# MATB43: Introduction to Analysis
# Lecture Notes

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Pre-reqs are MATA37. Instructor is John Scherk. If you find any problems in these notes, feel free to contact me at conconjoshua@gmail.com.

## Contents

# 1 Friday, January 5, 2018

## 1.1 Countability

Countability is all about the cardinality of sets

**Definition 1.1.** For sets $X, Y$, they have the same cardinality if there exists a bijection between them ($f : X \mapsto Y$ has an inverse as well). Equality of cardinality of 2 sets is written as $card(X) = card(Y)$

**Example 1.2.** $card(\{1, 2, 3, 4, 5\}) = card(\{a, b, c, d, e\})$

**Example 1.3.** $card(\mathbb{N}) = card(2\mathbb{N})$

This example is correct because $f : \mathbb{N} \mapsto 2\mathbb{N}, f(n) = 2n$ is a bijection.

**Example 1.4.** $card(\{\text{set of odd numbers}\}) = card(\mathbb{N})$

This example is correct because $g : \mathbb{N} \mapsto \{\text{set of odd numbers}\}$
$g(n) = 2n + 1, n \in \mathbb{N}$

**Definition 1.5.** $X$ is **finite** if for some $n \in \mathbb{N}$ there exists a bijection where $A = \{1, 2, ..., n\}$ $f : A \mapsto X$. So essentially $card(A) = card(X)$

Intuitively, we can label the elements of $X$ as $X = \{x_1, x_2, ..., x_n\}$

**Definition 1.6.** $X$ is **inifinite** if $X$ is not finite.

Examples of this include: $\mathbb{N}, 2\mathbb{N}, \mathbb{Q}, \mathbb{R}$
However, not all infinite sets have the same cardinality.

**Definition 1.7.** $X$ is **countable** if $card(X) = card(A)$ where $A \subseteq \mathbb{N}$

Again intuitively, we can label the elements of $X$ as $X = \{x_1, x_2, ..., x_n, ...\}$ (using elements of $\mathbb{N}$)

**Example 1.8.** $2\mathbb{N}$

**Example 1.9.** The set of odd numbers

**Example 1.10.** $\mathbb{Z}$

**Theorem 1.11.** *In general, if $Z = X \cup Y$ and $Y, X$ are both countable then $Z$ is countable as well*

*Proof.* label elements of $X = \{x_1, x_2, ..., x_n, ...\}$
label elements of $Y = \{y_1, y_2, ..., y_n, ...\}$
define $h : \mathbb{N} \mapsto \mathbb{Z}$ as, for $n \in \mathbb{N}$:

$$h(2n-1) = x_n$$

$$h(2n) = y_n$$

Then $h$ is a bijection, therefore $Z$ is also countable ■

**Example 1.12.** $\mathbb{N} \times \mathbb{N}$

This example is countable as we can label its elements in the following pattern.

$$(1,1) \mapsto 1$$

$$(2,1) \mapsto 2$$

$$(1,2) \mapsto 3$$

$$(1,3) \mapsto 4$$

$$(2,2) \mapsto 5$$

$$(3,1) \mapsto 6$$

$$(4,1) \mapsto 7$$

And so on, in this pattern. Intuitively, if you list out all the pairs of $\mathbb{N} \times \mathbb{N}$ like a matrix, this would create a sort of zig zag pattern.

**Proposition 1.13.** Suppose $X$ is countable and $Y \subset X$, then $Y$ is either finite (which also means that it is countable) or just countable

*Proof.* write $X = \{x_1, x_2, ..., x_n, ...\}$
let $j_1$ be the smallest index such that $x_{j_1} \in Y$
let $j_2$ be the smallest index such that $x_{j_2} \in Y, j_2 > j_1$
let $j_3$ be the smallest index such that $x_{j_3} \in Y, j_3 > j_2$
...
and so on.

Now either:
**This process terminates**, which means that $Y = \{x_{j_1}, ..., x_{j_n}\}$ for some

$j_1, ..., j_n$ which $Y$ is finite.

or:
**Y is countable**, since $Y = \{x_{j_1}, x_{j_2}, ..., x_{j_n}, ...\}$ ■

**Proposition 1.14.** $\mathbb{Q}$ is countable

*Proof.* Note that $\mathbb{Q} = \mathbb{Q}^- \cup \{0\} \cup \mathbb{Q}^+$
so we must check that $\mathbb{Q}^+$ is countable, note that

$$\mathbb{Q}^+ = \{\frac{m}{n} | m, n \in \mathbb{N}, n \neq 0\}$$

Since it is a pairing of two natural numbers $(m, n)$, this is a subset of $\mathbb{N} \times \mathbb{N}$, which means that $\mathbb{Q}^+$ is countable, we can say that same thing for $\mathbb{Q}^-$ as well, note that it is

$$\mathbb{Q}^+ = \{\frac{-m}{n} | m, n \in \mathbb{N}, n \neq 0\}$$

and since $\mathbb{Q}^+$ and $\mathbb{Q}^-$ are both countable and $\{0\}$ is a set of cardinality 1, therefore $\mathbb{Q}$ is countable ■

**Theorem 1.15.** *Let $S = \{s = (s_1, s_2, ...) | s_j = 0 \text{ or } 1 \forall j\}$ (note here that $s$ is a sequence). Then $S$ is not countable*

*Proof.* For proof by contradiction, suppose $S$ is countable.
We can then label the elements of $S$ as $S^1, S^2, ..., S^n, ... \in S$.
**So an example of this would be:**

$$S^1 = (s_1^1, s_2^1, s_3^1, ..., s_n^1, ...)$$
$$S^2 = (s_1^2, s_2^2, s_3^2, ..., s_n^2, ...)$$
$$...$$
$$S^m = (s_1^m, s_2^m, s_3^m, ..., s_n^m, ...)$$
$$...$$

And we will define a $t \in S$ as follows, let $t = (t_1, t_2, ..., t_m, ...)$

$$t_1 = \begin{cases} 0, & \text{if } S_1^1 = 1 \\ 1, & \text{if } S_1^1 = 0 \end{cases}$$

$$t_2 = \begin{cases} 0, & \text{if } S_2^2 = 1 \\ 1, & \text{if } S_2^2 = 0 \end{cases}$$
$$...$$
$$t_m = \begin{cases} 0, & \text{if } S_m^m = 1 \\ 1, & \text{if } S_m^m = 0 \end{cases}$$
$$...$$

Therefore $t \neq S^m, \forall m$. Therefore this is a contradiction as $t$ was not in the listed elements of $S$, therefore $S$ is not countable. ∎

**Corollary 1.16.** $\mathbb{R}$ is not countable

*Proof.* Regard $\mathbb{R}$ as a set of infinite decimal fractions.
Identify $s \in S$ with the decimal number $0.s_1 s_2 ... s_n ...$ so $S$ can be regarded as a subset of $\mathbb{R}$. If $\mathbb{R}$ were countable, then $S$ would be countable, therefore $\mathbb{R}$ must not be countable. ∎

## 1.2 Real Numbers

Examples of real numbers that are not rational include: $\pi, e$ (which are algebraic numbers), $\sqrt{2}, \sqrt{3}, \frac{\sqrt{5}+1}{2}$ (which are roots of polynomial equations with integer coefficients.)

## 1.3 Algebraic Numbers

The set of Algebraic Numbers is a set $\overline{\mathbb{Q}}$ such that

$$\overline{\mathbb{Q}} = \{x \in \mathbb{R} | x^n + a_{n-1}x^{n-1} + ... + a_0 = 0 \text{ for some } a_0, ..., a_{n-1} \in \mathbb{Z}\}$$

Note that $\mathbb{Q} \subset \overline{\mathbb{Q}}$, and in fact, $\overline{\mathbb{Q}}$ is countable.

if $x \in (\mathbb{R} \setminus \overline{\mathbb{Q}})$ then. $x$ is transcendental.

## 1.4 Bounds

**Definition 1.17.** $X \subset \mathbb{R}, X$ is **bounded above** if $\exists a \in \mathbb{R}$ such that $a \geq x, \forall x \in X$

**Definition 1.18.** $X \subset \mathbb{R}, X$ is **bounded below** if $\exists a \in \mathbb{R}$ such that $a \leq x, \forall x \in X$

**Example 1.19.** $X = \{x \in \mathbb{R} | x^2 \leq 2\}$ is bounded above by $\frac{3}{2}$ since $(\frac{3}{2})^2 \geq 2$

**Definition 1.20.** $a$ is the **least upper bound** of $X$ if $a$ is an upper bound of $X$ and if $b < a$, then there exists $x \in X$ such that $x > b$.

We write $a = lub(X)$ or $a = sup(X)$ if $a$ is the least upper bound of $X$

**Definition 1.21.** $a$ is the **greatest lower bound** of $X$ if $a$ is an lower bound of $X$ and if $b > a$, then there exists $x \in X$ such that $x < b$.

**Example 1.22.** if $X = \{x | x^2 \leq 2\}$, then $sup(X) = \sqrt{2} \notin \mathbb{Q}$

**Property 1.23.** if $X \subset \mathbb{R}$ is bounded above then there exists $a \in \mathbb{R}$, a least upper bound of X

**Theorem 1.24.** *given $a, b \in \mathbb{R}$ $a, b > 0$ $\exists n \in \mathbb{N}$ such that $na > b$ (This is known as the **archemedian property**)*

*Proof.* Suppose that $\forall n \in \mathbb{N}$, $na \leq b$ implies that $b$ is an upper bound for $X = \{na | n \in \mathbb{N}\}$.

Since $X$ is bounded above, let $c = sup(X)$, this implies that $c - a$ is not an upper bound for $X$, which further implies that there exists an $a \in \mathbb{N}$ such that $na > c - a$, and that

$$\begin{aligned} na &> c - a \\ na + a &> c \\ (n+1)a &> c \\ c < (n+1)a &\in X \end{aligned}$$

This is impossible since we've previously stated that $c$ is the upper bound. Therefore $X$ is not bounded above. ■

**Theorem 1.25.** *given $c, d \in \mathbb{R}$, there exists $q = \frac{m}{n} \in \mathbb{Q}$ such that $c < q < d$*

*Proof.* We want $c < \frac{m}{n} < d$ iff $nc < m < nd$.
let $\epsilon = d - c > 0$
pick $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$
pick $m$, such that $m > nc$
(since we also need $m < nd$, choose $m$ to be as small as possible such that $m - 1 \leq nc < m$)

$$m - 1 \leq nc \leq m$$
$$\frac{m}{n} - \frac{1}{n} = \frac{m-1}{n} \leq c \leq \frac{m}{n}$$

or

$$\frac{m}{n} \leq c + \frac{1}{n} < c + \epsilon = d$$

So now we have $c < \frac{m}{n} < d$ as desired.

$\blacksquare$

# 2 Monday, January 8, 2017

## 2.1 Sequences - Review

This section will be just review of MATA37 and will be concerning sequences of the real numbers $\{a_n\}$.

**Example 2.1.** $\{\frac{1}{n}\} = 1, \frac{1}{2}, \frac{1}{3}, ...,$

**Example 2.2.** $\{\frac{(-1)^n}{n}\} = -1, \frac{1}{2}, \frac{-1}{3}, ...,$ This sequence oscillates back and forth.

**Example 2.3.** $\{(-1)^n\} = -1, 1, -1, 1, ...,$ This sequence oscillates back and forth aswell.

**Example 2.4.** $\{x_1, x_2, ...\}$ where this sequence enumerates $\mathbb{Q}$. This sequence 'bounces around wildy'.

**Definition 2.5.** $x \in \mathbb{R}$ is the **limit of a sequence** $\{x_n\}$ (so that in converges to this number)
$\lim_{n\to\infty} x_n = a$, if given some tolerance $\epsilon > 0$.
There exists $N$, such that $n \geq N$ will lie in the interval $a - \epsilon, a + \epsilon$, aka $|x_n - a| < \epsilon$.

**Example 2.6.** $\lim_{n\to\infty} \frac{1}{n} = 0$

**Example 2.7.** $\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$

**Example 2.8.** example 2.4 and 2.3 both have no limit

**Proposition 2.9.** If a sequence $\{x_n\}$ has a limit, then this limit is unique, and the sequence is bounded.

**Proposition 2.10.** Supposed that $\{x_n\}, \{y_n\} \subset \mathbb{R}$ are convergent sequences, so that $\lim_{n\to\infty} x_n = a$, $\lim_{n\to\infty} y_n = b$ then:

1. $\{x_n + y_n\}$ converges and $\lim_{n\to\infty}(x_n + y_n) = a + b$
2. $\{x_n y_n\}$ converges and $\lim_{n\to\infty}(x_n y_n) = ab$
3. if $y_n \neq 0, \forall n$, and $b \neq 0$ then $\{\frac{x_n}{y_n}\}$ converges and $\lim_{n\to\infty}(\frac{x_n}{y_n}) = \frac{a}{b}$

**Definition 2.11.** A sequence is **monotone** if it is either increasing, or decreasing.

**Proposition 2.12.** A bounded monotone sequence converges.

*Proof.* Suppose that $\{x_n\}$ is a bounded increasing (monotone) sequence, such that

$$x_1 \leq x_2 \leq x_3 \leq ...$$

And there exists $A \in \mathbb{R}$ such that $x_n \leq A, \forall n$
Therefore, there exists a least upper bound $a \leq A$.

claim: $x_n \to a$ as $n \to \infty$

take $\epsilon > 0$, since $a$ is the least upper bound of $\{x_n\}$, there exists $N$ such that $a - \epsilon < x_N \leq a$. But $\{x_n\}$ is increasing. Therefore $\forall n \geq N, a - \epsilon < x_N \leq x_n \leq a$. This implies that $\lim_{n\to\infty} x_n = a$ ∎

# 3 Friday, January 12, 2017

## 3.1 Monotone Sequences

i.e. Sequences which are increasing or decreasing

**Proposition 3.1.** A bounded monotone sequence converges.

*Proof.* For an increasing sequence $\{x_n\}, \lim_{n\to\infty} x_n = sup(\{x_n\})$, suppose $\{x_n\}$ is decreasing and bounded below, then $\{-x_n\}$ is increasing and bounded above. This implies that $\lim_{n\to\infty} -x_n$ and $\lim_{n\to\infty} x_n$ both exist, and

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} -x_n$$

■

So now we're going to prove that Every sequence has a monotone subsequence along with other propositions on bounded monotone sequences. This will allow us to prove the Bolzano-Weierstrass Theorem, which states that every bounded sequence has a convergent subsequence, and this will help us prove the Cauchy property for sequences.

**Example 3.2.** Lets look at the following sequence:

$$x_n = 1 + \frac{1}{2} + ... + \frac{1}{n} - logn, n \geq 1$$

So we want to show that this converges, we'll show that if we show that $\{x_n\}$ is decreasing and is bounded below by 0.

The limit ($\lim_{n\to\infty} x_n$) is actually a mysterious number that we don't know too much about.

Recall: that $logn = \int_1^n \frac{dt}{t}$
So if we consider the space under the graph of $\frac{1}{t}$ from $n$ to $n+1$, we get the following inequality

$$\frac{1}{n+1} < log(n+1) - logn < \frac{1}{n}$$

Which implies that

$$\frac{1}{n+1} - log(n+1) + logn < 0$$

Now consider the following for an arbitrary $n$

$$x_{n+1} = x_n + (\frac{1}{n+1} + logn - log(n+1))$$

but since $\frac{1}{n+1} - log(n+1) + logn < 0$, this would mean that $x_{n+1} < x_n$, so this sequence is decreasing.

Now consider the following inequality derived from the recall block:

$$\sum_{n=1}^{m}(log(n+1) - logn) < \sum_{n=1}^{m}\frac{1}{n}$$

The left side of this inequality telescopes, giving us

$$log(m) < log(m+1) < 1 + \frac{1}{2} + ... + \frac{1}{m}$$

And since the left most side is greater than the right most side, that means that $x_m > 0, \forall m$.

## 3.2 Subsequences

**Definition 3.3.** Let $\{x_i\}$ be a sequence of the real numbers, pick a finite set of indices $j_1 < j_2 < ... < j_n < .....$
A **Subsequence** of $\{x_i\}$ is $\{x_{j_1}, x_{j_2}, ..., x_{j_n}, ...\}$

**Example 3.4.** Considering the sequence $\{x_n\} = \{1, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{n}, ...\}$

1. $\{\frac{1}{2n}\}$ is a subsequence of $\{x_n\}$.
2. $\{\frac{1}{2^n}\}$ is a subsequence of $\{x_n\}$.
3. $\{(-1)^n\}$ is also a subsequence of $\{x_n\}$, note that it does not converge, but has convergent subsequences of $\{(-1)^{2n}\}$ and $\{(-1)^{2n+1}\}$

**Definition 3.5.** Call a term $x_m$ **dominant** if $x_n \leq x_m, \forall n \geq m$

**Proposition 3.6.** Every real number sequence has a monotone subsequence

*Proof.* There are 2 cases:
Case 1: infinitely many dominant terms
let $\{x_{j_1}, x_{j_2}, ..., x_{j_n}, ...\}$ be the sequence of dominant terms. By definition, $x_{j_1} \geq x_{j_2} \geq ... \geq x_{j_n} \geq ...$
so $\{x_{j_n}\}$ is a decreasing sequence, which is monotone.

Case 2: only finitely many dominant terms
Pick an index $j$, so that $x_{j_1}$ is the first term beyond all dominant terms in the sequence ($\exists i, i < j_1, x_i$ is the last dominant term).
Since $x_{j_1}$ is not dominant, then $\exists j_2 > j_1$ where $x_{j_2} > x_{j_1}$
Since $x_{j_2}$ is not dominant, then $\exists j_3 > j_2$ where $x_{j_3} > x_{j_2}$
...
and so on
By induction, we construct an increasing subsequence $\{x_{j_m}\}$ of $\{x_n\}$.

By these 2 cases, every real number has a monotone sequence. ∎

**Theorem 3.7.** *(Bolzano-Weierstrass Theorem) every bounded sequence has a convergent subsequence*

*Proof.* Given a bounded sequence $\{x_n\}$ of the real numbers, there exists a monotone subsequence $\{x_{j_n}\}$.
The monotone subsequence $\{x_{j_n}\}$ is also bounded since $\{x_n\}$ is bounded.
This implies that $\lim_{n\to\infty} x_{j_n}$ exists. ∎

**Definition 3.8. Cauchy Property:** intuitively, in a convergent sequence, the terms get closer and closer as $n \to \infty$. More precisely:
$\{x_n\}$ is a real number sequence, given $\epsilon > 0, \exists N$ such that for $m, n > N, |x_m - x_n| < \epsilon$.

**Proposition 3.9.** Suppose that $\{x_n\}$ converges to $a$, then $\{x_n\}$ satisfies the Cauchy property.

*Proof.* given $\epsilon > 0$, then $\exists N$ such that $\forall n > N, |x_n - a| < \frac{\epsilon}{2}$, so if $m > N$ then $|x_m - a| < \frac{\epsilon}{2}$ as well.
This implies that

$$|x_m - x_n| = |x_m - a + a - x_n| \leq |x_m - a| + |x_n - a| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

∎

So now we want to show that if a sequence satisfies the Cauchy property, then it converges. We'll do this by first showing that a bounded sequence with the Cauchy property implies that sequence has a convergent subsequence. The second and final step is to show that the limit of a convergent subsequence is the limit of the original sequence.

**Proposition 3.10.** If $\{x_n\}$ satisfies the Cauchy property, then it's bounded.

*Proof.* By definition, there exists $N$ so that for $m, n > N, |x_m - x_n| < 1$ in particular, $\forall n > N, |x_n - x_{N+1}| < 1$, which gives us:

$$\begin{aligned} -1 <& x_n - x_{N+1} < 1 \\ x_{N+1} - 1 <& x_n < x_{N+1} + 1 \end{aligned}$$

Note: $x_{N+1}$ is fixed (a constant)

Let $A = min\{x_1, ...., x_N, x_{N+1} - 1\}$
Let $B = max\{x_1, ...., x_N, x_{N+1} - 1\}$
This implies that for all $n$, $A \leq x_n \leq B$, therefore Bounded. ∎

**Proposition 3.11.** A sequence with the Cauchy property is convergent.

*Proof.* Let $\{x_n\}$ be a sequence with the Cauchy property.
Since $\{x_n\}$ is bounded, by the Bolzano–Weiestrass Theorem, there exists a convergent subsequence $\{x_{j_m}\}$. Now let:

$$a = \lim_{m \to \infty} x_{j_m}$$

given $\epsilon, \exists M$ so that for all $m > M, |x_{j_m} - a| < \frac{\epsilon}{2}$.

Note that $j_m \geq m$

The Cauchy propert implies that $\exists N$ so that for all $m, n > N, |x_m - x_n| < \epsilon$.

We'll pick $P = max(N, M)$, then for $m, n > P$ we have:

$$\begin{aligned}|x_n - a| &= |x_n + x_{j_m} - x_{j_m} - a| \\ &= |x_n - x_{j_m}| + |x_{j_m} - a| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon\end{aligned}$$

■

**Example 3.12.**

$$x_n = \sum_{k=1}^{n} \frac{1}{k^2}$$

Verify the Cauchy property.

Take $m > n, x_m - x_n = \sum_{k-n+1}^{m} \frac{1}{k^2}$. Now $\frac{1}{k} < \frac{1}{k(k-1)} = \frac{1}{k-1} - \frac{1}{k}$. This implies that

$$\begin{aligned}&\implies \sum_{k-n+1}^{m} \frac{1}{k^2} < \sum_{k-n+1}^{m} (\frac{1}{k-1} - \frac{1}{k}) \\ &\overset{\text{telescoping}}{\implies} \sum_{k-n+1}^{m} \frac{1}{k^2} < \frac{1}{m} - \frac{1}{n} < \frac{1}{m} \\ &\implies (x_m - x_n) < \frac{1}{m}\end{aligned}$$

This implies that $x_n$ is convergent, in fact

$$\lim_{n\to\infty} x_n = \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$$