

MATB42: Multivariable Calculus II

Lecture Notes

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1 Friday, January 5, 2018

1.1 Fourier Expansions

In this section, we will focus on single variable calculus, (so where $f : \mathbb{R} \mapsto \mathbb{R}$)

Let us say that we have a function $f(x)$ and we want to approximate it. We can use an n th degree Taylor Polynomial, but this requires that $f(x)$ has at least n derivatives at some point x_0 and the k th derivative of f ($f^{(k)}(x)$) is determined by properties of f in some neighbourhood of x_0 , but what about outside this neighbourhood? How can we be certain of the approximation outside of this neighbour.

Our problem here is that Taylor Polynomial may only approximate "near" x_0

Now, consider the following function:

$$\Delta(x) = \begin{cases} 1, & [x] < x, [x] \text{ is odd} \\ 0, & [x] < x, [x] \text{ is even} \end{cases}$$

In this function, Taylor returns either 0 or 1 depending on your choice of x_0 and cannot work for an $x_0 = p \in \mathbb{Z}$. Therefore Taylor polynomials cannot reflect the true nature of this function. Taylor provides a "local" approximation, but we want a "global" approximation. We need an approximation that is more precise over an interval at the cost of being not as precise as precise at any particular x_0 .

Note that the example function is **periodic**.

Definition 1.1. A function $y = f(x)$ such that $f(x) = f(x + p), p \neq 0, \forall x$ is said to be **periodic** of period p

Example 1.2. The periodic function $\Delta(x)$ is of period 2.

What we want is a global approximation of a periodic function, and the Fourier Approximation will be periodic, so we can use it for exactly that.

Definition 1.3. A **trigonometric polynomial of degree N** is an expression of the form

$$\frac{a_0}{2} + \sum_{k=1}^N a_k \cos(kx) + b_k \sin(kx)$$

where the a_i, b_i are constants.

We know that $\sin(x)$ and $\cos(x)$ are the simplest periodic functions and repeat in intervals of 2π , so $\cos(kx)$ and $\sin(kx)$ have period $\frac{2\pi}{k}$, but the smallest shared period is 2π . If a trigonometric polynomial has period 2π and $f(x)$ has period p , then we must set $x = \frac{pt}{2\pi}$ to fix the period (where t is a variable).

So to approximate $y = f(x)$ by $F_N(x)$ for some N , we use the following equation:

$$F_N(x) = \frac{a_0}{2} + \sum_{k=1}^N a_k \cos(kx) + b_k \sin(kx)$$

Now we need to choose the a_k, b_k . We can define it in the following way:

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx \\ a_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx) dx, k = 1, 2, 3, \dots \\ b_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx) dx, k = 1, 2, 3, \dots \end{aligned}$$

When defined in this way, a_i, b_i are called the **Fourier Coefficients** of f over the interval $[-\pi, \pi]$ and we call $F_N(x)$ the **Fourier Polynomial of degree N** .

So why do we add the $\frac{a_0}{2}$? It is the average value of f over $[-\pi, \pi]$.

Note. *sometimes you will see a_0 used instead of $\frac{a_0}{2}$ in the Fourier polynomial where*

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx$$

Example 1.4. Consider $f(x) = \frac{-x}{2}$ over $[-\pi, \pi]$. Use Fourier Approximation.

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \left(-\frac{x}{2}\right) \cos(kx) dx \stackrel{\text{odd}}{=} 0$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \left(-\frac{x}{2}\right) dx \stackrel{\text{odd}}{=} 0$$

$$\begin{aligned} b_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx) dx \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} \left(-\frac{x}{2}\right) \sin(kx) dx \\ &\stackrel{\text{even}}{=} -\frac{1}{\pi} \int_{-\pi}^{\pi} x \sin(kx) dx \\ &\stackrel{\substack{\text{even} \\ u=x, dv=\sin(kx)dx}}{=} -\frac{1}{\pi} \left[-\frac{1}{k} x \cos(kx) + \frac{1}{k^2} \sin(kx) \right]_0^{\pi} \\ &= \frac{1}{\pi k} [\pi \cos(k\pi)] \\ &= \frac{1}{k} \cos(k\pi) \\ &= \frac{(-1)^k}{k} \end{aligned}$$

Thus we have:

$$F_N(x) = -\sin(x) + \frac{1}{2}\sin(2x) - \frac{1}{3}\sin(3x) + \frac{1}{4}\sin(4x) + \dots$$

$$F_1(x) = -\sin(x)$$

$$F_2(x) = -\sin(x) + \frac{1}{2}\sin(2x)$$

$$F_3(x) = -\sin(x) + \frac{1}{2}\sin(2x) - \frac{1}{3}\sin(3x)$$

...

And so on.

2 Monday, January 8, 2017

continuing from the last lecture...

Example 2.1. (continued from example 1.4)

$$f(x) = \frac{-x}{2}$$

$$F_N(x) = -\sin(x) + \frac{1}{2}\sin(2x) - \frac{1}{3}\sin(3x) + \frac{1}{4}\sin(4x) + \dots$$

This can be extended to a Fourier Series:

$$F_N(x) = \sum_{k=1}^{\infty} (-1)^k \frac{\sin(kx)}{k}$$

Definition 2.2. For $f : \mathbb{R} \mapsto \mathbb{R}$, the Fourier Series for f is

$$F(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kx) + b_k \sin(kx)$$

where a_i, b_i are Fourier coefficients.

The N th degree Fourier Polynomial can be regarded as the N th partial sum of the series.

We haven't talked about convergence yet, but for now, we will assume the series converges ($f(x) = F(x)$)

Definition 2.3. Function $a_k \cos(kx) + b_k \sin(kx)$ is the k th harmonic of f . The Fourier Series expresses f in terms of its harmonics.

Note. (*Looking at Harmonics in a Musical Sense*):

the 1st harmonic is the fundamental harmonic of f (the fundamental tone).

The 2nd harmonic is the first overtone.

(completely rewrite this amplitude section)

Definition 2.4. The amplitude of the k th harmonic is

$$A_k = \sqrt{(a_k)^2 + (b_k)^2}$$

And note that

$$a_k = A_k \sin \alpha, b_k = A_k \cos \alpha$$

Definition 2.5. The energy E of a periodic function f of period 2π is

$$E = \frac{1}{\pi} \int_{-\pi}^{\pi} [f(x)]^2 dx$$

So the energy of the k th harmonic is

$$E = \frac{1}{\pi} \int_{-\pi}^{\pi} [a_k \cos(kx) + b_k \sin(kx)]^2 dx = (a_k)^2 + (b_k)^2 = (A_k)^2$$

And the energy of the constant term is

$$\frac{1}{\pi} \int_{-\pi}^{\pi} [a_0]^2 dx = 2(a_0)^2$$

So we put $A_0 = \frac{1}{\sqrt{2}}a_0$.

For a "nice" periodic function, we have the following equation:

$$E = A_0^2 + A_1^2 + A_2^2 + \dots$$

This is known as the Energy Theorem, and comes from the study of periodic waves.

We can draw a graph of this as A_k^2 against k (This graph is known as the Energy Spectrum of f). It shows how the energy of f is distributed among its harmonics.

Note. Notice that

$$E = \frac{1}{\pi} \int_{-\pi}^{\pi} [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{k=1}^{\infty} (a_k^2 + b_k^2) \text{ Parseval's Equation}$$

Assume a function f of period 2π is the sum of a trigonometric series

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos(kx) + b_k \sin(kx)) \text{ on the interval } [-\pi, \pi]$$

Multiply by $\cos(mx)$ and integrate to get

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\cos(mx)dx &= \frac{a_0^2}{2} \int_{-\pi}^{\pi} \cos(mx)dx + \int_{-\pi}^{\pi} \left[\sum_{k=1}^{\infty} (a_k \cos(kx) + b_k \sin(kx)) \right] dx \\ &= \frac{a_0^2}{2} \int_{-\pi}^{\pi} \cos(mx)dx + \sum_{k=1}^{\infty} \left(a_k \int_{-\pi}^{\pi} \cos(kx)dx + b_k \int_{-\pi}^{\pi} \sin(kx)dx \right)\end{aligned}$$

Note. Recall the following trigonometric identities:

1. $\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$
2. $\cos A \sin B = \frac{1}{2}[\sin(A+B) - \sin(A-B)]$
3. $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$

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Continuing from where we left off.

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\cos(mx)dx &= \frac{a_0^2}{2} \int_{-\pi}^{\pi} \cos(mx)dx + \int_{-\pi}^{\pi} \left[\sum_{k=1}^{\infty} (a_k \cos(kx) + b_k \sin(kx)) \right] dx \\ &= \frac{a_0^2}{2} \int_{-\pi}^{\pi} \cos(mx)dx + \sum_{k=1}^{\infty} \left(a_k \int_{-\pi}^{\pi} \cos(kx)dx + b_k \int_{-\pi}^{\pi} \sin(kx)dx \right)\end{aligned}$$

We know the following from trigonometric identities

$$\int_{-\pi}^{\pi} \cos(kx)\cos(mx)dx = \begin{cases} 0, & k \neq m \\ \pi, & k = m \end{cases}$$

As well as the following from odd function properties

$$\begin{aligned}\int_{-\pi}^{\pi} \cos(kx)dx &= 0 \\ \int_{-\pi}^{\pi} \sin(kx)\cos(mx)dx &= 0\end{aligned}$$

So now we get

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\cos(mx)dx &= \frac{a_0^2}{2} \int_{-\pi}^{\pi} \cos(mx)dx + \sum_{k=1}^{\infty} \left(a_k \int_{-\pi}^{\pi} \cos(kx)dx + b_k \int_{-\pi}^{\pi} \sin(kx)dx \right) \\ &= a_m \pi \\ \implies a_m &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\cos(mx)dx\end{aligned}$$

Example 3.1. Lets take

$$f(x) = \begin{cases} 1, & 0 \leq x < \pi \\ -1, & -\pi \leq x < 0 \end{cases}$$

Note that this is an odd function, therefore $a_k = 0, \forall k \geq 0$. So now lets calculate b_k .

$$\begin{aligned}
b_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx) dx \\
&\stackrel{\text{even}}{=} \frac{2}{\pi} \int_0^{\pi} f(x) \sin(kx) dx \\
&= \frac{2}{\pi} \int_0^{\pi} \sin(kx) dx \\
&= \frac{2}{k\pi} [-\cos(kx)]_0^{\pi} \\
&= \begin{cases} \frac{4}{k\pi}, & k \text{ is odd} \\ 0, & k \text{ is even} \end{cases}
\end{aligned}$$

And now lets right out the Fourier Polynomial ($F_N(x)$)
if N is odd:

$$F_N(x) = \frac{4}{\pi} \sin(x) + \frac{4}{3\pi} \sin(3x) + \frac{4}{5\pi} \sin(5x) + \dots$$

if N is even:

$$F_N(x) = F_{N-1}(x)$$

We can also write it as a Fourier Series

$$F(x) = \frac{4}{\pi} \sum_{l=0}^{\infty} \frac{\sin((2l+1)x)}{2l+1}$$

The energy of the function is

$$E = \frac{1}{\pi} \int_{-\pi}^{\pi} [f(x)]^2 dx = \frac{2}{\pi} \int_0^{\pi} dx = 2$$

The amplitutde of the k th harmonic is

$$A_k = \sqrt{a_k^2 + b_k^2} = \sqrt{0 + \frac{16}{k^2\pi^2}} = \frac{4}{k\pi}$$

The energy of the k th harmonic is

$$A_k^2 = \frac{16}{k^2\pi^2}$$

Note that for this example, both the energy and the amplitude are 0 at an even k .

Lets now evaluate the energy spectrum:

$$k = 1, E = \frac{16}{\pi^2} \approx 1.62, \frac{1.62}{2} = 0.81 = 81\%$$

$$k = 3, E = \frac{16}{9\pi^2} \approx 0.18, \frac{0.18}{2} = 0.09 = 9\%$$

$$k = 5, E = \frac{16}{25\pi^2} \approx 0.06, \frac{0.06}{2} = 0.03 = 3\%$$

$$k = 7, E = \frac{16}{49\pi^2} \approx 0.03, \frac{0.03}{2} = 0.015 = 1.5\%$$

However, we do not need to exclusively work with the interval $[-\pi, \pi]$ we can even work over any interval of length 2π .

3.1 General Fourier Series (interval of length 2π)

$$a_k = \frac{1}{\pi} \int_c^{c+2\pi} f(x) \cos(kx) dx, k = 1, 2, 3, \dots$$

$$b_k = \frac{1}{\pi} \int_c^{c+2\pi} f(x) \sin(kx) dx, k = 1, 2, 3, \dots$$

What about for f if f has period p ?

$$f(x + p) = f(x), \forall x, \exists p \neq 0$$

We then substitute $x = \frac{pt}{2\pi}$ which gives a new function $f_p(t) = f(\frac{pt}{2\pi})$ with period 2π . So

$$f_p(t + 2\pi) = f(\frac{p}{2\pi}(t + 2\pi)) = f(\frac{pt}{2\pi} + p) = f(\frac{pt}{2\pi}) = f_p(t)$$

So how about the Fourier Expansion for $f_p(t)$? To find this, we must replace t by $\frac{2\pi x}{p}$ giving for $f(x)$.

$$F(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(\frac{2nx\pi}{p}\right) + b_k \sin\left(\frac{2nx\pi}{p}\right)$$

$$a_k = \frac{2}{p} \int_c^{c+p} f(x) \cos\left(\frac{2nx\pi}{p}\right) dx, k = 1, 2, 3, \dots$$

$$b_k = \frac{2}{p} \int_c^{c+p} f(x) \sin\left(\frac{2nx\pi}{p}\right) dx, k = 1, 2, 3, \dots$$

For any function defined on $[a, b]$, we can extend f to all of \mathbb{R} as a periodic function. Given a periodic function f_E from f of period $p = b - a$, we now have:

$$a_k = \frac{2}{b-a} \int_a^b f(x) \cos\left(\frac{2nx\pi}{b-a}\right) dx, k = 1, 2, 3, \dots$$

$$b_k = \frac{2}{b-a} \int_a^b f(x) \sin\left(\frac{2nx\pi}{b-a}\right) dx, k = 1, 2, 3, \dots$$

Example 3.2. Take the function $f(x) = x, 0 \leq x < 1$ and extend it with period 1. For $k \neq 0$:

$$\begin{aligned} a_k &= 2 \int_0^1 x \cos(2k\pi x) dx \\ &\stackrel{\text{parts}}{=} 2 \left[\frac{x \sin(2k\pi x)}{2\pi k} \right]_0^1 - \frac{2}{2k\pi} \int_0^1 \sin(2k\pi x) dx \\ &\stackrel{u=v, dv=\cos(2\pi kx)dx}{=} 0 \end{aligned}$$

$$a_0 = 2 \int_0^1 x dx = [x^2]_0^1$$

$$\begin{aligned} b_k &= 2 \int_0^1 x \sin(2k\pi x) dx \\ &\stackrel{\text{parts}}{=} 2 \left[\frac{-x \cos(2k\pi x)}{2\pi k} \right]_0^1 + \frac{1}{k\pi} \int_0^1 \cos(2k\pi x) dx \\ &= \frac{-\cos(2k\pi)}{k\pi} \\ &= \frac{-1}{k\pi} \end{aligned}$$

So the Fourier Series will be

$$F(x) = \frac{1}{2} - \frac{1}{\pi} \left[\sin(2\pi x) + \frac{\sin(4\pi x)}{2} + \frac{\sin(6\pi x)}{3} + \dots \right]$$

Example 3.3. $f(x) = |x|$, $-\pi < x \leq \pi$. Since $f(x)$ is even, $b_k = 0, \forall k \in \mathbb{N}$.

$$\begin{aligned}
 a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} |x| dx \stackrel{\text{even}}{=} \frac{2}{\pi} \int_0^{\pi} x dx = \pi \\
 a_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \cos(kx) dx \\
 &\stackrel{\text{even}}{=} \frac{2}{\pi} \int_0^{\pi} x \cos(kx) dx \\
 &\stackrel{\text{parts}}{=} \frac{2}{\pi} \left[\frac{x \sin(kx)}{k} + \frac{\cos(kx)}{k^2} \right]_0^{\pi} \\
 &\stackrel{u=x, dv=\cos(kx)dx}{=} \frac{2}{\pi k^2} (\cos(k\pi) - 1) \\
 &= \begin{cases} 0, & k \text{ is even} \\ \frac{-4}{\pi k^2}, & k \text{ is odd} \end{cases}
 \end{aligned}$$

So we end up with the Fourier Series

$$F(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{l=0}^{\infty} \frac{\cos((2l+1)x)}{(2l+1)^2}$$

Example 3.4. $f(x) = x$, $-\pi < x \leq \pi$. Note that since $f(x)$ is odd, $a_k = 0, \forall k \geq 0$

$$\begin{aligned}
 b_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin(kx) dx \\
 &\stackrel{\text{parts}}{=} \frac{1}{\pi} \left[\frac{-x \cos(kx)}{k} \right]_{-\pi}^{\pi} + \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(kx) dx \\
 &\stackrel{u=x, dv=\sin(kx)dx}{=} \frac{1}{\pi} \left[\frac{-\pi \cos(k\pi)}{k} + \frac{-\pi \cos(-k\pi)}{k} \right] \\
 &= \frac{-2}{k} \cos(k\pi) \\
 &= \frac{(-1)^{k+1} 2}{k}
 \end{aligned}$$

And the Fourier Series of this is

$$F(x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2}{k} \sin(kx)$$

To get a Fourier cosine series or a Fourier sine series, we need

- an f defined on the interval $[0, a]$, and we must extend this interval to also include $[-a, 0)$ to give an even or odd function on $[-a, a]$
- $f(-t) = f(t)$, $-a \leq t < 0$ for the even extension item $f(-t) = -f(t)$, $-a \leq t < 0$ for the odd extension

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Example 4.1. We want to express $f(x) = x, 0 \leq x < \pi$ as both a cosine series and a sine series.

cosine series For this, we need to extend the function as an even function, so we extend the function to $f(x) = |x|, -\pi < x \leq \pi$. This is a previous example that we computed before and gives us a cosine series

sine series For this, we must extend $f(x)$ as an odd function $f(x) = x, -\pi \geq x < \pi$. We've already seen from previous examples that is a sine series.

So for $f(x) = x, 0 \leq x < \pi$, the cosine series looks like this:

$$F(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{l=0}^{\infty} \frac{\cos((2l+1)x)}{(2l+1)^2}$$

And the sine series looks like this:

$$F(x) = \sum_{k=0}^{\infty} (-1)^{k+1} \frac{2}{k} \sin(kx)$$

Both of these series on the interval $[0, \pi)$ represent $f(x) = x$

Definition 4.2. A function $f(x)$ defined for $x \in [a, b]$ is **piece-wise continuous** if there exists a finite partition $P : a = t_0 < \dots < t_n = b$ such that f is continuous on $x \in (t_{i-1}, t_i), \forall i$ and both $\lim_{x \rightarrow t_{i-1}^+} f(x)$ and $\lim_{x \rightarrow t_i^-} f(x)$ both exist and are both finite.

Note. On the i th subinterval, $f(x)$ coincides with some $f_i(x)$ that is continuous on that subinterval.

Definition 4.3. If $f_i(x) \forall i$ has continuous 1st derivatives, $f(x)$ is called **piecewise smooth**

Definition 4.4. If $f_i(x) \forall i$ has continuous 2nd derivatives, $f(x)$ is called **piecewise very smooth**

Definition 4.5. The Fourier Series obtained from $f(x)$ converges to $f(x)$ if $f(x) = \lim_{N \rightarrow \infty} F_N(x)$

i.e.

$$f(x) = \lim_{N \rightarrow \infty} F_N(x) = \frac{a_0}{2} + \sum_{k=1}^N a_k \cos(kx) + b_k \sin(kx)$$

assuming a period of 2π (and can be adjusted for other periods). The a_k and b_k are the Fourier coefficients.

Theorem 4.6. *Let $f(x)$ be continuous and piece-wise very smooth for all x and let $f(x)$ have period 2π . Then the Fourier Series of $f(x)$ converges uniformly to $f(x)$, $\forall x$*

This helps with examples that have jump discontinuities.

Theorem 4.7. *Let $f(x)$ be defined and piece-wise very smooth for $x \in [-\pi, \pi]$ and let $f(x)$ be defined outside this interval to have period 2π . Then the Fourier Series of $f(x)$ converges uniformly to $f(x)$ in each interval containing no discontinuity of $f(x)$. At each discontinuity, x_0 , the series converges to*

$$\frac{1}{2} \left[\lim_{x \rightarrow x_0^+} f(x) + \lim_{x \rightarrow x_0^-} f(x) \right]$$

*This is the **Fundamental Theorem (for Fourier Series)**. The Theorem can be reinstated for f defined on any interval of length $p \neq 0$*

Example 4.8. 1. $f(x) = \frac{-x}{2}$ on the interval $[-\pi, \pi]$

$$2. f(x) = \begin{cases} 1, & 0 \leq x < \pi \\ -1, & -\pi \leq x < 0 \end{cases}$$

Note that they both are piece-wise continuous and both satisfy the previous theorem.

so (1.) converges to

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \sin(kx) = \begin{cases} -\frac{x}{2}, & x \in (-\pi, \pi) \\ 0, & x = \pm\pi \end{cases}$$

And (2.) converges to

$$\sum_{k=1}^{\infty} \frac{\sin((2k+1)x)}{2k+1} = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 < x < \pi \\ 0, & x = -\pi, 0, \pi \end{cases}$$

In (2.) set $x = \frac{\pi}{2}$ and we'll get

$$\begin{aligned} 1 &= \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{\sin((2k+1)\frac{\pi}{2})}{2k+1} \\ 1 &= \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k}{2k+1} \\ \frac{\pi}{4} &= \sum_{k=1}^{\infty} \frac{(-1)^k}{2k+1} \\ \frac{\pi}{4} &= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots \end{aligned}$$

So now we have $\frac{\pi}{4}$ represented by a series of numbers.

However, the domain of the function also play a role in the series.

Example 4.9. For $f(x), x \in [0, 1)$, we get

$$F(x) = \frac{1}{2} - \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{\sin((2k+1)x)}{2k+1}$$

The theorem still applies and we get $F(x) = x$ on the interval $[0, 1)$, and converge to $\frac{1}{2}$ at 0.

Example 4.10. For $f(x) = |x|, x \in [-\pi, \pi]$ we get.

$$F(x) = \frac{\pi}{2} - \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{\cos((2k+1)x)}{(2k+1)^2}$$

on $(0, 1), |x| = x$, and since f is piece-wise very smooth, we can write:

$$x = \frac{\pi}{2} - \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{\cos((2k+1)x)}{(2k+1)^2}$$

5 Friday, January 19, 2017

Definition 5.1. The orthonormal bases for \mathbb{R}^n

$$\{e_1, \dots, e_n\}$$

are all orthogonal to each other and are all unit vectors.

Each $v \in \mathbb{R}^n$ has a unique representation of $v = \lambda_1 e_1 + \dots + \lambda_n e_n$ where $\lambda_k = v \cdot e_k$

Given continuous functions f, g that map from $[-\pi, \pi] \mapsto \mathbb{R}$ we can define an inner product by

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x)dx$$

defined on all continuous real-valued functions defined on the interval $[-\pi, \pi]$

Example 5.2. The set of functions $\{\cos(kx), \sin(nx)\}_{n,k \in \mathbb{Z}}$ act like an orthonormal basis. since

$$\begin{aligned} \langle \cos(mx), \cos(nx) \rangle &= \begin{cases} 1, & \text{if } m = n \\ 0, & \text{otherwise} \end{cases} \\ \langle \sin(mx), \sin(nx) \rangle &= \begin{cases} 1, & \text{if } m = n \\ 0, & \text{otherwise} \end{cases} \\ \langle \sin(mx), \cos(nx) \rangle &= 0 \end{aligned}$$

for positive integers m, n

We can regard the Fourier Coefficients as the components of f under this basis. This explains why you don't need to recalculate the coefficients for each N for $F_N(x)$

Corollary 5.3. If a function f can be represented as a trigonometric polynomial, then that trigonometric polynomial is a Fourier Expansion of f , if the Fourier Series converges.

Example 5.4. What is the Fourier series of $f(x) = \cos^2(x) - \sin^2(x)$

We have trigonometric identities to solve this:

$$F(x) = \cos^2(x) - \sin^2(x) = \cos(2x)$$

Example 5.5.

$$\begin{aligned}
f(x) &= \cos^4(x) \\
&= (\cos^2(x))^2 \\
&= (\cos^2(x))^2 \\
&= \frac{1}{4}(1 + 2\cos(2x) + \cos^2(2x)) \\
&= \frac{1}{4}(1 + 2\cos(2x) + \frac{1}{2} + \frac{1}{2}\cos(4x)) = \frac{3}{8} + \frac{\cos(2x)}{2} + \frac{1}{8}\cos(4x) = F(x)
\end{aligned}$$

Definition 5.6. The **Total Square Error** of $g(x)$ relative to $f(x)$ is:

$$E = \int_{-\pi}^{\pi} [f(x) - g(x)]^2 dx$$

Note. if $f = g$, $E = 0$

We want a constant function $y = g_0$ so the square error is as small as possible

$$\begin{aligned}
E(g_0) &= \int_{-\pi}^{\pi} [f(x) - g_0]^2 dx \\
&= \int_{-\pi}^{\pi} f(x)^2 dx - 2g_0 \int_{-\pi}^{\pi} f(x) dx + g_0^2(2\pi)
\end{aligned}$$

And if we let A, B be constants such that

$$A = \int_{-\pi}^{\pi} f(x)^2 dx, B = \int_{-\pi}^{\pi} f(x) dx$$

Then

$$E(g_0) = A - 2Bg_0 + 2\pi g_0^2$$

With implies that $E(g_0)$ is a quad function in g having a minimum value when its derivative is equal to 0.

$$-2B + 4\pi g_0 = 0$$

$$g_0 = \frac{B}{2\pi} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{2}a_0$$

Therefore $\frac{1}{2}a_0$ is the best constant approximation in the sense of the square error. We can show that the best approximation by a trigonometric polynomial is the Fourier Polynomial.

Definition 5.7. If f is piece-wise continuous on $[-\pi, \pi]$ we see that

$$\frac{a_0}{2} + \sum_{k=1}^N a_k^2 + b_k^2 \leq \frac{1}{\pi} \int_{-\pi}^{\pi} [f(x)]^2 dx = \langle f, f \rangle$$

This is called **Bessel's Inequality** and it shows that $\sum_{k=1}^{\infty} a_k^2 + b_k^2$ converges

Theorem 5.8. (*Uniqueness Theorem*) Let $f(x), g(x)$ be piecewise continuous functions on the interval $[-\pi, \pi]$ and have all the same Fourier coefficients. Then $f(x) = g(x)$ except, perhaps, at discontinuities.

5.1 Vector-Valued Functions

We will now spend time with vector-valued functions ($f : \mathbb{R}^n \mapsto A$)

Definition 5.9. A **curve (or path)** in \mathbb{R}^n is a function

$$\gamma : [a, b] \subset \mathbb{R} \mapsto \mathbb{R}^n$$

We usually call the image of γ the curve and the function γ as the parameterization of the curve

Note. This curve has a direction given by γ and runs from $\gamma(a)$ to $\gamma(b)$, which are the endpoints. $\gamma(a)$ being the beginning and $\gamma(b)$ being the end

Think of $t, t \in [a, b]$ as a variable and $\gamma(t)$ as "tracing out" the curve in \mathbb{R} as t goes from a to b .

This interval be adapted to any (a, b) and even to \mathbb{R} itself.

$$\gamma(t) = (\gamma_1(t), \gamma_2(t), \dots, \gamma_n(t)), t \in [a, b]$$

Example 5.10. where $t \in \mathbb{R}$

$$\begin{aligned} \gamma(t) &= (x_0, y_0, z_0) + t(v_1, v_2, v_3) \\ &= x + tv \end{aligned}$$

This is a parametric representation of a line. Where x is a point and v is a direction vector

Definition 5.11. γ is **continuous** at $c \in (a, b)$ if $\lim_{t \rightarrow c} \gamma(t) = \gamma(c)$ iff the components $\gamma_i(t)$ for $i = 1, \dots, n$ are continuous at c

Definition 5.12. If γ is continuous at all points, γ is called a **Continuous Path**

Example 5.13. Consider a circle of radius 3 in \mathbb{R}^2 centred around the origin. A path of this circle is

$$\gamma(t) = (3\cos t, 3\sin t), t \in [0, 2\pi]$$

We say this curve is oriented in the counter clockwise direction

Example 5.14. The curve for $y = x^3 + 1$ is

$$\gamma(t) = (t, t^3 + 1), t \in \mathbb{R}$$

Example 5.15. In \mathbb{R}^3 , $\gamma(t) = (3\cos t, 3\sin t, t), t \in [0, 2\pi]$.

Since $x^2 + y^2 = 9$, this curve must live in the cylinder $x^2 + y^2 = 9$ and since $z = t$, the curve spirals upwards, this is called a helix

Example 5.16. $\gamma(t) = (t, t^2, t^3), t \in [-2, 2]$. This is a twisted cubic.

Example 5.17. Find the parameterization of the curve C of the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $y + z = 2$. The projection into the $x - y$ plane is the circle $x^2 + y^2 = 1, z = 0$, so

$$x = \cos t, y = \sin t$$

From the equation of the plane

$$z = 2 - y = 2 - \sin t$$

We now parameterize the curve C by

$$\gamma(t) = (\cos t, \sin t, 2 - \sin t), t \in [0, 2\pi]$$