MATB42: Multivariable Calculus II
Lecture Notes

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1.1 Fourier Expansions

In this section, we will focus on single variable calculus, (so where $f : \mathbb{R} \mapsto \mathbb{R}$)

Let us say that we have a function $f(x)$ and we want to approximate it. We can use an nth degree Taylor Polynomial, but this requires that $f(x)$ has at least n derivatives at some point x_0 and the kth derivative of f $(f^{(k)}(x))$ is determined by properties of f in some neighbourhood of x_0, but what about outside this neighbourhood? How can we be certain of the approximation outside of this neighbour.

Our problem here is that Taylor Polynomial may only approximate "near" x_0

Now, consider the following function:

$$\Delta(x) = \begin{cases} 1, & \lfloor x \rfloor < x, \lfloor x \rfloor \text{ is odd} \\ 0, & \lfloor x \rfloor < x, \lfloor x \rfloor \text{ is even} \end{cases}$$

In this function, Tayler returns either 0 or 1 depending on your choice of x_0 and cannot work for an $x_0 = p \in \mathbb{Z}$. Therefore Taylor polynomials cannot reflect the true nature of this function. Taylor provides a "local" approximation, but we want a "global" approximation. We need an approximation that is more precise over an interval at the cost of being not as precise as precise at any particular x_0.

Note that the example function is **periodic**.

Definition 1.1. A function $y = f(x)$ such that $f(x) = f(x + p), p \neq 0, \forall x$ is said to be **periodic** of period p

Example 1.2. The periodic function $\Delta(x)$ is of period 2.

What we want is a global approximation of a periodic function, and the Fourier Approximation will be periodic, so we can use it for exactly that.

Definition 1.3. A **trigonometric polynomial of degree** N is an expression of the form

$$\frac{a_0}{2} + \sum_{k-1}^{N} a_k cos(kx) + b_k sin(kx)$$

where the a_i, b_i are constants.

We know that $sin(x)$ and $cos(x)$ are the simplest periodic functions and repeat in intervals of 2π, so $cos(kx)$ and $sin(kx)$ have period $\frac{2\pi}{k}$, but the smallest shared period is 2π. If a trigonometric polynomial has period 2π and $f(x)$ has period p, then we must set $x = \frac{pt}{2\pi}$ to fix the period (where t is a variable).

So to approximate $y = f(x)$ by $F_N(x)$ for some N, we use the following equation:

$$F_N(x) = \frac{a_0}{2} + \sum_{k=1}^{N} a_k cos(kx) + b_k sin(kx)$$

Now we need to choose the a_k, b_k. We can define it in the following way:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)cos(kx)dx, k = 1, 2, 3, ...$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)sin(kx)dx, k = 1, 2, 3, ...$$

When defined in this way, a_i, b_i are called the **Fourier Coefficients** of f over the interval $[-\pi, \pi]$ and we call $F_N(x)$ the **Fourier Polynomial of degree** N.

So why do we add the $\frac{a_0}{2}$? It is the average value of f over $[-\pi, \pi]$.

Note. *sometimes you will see* a_0 *used instead of* $\frac{a_0}{2}$ *in the Fourier polynomial where*

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$$

Example 1.4. Consider $f(x) = \frac{-x}{2}$ over $[-\pi, \pi]$. Use Fourier Approximation.

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)cos(kx)dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(-\frac{x}{2})cos(kx)dx \overset{odd}{=} 0$$
$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(-\frac{x}{2})dx \overset{odd}{=} 0$$

$$\begin{aligned} b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)sin(kx)dx \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi}(-\frac{x}{2})sin(kx)dx \\ &\overset{even}{=} -\frac{1}{\pi}\int_{-\pi}^{\pi} xsin(kx)dx \\ &\underset{u=x,dv=sin(kx)dx}{\overset{even}{=}} -\frac{1}{\pi}[-\frac{1}{k}xcos(kx) + \frac{1}{k^2}sin(kx)]_0^{\pi} \\ &= \frac{1}{\pi k}[\pi cos(k\pi)] \\ &= \frac{1}{k}cos(k\pi) \\ &= \frac{(-1)^k}{k} \end{aligned}$$

Thus we have:

$$F_N(x) = -sin(x) + \frac{1}{2}sin(2x) - \frac{1}{3}sin(3x) + \frac{1}{4}sin(4x) + ...$$
$$F_1(x) = -sin(x)$$
$$F_2(x) = -sin(x) + \frac{1}{2}sin(2x)$$
$$F_3(x) = -sin(x) + \frac{1}{2}sin(2x) - \frac{1}{3}sin(3x)$$
$$...$$

And so on.

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continuing from the last lecture...

Example 2.1. (continued from example 1.4)
$f(x) = \frac{-x}{2}$
$F_N(x) = -sin(x) + \frac{1}{2}sin(2x) - \frac{1}{3}sin(3x) + \frac{1}{4}sin(4x) + ...$

This can be extended to a Fourier Series:

$$F_N(x) = \sum_{k=1}^{\infty}(-1)^k \frac{sin(kx)}{k}$$

Definition 2.2. For $f : \mathbb{R} \mapsto \mathbb{R}$, the Fourier Series for f is

$$F(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k cos(kx) + b_k sin(kx)$$

where a_i, b_i are Fourier coefficients.

The Nth degree Fourier Polynomial can be regarded as the Nth partial sum of the series.

We haven't talked about convergence yet, but for now, we will assume the series converges ($f(x) = F(x)$)

Definition 2.3. Function $a_k cos(kx) + b_k sin(kx)$ is the kth harmonic of f. The Fourier Series expresses f in terms of its harmonics.

Note. *(Looking at Harmonics in a Musical Sense):*
the 1st harmonic is the fundamental harmonic of f (the fundamental tone). The 2nd harmonic is the first overtone.

(completely rewrite this amplitude section)

Definition 2.4. The amplitude of the kth harmonic is

$$A_k = \sqrt{(a_k)^2 + (b_k)^2}$$

And note that

$$a_k = A_k sin\alpha, b_k = A_k cos\alpha$$

Definition 2.5. The energy E of a periodic function f of period 2π is

$$E = \frac{1}{\pi}\int_{-\pi}^{\pi}[f(x)]^2 dx$$

So the energy of the kth harmonic is

$$E = \frac{1}{\pi}\int_{-\pi}^{\pi}[a_k cos(kx) + b_k sin(kx)]^2 dx = (a_k)^2 + (b_k)^2 = (A_k)^2$$

And the energy of the constant term is

$$\frac{1}{\pi}\int_{-\pi}^{\pi}[a_0]^2 dx = 2(a_0)^2$$

So we put $A_0 = \frac{1}{\sqrt{2}}a_0$.

For a "nice" periodic function, we have the following equation:

$$E = A_0^2 + A_1^2 + A_2^2 + ...$$

This is known as the Energy Theorem, and comes from the study of periodic waves.

We can draw a graph of this as A_k^2 against k (This graph is known as the Energy Spectrum of f). It shows how the energy of f is distributed among its harmonics.

Note. *Notice that*

$$E = \frac{1}{\pi}\int_{-\pi}^{\pi}[f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{k=1}^{\infty}(a_k^2 + b_k^2) \textit{ Parseval's Equation}$$

Assume a function f of period 2π is the sum of a trigonometric series

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k cos(kx) + b_k sin(kx)) \text{ on the interval } [-\pi, \pi]$$

Multiply by $cos(mx)$ and integrate to get

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)cos(mx)dx &= \frac{a_0^2}{2}\int_{-\pi}^{\pi} cos(mx)dx + \int_{-\pi}^{\pi}[\sum_{k=1}^{\infty}(a_k cos(kx) + b_k sin(kx))]dx \\ &= \frac{a_0^2}{2}\int_{-\pi}^{\pi} cos(mx)dx + \sum_{k=1}^{\infty}(a_k \int_{-\pi}^{\pi} cos(kx)dx + b_k \int_{-\pi}^{\pi} sin(kx)dx)\end{aligned}$$

Note. *Recall the following trigonometric identities:*

1. $cosAcosB = \frac{1}{2}[cos(A+B) + cos(A-B)]$
2. $cosAsinB = \frac{1}{2}[sin(A+B) + sin(A-B)]$
3. $sinAsinB = \frac{1}{2}[cos(A-B) - cos(A+B)]$

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Continuing from where we left off.

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)cos(mx)dx &= \frac{a_0^2}{2}\int_{-\pi}^{\pi} cos(mx)dx + \int_{-\pi}^{\pi}[\sum_{k=1}^{\infty}(a_k cos(kx) + b_k sin(kx))]dx \\ &= \frac{a_0^2}{2}\int_{-\pi}^{\pi} cos(mx)dx + \sum_{k=1}^{\infty}(a_k \int_{-\pi}^{\pi} cos(kx)dx + b_k \int_{-\pi}^{\pi} sin(kx)dx)\end{aligned}$$

We know the following from trigonometric identities

$$\int_{-\pi}^{\pi} cos(kx)cos(mx)dx = \begin{cases} 0, & k \neq m \\ \pi, & k = m \end{cases}$$

As well as the following from odd function properties

$$\int_{-\pi}^{\pi} cos(kx)dx = 0$$

$$\int_{-\pi}^{\pi} sin(kx)cos(mx)dx = 0$$

So now we get

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)cos(mx)dx &= \frac{a_0^2}{2}\int_{-\pi}^{\pi} cos(mx)dx + \sum_{k=1}^{\infty}(a_k \int_{-\pi}^{\pi} cos(kx)dx + b_k \int_{-\pi}^{\pi} sin(kx)dx) \\ &= a_m\pi \\ &\implies a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)cos(mx)dx\end{aligned}$$

Example 3.1. Lets take

$$f(x) = \begin{cases} 1, & 0 \leq x < \pi \\ -1, & -\pi \leq x < 0 \end{cases}$$

Note that this is an odd function, therefore $a_k = 0, \forall k \geq 0$. So now lets calculate b_k.

$$
\begin{aligned}
b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)sin(kx)dx \\
&\overset{even}{=} \frac{2}{\pi}\int_{0}^{\pi} f(x)sin(kx)dx \\
&= \frac{2}{\pi}\int_{0}^{\pi} sin(kx)dx \\
&= \frac{2}{k\pi}[-cos(kx)]_0^{\pi} \\
&= \begin{cases} \frac{4}{k\pi}, & k \text{ is odd} \\ 0, & k \text{ is even} \end{cases}
\end{aligned}
$$

And now lets right out the Fourier Polynomial ($F_N(x)$)
<u>if N is odd:</u>

$$F_N(x) = \frac{4}{\pi}sin(x) + \frac{4}{3\pi}sin(3x) + \frac{4}{5\pi}sin(5x) + ...$$

<u>if N is even:</u>

$$F_N(x) = F_{N-1}(x)$$

We can also write it as a Fourier Series

$$F(x) = \frac{4}{\pi}\sum_{l=0}^{\infty}\frac{sin((2l+1)x)}{2l+1}$$

The energy of the function is

$$E = \frac{1}{\pi}\int_{-\pi}^{\pi}[f(x)]^2dx = \frac{2}{\pi}\int_0^{\pi} dx = 2$$

The amplitutde of the kth harmonic is

$$A_k = \sqrt{a_k^2 + b_k^2} = \sqrt{0 + \frac{16}{k^2\pi^2}} = \frac{4}{k\pi}$$

The energy of the kth harmonic is

$$A_k^2 = \frac{16}{k^2\pi^2}$$

Note that for this example, both the energy and the amplitude are 0 at an even k.

Lets now evaluate the energy spectrum:

$$k = 1, E = \frac{16}{\pi^2} \approx 1.62, \frac{1.62}{2} = 0.81 = 81\%$$

$$k = 3, E = \frac{16}{9\pi^2} \approx 0.18, \frac{0.18}{2} = 0.09 = 9\%$$

$$k = 5, E = \frac{16}{25\pi^2} \approx 0.06, \frac{0.06}{2} = 0.03 = 3\%$$

$$k = 7, E = \frac{16}{49\pi^2} \approx 0.03, \frac{0.03}{2} = 0.015 = 1.5\%$$

However, we do not need to exclusively work with the interval $[-\pi, \pi]$ we can even work over any interval of length 2π.

3.1 General Fourier Series (interval of length 2π)

$$a_k = \frac{1}{\pi}\int_c^{c+2\pi} f(x)cos(kx)dx, k = 1, 2, 3, ...$$

$$b_k = \frac{1}{\pi}\int_c^{c+2\pi} f(x)sin(kx)dx, k = 1, 2, 3, ...$$

What about for f if f has period p?

$$f(x + p) = f(x), \forall x, \exists p \neq 0$$

We then substitute $x = \frac{pt}{2\pi}$ which gives a new function $f_p(t) = f(\frac{pt}{2\pi})$ with period 2π. So

$$f_p(t + 2\pi) = f(\frac{p}{2\pi}(t + 2\pi)) = f(\frac{pt}{2\pi} + p) = f(\frac{pt}{2\pi}) = f_p(t)$$

So how about the Fourier Expansion for $f_p(t)$? To find this, we must replace t by $\frac{2\pi x}{p}$ giving for $f(x)$.

$$F(x) = \frac{a_0}{2} + \sum_{k-1}^{\infty} a_k cos(\frac{2nx\pi}{p}) + b_k sin(\frac{2nx\pi}{p})$$

$$a_k = \frac{2}{p}\int_c^{c+p} f(x)cos(\frac{2nx\pi}{p}x)dx, k = 1, 2, 3, ...$$
$$b_k = \frac{2}{p}\int_c^{c+p} f(x)sin(\frac{2nx\pi}{p}x)dx, k = 1, 2, 3, ...$$

For any function defined on $[a, b]$, we can extend f to all of $\mathbb{R}$ as a periodic function. Given a periodic function f_E from f of period $p = b - a$, we now have:

$$a_k = \frac{2}{b-a}\int_a^{b} f(x)cos(\frac{2nx\pi}{b-a}x)dx, k = 1, 2, 3, ...$$
$$b_k = \frac{2}{b-a}\int_a^{b} f(x)sin(\frac{2nx\pi}{b-a}x)dx, k = 1, 2, 3, ...$$

Example 3.2. Take the function $f(x) = x, 0 \leq x < 1$ and extend it with period 1. For $k \neq 0$:

$$\begin{aligned}
a_k &= 2\int_0^1 xcos(2k\pi x)dx \\
&\overset{parts}{\underset{u=v,dv=cos(2\pi kx)dx}{=}} 2[\frac{xsin(2k\pi x)}{2\pi k}]_0^1 - \frac{2}{2k\pi}\int_0^1 sin(2k\pi x)dx \\
&= 0
\end{aligned}$$

$$a_0 = 2\int_0^1 xdx = [x^2]_0^1$$

$$\begin{aligned}
b_k &= 2\int_0^1 xsin(2k\pi x)dx \\
&\overset{parts}{\underset{u=x,dv=sin(2k\pi x)dx}{=}} 2[\frac{-xcos(2k\pi x)}{2\pi k}]_0^1 + \frac{1}{k\pi}\int_0^1 cos(2k\pi x)dx \\
&= \frac{-cos(2k\pi)}{k\pi} \\
&= \frac{-1}{k\pi}
\end{aligned}$$

So the Fourier Series will be

$$F(x) = \frac{1}{2} - \frac{1}{\pi}[sin(2\pi x) + \frac{sin(4\pi x)}{2} + \frac{sin(6\pi x)}{3} + ...]$$

Example 3.3. $f(x) = |x|, -\pi < x \leq \pi$. Since $f(x)$ is even, $b_k = 0, \forall k \in \mathbb{N}$.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|dx \overset{even}{=} \frac{2}{\pi}\int_0^{\pi} xdx = \pi$$

$$\begin{aligned}
a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} |x|cos(kx)dx \\
&\overset{even}{=} \frac{2}{pi}\int_0^{\pi} xcos(kx)dx \\
&\underset{u=x,dv=cos(kx)dx}{\overset{parts}{=}} \frac{2}{\pi}[\frac{xsin(kx)}{k} + \frac{cos(kx)}{k^2}]_0^{\pi} \\
&= \frac{2}{\pi k^2}(cos(k\pi) - 1) \\
&= \begin{cases} 0, & k \text{ is even} \\ \frac{-4}{\pi k^2}, & k \text{ is odd} \end{cases}
\end{aligned}$$

So we end up with the Fourier Series

$$F(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{l=0}^{\infty} \frac{cos((2l+1)x)}{(2l+1)^2}$$

Example 3.4. $f(x) = x, -\pi < x \leq \pi$. Note that since $f(x)$ is odd, $a_k = 0, \forall k \geq 0$

$$\begin{aligned}
b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} xsin(kx)dx \\
&\underset{u=x,dv=sin(kx)dx}{\overset{parts}{=}} \frac{1}{\pi}[\frac{-xcos(kx)}{k}]_{-\pi}^{\pi} + \frac{1}{\pi}\int_{-\pi}^{\pi} cos(kx)dx \\
&= \frac{1}{\pi}[\frac{-\pi cos(k\pi)}{k} + \frac{-\pi cos(-k\pi)}{k}] \\
&= \frac{-2}{k}cos(k\pi) \\
&= \frac{(-1)^{k+1}2}{k}
\end{aligned}$$

And the Fourier Series of this is

$$F(x) = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{2}{k}sin(kx)$$

To get a Fourier cosine series or a Fourier sine series, we need

- an f defined on the interval $[0, a]$, and we must extend this interval to also include $[-a, 0)$ to give an even or odd function on $[-a, a]$
- $f(-t) = f(t), -a \leq t < 0$ for the even extension item$f(-t) = -f(t), -a \leq t < 0$ for the odd extension

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Example 4.1. We want to express $f(x) = x, 0 \leq x < \pi$ as both a cosine series and a sine series.

<u>cosine series</u> For this, we need to extend the function as an even function, so we extend the function to $f(x) = |x|, -\pi < x \leq x$. This is a previous example that we computated before and gives us a cosine series

<u>sine series</u> For this, we must extend $f(x)$ as an odd function $f(x) = x, -\pi \geq x < \pi$. We've already seen from previous examples that is a sine series.

So for $f(x) = x, 0 \leq x < pi$, the cosine series looks like this:

$$F(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{l=0}^{\infty}\frac{cos((2l+1)x)}{(2l+1)^2}$$

And the sine series looks like this:

$$F(x) = \sum_{k=0}^{\infty}(-1)^{k+1}\frac{2}{k}sin(kx)$$

Both of these series on the interval $[0, \pi)$ represent $f(x) = x$

Definition 4.2. A function $f(x)$ defined for $x \in [a, b]$ is **piece-wise continuous** if there exists a finite partition $P : a = t_0 < .... < t_n = b$ such that f is continuous on $x \in (t_{i=1}, t_i), \forall i$ and both $\lim_{x \to t_{i-1}^+} f(x)$ and $\lim_{x \to t_{i-1}^-} f(x)$ both exist and are both finite.

Note. *On the ith subinterval, $f(x)$ coincides with some $f_i(x)$ that is continuous on that subinterval.*

Definition 4.3. If $f_i(x)\forall i$ has continuous 1st derivatives, $f(x)$ is called **piecewise smooth**

Definition 4.4. If $f_i(x)\forall i$ has continuous 2nd derivatives, $f(x)$ is called **piecewise very smooth**

Definition 4.5. The Fourier Series obtained from $f(x)$ converges to $f(x)$ if $f(x) = \lim_{N \to \infty} F_N(x)$

i.e.

$$f(x) = \lim_{N\to\infty} F_N(x) = \frac{a_0}{2} + \sum_{k-1}^{N} a_k cos(kx) + b_k sin(kx)$$

assuming a period of 2π (and can be adjusted for other periods). The a_k and b_k are the Fourier coefficients.

Theorem 4.6. *Let $f(x)$ be continuous and piece-wise very smooth for all x and let $f(x)$ have period 2π. Then the Fourier Series of $f(x)$ converges uniformly to $f(x), \forall x$*

This helps with examples that have jump discontinuities.

Theorem 4.7. *Let $f(x)$ be defined and piece-wise very smooth for $x \in [-\pi, \pi]$ and let $f(x)$ be defined outside this interval to have period 2π. Then the Fourier Series of $f(x)$ converges uniformly to $f(x)$ in each interval containing no discontinuity of $f(x)$. At each discontinuity, x_0, the series converges to*

$$\frac{1}{2}[\lim_{x\to x_0^+} f(x) + \lim_{x\to x_0^-} f(x)]$$

*This is the **Fundamental Theorem (for Fourier Series)**. The Theorem can be reinstated for f defined on any interval of length $p \neq 0$*

Example 4.8. 1. $f(x) = \frac{-x}{2}$ on the interval $[-\pi, \pi]$

2. $f(x) = \begin{cases} 1, & 0 \leq x < \pi \\ -1, & -\pi \leq x < 0 \end{cases}$

Note that they both are piece-wise continuous and both satisfy the previous theorem.

so (1.) converges to

$$\sum_{k=1}^{\infty} \frac{(-1)}{k} sin(kx) = \begin{cases} -\frac{x}{2}, & x \in (-\pi, \pi) \\ 0, & x = \pm\pi \end{cases}$$

And (2.) converges to

$$\sum_{k=1}^{\infty} \frac{sin((2k+1)x)}{2k+1} = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 < x < \pi \\ 0, & x = -\pi, 0, \pi \end{cases}$$

In (2.) set $x = \frac{\pi}{2}$ and we'll get

$$1 = \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{sin((2k+1)\frac{\pi}{2})}{2k+1}$$
$$1 = \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{2k+1}$$
$$\frac{\pi}{4} = \sum_{k=1}^{\infty}\frac{(-1)^k}{2k+1}$$
$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ...$$

So now we have $\frac{\pi}{4}$ represented by a series of numbers.

However, the domain of the function also play a role in the series.

Example 4.9. For $f(x), x \in [0,1)$, we get

$$F(x) = \frac{1}{2} - \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{sin((2k+1)x)}{2k+1}$$

The theorem still applies and we get $F(x) = x$ on the interval $[0,1)$, and converge to $\frac{1}{2}$ at 0.

Example 4.10. For $f(x) = |x|, x \in [-\pi, \pi]$ we get.

$$F(x) = \frac{\pi}{2} - \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{cos((2k+1)x)}{(2k+1)^2}$$

on $(0,1), |x| = x$, and since f is piece-wise very smooth, we can write:

$$x = \frac{\pi}{2} - \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{cos((2k+1)x)}{(2k+1)^2}$$

5 Friday, January 19, 2018

Definition 5.1. The **orthonormal bases for** $\mathbb{R}^n$

$$\{e_1, ..., e_n\}$$

are all orthogonal to each other and are all unit vectors.

Each $v \in \mathbb{R}^n$ has a unique representation of $v = \lambda_1 e_1 + ... + \lambda_n e_n$ where $\lambda_k = v \cdot e_k$

Given continuous functions f, g that map from $[-\pi, \pi] \mapsto \mathbb{R}$ we can define an inner product by

$$< f, g >= frac1\pi \int_{-\pi}^{\pi} f(x)g(x)dx$$

defined on all continues real-valued functions defined on the interval $[-\pi, \pi]$

Example 5.2. The set of functions $\{cos(kx), sin(nx)\}_{n,k\in\mathbb{Z}}$ act like an orthonormal basis. since

$$< cos(mx), cos(nx) >= \begin{cases} 1, & \text{if } m = n \\ 0, & \text{otherwise} \end{cases}$$

$$< sin(mx), sin(nx) >= \begin{cases} 1, & \text{if } m = n \\ 0, & \text{otherwise} \end{cases}$$

$$< sin(mx), cos(nx) >= 0$$

for positive integers m, n

We can regard the Fourier Coefficients as the components of f under this basis. This explains why you dont need to recalculate the coefficients for each N for $F_N(x)$

Corollary 5.3. If a function f can be represented as a trigonometric polynomial, then that trigonometric polynomial is a Fourier Expansion of f, if the Fourier Series converges.

Example 5.4. What is the fourier series of $f(x) = cos^2(x) - sin^2(x)$

We have trigonometric identities to solve this:

$$F(x) = cos^2(x) - sin^2(x) = cos(2x)$$

Example 5.5.

$$
\begin{aligned}
f(x) &= cos^4(x) \\
&= (cos^2(x))^2 \\
&= (cos^2(x))^2 \\
&= \frac{1}{4}(1 + 2cos(2x) + cos^2(2x)) \\
&= \frac{1}{4}(1 + 2cos(2x) + \frac{1}{2} + \frac{1}{2}cos(4x)) \quad = \frac{3}{8} + \frac{cos(2x)}{2} + \frac{1}{8}cos(4x) = F(x)
\end{aligned}
$$

Definition 5.6. The **Total Square Error** of $g(x)$ relative to $f(x)$ is:

$$E = \int_{-\pi}^{\pi} [f(x) - g(x)]^2 dx$$

Note. *if* $f = g, E = 0$

We want a constant function $y = g_0$ so the square error is as small as possible

$$
\begin{aligned}
E(g_0) &= \int_{-\pi}^{\pi} [f(x) - g_0]^2 dx \\
&= \int_{-\pi}^{\pi} f(x)^2 dx - 2g_0 \int_{-\pi}^{\pi} f(x)dx + g_0^2(2\pi)
\end{aligned}
$$

And if we let A, B be constants such that

$$A = \int_{-\pi}^{\pi} f(x)^2 dx, B = \int_{-\pi}^{\pi} f(x)dx$$

Then

$$E(g_0) = A - 2Bg_0 + 2\pi g_0^2$$

With implies that $E(g_0)$ is a quad function in g having a minimum value when its derivative is equal to 0.

$$-2B + 4\pi g_0 = 0$$

$$g_0 = \frac{B}{2\pi} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)dx = \frac{1}{2}a_0$$

Therefore $\frac{1}{2}a_0$ is the best constant approximation in the sense of the square error. We can show that the best approximation by a trigonometric polynomial is the Fourier Polynomial.

Definition 5.7. If f is piece-wise continuous on $[-\pi, \pi]$ we see that

$$\frac{a_0}{2} + \sum_{k=1}^{N} a_k^2 + b_k^2 \leq \frac{1}{\pi} \int_{-\pi}^{\pi} [f(x)]^2 dx =< f, f >$$

This is called **Bessel's Inequality** and it shows that $\sum_{k=1}^{\infty} a_k^2 + b_k^2$ converges

Theorem 5.8. *(Uniqueness Theorem) Let $f(x), g(x)$ be piecewise continuous functions on the interval $[-\pi, \pi]$ and have all the same same Fourier coefficients. Then $f(x) = g(x)$ except, perhaps, at discontinuities.*

5.1 Vector-Valued Functions

We will now spend time with vector-valued functions ($f : \mathbb{R}^n \mapsto A$)

Definition 5.9. A **curve (or path)** in $\mathbb{R}^n$ is a function

$$\gamma : [a, b] \subset \mathbb{R} \mapsto \mathbb{R}^n$$

We usually call the image of γ the curve and the function γ as the parameterization of the curve

Note. *This curve has a direction given by γ and runs from $\gamma(a)$ to $\gamma(b)$, which are the endpoints. $\gamma(a)$ being the beginning and $\gamma(b)$ being the end*

Think of $t, t \in [a, b]$ as a variable and $\gamma(t)$ as "tracing out" the curve in $\mathbb{R}$ as t goes from a to b.
This interval be adapted to any (a, b) and even to $\mathbb{R}$ itself.

$$\gamma(t) = (\gamma_1(t), \gamma_2(t), ..., \gamma_n(t)), t \in [a, b]$$

Example 5.10. where $t \in \mathbb{R}$

$$\begin{aligned} \gamma(t) &= (x_0, y_0, z_0) + t(v_1, v_2, v_3) \\ &= x + tv \end{aligned}$$

This is a parametric representation of a line. Where x is a point and v is a direction vector

Definition 5.11. γ is **continuous** at $c \in (a, b)$ if $\lim_{t \to c} \gamma(t) = \gamma(c)$ iff the components $\gamma_i(t)$ for $i = 1, ..., n$ are continuous at c

Definition 5.12. If γ is continuous at all points, γ is called a **Continuous Path**

Example 5.13. Consider a circle of radius 3 in $\mathbb{R}^2$ centred around the origin. A path of this circle is

$$\gamma(t) = (3cost, 3sint), t \in [0, 2\pi]$$

We say this curve is oriented in the counter clockwise direction

Example 5.14. The curve for $y = x^3 + 1$ is

$$\gamma(t) = (t, t^3 + 1), t \in \mathbb{R}$$

Example 5.15. In $\mathbb{R}^3, \gamma(t) = (3cost, 3sint, t), t \in [0, 2\pi]$.

Since $x^2 + y^2 = 9$, this curve must live in the cylinder $x^2 + y^2 = 9$ and since $z = t$, the curve spirals upwards, this is called a helix

Example 5.16. $\gamma(t) = (t, t^2, t^3), t \in [-2, 2]$. This is a twisted cubic.

Example 5.17. Find the parameterization of the curve C of the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $y + z = 2$. The projection into the $x - y$ plane is the circle $x^2 + y^2 = 1, z = 0$, so

$$x = cost, y = sint$$

From the equation of the plane

$$z = 2 - y = 2 - sint$$

We now parameterize the curve C by

$$\gamma(t) = (cost, sint, 2 - sint), t \in [0, 2\pi]$$

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6.1 parameterization (continued)

Note. *For any given curve, there may be several possible parameterizations of that curve*

Example 6.1.

The 1st quadrant piece of the unit circle in a counter clock-wise direction, from $(1, 0)$ to $(0, 1)$

$$\gamma_1(t) = (cost, sint), t \in [0, \frac{\pi}{2}]$$

$$\gamma_2(t) = (cos(\frac{t}{2}), sin(\frac{t}{2})), t \in [0, \pi]$$

$$\gamma_3(t) = (\sqrt{1 - t^2}, t), t \in [0, 1]$$

All parameterize the same curve.

6.2 Derivatives

Definition 6.2. The **Derivative of a path** is

$$D(\gamma(t)) = (\gamma_1'(t), \gamma_2'(t), ..., \gamma_n'(t))$$

provided each $\gamma_i'(t)$ exists for $t = 1, ..., n$

We usually write $\gamma'(t) = (\gamma_1'(t), ..., \gamma_n'(t))$

Definition 6.3. If $\gamma'(t)$ exists, γ is called a **differentiable path**

Definition 6.4. If γ is a differentiable path with a continuous derivative, except at finitely many places, the image of γ is called a **piece-wise smooth curve**

Definition 6.5. $\gamma'(t)$ is the tanget v to the curve at the point $\gamma(t)$. Hence, the tanget line at $\gamma(t_0)$ is given by

$$\gamma(t_0) + \lambda\gamma'(t_0), \lambda \in \mathbb{R}$$

Note that $\gamma(t)$ must be smooth (class C^1) for t near t_0, be careful if $\gamma(t_0) = 0$

Remark. *If we think of $\gamma(t)$ representing the position of a particle at time t, we can regard $\gamma'(t)$ as its velocity and $\gamma''(t)$ as its acceleration.*

If $\gamma'(t)$ is the velocity, then $||\gamma'(t)||$ is the speed (magnitude of the velocity)

Example 6.6. Computing the period of a satellite when the radius of its orbit is known. The particle has constant speed s, and when $t = 0$, assume that $\gamma(t)$ is at the point $(r, 0)$.

So....

$$\gamma(t) = (rcos(kt), rsin(kt))$$

We need to find k.

$$\gamma'(t) = (-rksin(kt), rkcos(kt))$$

$$s = ||\gamma'(t)|| = \sqrt{r^2k^2(sin^2(kt) + cos^2(kt))} = rk$$

And this implies that $k = \frac{s}{r}$.

Note that $r, k \in \mathbb{R}^+ > 0$

$$\gamma''(t) = (-rk^2cos(kt), -rk^2sin(kt)) = -k^2\gamma(t)$$

Assuming particle is a satellite of mass m orbiting the earth of mass M.

Note.

$$F = ma = m(\gamma''(t)) = -mk^2\gamma(t)$$

$$F(\vec{v}) = \frac{-GmM\vec{v}}{||\vec{v}||^3}$$

where G is the gravitational constant and $\vec{v} = \gamma(t)$

$$-mk^2\gamma(t) = \frac{-GmM\gamma(t)}{r^3}$$

And this implies that $k^2 = \frac{GM}{r^3}$.

What is the period? $p = \frac{2\pi}{k}$ this implies that $k = \frac{2\pi}{p}$.

$$k^2 = (\frac{2\pi}{p})^2 = \frac{GM}{r^3}$$

This implies that

$$p^2 = \frac{4\pi^2}{GM}r^3$$

The period of a satellite squared is proportional to the radius of its orbit cubed (One of Kepler's Laws)

6.3 Path Integrals

Definition 6.7. The **path integral of** f or the **integral of** f **along the path** γ denoted

$$\int_\gamma f ds$$

is defined by

$$\int_\gamma f ds = \int_a^b f(\gamma(t))||\gamma'(t)||dt$$

Whenever $\gamma : [a,b] \mapsto \mathbb{R}^n$ is a smooth curve and the composite function $t \mapsto f(\gamma(t))$ is continuous on $[a,b]$

If $\gamma(t)$ is only piece-wise smooth or $f(\gamma(t))$ is only piece-wise continuous, we break $\gamma(t)$ into finitely many smooth pieces, and sum the integrals over the various pieces.

7 Friday, January 26, 2018

7.1 Path Integral (continued)

Why is this definition of a path integral a good definition?

We can think of speed $s = ||\gamma'(t)||$ as a rate of change in position relative to change in time t. Often we denote $ds = ||\gamma'(t)||dt$

If $f = 1$ then

$$\int_\gamma ds = \int_a^b ||\gamma'(t)||ds$$

Which is just the length of the curve.

Definition 7.1. The length of the curve is

$$\int_\gamma ds = \int_a^b ||\gamma'(t)||ds$$

The length of the path is the "total distance travelled."

Example 7.2. Find the length of a circle of radius 1 from 0 to 2π.

$$\gamma(t) = (cos(t), sin(t)), t \in [0, 2\pi]$$
$$\gamma'(t) = (-sin(t), cos(t))$$
$$||\gamma'(t)|| = \sqrt{sin^2(t) + cos^2(t)} = 1$$

$$s = \int_\gamma ds = \int_0^{2\pi} 1ds = 2\pi$$

Example 7.3. Find the arclength of $\gamma(t) = (sin(5t), cos(5t), \frac{10}{3}t^{1.5})$ from $t = 0$ to $t = 3$.

$$\gamma'(t) = (5cost, -5sint, 5\sqrt{t})$$

$$||\gamma'(t)|| = \sqrt{25cos^2(t) + 25sin^2(t), 25t} = 5\sqrt{1+t}$$

$$s = \int_\gamma ds = 5\int_0^3 \sqrt{1+t}dt = \frac{70}{3}$$

Example 7.4. $f(x, y, z) = xy - z^2$ along $\gamma(t) = (3t, -2t, \sqrt{3}t), t \in [0, 1]$

$$\gamma'(t) = (3, -2, \sqrt{3})$$
$$||\gamma'(t)|| = 4$$

$$\begin{aligned} f(\gamma(t)) &= f(3t, -2t, \sqrt{3}t) \\ &= (3t)(-2t) - (\sqrt{3}t)^2 \\ &= -6t^2 - 3t^2 = -9t^2 \end{aligned}$$

$$\int_\gamma f ds = \int_0^1 f(\gamma(t))||\gamma'(t)||dt = \int_0^1 (-9t^2)4dt = -36\int_0^1 t^2 dt = -12$$

Remark. *mass of a wire* (m) = *density* $D \times$ *length*

$$m = \int_\gamma dm = \int_\gamma D ds$$

Note. $\int_\gamma f ds$ *is independent of the choice of parameterization*

Note. *If a curve in* $\mathbb{R}^2$ *is given by* $y = f(x)$ *and* x *is in the domain of* $[a, b]$*, then the arclength is*

$$\int_a^b \sqrt{1 + f'(x)^2} dx$$

7.2 Vector Fields

Definition 7.5. A function $F : A \subset \mathbb{R}^n \mapsto \mathbb{R}^n$ is called a **vector field** where

$$F(x) = (F_1(x), ..., F_n(x))$$

Definition 7.6. a function $f : \mathbb{R}^n \mapsto \mathbb{R}$ is called a **scalar field**, the components of a vector field are scalar fields.

We think of a vector field F as attaching to each point x a vector $f(x)$

Example 7.7. Newtons Gravitational Law

$$F(x) = -\frac{mMG}{||x||^3} x$$

"pull of a satellite towards the earth"

The strength of the force depends on d, the distance away. Force is in the opposite direction to the position vector.

$$F(x) = F(x, y, z) = (-\frac{mMG}{||(x, y, z)||^3} x, -\frac{mMG}{||(x, y, z)||^3} y, -\frac{mMG}{||(x, y, z)||^3} z)$$

Example 7.8. Put a toy boat into a stream, a velocity field is the velocity at each point of the stream.

Let the path be $\gamma(t)$, then $\gamma'(t) = F(\gamma(t)), \forall t$

Definition 7.9. If F is a vector field, a **flowline** for F is a path $\gamma(t)$ such that $\gamma'(t) = F(\gamma(t))$

Definition 7.10. The **flow of a vector field** is the family of all flow lines

Example 7.11. $F(x, y) = (3, 4)$, vector $(3, 4)$ at every point will be straight lines. Want to know flow line at $(1, 2)$

$$\begin{aligned} \gamma(t) &= (x(t), y(t)) \\ \gamma'(t) &= F(\gamma(t)) = (3, 4) \\ \gamma(t) &= (3t + 1, 4t + 2) \end{aligned}$$

as it passes through $(1, 2)$.

8 Monday, January 29, 2018

8.1 Line Integrals

Example 8.1. For time t, force $F = F(\gamma(t)) \cdot \frac{\gamma'(t)}{||\gamma'(t)||} = F(\gamma(t)) \cdot T(\gamma(t))$

$$\Delta w = F(\gamma(t)) \cdot T(\gamma(t))\Delta s$$

$$w = \int_\gamma F \cdot T ds$$

Definition 8.2. Let F be a vector field in $\mathbb{R}^n$ and let $\gamma : [a, b] \mapsto \mathbb{R}^n$ be a smooth curve in $\mathbb{R}^n$. The **line integral** of F over γ, denoted $\int_\gamma F ds$ is defined by

$$\int_\gamma F \cdot ds = \int_\gamma (F \cdot T) ds$$

Remark. *$(F \cdot T)$ is a scalar field, and as produced before:*

$$\int_\gamma (F \cdot T) ds = \int_a^b F(\gamma(t)) \cdot \gamma'(t) dt$$

Remark. *Within signs, the line integral is independent of the parameterization. ie.*

$$\int_\gamma F \cdot ds = -\int_{-\gamma} F \cdot ds$$

Remark. *We can work over piecewise smooth curves by dividing it into smooth sub intervals and summing them up.*

Example 8.3. $F(x,y) = (y, x^2), \gamma(t) = (t^2, t^3), t \in [0,1]$

$$F(\gamma(t)) = (t^3, t^4)$$

$$\gamma'(t) = (2t, 3t^2)$$

$$\begin{aligned}\int_\gamma F \cdot ds &= \int_0^1 (2t^4 + 3t^6)dt \\ &= [\frac{2}{5}t^5 + \frac{3}{7}t^7]_0^1 \\ &= \frac{2}{5} + \frac{3}{7} = \frac{29}{35}\end{aligned}$$

Example 8.4.

$$F(x,y) = (2y - x, -y)$$

$$\gamma(t) = (-cost, sint), t \in [0, \pi]$$

$$\gamma_0(t) = (cost, sint), t \in [0, \pi]$$

$$\begin{aligned}\int_\gamma F ds &= \int_0^\pi F(\gamma(t)) \cdot \gamma'(t)dt \\ &= \int_0^\pi (2sin^2t + costsint - sintcost)dt \\ &= 2\int_0^\pi (\frac{1}{2} - \frac{1}{2}cos(2t))dt = \pi\end{aligned}$$

$$\begin{aligned}\int_{\gamma_0} F ds &= \int_0^\pi F(\gamma_0(t)) \cdot \gamma_0'(t)dt \\ &= \int_0^\pi (-2sin^2t + costsint - sintcost)dt \\ &= -2\int_0^\pi (\frac{1}{2} - \frac{1}{2}cos(2t))dt = -\pi\end{aligned}$$

Note. *Notice from the previous example that*

$$\int_\gamma F ds = \int_0^\pi (2sint + cost)(sint) - (sint)(cost)$$

we notice this is in the form

$$A = \int_a^b F_1(\gamma(t))\gamma_1'(t) + ... + F_n(\gamma(t))\gamma_n'(t)dt$$

We introduce a new notation that can take advantage of this form and rewrite A as

$$\int_a^b F_1 dx_1 + ... + F_n dx_n$$

where $dx_i = \gamma'(t)dt, i = 1, ..., n$

Definition 8.5. A (first-order) **differential form** (1-form) in $\mathbb{R}^n$ is an expression of the form

$$f_1 dx_1 + ... + f_n dx_n$$

Definition 8.6. Let $\omega = f_1 dx_1 + ... + f_n dx_n$ be a differential form in $\mathbb{R}^n$ and let $\gamma : [a, b] \mapsto \mathbb{R}^n$ be a smooth curve in $\mathbb{R}^n$. The integral of ω over γ, denoted

$$\int_\gamma \omega$$

is defined by

$$\int_\gamma \omega = \int_\gamma F \cdot ds$$

where $F = (f_1, ..., f_n)$

So

$$\begin{aligned}\int_\gamma \omega &= \int_a^b [f_1(\gamma(t))\gamma_1'(t) + ... + f_n(\gamma(t))\gamma_n'(t)]dt \\ &= \int_a^b F(\gamma(t))\gamma'(t)dt = \int_\gamma F \cdot ds\end{aligned}$$

where $F = (f_1, ..., f_n)$

We can think of $F = (f_1, ..., f_n)$ and $ds = (dx_1, ..., dx_n)$ where $dx_i = \frac{d(x_i(t))}{dt}dt, i = 1, ..., n$ which means the dot product notation makes sense.

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Example 8.7.

$$\int_\gamma zx^2dx + yzdy + arcsin(x)dz$$

where $\gamma(t) = (t, t^2, 1), t \in [0, 1]$

$$\begin{aligned}\int_\gamma \omega &= \int_0^1 [(t^2)(1) + (t^2)(2t) + (0)(arcsint)]dt \\ &= \int_0^1 (t^2 + 2t^3)dt \\ &= [\frac{1}{3}t^3 + \frac{1}{2}t^4]_0^1 \\ &= \frac{5}{6}\end{aligned}$$

Given a vector field $F = (f_1, ..., f_n)$ we have the (1-form) $\omega = f_1dx_1 + ... + f_ndx_n$ and vice versa

Definition 8.8. Let $f : \mathbb{R}^n \mapsto \mathbb{R}$. The differential form corresponding to the vector field ∇f, denoted df, is called the **total differential of** f