

MATB43: Introduction to Analysis

Lecture Notes

Joshua Concon

University of Toronto Scarborough – Winter 2018

Pre-reqs are MATA37. Instructor is John Scherk. If you find any problems in these notes, feel free to contact me at conconjoshua@gmail.com.

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1 Friday, January 5, 2018

1.1 Countability

Countability is all about the cardinality of sets

Definition 1.1. For sets X, Y , they have the same cardinality if there exists a bijection between them ($f : X \mapsto Y$ has an inverse as well). Equality of cardinality of 2 sets is written as $\text{card}(X) = \text{card}(Y)$

Example 1.2. $\text{card}(\{1, 2, 3, 4, 5\}) = \text{card}(\{a, b, c, d, e\})$

Example 1.3. $\text{card}(\mathbb{N}) = \text{card}(2\mathbb{N})$

This example is correct because $f : \mathbb{N} \mapsto 2\mathbb{N}, f(n) = 2n$ is a bijection.

Example 1.4. $\text{card}(\{\text{set of odd numbers}\}) = \text{card}(\mathbb{N})$

This example is correct because $g : \mathbb{N} \mapsto \{\text{set of odd numbers}\}$
 $g(n) = 2n + 1, n \in \mathbb{N}$

Definition 1.5. X is **finite** if for some $n \in \mathbb{N}$ there exists a bijection where $A = \{1, 2, \dots, n\} \xrightarrow{f} X$. So essentially $\text{card}(A) = \text{card}(X)$

Intuitively, we can label the elements of X as $X = \{x_1, x_2, \dots, x_n\}$

Definition 1.6. X is **infinite** if X is not finite.

Examples of this include: $\mathbb{N}, 2\mathbb{N}, \mathbb{Q}, \mathbb{R}$

However, not all infinite sets have the same cardinality.

Definition 1.7. X is **countable** if $\text{card}(X) = \text{card}(A)$ where $A \subseteq \mathbb{N}$

Again intuitively, we can label the elements of X as $X = \{x_1, x_2, \dots, x_n, \dots\}$
 (using elements of \mathbb{N})

Example 1.8. $2\mathbb{N}$

Example 1.9. The set of odd numbers

Example 1.10. \mathbb{Z}

Theorem 1.11. *In general, if $Z = X \cup Y$ and X, Y are both countable then Z is countable as well*

Proof. label elements of $X = \{x_1, x_2, \dots, x_n, \dots\}$
 label elements of $Y = \{y_1, y_2, \dots, y_n, \dots\}$
 define $h : \mathbb{N} \mapsto \mathbb{Z}$ as, for $n \in \mathbb{N}$:

$$h(2n - 1) = x_n$$

$$h(2n) = y_n$$

Then h is a bijection, therefore Z is also countable ■

Example 1.12. $\mathbb{N} \times \mathbb{N}$

This example is countable as we can label its elements in the following pattern.

$$(1, 1) \mapsto 1$$

$$(2, 1) \mapsto 2$$

$$(1, 2) \mapsto 3$$

$$(1, 3) \mapsto 4$$

$$(2, 2) \mapsto 5$$

$$(3, 1) \mapsto 6$$

$$(4, 1) \mapsto 7$$

And so on, in this pattern. Intuitively, if you list out all the pairs of $\mathbb{N} \times \mathbb{N}$ like a matrix, this would create a sort of zig zag pattern.

Proposition 1.13. Suppose X is countable and $Y \subset X$, then Y is either finite (which also means that it is countable) or just countable

Proof. write $X = \{x_1, x_2, \dots, x_n, \dots\}$
 let j_1 be the smallest index such that $x_{j_1} \in Y$
 let j_2 be the smallest index such that $x_{j_2} \in Y, j_2 > j_1$
 let j_3 be the smallest index such that $x_{j_3} \in Y, j_3 > j_2$
 ...
 and so on.

Now either:

This process terminates, which means that $Y = \{x_{j_1}, \dots, x_{j_n}\}$ for some

j_1, \dots, j_n which Y is finite.

or:

Y is countable, since $Y = \{x_{j_1}, x_{j_2}, \dots, x_{j_n}, \dots\}$ ■

Proposition 1.14. \mathbb{Q} is countable

Proof. Note that $\mathbb{Q} = \mathbb{Q}^- \cup \{0\} \cup \mathbb{Q}^+$
so we must check that \mathbb{Q}^+ is countable, note that

$$\mathbb{Q}^+ = \left\{ \frac{m}{n} \mid m, n \in \mathbb{N}, n \neq 0 \right\}$$

Since it is a pairing of two natural numbers (m, n) , this is a subset of $\mathbb{N} \times \mathbb{N}$, which means that \mathbb{Q}^+ is countable, we can say the same thing for \mathbb{Q}^- as well, note that it is

$$\mathbb{Q}^+ = \left\{ \frac{-m}{n} \mid m, n \in \mathbb{N}, n \neq 0 \right\}$$

and since \mathbb{Q}^+ and \mathbb{Q}^- are both countable and $\{0\}$ is a set of cardinality 1, therefore \mathbb{Q} is countable ■

Theorem 1.15. Let $S = \{s = (s_1, s_2, \dots) \mid s_j = 0 \text{ or } 1 \forall j\}$ (note here that s is a sequence). Then S is not countable

Proof. For proof by contradiction, suppose S is countable.
We can then label the elements of S as $S^1, S^2, \dots, S^n, \dots \in S$.

So an example of this would be:

$$\begin{aligned} S^1 &= (s_1^1, s_2^1, s_3^1, \dots, s_n^1, \dots) \\ S^2 &= (s_1^2, s_2^2, s_3^2, \dots, s_n^2, \dots) \\ &\dots \\ S^m &= (s_1^m, s_2^m, s_3^m, \dots, s_n^m, \dots) \\ &\dots \end{aligned}$$

And we will define a $t \in S$ as follows, let $t = (t_1, t_2, \dots, t_m, \dots)$

$$t_1 = \begin{cases} 0, & \text{if } S_1^1 = 1 \\ 1, & \text{if } S_1^1 = 0 \end{cases}$$

$$t_2 = \begin{cases} 0, & \text{if } S_2^2 = 1 \\ 1, & \text{if } S_2^2 = 0 \end{cases}$$

...

$$t_m = \begin{cases} 0, & \text{if } S_m^m = 1 \\ 1, & \text{if } S_m^m = 0 \end{cases}$$

...

Therefore $t \neq S^m, \forall m$. Therefore this is a contradiction as t was not in the listed elements of S , therefore S is not countable. ■

Corollary 1.16. \mathbb{R} is not countable

Proof. Regard \mathbb{R} as a set of infinite decimal fractions.

Identify $s \in S$ with the decimal number $0.s_1s_2...s_n...$ so S can be regarded as a subset of \mathbb{R} . If \mathbb{R} were countable, then S would be countable, therefore \mathbb{R} must not be countable. ■

1.2 Real Numbers

Examples of real numbers that are not rational include: π, e (which are algebraic numbers), $\sqrt{2}, \sqrt{3}, \frac{\sqrt{5}+1}{2}$ (which are roots of polynomial equations with integer coefficients.)

1.3 Algebraic Numbers

The set of Algebraic Numbers is a set $\overline{\mathbb{Q}}$ such that

$$\overline{\mathbb{Q}} = \{x \in \mathbb{R} \mid x^n + a_{n-1}x^{n-1} + \dots + a_0 = 0 \text{ for some } a_0, \dots, a_{n-1} \in \mathbb{Z}\}$$

Note that $\mathbb{Q} \subset \overline{\mathbb{Q}}$, and in fact, $\overline{\mathbb{Q}}$ is countable.

if $x \in (\mathbb{R} \setminus \overline{\mathbb{Q}})$ then. x is transcendental.

1.4 Bounds

Definition 1.17. $X \subset \mathbb{R}$, X is **bounded above** if $\exists a \in \mathbb{R}$ such that $a \geq x, \forall x \in X$

Definition 1.18. $X \subset \mathbb{R}$, X is **bounded below** if $\exists a \in \mathbb{R}$ such that $a \leq x, \forall x \in X$

Example 1.19. $X = \{x \in \mathbb{R} | x^2 \leq 2\}$ is bounded above by $\frac{3}{2}$ since $(\frac{3}{2})^2 \geq 2$

Definition 1.20. a is the **least upper bound** of X if a is an upper bound of X and if $b < a$, then there exists $x \in X$ such that $x > b$.

We write $a = \text{lub}(X)$ or $a = \text{sup}(X)$ if a is the least upper bound of X

Definition 1.21. a is the **greatest lower bound** of X if a is a lower bound of X and if $b > a$, then there exists $x \in X$ such that $x < b$.

Example 1.22. if $X = \{x | x^2 \leq 2\}$, then $\text{sup}(X) = \sqrt{2} \notin \mathbb{Q}$

Property 1.23. if $X \subset \mathbb{R}$ is bounded above then there exists $a \in \mathbb{R}$, a least upper bound of X

Theorem 1.24. given $a, b \in \mathbb{R}$ $a, b > 0$ $\exists n \in \mathbb{N}$ such that $na > b$ (This is known as the **archimedean property**)

Proof. Suppose that $\forall n \in \mathbb{N}$, $na \leq b$ implies that b is an upper bound for $X = \{na | n \in \mathbb{N}\}$.

Since X is bounded above, let $c = \text{sup}(X)$, this implies that $c - a$ is not an upper bound for X , which further implies that there exists an $n \in \mathbb{N}$ such that $na > c - a$, and that

$$\begin{aligned} na &> c - a \\ na + a &> c \\ (n+1)a &> c \\ c &< (n+1)a \in X \end{aligned}$$

This is impossible since we've previously stated that c is the upper bound. Therefore X is not bounded above. ■

Theorem 1.25. given $c, d \in \mathbb{R}$, there exists $q = \frac{m}{n} \in \mathbb{Q}$ such that $c < q < d$

Proof. We want $c < \frac{m}{n} < d$ iff $nc < m < nd$.

let $\epsilon = d - c > 0$

pick $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$

pick m , such that $m > nc$

(since we also need $m < nd$, choose m to be as small as possible such that $m - 1 \leq nc < m$)

$$\begin{aligned} m - 1 &\leq nc \leq m \\ \frac{m}{n} - \frac{1}{n} &= \frac{m - 1}{n} \leq c \leq \frac{m}{n} \end{aligned}$$

or

$$\frac{m}{n} \leq c + \frac{1}{n} < c + \epsilon = d$$

So now we have $c < \frac{m}{n} < d$ as desired.

■

2 Monday, January 8, 2017

2.1 Sequences - Review

This section will be just review of MATA37 and will be concerning sequences of the real numbers $\{a_n\}$.

Example 2.1. $\{\frac{1}{n}\} = 1, \frac{1}{2}, \frac{1}{3}, \dots$,

Example 2.2. $\{\frac{(-1)^n}{n}\} = -1, \frac{1}{2}, -\frac{1}{3}, \dots$, This sequence oscillates back and forth.

Example 2.3. $\{(-1)^n\} = -1, 1, -1, 1, \dots$, This sequence oscillates back and forth as well.

Example 2.4. $\{x_1, x_2, \dots\}$ where this sequence enumerates \mathbb{Q} . This sequence 'bounces around wildy'.

Definition 2.5. $x \in \mathbb{R}$ is the **limit of a sequence** $\{x_n\}$ (so that it converges to this number)

$\lim_{n \rightarrow \infty} x_n = a$, if given some tolerance $\epsilon > 0$.

There exists N , such that $n \geq N$ will lie in the interval $a - \epsilon, a + \epsilon$, aka $|x_n - a| < \epsilon$.

Example 2.6. $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$

Example 2.7. $\lim_{n \rightarrow \infty} \frac{(-1)^n}{n} = 0$

Example 2.8. example 2.4 and 2.3 both have no limit

Proposition 2.9. If a sequence $\{x_n\}$ has a limit, then this limit is unique, and the sequence is bounded.

Proposition 2.10. Supposed that $\{x_n\}, \{y_n\} \subset \mathbb{R}$ are convergent sequences, so that $\lim_{n \rightarrow \infty} x_n = a$, $\lim_{n \rightarrow \infty} y_n = b$ then:

1. $\{x_n + y_n\}$ converges and $\lim_{n \rightarrow \infty} (x_n + y_n) = a + b$
2. $\{x_n y_n\}$ converges and $\lim_{n \rightarrow \infty} (x_n y_n) = ab$
3. if $y_n \neq 0, \forall n$, and $b \neq 0$ then $\{\frac{x_n}{y_n}\}$ converges and $\lim_{n \rightarrow \infty} (\frac{x_n}{y_n}) = \frac{a}{b}$

Definition 2.11. A sequence is **monotone** if it is either increasing, or decreasing.

Proposition 2.12. A bounded monotone sequence converges.

Proof. Suppose that $\{x_n\}$ is a bounded increasing (monotone) sequence, such that

$$x_1 \leq x_2 \leq x_3 \leq \dots$$

And there exists $A \in \mathbb{R}$ such that $x_n \leq A, \forall n$

Therefore, there exists a least upper bound $a \leq A$.

claim: $x_n \rightarrow a$ as $n \rightarrow \infty$

take $\epsilon > 0$, since a is the least upper bound of $\{x_n\}$, there exists N such that $a - \epsilon < x_N \leq a$. But $\{x_n\}$ is increasing. Therefore $\forall n \geq N, a - \epsilon < x_N \leq x_n \leq a$. This implies that $\lim_{n \rightarrow \infty} x_n = a$ ■

3 Friday, January 12, 2017

3.1 Monotone Sequences

i.e. Sequences which are increasing or decreasing

Proposition 3.1. A bounded monotone sequence converges.

Proof. For an increasing sequence $\{x_n\}$, $\lim_{n \rightarrow \infty} x_n = \sup(\{x_n\})$, suppose $\{x_n\}$ is decreasing and bounded below, then $\{-x_n\}$ is increasing and bounded above. This implies that $\lim_{n \rightarrow \infty} -x_n$ and $\lim_{n \rightarrow \infty} x_n$ both exist, and

$$\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} -x_n$$

■

So now we're going to prove that Every sequence has a monotone subsequence along with other propositions on bounded monotone sequences. This will allow us to prove the Bolzano-Weierstrass Theorem, which states that every bounded sequence has a convergent subsequence, and this will help us prove the Cauchy property for sequences.

Example 3.2. Lets look at the following sequence:

$$x_n = 1 + \frac{1}{2} + \dots + \frac{1}{n} - \log n, n \geq 1$$

So we want to show that this converges, we'll show that if we show that $\{x_n\}$ is decreasing and is bounded below by 0.

The limit ($\lim_{n \rightarrow \infty} x_n$) is actually a mysterious number that we don't know too much about.

Recall: that $\log n = \int_1^n \frac{dt}{t}$

So if we consider the space under the graph of $\frac{1}{t}$ from n to $n+1$, we get the following inequality

$$\frac{1}{n+1} < \log(n+1) - \log n < \frac{1}{n}$$

Which implies that

$$\frac{1}{n+1} - \log(n+1) + \log n < 0$$

Now consider the following for an arbitrary n

$$x_{n+1} = x_n + \left(\frac{1}{n+1} + \log n - \log(n+1)\right)$$

but since $\frac{1}{n+1} - \log(n+1) + \log n < 0$, this would mean that $x_{n+1} < x_n$, so this sequence is decreasing.

Now consider the following inequality derived from the recall block:

$$\sum_{n=1}^m (\log(n+1) - \log n) < \sum_{n=1}^m \frac{1}{n}$$

The left side of this inequality telescopes, giving us

$$\log(m) < \log(m+1) < 1 + \frac{1}{2} + \dots + \frac{1}{m}$$

And since the left most side is greater than the right most side, that means that $x_m > 0, \forall m$.

3.2 Subsequences

Definition 3.3. Let $\{x_i\}$ be a sequence of the real numbers, pick a finite set of indices $j_1 < j_2 < \dots < j_n < \dots$

A **Subsequence** of $\{x_i\}$ is $\{x_{j_1}, x_{j_2}, \dots, x_{j_n}, \dots\}$

Example 3.4. Considering the sequence $\{x_n\} = \{1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots\}$

1. $\{\frac{1}{2^n}\}$ is a subsequence of $\{x_n\}$.
2. $\{\frac{1}{2^n}\}$ is a subsequence of $\{x_n\}$.
3. $\{(-1)^n\}$ is also a subsequence of $\{x_n\}$, note that it does not converge, but has convergent subsequences of $\{(-1)^{2n}\}$ and $\{(-1)^{2n+1}\}$

Definition 3.5. Call a term x_m **dominant** if $x_n \leq x_m, \forall n \geq m$

Proposition 3.6. Every real number sequence has a monotone subsequence

Proof. There are 2 cases:

Case 1: infinitely many dominant terms

let $\{x_{j_1}, x_{j_2}, \dots, x_{j_n}, \dots\}$ be the sequence of dominant terms. By definition,
 $x_{j_1} \geq x_{j_2} \geq \dots \geq x_{j_n} \geq \dots$
 so $\{x_{j_n}\}$ is a decreasing sequence, which is monotone.

Case 2: only finitely many dominant terms

Pick an index j , so that x_{j_1} is the first term beyond all dominant terms in the sequence ($\exists i, i < j_1, x_i$ is the last dominant term).

Since x_{j_1} is not dominant, then $\exists j_2 > j_1$ where $x_{j_2} > x_{j_1}$

Since x_{j_2} is not dominant, then $\exists j_3 > j_2$ where $x_{j_3} > x_{j_2}$

...

and so on

By induction, we construct an increasing subsequence $\{x_{j_m}\}$ of $\{x_n\}$.

By these 2 cases, every real number has a monotone sequence. ■

Theorem 3.7. (*Bolzano-Weierstrass Theorem*) every bounded sequence has a convergent subsequence

Proof. Given a bounded sequence $\{x_n\}$ of the real numbers, there exists a monotone subsequence $\{x_{j_n}\}$.

The monotone subsequence $\{x_{j_n}\}$ is also bounded since $\{x_n\}$ is bounded.

This implies that $\lim_{n \rightarrow \infty} x_{j_n}$ exists. ■

Definition 3.8. Cauchy Property: intuitively, in a convergent sequence, the terms get closer and closer as $n \rightarrow \infty$. More precisely:

$\{x_n\}$ is a real number sequence, given $\epsilon > 0$, $\exists N$ such that for $m, n > N$, $|x_m - x_n| < \epsilon$.

Proposition 3.9. Suppose that $\{x_n\}$ converges to a , then $\{x_n\}$ satisfies the Cauchy property.

Proof. given $\epsilon > 0$, then $\exists N$ such that $\forall n > N, |x_n - a| < \frac{\epsilon}{2}$, so if $m > N$ then $|x_m - a| < \frac{\epsilon}{2}$ as well.

This implies that

$$|x_m - x_n| = |x_m - a + a - x_n| \leq |x_m - a| + |x_n - a| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

■

So now we want to show that if a sequence satisfies the Cauchy property, then it converges. We'll do this by first showing that a bounded sequence with the Cauchy property implies that sequence has a convergent subsequence. The second and final step is to show that the limit of a convergent subsequence is the limit of the original sequence.

Proposition 3.10. If $\{x_n\}$ satisfies the Cauchy property, then it's bounded.

Proof. By definition, there exists N so that for $m, n > N$, $|x_m - x_n| < 1$ in particular, $\forall n > N$, $|x_n - x_{N+1}| < 1$, which gives us:

$$\begin{aligned} -1 &< x_n - x_{N+1} < 1 \\ x_{N+1} - 1 &< x_n < x_{N+1} + 1 \end{aligned}$$

Note: x_{N+1} is fixed (a constant)

Let $A = \min\{x_1, \dots, x_N, x_{N+1} - 1\}$

Let $B = \max\{x_1, \dots, x_N, x_{N+1} + 1\}$

This implies that for all n , $A \leq x_n \leq B$, therefore Bounded. ■

Proposition 3.11. A sequence with the Cauchy property is convergent.

Proof. Let $\{x_n\}$ be a sequence with the Cauchy property.

Since $\{x_n\}$ is bounded, by the Bolzano–Weierstrass Theorem, there exists a convergent subsequence $\{x_{j_m}\}$. Now let:

$$a = \lim_{m \rightarrow \infty} x_{j_m}$$

given ϵ , $\exists M$ so that for all $m > M$, $|x_{j_m} - a| < \frac{\epsilon}{2}$.

Note that $j_m \geq m$

The Cauchy property implies that $\exists N$ so that for all $m, n > N$, $|x_m - x_n| < \epsilon$.

We'll pick $P = \max(N, M)$, then for $m, n > P$ we have:

$$\begin{aligned}
|x_n - a| &= |x_n + x_{j_m} - x_{j_m} - a| \\
&= |x_n - x_{j_m}| + |x_{j_m} - a| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

■

Example 3.12.

$$x_n = \sum_{k=1}^n \frac{1}{k^2}$$

Verify the Cauchy property.

Take $m > n$, $x_m - x_n = \sum_{k=n+1}^m \frac{1}{k^2}$. Now $\frac{1}{k} < \frac{1}{k(k-1)} = \frac{1}{k-1} - \frac{1}{k}$. This implies that

$$\begin{aligned}
&\Rightarrow \sum_{k=n+1}^m \frac{1}{k^2} < \sum_{k=n+1}^m \left(\frac{1}{k-1} - \frac{1}{k} \right) \\
&\stackrel{\text{telescoping}}{\Rightarrow} \sum_{k=n+1}^m \frac{1}{k^2} < \frac{1}{m} - \frac{1}{n} < \frac{1}{m} \\
&\Rightarrow (x_m - x_n) < \frac{1}{m}
\end{aligned}$$

This implies that x_n is convergent, in fact

$$\lim_{n \rightarrow \infty} x_n = \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$$