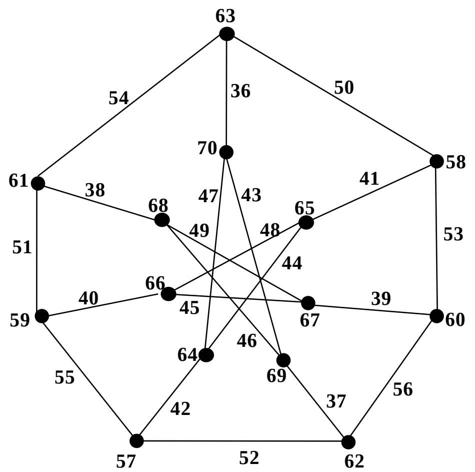
for all i , $j = nt, nt+1, nt+2, \ldots, nt+(n-1)$; where $t = 0,1,2,\ldots r-1$.

Hence r P(n,3) is a vertex antimagic graph.

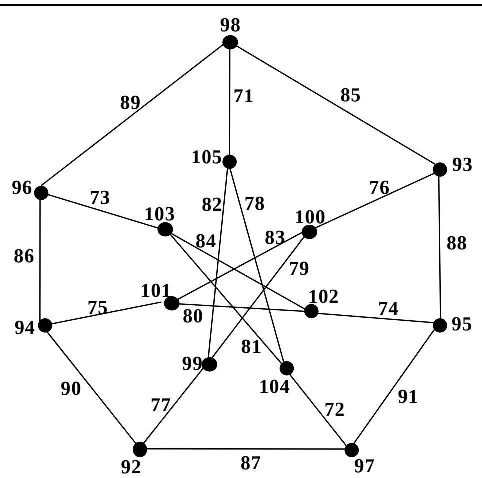
Example: This theorem is illustrated in Fig 2.3.2.



$W(V_1) = \{55,56,57,\ldots,68\}$



$W(V_2) = \{195,196,197,\ldots,208\}$



$W(V_3) = \{335,336,337,\ldots,348\}$

We can observe that $w(u_i) \neq w(u_j)$; $w(v_i) \neq w(v_j)$ and $w(u_i) \neq w(v_j)$ for any i,j

Fig 2.3.2 - VATL of 3 P(7,3)

Theorem 2.3.3

For n odd, $n \geq 9$, $r\,P(n,4)$ is a vertex antimagic graph.

Proof

Consider the labeling $f^t : V(rP(n,4)) \cup E(rP(n,4)) \to \{1,2,\ldots,5nr\}$ defined as follows:

We consider two possible cases.

Case 1: If $n \equiv 1 \pmod 4$

For $i = nt, nt+1, nt+2, \ldots, nt+(n-1)$; where $t = 0,1,2,\ldots r-1$,

$$f(u_i) = \begin{cases} \frac{1}{4}(16n - i + 21nt), & \text{for } i - nt \equiv 0 \pmod 4, \\ \frac{1}{4}(13n - i + 21nt), & \text{for } i - nt \equiv 1 \pmod 4, \\ \frac{1}{4}(14n - i + 21nt), & \text{for } i - nt \equiv 2 \pmod 4, \\ \frac{1}{4}(15n - i + 21nt), & \text{for } i - nt \equiv 3 \pmod 4. \end{cases}$$

$$f(v_i) = \begin{cases} \frac{1}{4}(20n - i + 21nt), & \text{for } i - nt \equiv 0 \pmod 4, \\ \frac{1}{4}(17n - i + 21nt), & \text{for } i - nt \equiv 1 \pmod 4, \\ \frac{1}{4}(18n - i + 21nt), & \text{for } i - nt \equiv 2 \pmod 4, \\ \frac{1}{4}(19n - i + 21nt), & \text{for } i - nt \equiv 3 \pmod 4. \end{cases}$$

$$f(u_i v_i) = \begin{cases} \frac{1}{4}(i+19nt+4), & for\ \ i-nt \equiv 0 \pmod 4, \\ \frac{1}{4}(3n+i+19nt+4), & for\ \ i-nt \equiv 1 \pmod 4, \\ \frac{1}{4}(2n+i+19nt+4), & for\ \ i-nt \equiv 2 \pmod 4, \\ \frac{1}{4}(n+i+19nt+4), & for\ \ i-nt \equiv 3 \pmod 4. \end{cases}$$

$$f(u_i u_{i+1}) = \begin{cases} 2n+5nt+1, & for\ \ i-nt = 0, \\ \frac{1}{2}(6n-i+11nt+2), & for\ \ i-nt \equiv 0 \pmod 2, i-nt \neq 0, \\ \frac{1}{2}(5n-i+11nt+2), & for\ \ i-nt \equiv 1 \pmod 2. \end{cases}$$

For labeling edges $v_i v_{i+4}$, we have the following subcases:

Case (i) $n \equiv 1 \pmod 8$

$$f(v_i v_{i+4}) = \begin{cases} \frac{1}{8}(9n-i+41nt+8), & for\ \ i-nt \equiv 1 \pmod 8, \\ \frac{1}{8}(10n-i+41nt+8), & for\ \ i-nt \equiv 2 \pmod 8, \\ \frac{1}{8}(11n-i+41nt+8), & for\ \ i-nt \equiv 3 \pmod 8, \\ \frac{1}{8}(12n-i+41nt+8), & for\ \ i-nt \equiv 4 \pmod 8, \\ \frac{1}{8}(13n-i+41nt+8), & for\ \ i-nt \equiv 5 \pmod 8, \\ \frac{1}{8}(14n-i+41nt+8), & for\ \ i-nt \equiv 6 \pmod 8, \\ \frac{1}{8}(15n-i+41nt+8), & for\ \ i-nt \equiv 7 \pmod 8, \\ \frac{1}{8}(16n-i+41nt+8), & for\ \ i-nt \equiv 0 \pmod 8, i-nt \neq 0. \end{cases}$$

$f(v_i v_{i+4}) = n+5nt+1$ for $i-nt = 0$.

Case (ii) $n \equiv 5 \pmod 8$

$$f(v_i v_{i+4}) = \begin{cases} \frac{1}{8}(13n - i + 41nt + 8), & for\ \ i - nt \equiv 1 \pmod 8, \\ \frac{1}{8}(10n - i + 41nt + 8), & for\ \ i - nt \equiv 2 \pmod 8, \\ \frac{1}{8}(15n - i + 41nt + 8), & for\ \ i - nt \equiv 3 \pmod 8, \\ \frac{1}{8}(12n - i + 41nt + 8), & for\ \ i - nt \equiv 4 \pmod 8, \\ \frac{1}{8}(9n - i + 41nt + 8), & for\ \ i - nt \equiv 5 \pmod 8, \\ \frac{1}{8}(14n - i + 41nt + 8), & for\ \ i - nt \equiv 6 \pmod 8, \\ \frac{1}{8}(11n - i + 41nt + 8), & for\ \ i - nt \equiv 7 \pmod 8, \\ \frac{1}{8}(16n - i + 41nt + 8), & for\ \ i - nt \equiv 0 \pmod 8, i - nt \neq 0. \end{cases}$$

$f(v_i v_{i+4}) = n+5nt+1$ for $i\text{-}nt = 0$.

Now, let us evaluate the weights of u_i 's and v_i 's .

For any vertex $v \in V$, its weight $w(v) = f(v) + \sum_{u \in N(v)} f(uv)$.

Thus $w(u_i) = f(u_i) + f(u_i u_{i+1}) + f(u_{i-1} u_i) + f(u_i v_i)$

and $w(v_i) = f(v_i) + f(u_i v_i) + f(v_i v_{i+4}) + f(v_{n-4+i} v_i)$

which are derived for all possible cases and observed as follows:

For $i = nt, nt+1, nt+2, \ldots, nt+(n-1)$; where $t = 0,1,2,\ldots r-1$,

$$w(u_i) = \begin{cases} \frac{1}{2}(17n+40nt+7), & for\ \ i-nt=0\,, \\ \frac{1}{2}(17n+40nt+5), & for\ \ i-nt=1\,, \\ \frac{1}{2}(19n-2i+42nt+7), & for\ \ i-nt\neq 0,\ \ i-nt\neq 1\,. \end{cases}$$

$$w(v_i) = \begin{cases} \frac{1}{4}(30n+80nt+14), & for\ \ i-nt=0\,, \\ \frac{1}{4}(30n+80nt+10), & for\ \ i-nt=4\,, \\ \frac{1}{4}(34n-i+81nt+14), & for\ \ i-nt\equiv 0\ (\mathrm{mod}\ 4), i-nt\neq 0, i-nt\neq 4, \\ \frac{1}{4}(31n-i+81nt+14), & for\ \ i-nt\equiv 1\ (\mathrm{mod}\ 4)\,, \\ \frac{1}{4}(32n-i+81nt+14), & for\ \ i-nt\equiv 2\ (\mathrm{mod}\ 4)\,, \\ \frac{1}{4}(33n-i+81nt+14), & for\ \ i-nt\equiv 3\ (\mathrm{mod}\ 4)\,. \end{cases}$$

We can prove $w(u_i) \neq w(u_j)$; $w(v_i) \neq w(v_j)$ and $w(u_i) \neq w(v_j)$ for all i, j .

For instance consider $i\text{-}nt \neq 0$ and $\neq 1$.

Then $w(u_i) = \frac{1}{2}\ (19n-2i+42nt+7) \neq \frac{1}{2}\ (19n-2j+42nt+7) = w(u_j)$.

Similarly, consider $i\text{-}nt \equiv 3$ (mod4) .

Then $w(v_i) = \frac{1}{4}\ (33n-i+81nt+14) \neq \frac{1}{4}\ (33n-j+81nt+14) = w(v_j)$.

Thus $w(u_i) \neq w(u_j)$; $w(v_i) \neq w(v_j)$ and $w(u_i) \neq w(v_j)$

for all $i, j = nt, nt+1, nt+2, \ldots, nt+(n-1)$; where $t = 0,1,2,\ldots r-1$.

Hence r P(n,4) is a vertex antimagic graph.

Case 2: If $n \equiv 3 \pmod 4$

For $i = nt, nt+1, nt+2, \ldots, nt+(n-1)$; where $t = 0,1,2,\ldots r-1$,

$$f(u_i) = \begin{cases} \frac{1}{4}(16n - i + 21nt), & for\ \ i - nt \equiv 0 \ (\text{mod } 4), \\ \frac{1}{4}(15n - i + 21nt), & for\ \ i - nt \equiv 1 \ (\text{mod } 4), \\ \frac{1}{4}(14n - i + 21nt), & for\ \ i - nt \equiv 2 \ (\text{mod } 4), \\ \frac{1}{4}(13n - i + 21nt), & for\ \ i - nt \equiv 3 \ (\text{mod } 4). \end{cases}$$

$$f(v_i) = \begin{cases} \frac{1}{4}(20n - i + 21nt), & for\ \ i - nt \equiv 0 \ (\text{mod } 4), \\ \frac{1}{4}(19n - i + 21nt), & for\ \ i - nt \equiv 1 \ (\text{mod } 4), \\ \frac{1}{4}(18n - i + 21nt), & for\ \ i - nt \equiv 2 \ (\text{mod } 4), \\ \frac{1}{4}(17n - i + 21nt), & for\ \ i - nt \equiv 3 \ (\text{mod } 4). \end{cases}$$

$$f(u_i v_i) = \begin{cases} \frac{1}{4}(i + 19nt + 4), & for\ \ i - nt \equiv 0 \ (\text{mod } 4), \\ \frac{1}{4}(n + i + 19nt + 4), & for\ \ i - nt \equiv 1 \ (\text{mod } 4), \\ \frac{1}{4}(2n + i + 19nt + 4), & for\ \ i - nt \equiv 2 \ (\text{mod } 4), \\ \frac{1}{4}(3n + i + 19nt + 4), & for\ \ i - nt \equiv 3 \ (\text{mod } 4). \end{cases}$$

$$f(u_i u_{i+1}) = \begin{cases} 2n+5nt+1, & \text{for } i-nt=0, \\ \frac{1}{2}(6n-i+11nt+2), & \text{for } i-nt \equiv 0 \pmod{2}, i-nt \neq 0, \\ \frac{1}{2}(5n-i+11nt+2), & \text{for } i-nt \equiv 1 \pmod{2}. \end{cases}$$

For labeling edges $v_i v_{i+4}$, we have the following subcases:

Case (i) $\boldsymbol{n \equiv 3 \pmod 8}$

$$f(v_i v_{i+4}) = \begin{cases} \frac{1}{8}(11n-i+41nt+8), & \text{for } i-nt \equiv 1 \pmod{8}, \\ \frac{1}{8}(14n-i+41nt+8), & \text{for } i-nt \equiv 2 \pmod{8}, \\ \frac{1}{8}(9n-i+41nt+8), & \text{for } i-nt \equiv 3 \pmod{8}, \\ \frac{1}{8}(12n-i+41nt+8), & \text{for } i-nt \equiv 4 \pmod{8}, \\ \frac{1}{8}(15n-i+41nt+8), & \text{for } i-nt \equiv 5 \pmod{8}, \\ \frac{1}{8}(10n-i+41nt+8), & \text{for } i-nt \equiv 6 \pmod{8}, \\ \frac{1}{8}(13n-i+41nt+8), & \text{for } i-nt \equiv 7 \pmod{8}, \\ \frac{1}{8}(16n-i+41nt+8), & \text{for } i-nt \equiv 0 \pmod{8}, i-nt \neq 0. \end{cases}$$

$f(v_i v_{i+4}) = n+5nt+1$ for $i-nt = 0$.

Case (ii) $n \equiv 7$ (mod8)

$$f(v_i v_{i+4}) = \begin{cases} \frac{1}{8}(15n - i + 41nt + 8), & for \;\; i - nt \equiv 1 \pmod 8, \\ \frac{1}{8}(14n - i + 41nt + 8), & for \;\; i - nt \equiv 2 \pmod 8, \\ \frac{1}{8}(13n - i + 41nt + 8), & for \;\; i - nt \equiv 3 \pmod 8, \\ \frac{1}{8}(12n - i + 41nt + 8), & for \;\; i - nt \equiv 4 \pmod 8, \\ \frac{1}{8}(11n - i + 41nt + 8), & for \;\; i - nt \equiv 5 \pmod 8, \\ \frac{1}{8}(10n - i + 41nt + 8), & for \;\; i - nt \equiv 6 \pmod 8, \\ \frac{1}{8}(9n - i + 41nt + 8), & for \;\; i - nt \equiv 7 \pmod 8, \\ \frac{1}{8}(16n - i + 41nt + 8), & for \;\; i - nt \equiv 0 \pmod 8, i - nt \neq 0. \end{cases}$$

$f(v_i v_{i+4}) = n+5nt+1$ for $i-nt = 0$.

Now, let us evaluate the weights of u_i ‘s and v_i ‘s .

For any vertex $v \in V$, its weight $w(v) = f(v) + \sum_{u \epsilon N(v)} f(uv)$.

Thus $w(u_i) = f(u_i) + f(u_i u_{i+1}) + f(u_{i-1} u_i) + f(u_i v_i)$

and $w(v_i) = f(v_i) + f(u_i v_i) + f(v_i v_{i+4}) + f(v_{n-4+i} v_i)$

which are derived for all possible cases and observed as follows:

For $i = nt, nt+1, nt+2, \ldots, nt+(n-1)$; where $t = 0,1,2,\ldots r-1$,

$$w(u_i) = \begin{cases} \frac{1}{2}(17n+40nt+7), & for \;\; i-nt=0\,, \\ \frac{1}{2}(17n+40nt+5), & for \;\; i-nt=1\,, \\ \frac{1}{2}(19n-2i+42nt+7), & for \;\; i-nt\neq 0,\;\; i-nt\neq 1\,. \end{cases}$$

$$w(v_i) = \begin{cases} \frac{1}{4}(30n+80nt+14), & for \;\; i-nt=0\,, \\ \frac{1}{4}(30n+80nt+10), & for \;\; i-nt=4\,, \\ \frac{1}{4}(34n-i+81nt+14), & for \;\; i-nt\equiv 0 \pmod 4, i-nt\neq 0, i-nt\neq 4, \\ \frac{1}{4}(33n-i+81nt+14), & for \;\; i-nt\equiv 1 \pmod 4\,, \\ \frac{1}{4}(32n-i+81nt+14), & for \;\; i-nt\equiv 2 \pmod 4\,, \\ \frac{1}{4}(31n-i+81nt+14), & for \;\; i-nt\equiv 3 \pmod 4\,. \end{cases}$$

We can prove $w(u_i) \neq w(u_j)$; $w(v_i) \neq w(v_j)$ and $w(u_i) \neq w(v_j)$ for all i, j .

For instance consider $i\text{-}nt \neq 0$ and $\neq 1$.

Then $w(u_i) = \frac{1}{2}\,(19n - 2i + 42nt + 7) \neq \frac{1}{2}\,(19n - 2j + 42nt + 7) = w(u_j)$.

Similarly, consider $i\text{-}nt \equiv 2 \pmod 4$.

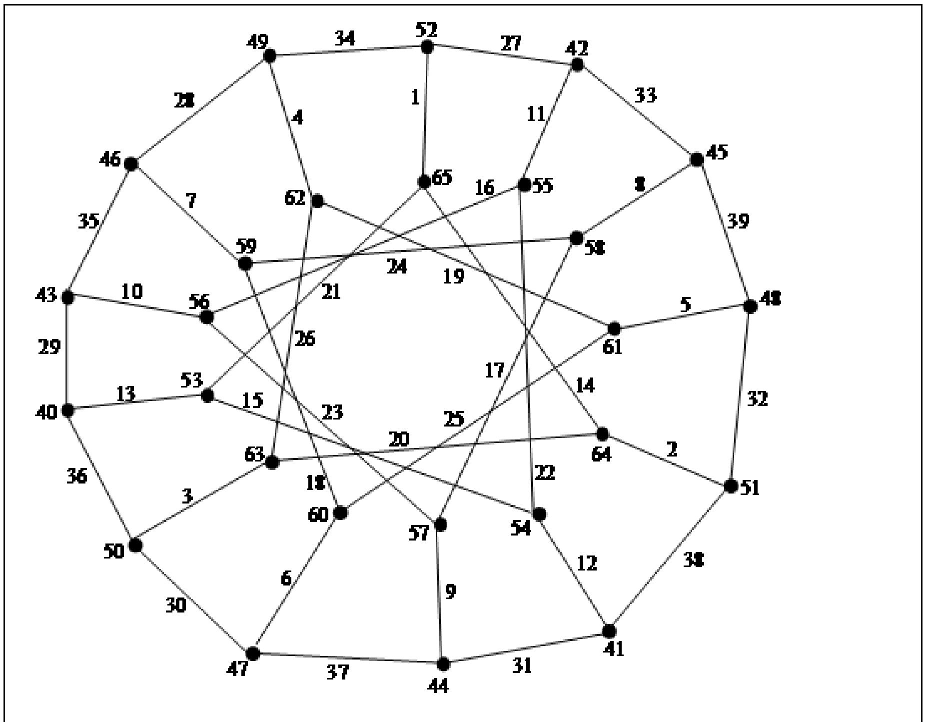
Then $w(v_i) = \frac{1}{4}\,(32n - i + 81nt + 14) \neq \frac{1}{4}\,(32n - j + 81nt + 14) = w(v_j)$.

Thus $w(u_i) \neq w(u_j)$; $w(v_i) \neq w(v_j)$ and $w(u_i) \neq w(v_j)$

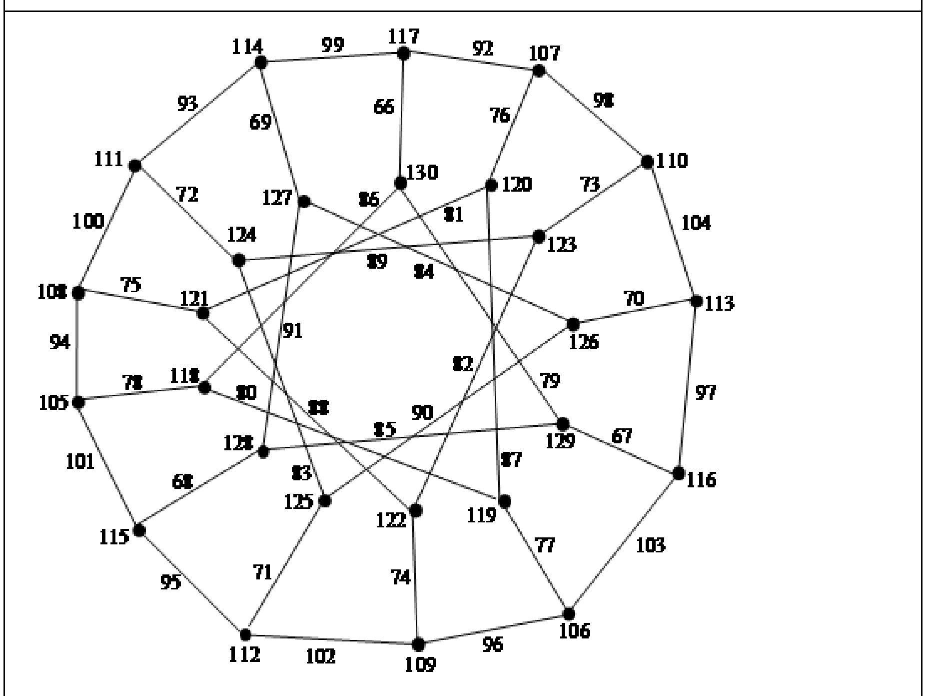
for all i , $j = nt, nt+1, nt+2, \ldots, nt+(n-1)$; where $t = 0,1,2,\ldots r-1$.

Hence $r\,P(n,4)$ is a vertex antimagic graph.

Example: This theorem is illustrated in Fig 2.3.3.



W(V_1) = {100,101,102,…,125}



W(V_2) = {360,361,362,…,385}

We can observe that $w(u_i) \neq w(u_j)$; $w(v_i) \neq w(v_j)$ and $w(u_i) \neq w(v_j)$ for any i,j

Fig 2.3.3 - VATL of 2 P(13,4)

Theorem 2.3.4

For $n \geq 3$, $1 \leq m \leq \left\lfloor \frac{n-1}{2} \right\rfloor$, $rP(n,m)$ is a vertex antimagic total graph.

Proof

Consider the labeling $f^t : V(rP(n,m)) \cup E(rP(n,m)) \rightarrow \{1,2,\ldots 5nr\}$ defined as :

For $i = nt, nt+1, nt+2, \ldots, nt+(n-1)$; where $t = 0,1,2,\ldots r-1$,

$f(u_i) = 4nt+3n+1+i, \quad nt \leq i \leq nt+(n-1)$

$$f(v_i) = \begin{cases} 4nt+n+m+i, & nt \leq i \leq n(t+1)-m \\ 4nt+m+i, & n(t+1)-m+1 \leq i \leq n(t+1)-1 \end{cases}$$

$f(u_i u_{i+1}) = 4nt+1+i, \quad nt \leq i \leq nt+(n-1)$

$$f(u_i v_i) = \begin{cases} 5nt+4n+1, & i = nt \\ 6nt+5n+1-i, & nt+1 \leq i \leq n(t+1)-1 \end{cases}$$

$$f(v_i v_{i+m}) = \begin{cases} 4nt+2n+m+i, & nt \leq i \leq n(t+1)-m \\ 4nt+n+m+i, & n(t+1)-m+1 \leq i \leq n(t+1)-1 \end{cases}$$

Now, let us evaluate the weights of u_i 's and v_i 's .

For any vertex $v \in V$, its weight $w(v) = f(v) + \sum_{u \in N(v)} f(uv)$.

Consider $w(u_i) = f(u_i) + f(u_i u_{i+1}) + f(u_{i-1} u_i) + f(u_i v_i)$

Case 1: For $i = nt+1, nt+2, \ldots, nt+(n-1)$; where $t = 0,1,2,\ldots r-1$,

$w(u_i) = \{4nt+3n+1+i\}+\{4nt+1+i\}+\{4nt+1+(i-1)\}+\{6nt+5n+1-i\} = 18nt+8n+2i+3$

Case 2: For $i = nt$,

$w(u_i) = \{4nt+3n+1+i\}+\{4nt+1+i\}+\{4nt+1+(t+1)n-1\}+\{5nt+4n+1\} = 18nt+8n+2i+3$

Thus $w(u_i) = 18nt+8n+2i+3$ for all $i = nt, nt+1, nt+2, \ldots, nt+(n-1)$; $t = 0,1,2,\ldots r-1$

Consider $w(v_i) = f(v_i) + f(u_i v_i) + f(v_i v_{i+m}) + f(v_{n-m+i} v_i)$

Case 1: For $i = nt$

$w(v_i)=\{4nt+n+m+i\}+\{5nt+4n+1\}+\{4nt+2n+m+i\}+\{4nt+2n+m+(n-m+i)\}$

$= 18nt+10n+2m+2i+1$

Case 2: For $nt+1 \leq i \leq n(t+1)-m$

$w(v_i)=\{4nt+n+m+i\}+\{6nt+5n+1-i\}+\{4nt+2n+m+i\}+\{4nt+n+m+(n-m+i)\}$

$= 8nt+10n+2m+2i+1$

Case 3: For $n(t+1)-m+1 \leq i \leq n(t+1)-1$

$w(v_i) = \{4nt+m+i\}+\{6nt+5n+1-i\}+\{4nt+n+m+i\}+\{4nt+n+m+(n-m+i)\}$

$= 18nt+8n+2m+2i+1$

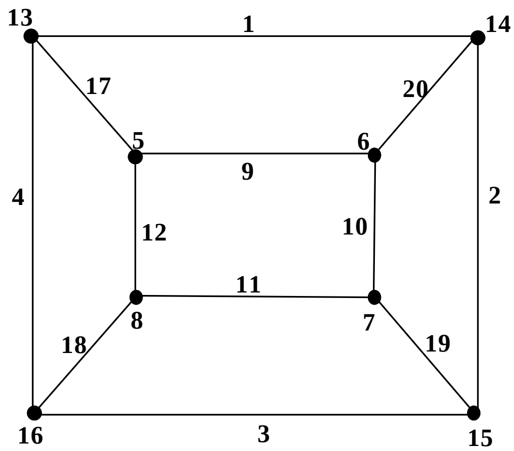
Thus $w(v_i) = \begin{cases} 18nt + 10n + 2m + 2i + 1, & nt \leq i \leq n(t+1) - m \\ 18nt + 8n + 2m + 2i + 1, & n(t+1) - m + 1 \leq i \leq n(t+1) - 1 \end{cases}$

where $t = 0,1,2,\ldots r-1$

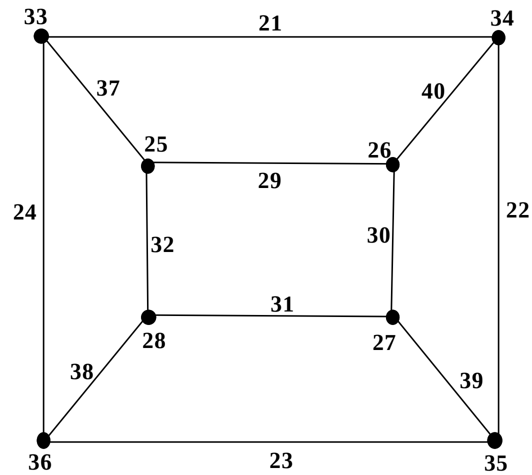
$w(u_i) \neq w(u_j)$; $w(v_i) \neq w(v_j)$ and $w(u_i) \neq w(v_j)$ for any $i,j \in \{nt, nt+1, \ldots, nt+(n-1)\}$.

Hence f is a *vertex antimagic total labeling* and $rP(n,m)$ is a vertex antimagic total graph.

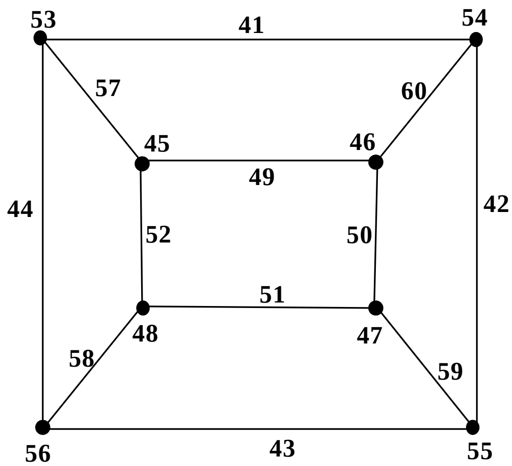
Example: This theorem is illustrated in Fig 2.3.4.



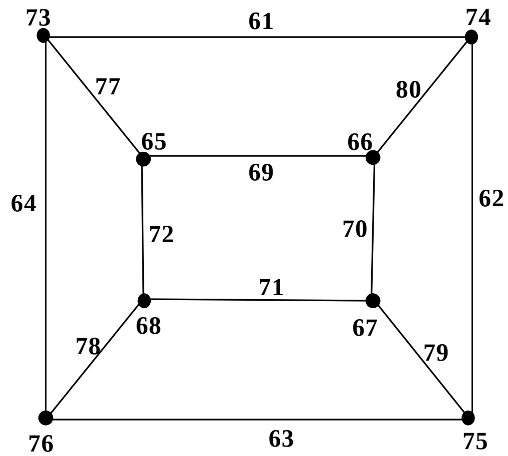
W(V_1) = {35,37,39,…,49}



W(V_2) = {115,117,119,…,129}



W(V_3) = {195,197,199,…,209}



W(V_4) = {275,277,279,…,289}

We can observe that $w(u_i) \neq w(u_j)$; $w(v_i) \neq w(v_j)$ and $w(u_i) \neq w(v_j)$ for any i,j

Fig 2.3.4 - VATL of 4P(4,1)