MATB43: Introduction to Analysis
Lecture Notes

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Pre-reqs are MATA37. Instructor is John Scherk. If you find any problems in these notes, feel free to contact me at conconjoshua@gmail.com.

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1 Friday, January 5, 2018

1.1 Countability

Countability is all about the cardinality of sets

Definition 1.1. For sets X, Y, they have the same cardinality if there exists a bijection between them ($f : X \mapsto Y$ has an inverse as well). Equality of cardinality of 2 sets is written as $card(X) = card(Y)$

Example 1.2. $card(\{1, 2, 3, 4, 5\}) = card(\{a, b, c, d, e\})$

Example 1.3. $card(\mathbb{N}) = card(2\mathbb{N})$

This example is correct because $f : \mathbb{N} \mapsto 2\mathbb{N}, f(n) = 2n$ is a bijection.

Example 1.4. $card(\{\text{set of odd numbers}\}) = card(\mathbb{N})$

This example is correct because $g : \mathbb{N} \mapsto \{\text{set of odd numbers}\}$
$g(n) = 2n + 1, n \in \mathbb{N}$

Definition 1.5. X is **finite** if for some $n \in \mathbb{N}$ there exists a bijection where $A = \{1, 2, ..., n\}$ $f : A \mapsto X$. So essentially $card(A) = card(X)$

Intuitively, we can label the elements of X as $X = \{x_1, x_2, ..., x_n\}$

Definition 1.6. X is **inifinite** if X is not finite.

Examples of this include: $\mathbb{N}, 2\mathbb{N}, \mathbb{Q}, \mathbb{R}$
However, not all infinite sets have the same cardinality.

Definition 1.7. X is **countable** if $card(X) = card(A)$ where $A \subseteq \mathbb{N}$

Again intuitively, we can label the elements of X as $X = \{x_1, x_2, ..., x_n, ...\}$ (using elements of $\mathbb{N}$)

Example 1.8. $2\mathbb{N}$

Example 1.9. The set of odd numbers

Example 1.10. $\mathbb{Z}$

Theorem 1.11. *In general, if $Z = X \cup Y$ and Y, X are both countable then Z is countable as well*

Proof. label elements of $X = \{x_1, x_2, ..., x_n, ...\}$
label elements of $Y = \{y_1, y_2, ..., y_n, ...\}$
define $h : \mathbb{N} \mapsto \mathbb{Z}$ as, for $n \in \mathbb{N}$:

$$h(2n-1) = x_n$$

$$h(2n) = y_n$$

Then h is a bijection, therefore Z is also countable ■

Example 1.12. $\mathbb{N} \times \mathbb{N}$

This example is countable as we can label its elements in the following pattern.

$$(1,1) \mapsto 1$$

$$(2,1) \mapsto 2$$

$$(1,2) \mapsto 3$$

$$(1,3) \mapsto 4$$

$$(2,2) \mapsto 5$$

$$(3,1) \mapsto 6$$

$$(4,1) \mapsto 7$$

And so on, in this pattern. Intuitively, if you list out all the pairs of $\mathbb{N} \times \mathbb{N}$ like a matrix, this would create a sort of zig zag pattern.

Proposition 1.13. Suppose X is countable and $Y \subset X$, then Y is either finite (which also means that it is countable) or just countable

Proof. write $X = \{x_1, x_2, ..., x_n, ...\}$
let j_1 be the smallest index such that $x_{j_1} \in Y$
let j_2 be the smallest index such that $x_{j_2} \in Y, j_2 > j_1$
let j_3 be the smallest index such that $x_{j_3} \in Y, j_3 > j_2$
...
and so on.

Now either:
This process terminates, which means that $Y = \{x_{j_1}, ..., x_{j_n}\}$ for some

$j_1, ..., j_n$ which Y is finite.

or:
Y is countable, since $Y = \{x_{j_1}, x_{j_2}, ..., x_{j_n}, ...\}$ ■

Proposition 1.14. $\mathbb{Q}$ is countable

Proof. Note that $\mathbb{Q} = \mathbb{Q}^- \cup \{0\} \cup \mathbb{Q}^+$
so we must check that $\mathbb{Q}^+$ is countable, note that

$$\mathbb{Q}^+ = \{\frac{m}{n} | m, n \in \mathbb{N}, n \neq 0\}$$

Since it is a pairing of two natural numbers (m, n), this is a subset of $\mathbb{N} \times \mathbb{N}$, which means that $\mathbb{Q}^+$ is countable, we can say that same thing for $\mathbb{Q}^-$ as well, note that it is

$$\mathbb{Q}^+ = \{\frac{-m}{n} | m, n \in \mathbb{N}, n \neq 0\}$$

and since $\mathbb{Q}^+$ and $\mathbb{Q}^-$ are both countable and $\{0\}$ is a set of cardinality 1, therefore $\mathbb{Q}$ is countable ■

Theorem 1.15. *Let $S = \{s = (s_1, s_2, ...) | s_j = 0 \text{ or } 1 \forall j\}$ (note here that s is a sequence). Then S is not countable*

Proof. For proof by contradiction, suppose S is countable.
We can then label the elements of S as $S^1, S^2, ..., S^n, ... \in S$.
So an example of this would be:

$$S^1 = (s_1^1, s_2^1, s_3^1, ..., s_n^1, ...)$$
$$S^2 = (s_1^2, s_2^2, s_3^2, ..., s_n^2, ...)$$
$$...$$
$$S^m = (s_1^m, s_2^m, s_3^m, ..., s_n^m, ...)$$
$$...$$

And we will define a $t \in S$ as follows, let $t = (t_1, t_2, ..., t_m, ...)$

$$t_1 = \begin{cases} 0, & \text{if } S_1^1 = 1 \\ 1, & \text{if } S_1^1 = 0 \end{cases}$$

$$t_2 = \begin{cases} 0, & \text{if } S_2^2 = 1 \\ 1, & \text{if } S_2^2 = 0 \end{cases}$$
$$\dots$$
$$t_m = \begin{cases} 0, & \text{if } S_m^m = 1 \\ 1, & \text{if } S_m^m = 0 \end{cases}$$
$$\dots$$

Therefore $t \neq S^m, \forall m$. Therefore this is a contradiction as t was not in the listed elements of S, therefore S is not countable. ■

Corollary 1.16. $\mathbb{R}$ is not countable

Proof. Regard $\mathbb{R}$ as a set of infinite decimal fractions.
Identify $s \in S$ with the decimal number $0.s_1 s_2 ... s_n ...$ so S can be regarded as a subset of $\mathbb{R}$. If $\mathbb{R}$ were countable, then S would be countable, therefore $\mathbb{R}$ must not be countable. ■

1.2 Real Numbers

Examples of real numbers that are not rational include: π, e (which are algebraic numbers), $\sqrt{2}, \sqrt{3}, \frac{\sqrt{5}+1}{2}$ (which are roots of polynomial equations with integer coefficients.)

1.3 Algebraic Numbers

The set of Algebraic Numbers is a set $\overline{\mathbb{Q}}$ such that

$$\overline{\mathbb{Q}} = \{x \in \mathbb{R} | x^n + a_{n-1}x^{n-1} + ... + a_0 = 0 \text{ for some } a_0, ..., a_{n-1} \in \mathbb{Z}\}$$

Note that $\mathbb{Q} \subset \overline{\mathbb{Q}}$, and in fact, $\overline{\mathbb{Q}}$ is countable.

if $x \in (\mathbb{R} \setminus \overline{\mathbb{Q}})$ then. x is transcendental.

1.4 Bounds

Definition 1.17. $X \subset \mathbb{R}, X$ is **bounded above** if $\exists a \in \mathbb{R}$ such that $a \geq x, \forall x \in X$

Definition 1.18. $X \subset \mathbb{R}, X$ is **bounded below** if $\exists a \in \mathbb{R}$ such that $a \leq x, \forall x \in X$

Example 1.19. $X = \{x \in \mathbb{R} | x^2 \leq 2\}$ is bounded above by $\frac{3}{2}$ since $(\frac{3}{2})^2 \geq 2$

Definition 1.20. a is the **least upper bound** of X if a is an upper bound of X and if $b < a$, then there exists $x \in X$ such that $x > b$.

We write $a = lub(X)$ or $a = sup(X)$ if a is the least upper bound of X

Definition 1.21. a is the **greatest lower bound** of X if a is an lower bound of X and if $b > a$, then there exists $x \in X$ such that $x < b$.

Example 1.22. if $X = \{x | x^2 \leq 2\}$, then $sup(X) = \sqrt{2} \notin \mathbb{Q}$

Property 1.23. if $X \subset \mathbb{R}$ is bounded above then there exists $a \in \mathbb{R}$, a least upper bound of X

Theorem 1.24. *given $a, b \in \mathbb{R}$ $a, b > 0$ $\exists n \in \mathbb{N}$ such that $na > b$ (This is known as the **archemedian property**)*

Proof. Suppose that $\forall n \in \mathbb{N}$, $na \leq b$ implies that b is an upper bound for $X = \{na | n \in \mathbb{N}\}$.

Since X is bounded above, let $c = sup(X)$, this implies that $c - a$ is not an upper bound for X, which further implies that there exists an $a \in \mathbb{N}$ such that $na > c - a$, and that

$$\begin{aligned} na &> c - a \\ na + a &> c \\ (n+1)a &> c \\ c < (n+1)a &\in X \end{aligned}$$

This is impossible since we've previously stated that c is the upper bound. Therefore X is not bounded above. ■

Theorem 1.25. *given $c, d \in \mathbb{R}$, there exists $q = \frac{m}{n} \in \mathbb{Q}$ such that $c < q < d$*

Proof. We want $c < \frac{m}{n} < d$ iff $nc < m < nd$.
let $\epsilon = d - c > 0$
pick $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$
pick m, such that $m > nc$
(since we also need $m < nd$, choose m to be as small as possible such that $m - 1 \leq nc < m$)

$$m - 1 \leq nc \leq m$$
$$\frac{m}{n} - \frac{1}{n} = \frac{m-1}{n} \leq c \leq \frac{m}{n}$$

or

$$\frac{m}{n} \leq c + \frac{1}{n} < c + \epsilon = d$$

So now we have $c < \frac{m}{n} < d$ as desired.

■

2 Monday, January 8, 2018

2.1 Sequences - Review

This section will be just review of MATA37 and will be concerning sequences of the real numbers $\{a_n\}$.

Example 2.1. $\{\frac{1}{n}\} = 1, \frac{1}{2}, \frac{1}{3}, ...,$

Example 2.2. $\{\frac{(-1)^n}{n}\} = -1, \frac{1}{2}, \frac{-1}{3}, ...,$ This sequence oscillates back and forth.

Example 2.3. $\{(-1)^n\} = -1, 1, -1, 1, ...,$ This sequence oscillates back and forth aswell.

Example 2.4. $\{x_1, x_2, ...\}$ where this sequence enumerates $\mathbb{Q}$. This sequence 'bounces around wildy'.

Definition 2.5. $x \in \mathbb{R}$ is the **limit of a sequence** $\{x_n\}$ (so that in converges to this number)
$\lim_{n\to\infty} x_n = a$, if given some tolerance $\epsilon > 0$.
There exists N, such that $n \geq N$ will lie in the interval $a - \epsilon, a + \epsilon$, aka $|x_n - a| < \epsilon$.

Example 2.6. $\lim_{n\to\infty} \frac{1}{n} = 0$

Example 2.7. $\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$

Example 2.8. example 2.4 and 2.3 both have no limit

Proposition 2.9. If a sequence $\{x_n\}$ has a limit, then this limit is unique, and the sequence is bounded.

Proposition 2.10. Supposed that $\{x_n\}, \{y_n\} \subset \mathbb{R}$ are convergent sequences, so that $\lim_{n\to\infty} x_n = a$, $\lim_{n\to\infty} y_n = b$ then:

1. $\{x_n + y_n\}$ converges and $\lim_{n\to\infty}(x_n + y_n) = a + b$
2. $\{x_n y_n\}$ converges and $\lim_{n\to\infty}(x_n y_n) = ab$
3. if $y_n \neq 0, \forall n$, and $b \neq 0$ then $\{\frac{x_n}{y_n}\}$ converges and $\lim_{n\to\infty}(\frac{x_n}{y_n}) = \frac{a}{b}$

Definition 2.11. A sequence is **monotone** if it is either increasing, or decreasing.

Proposition 2.12. A bounded monotone sequence converges.

Proof. Suppose that $\{x_n\}$ is a bounded increasing (monotone) sequence, such that

$$x_1 \leq x_2 \leq x_3 \leq ...$$

And there exists $A \in \mathbb{R}$ such that $x_n \leq A, \forall n$
Therefore, there exists a least upper bound $a \leq A$.

<u>claim:</u> $x_n \to a$ as $n \to \infty$

take $\epsilon > 0$, since a is the least upper bound of $\{x_n\}$, there exists N such that $a - \epsilon < x_N \leq a$. But $\{x_n\}$ is increasing. Therefore $\forall n \geq N, a - \epsilon < x_N \leq x_n \leq a$. This implies that $\lim_{n\to\infty} x_n = a$ ■

3 Friday, January 12, 2018

3.1 Monotone Sequences

i.e. Sequences which are increasing or decreasing

Proposition 3.1. A bounded monotone sequence converges.

Proof. For an increasing sequence $\{x_n\}, \lim_{n\to\infty} x_n = sup(\{x_n\})$, suppose $\{x_n\}$ is decreasing and bounded below, then $\{-x_n\}$ is increasing and bounded above. This implies that $\lim_{n\to\infty} -x_n$ and $\lim_{n\to\infty} x_n$ both exist, and

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} -x_n$$

∎

So now we're going to prove that Every sequence has a monotone subsequence along with other propositions on bounded monotone sequences. This will allow us to prove the Bolzano-Weierstrass Theorem, which states that every bounded sequence has a convergent subsequence, and this will help us prove the Cauchy property for sequences.

Example 3.2. Lets look at the following sequence:

$$x_n = 1 + \frac{1}{2} + ... + \frac{1}{n} - logn, n \geq 1$$

So we want to show that this converges, we'll show that if we show that $\{x_n\}$ is decreasing and is bounded below by 0.

The limit ($\lim_{n\to\infty} x_n$) is actually a mysterious number that we don't know too much about.

Recall: that $logn = \int_1^n \frac{dt}{t}$
So if we consider the space under the graph of $\frac{1}{t}$ from n to $n+1$, we get the following inequality

$$\frac{1}{n+1} < log(n+1) - logn < \frac{1}{n}$$

Which implies that

$$\frac{1}{n+1} - log(n+1) + logn < 0$$

Now consider the following for an arbitrary n

$$x_{n+1} = x_n + (\frac{1}{n+1} + logn - log(n+1))$$

but since $\frac{1}{n+1} - log(n+1) + logn < 0$, this would mean that $x_{n+1} < x_n$, so this sequence is decreasing.

Now consider the following inequality derived from the recall block:

$$\sum_{n=1}^{m}(log(n+1) - logn) < \sum_{n=1}^{m}\frac{1}{n}$$

The left side of this inequality telescopes, giving us

$$log(m) < log(m+1) < 1 + \frac{1}{2} + ... + \frac{1}{m}$$

And since the left most side is greater than the right most side, that means that $x_m > 0, \forall m$.

3.2 Subsequences

Definition 3.3. Let $\{x_i\}$ be a sequence of the real numbers, pick a finite set of indices $j_1 < j_2 < ... < j_n < .....$
A **Subsequence** of $\{x_i\}$ is $\{x_{j_1}, x_{j_2}, ..., x_{j_n}, ...\}$

Example 3.4. Considering the sequence $\{x_n\} = \{1, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{n}, ...\}$

1. $\{\frac{1}{2n}\}$ is a subsequence of $\{x_n\}$.
2. $\{\frac{1}{2^n}\}$ is a subsequence of $\{x_n\}$.
3. $\{(-1)^n\}$ is also a subsequence of $\{x_n\}$, note that it does not converge, but has convergent subsequences of $\{(-1)^{2n}\}$ and $\{(-1)^{2n+1}\}$

Definition 3.5. Call a term x_m **dominant** if $x_n \leq x_m, \forall n \geq m$

Proposition 3.6. Every real number sequence has a monotone subsequence

Proof. There are 2 cases:
Case 1: infinitely many dominant terms
let $\{x_{j_1}, x_{j_2}, ..., x_{j_n}, ...\}$ be the sequence of dominant terms. By definition, $x_{j_1} \geq x_{j_2} \geq ... \geq x_{j_n} \geq ...$
so $\{x_{j_n}\}$ is a decreasing sequence, which is monotone.

Case 2: only finitely many dominant terms
Pick an index j, so that x_{j_1} is the first term beyond all dominant terms in the sequence ($\exists i, i < j_1, x_i$ is the last dominant term).
Since x_{j_1} is not dominant, then $\exists j_2 > j_1$ where $x_{j_2} > x_{j_1}$
Since x_{j_2} is not dominant, then $\exists j_3 > j_2$ where $x_{j_3} > x_{j_2}$
...
and so on
By induction, we construct an increasing subsequence $\{x_{j_m}\}$ of $\{x_n\}$.

By these 2 cases, every real number has a monotone sequence. ■

Theorem 3.7. *(Bolzano-Weierstrass Theorem) every bounded sequence has a convergent subsequence*

Proof. Given a bounded sequence $\{x_n\}$ of the real numbers, there exists a monotone subsequence $\{x_{j_n}\}$.
The monotone subsequence $\{x_{j_n}\}$ is also bounded since $\{x_n\}$ is bounded.
This implies that $\lim_{n\to\infty} x_{j_n}$ exists. ■

Definition 3.8. Cauchy Property: intuitively, in a convergent sequence, the terms get closer and closer as $n \to \infty$. More precisely:
$\{x_n\}$ is a real number sequence, given $\epsilon > 0, \exists N$ such that for $m, n > N, |x_m - x_n| < \epsilon$.

Proposition 3.9. Suppose that $\{x_n\}$ converges to a, then $\{x_n\}$ satisfies the Cauchy property.

Proof. given $\epsilon > 0$, then $\exists N$ such that $\forall n > N, |x_n - a| < \frac{\epsilon}{2}$, so if $m > N$ then $|x_m - a| < \frac{\epsilon}{2}$ as well.
This implies that

$$|x_m - x_n| = |x_m - a + a - x_n| \leq |x_m - a| + |x_n - a| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

■

So now we want to show that if a sequence satisfies the Cauchy property, then it converges. We'll do this by first showing that a bounded sequence with the Cauchy property implies that sequence has a convergent subsequence. The second and final step is to show that the limit of a convergent subsequence is the limit of the original sequence.

Proposition 3.10. If $\{x_n\}$ satisfies the Cauchy property, then it's bounded.

Proof. By definition, there exists N so that for $m, n > N, |x_m - x_n| < 1$ in particular, $\forall n > N, |x_n - x_{N+1}| < 1$, which gives us:

$$\begin{aligned} -1 <& x_n - x_{N+1} < 1 \\ x_{N+1} - 1 <& x_n < x_{N+1} + 1 \end{aligned}$$

<u>Note:</u> x_{N+1} is fixed (a constant)

Let $A = min\{x_1, ..., x_N, x_{N+1} - 1\}$
Let $B = max\{x_1, ..., x_N, x_{N+1} - 1\}$
This implies that for all n, $A \leq x_n \leq B$, therefore Bounded.

■

Proposition 3.11. A sequence with the Cauchy property is convergent.

Proof. Let $\{x_n\}$ be a sequence with the Cauchy property.
Since $\{x_n\}$ is bounded, by the Bolzano–Weiestrass Theorem, there exists a convergent subsequence $\{x_{j_m}\}$. Now let:

$$a = \lim_{m \to \infty} x_{j_m}$$

given $\epsilon, \exists M$ so that for all $m > M, |x_{j_m} - a| < \frac{\epsilon}{2}$.

<u>Note that $j_m \geq m$</u>

The Cauchy propert implies that $\exists N$ so that for all $m, n > N, |x_m - x_n| < \epsilon$.

We'll pick $P = max(N, M)$, then for $m, n > P$ we have:

$$\begin{aligned}
|x_n - a| &= |x_n + x_{j_m} - x_{j_m} - a| \\
&= |x_n - x_{j_m}| + |x_{j_m} - a| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

■

Example 3.12.

$$x_n = \sum_{k=1}^{n} \frac{1}{k^2}$$

Verify the Cauchy property.

Take $m > n, x_m - x_n = \sum_{k-n+1}^{m} \frac{1}{k^2}$. Now $\frac{1}{k} < \frac{1}{k(k-1)} = \frac{1}{k-1} - \frac{1}{k}$. This implies that

$$\begin{aligned}
&\implies \sum_{k-n+1}^{m} \frac{1}{k^2} < \sum_{k-n+1}^{m} (\frac{1}{k-1} - \frac{1}{k}) \\
&\overset{\text{telescoping}}{\implies} \sum_{k-n+1}^{m} \frac{1}{k^2} < \frac{1}{m} - \frac{1}{n} < \frac{1}{m} \\
&\implies (x_m - x_n) < \frac{1}{m}
\end{aligned}$$

This implies that x_n is convergent, in fact

$$\lim_{n\to\infty} x_n = \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$$

4 Monday, January 15, 2018

4.1 Cauchy

Remark. *Let $\{x_n\}$ be a sequence of real numbers. Given $\epsilon > 0$, there exists N such that for $m, n > N, |x_m - x_n| < \epsilon$. If $\{x_n\}$ converges then $\{x_n\}$ satisfies the cauchy property.*

The converse also holds. So if $\{x_n\}$ satisfies the cauchy property, then it converges. But why? It holds because:

1. $\{x_n\}$ must be bounded
2. Therefore it has a convergent subsequence $\{x_{j_m}\}$, assume $\{x_{j_m}\}$ converges to a.
3. Then $\{x_n\}$ converges to a as $n \to \infty$

4.2 Series

This section will be focused on

Definition 4.1.

$$\sum_{n=0}^{\infty} a_n$$

which is an **infinite sum** of real numbers

Definition 4.2. The following is a **partial sum**:

$$S_n = \sum_{k=0}^{n} a_k$$

Definition 4.3. The series $\sum_{n=0}^{\infty} a_n$ converges if $\lim_{n\to\infty} S_n$ exists. If $\lim_{n\to\infty} S_n = a$, then we write $a = \sum_{n=0}^{\infty} a_n$ as the **Sum of the series**. However, if $\lim_{n\to\infty} S_n$ does not exist, then the series diverges.

Example 4.4. The geometric series $\sum_{n=0}^{\infty} a^n$ converges if $|a| < 1$ and diverges otherwise. If the series does converge, it converges to:

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-n}$$

Property 4.5. If $\sum_{n=0}^{\infty} a_n$, $\sum_{n=0}^{\infty} b_n$ both converge, then

$$\sum_{n=0}^{\infty} a_n + \sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty}(a_n + b_n)$$

converges as well.

Property 4.6. $\forall c$, if $\sum_{n=0}^{\infty} a_n$ converges, then

$$c\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} c(a_n)$$

Also converges

Definition 4.7. The **Cauchy Criterion** states that $\sum_{n=0}^{\infty} a_n$ converges iff $\forall \epsilon > 0, \exists N$ such that $m > n > N$

$$|S_m - S_n| = |\sum_{k=n+1}^{m} a_k| < \epsilon$$

Example 4.8. Apply Cauchy Criterion to $\sum_{n=0}^{\infty} \frac{1}{n^2}$ and $\sum_{n=0}^{\infty} \frac{1}{n(n+1)}$

To prove they converge, we first need the following proposition:

Proposition 4.9. Suppose $\sum_{n=0}^{\infty} a_n$ converges, then $a_n \to 0$ as $n \to \infty$

Proof. (Using the Cauchy Criterion)
Given $\epsilon > 0$, there exists N such that for $m > n > N, |\sum_{k=n+1}^{m} a_k| < \epsilon$, take $m = n + 1$, this implies that $|a_{n+1}| < \epsilon$ for all $n > N$, this then implies that $a_n \to 0$ ■

Proposition 4.10. Suppose $a_n \geq 0$ then $\sum_{n=0}^{\infty} a_n$ converges iff $\{S_n\}$ is bounded above.

Proof. Since $a_n \geq 0, \forall n$, then $\{S_n\}$ is increasing. Therefore $\lim_{N\to\infty} S_N$ exists iff $\{S_n\}$ is bounded above. ■

Example 4.11.

$$\sum_{n=0}^{\infty} \frac{1}{n}, S_N = 1 + ... + \frac{1}{N} > log(N), log(N) \to \infty \text{ as } n \to \infty$$

4.2.1 Convergence Tests

Definition 4.12. The Integral Test: let f be a function defined for $x \geq 1$ and is integrable. Let $a_n = f(n), \forall n \in \mathbb{N}$ then $sum_{n=0}^{\infty} a_n$ converges iff

$$\int_1^{\infty} f(x)dx = \lim_{y \to \infty} \int_1^y f(x)dx$$

exists

Example 4.13. Consider $f(x) = x^a, a \in \mathbb{R}, a \neq -1$

$$\int_1^y x^a dx = \left. \frac{x^{a+1}}{a+1} \right|_1^y = \frac{x^{y+1}}{y+1} - \frac{1}{a+1}$$

So now we know that $\int_1^y x^a dx$ exists iff $a < -1$, so equivalently:

$$\sum_{a=1}^{\infty} \frac{1}{n^a} \text{ converges iff } a > 1$$

5 Friday, January 19, 2018

5.1 Series (continued)

5.1.1 Convergence Tests

Definition 5.1. The **Comparison Test** is: Given a series $\sum_{k=1}^{\infty} a_k$ converges and $a_k \geq 0$. A series $\sum_{k=1}^{\infty} b_k$ converges if $|b_k| \leq a_k, \forall k$

Definition 5.2. The **Ratio Test** is given a series $\sum_{k=1}^{\infty} a_k$, consider

$$\lim_{k\to\infty} |\frac{a_{k+1}}{a_k}| = r$$

if $r = 1$, the test is inconclusive, if $r > 1$ the series diverges, if $if r < 1$ the series converges.

Example 5.3. Consider $\sum_{k=1}^{\infty} \frac{c^k}{k!}, c \in \mathbb{R}$
So we will apply to the ratio test to this:

$$\begin{aligned}\lim_{k\to\infty} |\frac{(c^{k+1}/(k+1)!)}{(c^k/k!)}| &= \lim_{k\to\infty} |\frac{c^{k+1}k!}{c^k(k+1)!}| \\ &= \lim_{k\to\infty} \frac{c}{k+1} \\ &= \end{aligned}$$

Therefore this series converges

Note.

$$\sum_{k=1}^{\infty} \frac{c^k}{k!} = e^c$$

Series that contain both positive and negative terms can behave strangely

Example 5.4.

$$\sum_{k=1}^{\infty} \frac{(-1)^n}{n} = log(2)$$

Definition 5.5. $\sum_{k=1}^{\infty} a_k$ **converges absolutely** if $\sum_{k=1}^{\infty} |a_k|$ converges

Definition 5.6. $\sum_{k=1}^{\infty} a_k$ **converges conditionally** if it converges but does not converge absolutely

Example 5.7. $\sum_{k=1}^{\infty} \frac{c^k}{k!}$ converges absolutely

Example 5.8. $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ converges conditionally since $\sum_{k=1}^{\infty} \frac{1}{k}$

Series that converge absolutely act like finite sums, you can rearrange the order of their terms and get the same sum, but this is not the case for conditional convergent series.

Proposition 5.9. If a series converges absolutely, then it converges.

Proof. Use the Cauchy Criterion. Given $\sum_{k=1}^{\infty} a_k$ converges absolutely, suppose $\epsilon > 0$, we have the following inequality because of the triangle inequality

$$|\sum_{k=n+1}^{m} a_k| \leq |\sum_{k=n+1}^{m} (|a_k|)|$$

now $\sum_{k=1}^{\infty} |a_k|$ converges, therefore, this satisfies the Cauchy Criterion, and so

$$\exists N | \forall m, n > N, \sum_{k=n+1}^{m} |a_k| < \epsilon \implies |\sum_{k=n+1}^{m} a_k| < \epsilon$$

Therefore $\sum_{k=1}^{\infty} a_k$ satisfies Cauchy Criterion and converges ■

Definition 5.10. Given $\sum_{k=1}^{\infty} a_k$ as a series of positive and negative terms, let:

$$a_k^+ = \begin{cases} a_k, & \text{if } a_k \geq 0 \\ 0, & \text{if } a_k < 0 \end{cases}$$

$$a_k^- = \begin{cases} a_k, & \text{if } a_k \leq 0 \\ 0, & \text{if } a_k > 0 \end{cases}$$

Note. $a_k = a_k^+ + a_k^-$, $|a_k| = a_k^+ - a_k^-, \forall k$

Proposition 5.11. $\sum_{k=1}^{\infty} a_k$ converges absolutely iff $\sum_{k=1}^{\infty} a_k^+$ and $\sum_{k=1}^{\infty} a_k^-$ both converge

Proof. So if we know that

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} a_k^+ + \sum_{k=1}^{\infty} a_k^-$$

$$\sum_{k=1}^{\infty} |a_k| = \sum_{k=1}^{\infty} a_k^+ - \sum_{k=1}^{\infty} a_k^-$$

This implies the following 2 equations:

$$\sum_{k=1}^{\infty} |a_k| + \sum_{k=1}^{\infty} a_k = 2\sum_{k=1}^{\infty} a_k^+$$

$$-\sum_{k=1}^{\infty} |a_k| + \sum_{k=1}^{\infty} a_k = 2\sum_{k=1}^{\infty} a_k^-$$

And we know that these statements only hold true of they all converge. ■

Proposition 5.12. if $\sum_{k=1}^{\infty} a_k$ converges conditionally then $\sum_{k=1}^{\infty} a_k^+$, $\sum_{k=1}^{\infty} a_k^-$ both diverge.

Proof. Suppose only one of $\sum_{k=1}^{\infty} a_k^+, \sum_{k=1}^{\infty} a_k^-$ diverges, this implies that $\sum_{k=1}^{\infty} |a_k|, \sum_{k=1}^{\infty} a_k$ diverge.
So conditional convergence must imply that both $\sum_{k=1}^{\infty} a_k^+, \sum_{k=1}^{\infty} a_k^-$ diverge since we need $\sum_{k=1}^{\infty} |a_k|$ to diverge, but $\sum_{k=1}^{\infty} a_k$ to converge. We get this from looking at the following equations from the previous proof:

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} a_k^+ + \sum_{k=1}^{\infty} a_k^-$$

$$\sum_{k=1}^{\infty} |a_k| = \sum_{k=1}^{\infty} a_k^+ - \sum_{k=1}^{\infty} a_k^-$$

■

Remark. *Suppose $a_k \geq 0, \forall k$ and $\sum_{k=1}^{\infty} a_k$ converges, then since $a_k \geq 0, |a_k| = a_k$ so the series converges absolutely.*

Now if we change the signs of the terms arbitrarily, the new series still converges.

5.2 Alternating Series

Definition 5.13. Suppose $a_k \geq a_{k+1} \geq 0, \forall k \geq 1$ then

$$\sum_{k=1}^{\infty}(-1)^{k+1}a_k$$

converges, this is an example of an **Alternating Series**

Example 5.14. $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$

Definition 5.15.

$$S_n = \sum_{k=1}^{\infty}(-1)^{k+1}a_k$$

Is the **Partial Sum of an Alternating Series**

Partial Sums of alternating series also have the following properties:

Property 5.16. $S_2 \leq S_4 \leq ... \leq S_{2k}$

Proof. We know that $S_{2k+2} = S_{2k} + (a_{2k+1} - a_{2k+2})$ and that $a_{2k+1} \geq a_{2k+2}$, this implies that $a_{2k+1} - a_{2k+2} \geq 0$, therefore $S_{2k+2} \geq S_{2k}$ ∎

Property 5.17. $S_1 \geq S_3 \geq ... \geq S_{2k+1}$

Proof. We know that $S_{2k+1} = S_{2k-1} + (-a_{2k} + a_{2k+1})$ and that $a_{2k} \geq a_{2k+1}$, this implies that $-a_{2k} + a_{2k+1} \leq 0$, therefore $S_{2k+1} \geq S_{2k-1}$ ∎

Property 5.18. for an even l and an odd m, $S_l \leq S_m$

Proof. if $l = 2k, m = 2j+1, k, j \in \mathbb{N}$. Pick i such that $i \geq j, k$ so that $2i \geq 2k = l$ and $2i+1 \geq 2j+1 = m$. This implies that $S_{2i} \geq S_l, S_{2i+1} \leq S_m$. So now

$$S_{2i+1} = S_{2i} + a_{2i+1}$$

But since $a_{2i+1} \geq 0$, this implies that $S_{2i+1 \geq S_{2i}}$ ∎

And now, we'll prove that alternating series where a_k decreases converges:

Proof. Now $S_2, S_4, ..., S_{2k}, ...$ forms an increasing sequence, bounded above by all S_{2k+1}.

Let $a = \underset{k}{sup}(\{S_{2k}\})$, which implies that $a \leq S_{2k+1}$

Similarly, $S_1, S_3, ..., S_{2k+1}$ forms a decreasing sequence, bounded below by all S_{2k}, a is also a lower bound of $\{S_{2k+1}\}$. Let $b = \underset{k}{inf}(\{S_{2k+1}\})$ ∎

6 Monday, January 22, 2018

6.1 Alternating continued

$$\sum_{k=1}^{\infty} a_k \text{ converges absolutely if } \sum_{k=1}^{\infty} |a_k| \text{ converges}$$

$$\sum_{k=1}^{\infty} a_k \text{ converges conditionally if it converges but not absolutely}$$

Example 6.1. The alternating harmonic series converges, but not absolutely.

6.2 Rearrangement

Because of this, we can rearrange the terms in the series to get it to converge to any number we want.

Example 6.2. For the alternating harmonic series, we separate all the terms into 2 groups, those that are positive and those that are negative, and we can add them in a way that makes the series converge to, let's say, 2.

For example, we can add up all the positive terms until we get to $\frac{1}{15}$, where adding $\frac{1}{15}$ brings the sum over 2 and not adding $\frac{1}{15}$ keeps the sum below 2, so we set S_1 to be all the decreasing positive terms until $\frac{1}{15}$, and then we will add increasing negative terms until the sum is below 2 to get T_1, and we add decreasing terms until the sum is above 2 and so on and so forth. This makes $S_{k+1} < S_k, \forall k$, $T_{k+1} > T_k, \forall k$.

Theorem 6.3. *If $sum_{k=1}^{\infty} a_k$ converges conditionally, then for any $b \in \mathbb{R}$, there exists an rearrangement for the series where its sum is b.*

<u>Recall:</u> Series of positive and negative terms of $\sum a_k$

$$a_k^+ = \begin{cases} a_k, & \text{if } a_k \geq 0 \\ 0, & \text{if } a_k < 0 \end{cases}$$

$$a_k^- = \begin{cases} a_k, & \text{if } a_k \leq 0 \\ 0, & \text{if } a_k > 0 \end{cases}$$

and $a_k = a_k^+ + a_k^-$, and if $\sum a_k$ is convergent, the both $\sum a_k^+$ and $\sum a_k^-$ diverge. So either the positive sum approaches infinity or the negative sum approaches negative infinity as more of their terms are added together, and at the same time, a_k approaches 0.

Proof. Choose an N such that

$$\sum_{k=1}^{N_1} a_k^+ > b > \sum_{k=1}^{N_1-1} a_k^+$$

Set $S_1 = \sum_{k=1}^{N_1} a_k^+, S_1 - b < a_{N_1}^+$
Choose M_1 so that

$$T_1 = S_1 + \sum_{k=1}^{M_1} a_k^- < b \leq S_1 + \sum_{k=1}^{M_1-1} a_k^-$$

And continue this pattern to get the rearranged series:

$$a_1^+, ..., a_{N_1}^+, a_1^-, ...a_{M_1}^-, a_{N_1+1}^+, ...a_{N_2}^+, ...$$

With $|S_k - b| < a_{N_k}^+, |T_k - b| < -a_{M_k}^-$

$$a_{N_k}^+, a_{M_k}^- \text{ as } k \to \infty$$

So $S_k \to b$ from above and $T_k \to b$ from below. ∎

Theorem 6.4. *Suppose that $\sum_{k=1}^{\infty} a_k$ is absolutely convergent and $b = \sum_{k=1}^{\infty} a_k$ then any rearrangement of this series will also have the sum of b.*

7 Monday, January 29, 2018

7.1 Rearrangement (continued)

Theorem 7.1. *Suppose $\sum_{k=1}^{\infty} a_k$ is absolutely convergent, let $\sum_{k=1}^{\infty} b_k$ be a rearrangement, then it has the same sum as the original series*

Proof. Let $a = \sum_{k=1}^{\infty} a_k$
Pick $\epsilon > 0$
Since $\sum_{k=1}^{\infty} |a_k|$ converges, then there exists an M such that $\sum_{k=M+1}^{\infty} |a_k| = \sum_{k=1}^{\infty} |a_k| - \sum_{k=1}^{M} |a_k| < \frac{\epsilon}{2}$ Now let

$$\{a_1, a_2, ...a_M\} \subset \{b_1, b_2, ...b_N\} \text{ for some } N \geq M$$

Be true for some sequence $\{b_k\}$. This implies that

$$(\{b_1, b_2, ...b_N\} \setminus \{a_1, a_2, ...a_M\}) \subset \{a_{M+1}, a_{M+2}, ...\}$$

Let us set $s_M = \sum_{k=1}^{M} a_k, t_N = \sum_{k=1}^{N} b_k$

$$|t_N - s_M| \leq \sum_{k=M+1}^{M} a_k < \frac{\epsilon}{2}$$

Now

$$\begin{aligned} |a - t_N| &\leq |s_N - a| + |s_M - t_N| \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

Therefore $\sum_{k=1}^{\infty} b_k = a = \sum_{k=1}^{\infty} a_k$ ■

Theorem 7.2. *e is irrational*

Proof. We know that $e = \sum_{k=0}^{\infty} \frac{1}{k!}$. Suppose that e is rational, then $e = \frac{a}{b}, a, b \in \mathbb{Z}, b \neq 0$. Then.

$$(b+1)! \sum_{k=0}^{b+1} \frac{1}{k!} \in \mathbb{N}, (b+1)!e \in \mathbb{N}$$

This implies that

$$
\begin{aligned}
&\implies (b+1)! \sum_{k=b+2}^{\infty} \frac{1}{k!} \in \mathbb{N} \\
&= (b+1)!\left(\frac{1}{(b+2)!} + \frac{1}{(b+3)!} + ...\right) \\
&= \left(\frac{1}{(b+2)} + \frac{1}{(b+2)(b+3)} + ...\right) \\
&< \left(\frac{1}{(b+2)} + \frac{1}{(b+2)^2} + ...\right) \\
&= \sum_{k=1}^{\infty} \frac{1}{(b+2)^k} \\
&= \frac{1}{(b+2)}\left(\frac{1}{1-\frac{1}{b+2}}\right) \\
&= \frac{1}{(b+2)}\left(\frac{b+2}{(b+1)}\right) = \frac{1}{(b+1)} < 1
\end{aligned}
$$

This is a contradiction, therefore e must be irrational

■

7.2 Power Series

Another convergence test: The root test

Definition 7.3. (The root test) given $\sum_{k=1}^{\infty} a_k$, consider $\lim_{k\to\infty} |a_k|^{\frac{1}{k}}$, if $\lim_{k\to\infty} |a_k|^{\frac{1}{k}}$ exists and it's less than 1, then it converges, if it's greater than 1, then it diverges.

We can apply this test to power series.

Theorem 7.4. *Given $\sum_{k=1}^{\infty} a_k x^k$, let $c = \lim_{k\to\infty} |a_k|^{\frac{1}{k}}$, $R = \frac{1}{c}, c \neq 0$ then $\sum_{k=1}^{\infty} a_k x^k$ converges for $|x| < R$ if $R < 1$ and $\sum_{k=1}^{\infty} a_k x^k$ diverges if $|x| > R$*

So basically, for $x \in (-R, R)$ the series converges, and for $x \notin [-R, R]$ the series diverges. $x = \pm R$ is still unknown

Proof. apply the root test

$$\begin{aligned}\lim_{k\to\infty}(|a_k x^k|)^{\frac{1}{k}} &= \lim_{k\to\infty}|a_k|^{\frac{1}{k}}|x| \\ &= |x|\lim_{k\to\infty}|a_k|^{\frac{1}{k}} \\ &= |x|\cdot c\end{aligned}$$

The root test implies that the series converges if $|x| \cdot c < 1$ and this implies that $|x| < \frac{1}{c} = R, c \neq 0$ And the series diverges if $|x|c > 1$ which happens iff $|x| > R$

■

Remark. *if* $c = 0$ $(R = \infty)$ *then* $\sum_{k=0}^{\infty} a_k x^k$ *converges for all* $x \in \mathbb{R}$.

if $\lim_k |a_k|^{\frac{1}{k}}$ *then series only converges for* $x = 0$

Example 7.5. $sum_{k=0}^{\infty}\frac{x^k}{k!}, R = \infty$

Example 7.6. $sum_{k=0}^{\infty}x^k, R = 1$, if $x = \pm 1$ then the series diverges

Example 7.7. $sum_{k=1}^{\infty}\frac{x^k}{k}$ apply the ratio test, $R = 1$, $|(\frac{x^{k+1}}{k+1})(\frac{k}{x^k})| = |x|\frac{k}{k+1}$ so when $x = 1$, $sum_{k=1}^{\infty}\frac{x^k}{k}$ diverges and when $x = -1$, $sum_{k=1}^{\infty}\frac{x^k}{k}$ converges

Example 7.8. $sum_{k=1}^{\infty}\frac{x^k}{k^2}, R = 1$ and this converges for $x = \pm 1$

8 Friday, February 2, 2018

8.1 Pointwise and Uniform Convergence

Consider a sequence of functions $f_1, ..., f_k, ...$ defined on $S \subset \mathbb{R}$

Definition 8.1. $\{f_k\}$ converges **pointwise** to a function f on S if for any $x_0 \in S, \{f_k(x_0)\} \to f(x_0)$

Example 8.2. $f_n(x) = \frac{sin(nx)}{x}, x \in \mathbb{R}$

We know that $\lim_{n\to\infty} \frac{sin(nx_0)}{n} = 0, x_0 \in \mathbb{R}$ as $\frac{|sin(nx_0)|}{n} \leq \frac{1}{n} \to 0$ as n approaches infinity.

if $f(x) = 0$ for all $x \in \mathbb{R}$, then f_n converges pointwise to f.

Example 8.3. $f_n(x) = \frac{nx}{1+n^2x^2}, x \geq 0,\ f(x) = 0$, for $x \geq 0$
f_n converges pointwise to f.

This is because $f_n(x) < \frac{nx}{n^2x^2} = \frac{1}{nx}, x > 0$ so for $x_0 > 0, 0 < f_n(x_0) < \frac{1}{nx_0} \to 0$, as n approaches infinity.

For $x_0 = 0, f_n(x_0) \to f(x_0) = 0$

Note. *For* $x_0 = \frac{1}{n}, f_n(\frac{1}{n}) = \frac{1}{2}$

Definition 8.4. A sequence of functions $\{f_n\}$ converges to a function f **uniformly** on S if given $\epsilon > 0$, there exists N such that for $n \geq N$

$$|f_n(x_0) = f(x_0)| < \epsilon, \forall x_0 \in S$$

Example 8.5. Consider $f_n(x) = 1 + x + ... + x^n$,$f(x) = \frac{1}{1-x}$ if $|x| < 1$.

We know that $f_n(x_0) \to f(x_0)$ for any $x_0 \in (-1, 1)$

Note. $f_n \to f$ *uniformly, means that sequences* $f_n(x_0) \to f(x_0)$ *"at the same rate", for all* x_0

Example 8.6. $f_n(x) = \frac{nx}{1+n^2x^2} < \frac{1}{nx}, x \geq 0$, take $\epsilon = \frac{1}{4}$ (we're looking for how big n has to be for the function f_n to be less than $\frac{1}{4}$)

if $x_0 = \frac{1}{10}$, then $\frac{1}{n10^{-1}} = \frac{10}{n} < \frac{1}{4}$, for $n > 40$

if $x_0 = \frac{1}{100}$, then $\frac{1}{n10^{-2}} = \frac{100}{n} < \frac{1}{4}$, for $n > 400$

if $x_0 = 10^{-6}$, then $\frac{1}{n10^{-6}} = \frac{10^6}{n} < \frac{1}{4}$, for $n > 4 \times 10^6$
The differences in the rate of convergence is exponential the bigger the n. So $f_n(\frac{1}{10})$, $f_n(\frac{1}{100})$, $f_n(10^{-6})$, do not converge at the same rate.

Example 8.7. $S = [1, 0]$, $f_n(x) = x^n$,

$$f(x) = \begin{cases} 0, & 0 \leq x < 1 \\ 1, & x = 1 \end{cases}$$

if we take different values of n for f_n, they seem to drop down steeper as they go close to 1, but also rise up steeper as they approach closer to 1, there is a bit of a dip.

$\{x_0^n\} \to 0, 0 \leq x_0 < 1$. f_n converges pointwise to f.
Note that f_n is continuous for all of n, but f is not continuous.

Example 8.8. verify that $f_n \to f$ uniformly in example 1.
$f_n(x) = \frac{sin(nx)}{n}, f(x) = 0, \forall x \in \mathbb{R}$

given $\epsilon > 0$,

$$|\frac{sin(nx)}{n}| < \frac{1}{n}$$

So choose N, with $\frac{1}{N} < \epsilon$, this implies that for all $n \geq N, |\frac{sin(nx)}{n}| < \frac{1}{n} < \epsilon, \forall x \in \mathbb{R}$

Theorem 8.9. *Suppose $f_n \to f$ uniformly on $S \subset \mathbb{R}$, If f_n is continuous for all n, then f is continuous*

Proof. Need to show f is continuous at any $x_0 \in S$, take $\epsilon > 0$, we need to find $\delta > 0$ so that for $|x - x_0| < \delta$, that $|f(x) - f(x_0)| < \epsilon$

Since $f_n \to f$ uniformly, there exists N so that for $n \geq N$, $|f_n(x) - f(x)| < \frac{\epsilon}{3}$ for all $x \in S$

f_N is continuous at x_0, therefore there exists a $\delta > 0$ such that for $|x - x_0| < \delta$ then $|f_N(x) - f_N(x_0)| < \frac{\epsilon}{3}$

now

$$\begin{aligned} |f(x) - f(x_0)| &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f(x_0)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \end{aligned}$$

This is because of the uniform convergence for $|f(x) - f_N(x)|, |f_N(x_0) - f(x_0)|$ and continuity of F_N for $|f_N(x) - f_N(x_0)|$

this implies that for $|x - x_0| < \delta$, that $|f(x) - f(x_0)| < \epsilon$. ■

Example 8.10. $S = [1, 0]$, $f_n(x) = x^n$,

$$f(x) = \begin{cases} 0, & 0 \leq x < 1 \\ 1, & x = 1 \end{cases}$$

f_n does not converge uniformly in this example since f is not continuous at 1

Note. *When he have shown that the partial sums of a power series converge uniformly, then this theorem will show that the power series is continuous*

8.2 Integrals

Suppose $f_n \to f$ uniformly on an interval $[a, b] \subset \mathbb{R}$ and suppose f_n is integrable for all n, we want to show that f is integrable

$$\int_a^b f(x)dx = \lim_{n\to\infty} \int_a^b f_n(x)dx$$

we're gonna assume f_n are continuous on $[a, b]$ for all n which implies that f_n is integrable on $[a, b]$, but then f continuous on $[a, b]$, this implies that f is integrable.

Recall: if f, g on $[a, b]$, they are integrable if $f \leq g$

This implies that

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx$$

which implies that for any f integrable on $[a, b]$

$$|\int_a^b f(x)dx| \leq \int_a^b |f(x)|dx$$

By triangle inequality.

Theorem 8.11. *$f_n \to f$ uniformly on $[a, b]$, f_n is continuous for all n, this implies that*

$$\int_a^b f(x)dx = \lim_{n\to\infty} \int_a^b f_n(x)dx$$

Proof. Given $\epsilon > 0$, since $f_n \to f$ uniformly, there exists N such that for $n \geq N$

$$|f_n(x) - f(x)| < \frac{\epsilon}{b-a}$$

for all $x \in [a, b]$, then

$$\begin{aligned}
|\int_a^b f(x)dx - \int_a^b f_n(x)dx| &= |\int_a^b (f(x) - f_n(x))dx| \\
&\leq \int_a^b f_n(x) - f(x)|dx \\
&\leq \int_a^b \frac{\epsilon}{b-a} dx = \epsilon
\end{aligned}$$

And this implies that

$$\int_a^b f(x)dx = \lim_{n\to\infty} \int_a^b f_n(x)dx$$

■

9 Monday, February 5, 2018

9.1 Convergence

Definition 9.1. Sequence of functions $\{f_n(x)\}, x \in S \subset \mathbb{R}$ converges pointwise to $f(x)$ on S if for any $x_0 \in S$, the sequence $\{f_n(x_0)\} \to f(x_0)$ as $n \to \infty$

$f_n(x_0)$ uniformly converges if $\epsilon > 0$, there exists N such that $\forall x \in S$, $|f_n(x) - f(x)| < \epsilon$ if $n > N$

Example 9.2. on $[0,1], f_n(x) = \frac{x}{n} \leq \frac{1}{n}$. For all x, pick N given ϵ, $\frac{1}{N} < \epsilon$, then for all $n \geq N$, $f_n(x) \leq \frac{1}{n} \leq \frac{1}{N} < \epsilon$ for all x, this implies that on $[0,1]$, $f_n \to f$ uniformly

Example 9.3. Does $f_n(x)$ from the above example converge uniformly for $x \in [0, \infty)$? No.

Given $\epsilon > 0$, for any N, we can find x_0 with $\frac{x_0}{N} \geq \epsilon$, $f_n(x_0) \geq \epsilon$

Theorem 9.4. *Suppose $f_n \to f$ uniformly on S, f_n continuous for all n, then f continuous on S*

Example 9.5. $f_n(x) = \frac{1}{1+x^n}$

$$\lim_{n\to\infty} = \begin{cases} \frac{1}{2}, & x = 1 \\ 1, & x \in [0,1) \end{cases}$$

Since f is not continuous at 1, convergence not uniform by the theorem.

Theorem 9.6. *f_n continuous on $[a,b] \in \mathbb{R}$, $f_n \to f$ uniformly on $[a,b]$ then that implies that*

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx$$

Example 9.7. $f(x) = \sum_{k=0}^{\infty} x^k, |x| < 1$ set $S_n(x) = \sum_{k=0}^{n} x^k$ then $S_n \to f$ pointwise, pick $a \in (0,1)$

$$|S_m - S_n| = \sum_{k=n+1}^{m} x^k \leq \sum_{k=n+1}^{m} |x|^k \leq \sum_{k=n+1}^{m} a^k$$

if $|x| \leq a$, since $\sum_{k=0}^{\infty} a^k$ converges it satisfies the cauchy criterion. So, given $\epsilon > 0$, there exists N, such that for $m > n > N$ that

$$\sum_{k=n+1}^{m} a^k < \epsilon$$

$$\implies |S_m(x) - S_n(x)| < \epsilon$$

for $m, n > N$ and for all x.

So a uniform definition version of the Cauchy Criterion is needed. A Uniform Cauchy Criterion perhaps.

Theorem 9.8. *$\{f_n\}$ defined on S, satisfies the Uniform Cauchy Criterion if given $\epsilon > 0$, there exists N such that for $m, n > N$, we have $|f_n(x) - f_m(x)| < \epsilon$ for all $x \in S$.*

Theorem 9.9. *Suppose $\{f_n(x)\}$ satisfies the uniform Cauchy Criterion on S then for $x_0 \in S$, $\{f_n(x)\}$ satisfies the the cauchy property and therefore $\lim_{n\to\infty} f_n(x_0)$ exists, if $f(x_0) = \lim_{n\to\infty} f_n(x_0)$ then $f_n \to f$ uniformly on S*

Example 9.10. partial sums $S_n(x)$ of a geometric series satisfies Uniform Cauchy Criterion on $[-a, a]$ where $a \in [0, 1]$, $S_n \to f$ uniformly on $[-a, a]$.

10 Friday, February 9, 2018

10.1 Power Series

Suppose $\sum_{k=0}^{\infty} a_k x^k$ with a radius of convergence R, then the partial sums $S_n(x) = \sum_{k=0}^{n} a_k x^k$ converge uniformly to f on $[-a, a]$ for $a \in [0, R]$. Such an example is a geometric series.

So our plan is to define the Uniform Cauchy Criterion and prove that the sequence converges uniformly, then we will introduce the Weierstrass M-Test and the prove the theorem.

Definition 10.1. Uniform Cauchy Criterion given a sequence of functions $\{f_k\}$ defined on $S \subset \mathbb{R}$, $\{f_k\}$ is uniformly cauchy, if given $\epsilon > 0$, there exists N such that for $m, n > N$, $|f_m - f_n| < \epsilon$ for all $x \in S$

Remark: So for each $x_0 \in S$, iff follows that the sequence $\{f_k\}$ is a cauchy sequence $\implies \{f_k(x_0)\}$ converges

Proof. given $\epsilon > 0$ there exists N such that for $m, n > N$ then $|f_m(x) - f_n(x)| < \frac{\epsilon}{2}$ equivalently, $f_n(x) \in (f_m(x) - \frac{\epsilon}{2}, f_m(x) + \frac{\epsilon}{2})$

$$a_n \in (c, d) \implies \lim_{n \to \infty} a_n \in [c, d]$$

but

$$[f_m(x) - \frac{\epsilon}{2}, f_m(x) + \frac{\epsilon}{2}] \subset (f_m(x) - \epsilon, f_m(x) + \epsilon)$$

$$\implies |f_m(x) - f(x)| < \epsilon \text{ therefore cauchy convergent}$$

■

Definition 10.2. Weierstrass M-Test $\{f_k\}$, $k \geq 0$ defined on $S \subset \mathbb{R}$, such that $|f_k(x)| \leq M_k$ for some $M_k \geq 0$ where $\sum_k M_k < \infty$ then $\sum_{k=0}^{\infty} f_k(x)^k$ converges uniformly on S.

Example 10.3. $f_k(x) = a_k x^k$

Proof. Show that $S_n(x) = \sum_{k=0}^{n} f_k(x)$ satisfy uniform cauchy criterion, now

$$|S_n(x) - S_m(x)| = |\sum_{k=n+1}^{m} f_k(x)| \leq \sum_{k=n+1}^{m} |f(x)| \leq \sum_{k=n+1}^{m} M_k, \forall x \in S$$

since $\sum_{k=0}^{\infty} M_k$ converges it satisfies the cauchy criterion so given $\epsilon > 0$, there exists N such that for $m > n > N \implies |S_n(x) - S_m(x)| < \epsilon$ for $x \in S$ therefore $\{S_n(x)\}$ satisfies the Uniform Cauchy Criterion $\implies S_n \to \sum_{k=0}^{\infty} f_k(x)$ uniformly ■

Theorem 10.4. *Suppose $f(x) = \sum_{k=0}^{\infty} a_k x^k$ has radius of convergence $R > 0$, let $S_n(x) = \sum_{k=0}^{n} a_k x^k$ then $S_n \to f$ uniformly on $[-a, a]$, $0 < a < R$*

Proof. $|a_k x^k| < |a_k| x^k$ for $|x^k| < a$, let $M_k = |a_k| x^k$
Since $\sum_{k=0}^{\infty} a_k a^k$ converges, $\sum_{k=0}^{\infty} < \infty$, Weierstrass converges uniformly on $[-a, a]$

Remark: We need to use $a < R$ because if $a = R$, then $\sum_{k=0}^{\infty} a_k R^k$ might not converge

to see that f is continuous at $x_0 \in (-R, R)$, pick $a > 0$ such that $x_0 \in [-a, a]$ then uniform convergence $\implies f$ is continuous at x_0 ■

Proposition 10.5. Suppose that the series $\sum_{k=0}^{\infty} a_k x^k$ has a radius of convergence R, then

$$\sum_{n=0}^{\infty} n a_n x^{n-1}, \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

also have radius of convergence R

Proof. $\sum_{n=0}^{\infty} n a_n x^{n-1}$ has some radius of convergence as

$$x \sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} n a_n x^n$$

So the radius of convergence is

$$\begin{aligned}
&= \lim_{n\to\infty} |n a_n|^{\frac{1}{n}} = \lim_{n\to\infty} n^{\frac{1}{n}} |a_n|^{\frac{1}{n}} \\
&= \lim_{n\to\infty} n^{\frac{1}{n}} \cdot \lim_{n\to\infty} |a_n|^{\frac{1}{n}} \\
&= 1 \cdot c = \frac{1}{R}
\end{aligned}$$

The Radius of Convergence of $\sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$ is

$$\begin{aligned}
&= |\frac{a_n}{n+1}|^{\frac{1}{n+1}} \\
&= |a_n|^{\frac{1}{n+1}} (\frac{1}{(n+1)^{n+1}})_{n\to\infty} \\
&\to \frac{1}{R}
\end{aligned}$$

∎

Theorem 10.6. $\int_0^x f(t)dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}, |x| < R$

Proof. Since $S_n \to f$ uniformly

$$\int_0^x f(t)dt = \lim_{n\to\infty} \int_0^x S_n dt$$

now

$$\int_0^x S_n dt = \sum_{k=0}^{n} \frac{a_k x^{k+1}}{k+1} \overset{\text{uniformly}}{\to} \sum_{k=0}^{\infty} \frac{a_k x^{k+1}}{k+1}$$

$$\implies \int_0^x f(t)dt = \sum_{k=0}^{\infty} \frac{a_k x^{k+1}}{k+1}$$

∎

Theorem 10.7. *let $f(x) = \sum_{k=0}^{\infty} a_k x^k$ has radius of convergence $R > 0$ then f is differentiable and*

$$f'(x) = \sum_{k=1}^{\infty} k a_k x^{k-1}, x \in (-R, R)$$

The previous theorem implies that

$$\int_0^x g(t)dt = \sum_{k=1}^{\infty} a_k x^k = f(x) - a_0$$

want to apply fundamental theorem. So take R_1 where $0 < R_1 < R$ and then

$$f(x) = \int_0^x g(t)dt - \int_{-R_1}^{c} g(t)dt + a_0 = \int_{-R_1}^{x} g(t)dt$$

So the fundamental theorem implies that $f'(x) = g(x)$ for $|x| < R_1$

10 FRIDAY, FEBRUARY 9, 2018

Example 10.8.

$$\frac{1}{1-x} = 1 + x. + x^s + ... = \sum_{k=0}^{\infty} x^k$$

$$\int_0^x \frac{dt}{1-t} = -log(1-x)$$

but

$$\int_0^x \sum_{k=0}^{\infty} t^k dt = \sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1}$$

so $\sum_{k=0}^{\infty} \frac{x^k}{k}$ must be $-log(1-x), |x| < 1$
here we replace x by $-x$ and then

$$-log(1-x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k}$$

$$log(1-x) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1} x^k}{k}$$

11 Monday, February 12, 2018

11.1 Power Series (cont'd)

Definition 11.1. A **Power Series** is a series of the form

$$f(x) = \sum_{k=0}^{\infty} a_k x^k$$

and if it has a radius of convergence of $R > 0$ then f is continuous and differentiable on $(-R, R)$ with

$$f(x) = \sum_{k=0}^{\infty} k x^{k-1}$$

and

$$\int_0^x f(t)dt = \sum_{k=0}^{\infty} a_k \frac{x^{k+1}}{k+1}, x \in (-R, R)$$

Example 11.2. $\frac{1}{1-x} = sum_{k=0}^{\infty} x^k, |x| < 1$

$$\begin{aligned} \int_0^x \frac{dt}{1-t} &= -log(1-x) \\ &= sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1} \\ &\implies log(1-x) = -\sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1}, |x| < 1 \\ &\implies log(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k} \end{aligned}$$

What happens at $x = 1$? Both sides define functions which are continuous at 1.

Theorem 11.3. ***Abel's Theorem:*** *Suppose $f(x) = \sum_{k=0}^{\infty} a_k x^k$, radius of convergence is $R > 0$, if the series converges at R or $-R$ then f is continuous at R or $-R$.*

Definition 11.4. Exponential Function: $exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} = e^k$ converges for all real values of x
<u>properties:</u>

1.
$$exp'(x) = exp(x)$$

Suppose $f(x) = \sum_{k=0}^{\infty} a_k x^k$ converges for $|x| < R$, and $f'(x) = f(x)$, this implies that $f^n(x) = f(x)$ and that $f^n(0) = a_n$, which also implies that $a_n = \frac{a_0}{n!}$ which implies that $f(x) = a_0 exp(x)$
(note: the solutions for $\frac{dy}{dx} = y$ are $c \cdot exp(x), c \in \mathbb{R}$)

2.
$$exp(a+b) = exp(a)exp(b) \quad a, b \in \mathbb{R}$$

let $f(x) = exp(a+x)$

$$\implies f'(x) = cf(x), c \in \mathbb{R}$$

$$exp(a+x) = c \cdot exp(x)$$

in particular for $x = 0$,

$$exp(a) = c \cdot exp(0) \implies exp(a) = c \cdot 1 = c$$

$$\implies exp(a+x) = exp(a)exp(x), \forall x \in \mathbb{R}$$

define $e = exp(1) = \sum_{k=0}^{\infty} \frac{1}{k!}$

$$e^m = exp(m), m \in \mathbb{N}$$

$l \in \mathbb{N}$ then, $exp(\frac{m}{l})^l = exp(m) = e^m$

$$\implies exp(\frac{m}{l}) = e^{\frac{m}{l}}$$

$$\implies e^q = exp(q), q \in \mathbb{Q}$$

since $exp'(x) = exp(x) > 0 \implies exp$ is increasing.
For $r \in \mathbb{R}$, define $e^r = sup\{e^q | q \in \mathbb{Q}, q < r\} = exp(r)$

$$e^x = exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

Properties:

1. e^x is strictly increasing

   Proof. The mean value theorem states that:

   $$e^b - e^a = (b-a)e^c$$

   for some $c \in (a,b)$, and since $(b-a)e^c$ is always above 0, then $e^b > e^a$ ■

2. $\lim_{x\to-\infty} e^x = 0$

   Proof.
   $$e^x > 1 + x$$
   so $x \to \infty \implies e^x \to \infty \implies e^{-x} = \frac{1}{e^x} \to 0$ as $x \to \infty$ ■

12 Friday, March 2, 2018

12.1 Open and closed subsets of the real numbers

Definition 12.1. $U \subset \mathbb{R}$ is an open set if, for any $x \in U$, there exists $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset U$

Example 12.2. $\mathbb{R}$ is an open set

Example 12.3. $(0, 1)$ is an open set

Example 12.4. $\mathbb{R} \setminus \{0\}$ is an open set

Example 12.5. $[0, 1)$ is an open set

Proposition 12.6. Suppose $U_a, a \in A$ is a collection of open sets $\implies$

$$U = \bigcup_{a \in A} U_a \text{ is open}$$

Proof. Suppose $x \in U$, then $x \in U_a$ for some $a \in A$. Since U_a is open, there exists a neighbourhood $(x - \epsilon, x + \epsilon) \subset U_a$ for some $\epsilon > 0$. Therefore $(x - \epsilon, x + \epsilon) \subset U = \bigcup_{a \in A} U_a$ ■

Proposition 12.7. Suppose $U_1, ..., U_n$ are open sets then

$$U = \bigcap_{i=1}^{m} U_i \text{ is open}$$

Proof. Suppose $x \in U = \bigcap_{i=1}^{m} U_i \implies x \in U_i$ for each i, U_i is open $\implies$ there exists a neighbourhood $(x - \epsilon_i, x + \epsilon_i), \epsilon_i > 0$

Take $\epsilon = sup(\epsilon_i), \forall i, (i \in [1, m])$ so $\epsilon > 0$.
(Since $\forall \epsilon$ sits inside all sets, openness is proven).
for all i,

$$(x - \epsilon, x + \epsilon) \subset (x - \epsilon_i x + \epsilon_i) \subset U_i$$

$$\implies (x - \epsilon, x + \epsilon) \subset \bigcap_{i=1}^{m} U_i$$

■

Example 12.8. let $U_i = (-\frac{1}{i}, \frac{1}{i}), i \in \mathbb{N}$

$$\implies \bigcap_{i\in\mathbb{N}} U_i = \{0\} \text{ which contains no neighbourhood, therefore not open}$$

So infinite examples may not work

Proposition 12.9. Suppose $U \subset \mathbb{R}$ is open, then U is the union of a finite or countablecollection of pairwise disjoint open intervals (take two open intervals, and they cannot intersect).

Proof. Define a relation on U, $a, b \in U, a \sim b$ if all points between a, b lie in U. We must prove $\sim$ is reflexive, symmetric, and transitive to show that it is also an equivalence relation. Assume $a < b < c$, if all the points in $a\ \ b$ lie in U, if all the points in $b\ \ c$ lie in U, and if all the points in $a\ \ c$ lie in U, then $\sim$ is an equivalence relation in U.
Equivalence classes are clearly an open interval and they are pairwise disjoint. Each open interval contains a relational number, list these numbers, there are countably many or finitely many in the list. ∎

Definition 12.10. A set $S \subset \mathbb{R}$ is **Closed** if $U = \mathbb{R} \setminus S$ is open

Example 12.11. $[a, b]$ is closed because it's compliment is open

Example 12.12. $[b, \infty]$ is closed because it's compliment is open

Example 12.13. $(0, 1]$ is neither

Proposition 12.14. $S_a, a \in A$ is a collection of closed sets.

$$\implies \bigcap_{a\in A} S_a \text{ is closed}$$

Proof. let $U_a = \mathbb{R} \setminus S_a$, and since $\mathbb{R} \setminus \bigcap_{a\in A} S_a$ is open, then S must be closed ∎

Proposition 12.15. $S_1, ..., S_m$ is closed $\implies S = \bigcup_{i=1}^m S_i$ is also closed

Proof. let $U_i = \mathbb{R} \setminus S_i, i \in \mathbb{N}$.

$$\mathbb{R} \setminus \bigcap_{i=1}^{m} U_i = \bigcup_{i=1}^{m} S_i$$

$$\bigcap_{i=1}^{m} U_i \text{ is open} \implies \bigcup_{i=1}^{m} S_i \text{ is closed}$$

∎

13 Monday, March 5, 2018

13.1 Topology of the Real Numbers

Definition 13.1. The **The neighbourhood of** $a \in \mathbb{R}$ is $(a-\epsilon, a+\epsilon)$ for some $\epsilon > 0$

Definition 13.2. The **Open Set** is a set U such that, for any $a \in U$, there exists a neighbourhood of a in U, i.e. There exists $\epsilon > 0$ with $(a-\epsilon, a+\epsilon) \subset U$.

Example 13.3. Examples of an open interval include:

$$(c,d), (c,\infty), (-\infty,d), \mathbb{R}, \varnothing$$

properties of the open set:

1. if U_α is open, $\alpha \in A$, then $U = \bigcup_{\alpha \in A} U_\alpha$ is open
2. If $U_1, ..., U_n$ is open, then $U = \bigcap_{i=1}^n U_i$ is open (but this is not true for infinite sets.)
3. If $U \subset \mathbb{R}$ is open, then U is the union of a finite or countable collection of disjoint open intervals.

Clearly, we are also interested in a set $S \subset \mathbb{R}$ with the property: if $\{a_n\}_{n=1}^\infty \subset S$, $\{a_n\}$ converges to $a \in \mathbb{R}$ then $a \in S$ as well.

Definition 13.4. $S \subset \mathbb{R}$ is closed if $U = \mathbb{R} \setminus S$ is open.

Recall $\{a_n\}$ and cauchy sequences

Proposition 13.5. S is closed iff the limit of every cauchy sequence in S lies in S.

Proof. First, suppose S is closed, therefore $U = \mathbb{R} \setminus S$ is open. Let $\{a_n\} \subset S$ be a cauchy sequence with it's limit being a. Suppose $a \notin S$, that must mean that $a \in U$ and since U is open, for some $\epsilon > 0$, $(a-\epsilon, a+\epsilon) \subset U$ but since $a_n \to a$, there exists N such that for $n > N$, $a_n \in (a-\epsilon, a+\epsilon) \subset U$, but this is a contradiction as $a_n \in S = \mathbb{R} \setminus U$ for all n.

Conversely, suppose that for any cauchy sequence $\{a_n\} \subset S, \lim_{n\to\infty} a_n \in S$. We want to show that S is closed, but for the sake of contradiction, we shall

assume S is not closed, this means that $U = \mathbb{R} \setminus S$ is not open. i.e. there exists $a \in U$ such that no neighbourhood of a exists in U.
For any $n > 0$, there exists $a \in (a - \frac{1}{n}, a + \frac{1}{n}), a_n \notin U \to a_n S$. Now $a_n \to a$ as $n \to \infty$ and $\{a_n\} \subset S$, but since $a \in U = \mathbb{R} \setminus S$, this is a contradiction which implies that U is open, which implies that S is closed. ■

properties of closed sets

1. if $S_a, a \in A$, is closed, then $S = \bigcap_{a \in A} S_a$ is also closed
2. if $S_1, ..., S_n$ is closed, then $\bigcup_{j=1}^{n} S_j$ is also closed
3. $\varnothing, \mathbb{R}$ are both closed

13.2 Boundary Points

Definition 13.6. For $S \in \mathbb{R}, a \in S$, Call a a boundary point if every neighbourhood of a is both in and not in the set S. (Notation: ∂S)

Example 13.7. For $S = (0, 1]$ then 0,1 are boundary points

Theorem 13.8. *if $a \in (0, 1)$ then there exists a neighbourhood such that $a \in S \implies a \notin \partial S$*

Example 13.9. $S = \mathbb{Q} \subset \mathbb{R}$. $\partial S = \mathbb{R}$, because if $a \in \mathbb{Q}$, then it's neighbourhood would include both rational and irrational numbers.

14 Friday, March 9, 2018

14.1 Boundary Points

Example 14.1.

$$S_n = [-1 + \frac{1}{n}, 1 - \frac{1}{n}]$$

$$S = \bigcup_{n=1}^{\infty} S_n = (-1, 1)$$

given $a \in (-1,1)$, for n large enough, $a \in [-1 + \frac{1}{n}, 1 - \frac{1}{n}]$. $-1 \notin S, 1 \notin S$

Proposition 14.2. $U \subset \mathbb{R}$ is open iff $U \cap \partial U = \varnothing$

Proof. Suppose U is open, $a \in U$
$\implies$ there exists a neighbourhood $(a - \epsilon, a + \epsilon) \subset U$, if $a \in \partial U$ as well, then this neighbourhood would also contain points in $\mathbb{R} \setminus U, U \cap \partial U = \varnothing$

Conversely, Suppose $U \in \mathbb{R}, U \cap \partial U = \varnothing$, take $a \in U$, then $a \notin \partial U$
$\implies$ There exists a neighbourhood $(a - \epsilon, a + \epsilon)$ of a which lies in U.
$\implies$ U is open ■

Remark. *For $S \subset \mathbb{R}, T = \mathbb{R} \setminus S$, then $\partial S = \partial T$.*

Theorem 14.3. *If $a \in \partial S$, then every neighbourhood of a includes points in S and in T, where $a \in \partial T$*

Proof. Let $U = \mathbb{R} \setminus S$, S is closed $\iff$ U is open. So know we know that $U \cap \partial U = \varnothing \iff S \supset \partial S = \partial U$ ■

Definition 14.4. $a \in S$ is an **isolated point** of S, if there exists an neighbourhood $(a - \epsilon, a + \epsilon)$ for a such that

$$(a - \epsilon, a + \epsilon) \cap S = \{a\}$$

Definition 14.5. $a \in \mathbb{R}$ is an accumulation point of S if every neighbourhood of a contains infinitely many points in S.

Theorem 14.6. *If $a \in \partial S$ then a is either an accumulation point of S or an isolated point in S*

15 Monday, March 12, 2018

15.1 Topology of the real numbers

Definition 15.1. Open sets: Every $a \in S$ has a neighbourhood that is a subset of S.

Definition 15.2. Closed sets: The limit of a cauchy sequence in S lies in S.

Definition 15.3. Boundary of $S(\partial S)$: $a \in \partial S$ if every neighbourhood of A meets S and $\mathbb{R} \setminus S$.
$a \in \partial S$ is either an isolated point of S or an accumulation point of S.
$a \in \partial S$ is an isolated point of S if there is a neighbourhood of a, which only meets S at a
$a \in \partial S$ is an accumulation point of S if every neighbourhood of a contains infinitely many points of S.

Example 15.4. on the interval $[0, 1] \cup \{2\}, 2$ is an isolated point

Proposition 15.5. If $S \subset \mathbb{R}$ is a bounded infinite set, then S has a accumulation point.

Proof. Let $\{a_k\} \subset S$ be an infinite sequence. Since $\{a_k\}$ is bounded, it has a convergent subsequence: $\{a_{k_j}\} \to a \in \mathbb{R}$
then a is an accumulation point, since every neighbourhood of a has infinitely many points in the subsequence ■

Example 15.6. $S = \{\frac{1}{n}\}_{n=1}^{\infty}$. Here, 0 is an accumulation point.

Definition 15.7. Alternative definition for an Open Set: U is open iff U is disjoint $\implies U \cap (\partial U) = \varnothing$

Definition 15.8. Alternative definition for a Closed Set: S is closed iff $S \supset \partial S$

Definition 15.9. Closure of S: $\overline{S} = S \cup \partial S$, if S is closed, then $\overline{S} = S$

Example 15.10. Let $S = (0, 1)$, then $\overline{S} = [0, 1]$

Proposition 15.11. $\overline{S}$ is closed

Proof. We want to show that $\overline{S} = S$, and S is an open set.
$U = \mathbb{R} \setminus S$ we want to show that U is open.
If $a \in U$, then $a \notin S, a \notin \partial S$ then there exists a neighbourhood $(a-\epsilon, a+\epsilon) \subset \mathbb{R} \setminus \overline{S} = U$ This implies that U is open and $\overline{S}$ is closed. ∎

This implies that there exists an a such that the neighbourhood around a

So $(a - \epsilon, a + \epsilon) \to \forall a$ then given $S \subset \mathbb{R}$, suppose $a \in S$ and $a \notin \partial S$
$\implies$ There exists a neighbourhood of $a \subset S$

Definition 15.12. If $a \in S$ and there exists a neighbourhood of a, $(a-\epsilon, a+\epsilon) \subset S$, then a is an **interior point** of S.

Example 15.13. For the interval $[0, 1]$, only 0 and 1 are not interior points

Example 15.14. S is open $\implies$ every point in S is an interior point

Example 15.15. $S = \{\frac{1}{n}\}_{n=1}^{\infty}$ has no interior points

Definition 15.16. A set $S \subset \mathbb{R}$ is compact if every sequence $\{a_n\} \subset S$ has a convergent subsequence who's limit lies in S

Example 15.17. $[0, 2]$ is compact, as every sequence is bounded, therefore there exists a convergent subsequence since $[0, 2]$ is closed, the limit lies in $[0, 2]$

Example 15.18. $S = [0, \infty)$, take $a_n = n, n \geq 1$, since $a_n \to \infty$, there is no convergent subsequence, therefore S is not compact.

Example 15.19. $S = (0, 3)$. Take $a_n = 3 - \frac{1}{n}, n \geq 1$, then $a_n \to 3$. This means than S is not compact

Proposition 15.20. S is compact iff S is closed and bounded

Proof. Suppose S is closed and bounded, then if $\{a_n\} \subset S, \{a_n\}$ is bounded.
$\implies$ There exists a convergent subsequence $\{a_{n_k}\}_k \to a$. Since S is closed, $a \in S \implies S$ is compact.

Conversely, assuming S is compact.

S is not bounded, then there exists a sequence $\{a_n\}$, such that $a_n \to \infty$ or $a_n \to -\infty$ as $n \to \infty$
$\implies$ no subsequence converges either ∎

16 Friday, March 16, 2018

16.1 Compact Sets

Proof. (1) K is compact and bounded $\iff$ (2) Any sequence in K has a subsequence, which converges for a point on K.

(1) $\implies$ (2) follows from bolzano-weierstrass and the definition of a closed set

(2) $\implies$ (1):
(2) implies that K is closed, by definition of a closed set. Why is K bounded? Suppose K is not bounded then there exists a sequence $a_n \in K$ such that $|a_n| > n, \forall n \implies \{a_n\}$ does not converge.
Suppose $K_1 \supseteq K_2 \supseteq ... \supseteq K_n$, then K_n is compact for all N, for example, $K_n = [0, \frac{1}{n}]$
$\bigcap_{n=1}^{\infty} K_n = \{0\}$ ■

Proposition 16.1. let $K = \bigcap_{n=1}^{\infty} K_n$, then K is compact and K is not empty

Proof. Since K_n is closed for all n, therefore $K = \bigcap_{n=1}^{\infty} K_n$ is closed. Since $K \subset K_1$, which is bounded, therefore K is bounded too. These all imply that K is compact.

We now need to show that K is nonempty

pick $a_n \in K_n, n \geq 1$, notice that for $m \geq n, a_m \in K_m \subset K_n$, in particular, $\{a_n\} \in K$
$\implies \{a_n\}$ has a subsequence $\{a_n\}_{j=1}^{\infty}$ such that $a_{n_j} \to a \in K_i$ as $j \to \infty$ now for $j \geq m, nj \geq j \geq m$
$\implies a_{n_j} \in K_m \implies a \in \bigcap_{n=1}^{\infty} K_n = k$
$\implies k \neq \varnothing$ ■

Example 16.2. $S_n : [n, \infty), n \geq 1$
$\bigcap_{n=1}^{\infty} S_n = \varnothing, S_{n+1} \subset S_n$, S_n is closed, but not bounded

Example 16.3. Let $S = (0, \frac{1}{n})$

$$S_{n+1} \subset S_n, \bigcap_{n=1}^{\infty} S_n = \varnothing$$

Then there exists n, with $\frac{1}{n} < a \Longrightarrow a \notin S_n \Longrightarrow a \notin \bigcap_{n=1}^{\infty} S_n$

16.2 Cantor Set

$$S_0 = [0, 1]$$

if we remove the middle third of the set we get:

$$S_1 = S_0 \setminus (\frac{1}{3}, \frac{2}{3}) = [0, \frac{1}{3}) \cup (\frac{2}{3}, 1]$$

and for S_2 we will remove the middle thirds of S_1. So this results in the pattern where S_n is a union of closed intervals: to obtain S_{n+1}, remove middle thirds of each interval $\Longrightarrow$ the number of intervals in S_n

The length of each interval in S_{n+1} is one third the length of each interval in S_n, so $S_0 \supset S_1 \supset ... \supset S_n \supset ...$

Definition 16.4. Cantor Set : $S = \bigcap_{n=1}^{\infty} S_n$ by proposition, S is compact and nonempty

Remark. *in fact, endpoints of the intervals in S_n all belong to S.*

There are 2^n separate continuous intervals in S_n with each interval being of length $(\frac{1}{3})^n$

The length of S_n is $(\frac{2}{3})^n$, length of S_{n+1} is $[0, 1] \setminus S_n = 1 - (\frac{2}{3})^n$

16.2.1 Description of Points in S

Take an infinite sequence $\{a_n\}$ where a_n is either 1 or 0 for $n \geq 1$, we associate to $\{a_n\}$ as a point in S

If $a_1 = \begin{cases} 0, & \text{if we picked the first interval of } S_1([0, \frac{1}{3}]) \\ 1, & \text{if we picked the second interval of } S_1([\frac{2}{3}, 1]) \end{cases}$

If $a_2 = \begin{cases} 0, & \text{if we picked the first subinterval of second interval of } S_2 \\ 1, & \text{if we picked the second subinterval of second interval of } S_2 \end{cases}$

...

If $a_{n+1} = \begin{cases} 0, & \text{if we picked the first interval of previous interval} \\ 1, & \text{if we picked the second interval of previous interval} \end{cases}$

This obtains a sequence of rested closed intervals, length of the nth interval is $(\frac{1}{3})^n$

Intersection of sequence of intervals consist of 1 point: $x_n \in S$

$$a = \{a_n\} \to x_a \in S$$

this obtains a one to one correspondence between S and the set of sequences.

Recall: Set of sequences of 0s and 1s is uncountable

16 FRIDAY, MARCH 16, 2018

17 Monday, March 19, 2018

17.1 Topology of real numbers

Example 17.1. Open sets $U_1 \supseteq U_2 \supseteq ... \supseteq U_n \supseteq ....$ such that $\bigcap U_n \neq \varnothing$, is it closed?

Take $U_n = (-\frac{1}{n}, \frac{1}{n})$. $\bigcap_n^\infty U_n = \{0\}$, this is nonempty and closed.

Example 17.2. Let $U_1 \subseteq U_2 \subseteq ... \subseteq U_n \subseteq ...$ be open sets and suppose $S_n = \mathbb{R} \setminus U_n$ bounded, $S_n \neq \varnothing$
Claim: $U = \bigcup_{n=1}^\infty U_n \neq \mathbb{R}$.
We're looking for an example of an open set U such that $S = \mathbb{R} \setminus U$ is bounded and nonempty.

$U = (-\infty, 0) \cup (1, \infty)$
$S = [0, 1]$

(Equivalent claim:) $\mathbb{R} \setminus U = \cap_{n=1}^\infty S_n$ where $S_1 \supseteq S_2 \supseteq ... \supseteq S_n \supseteq ...$ and S_n is bounded and closed $\implies S_n$ compact

17.2 More on the exponential function

Example 17.3.

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

1. convergence of the integral if $\int_0^\infty e^{-x^2} dx$ converges then

$$\int_{-\infty}^{\infty} e^{-x^2} dx = 2\int_0^{\infty} e^{-x^2} dx = \int_0^{\infty} e^{-x^2} dx + \int_{-\infty}^{0} e^{-x^2} dx$$

$$e^{-x^2} \to 0 \text{ as } x \to \infty$$

$$e^t = 1 + t + \frac{t^2}{2} + ... \implies e^t > t, \forall t \geq 0$$

$$e^{x^2} > x^2 \implies e^{-x^2} < x^{-2}$$

$$\int_1^a e^{-x^2}dx < \int_1^a x^{-2}dx = -[\frac{1}{x}]_1^a = 1 - \frac{1}{a} \to 1, a > 1$$

So therefore

$$\lim_{a\to\infty}\int_1^a e^{-x^2}dx \text{ exists}$$

i.e. $\int_1^\infty e^{-x^2}dx$ converges $\implies \int_0^\infty e^{-x^2}dx$ converges

Example 17.4. Compute

$$\begin{aligned}(\int_{-\infty}^{\infty} e^{-x^2}dx) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2}\cdot e^{-y^2}dxdy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\end{aligned}$$

Here we use polar coordinates:

$$r^2 = x^2 + y^2, dxdy = rdrd\theta$$

$$\int_0^{2\pi}\int_0^{\infty} e^{-r^2}rdrd\theta$$

And since

$$\frac{d}{dr}e^{-r^2} = -2re^{-r^2} \implies$$

$$\int_0^{2\pi}\int_0^{\infty} e^{-r^2}rdrd\theta = 2\pi(\int_0^{\infty} e^{-r^2}rdr) = \pi$$

18 Friday, April 6, 2018

18.1 the Limit of the Supremum and the Limit of the Infimum

Note. *If S_n is not bounded above, then the limit of the Supremum of S_n is ∞*
If S_n is not bounded below, then the limit of the Infimum of S_n is $-\infty$

Consider the bounded sequence $\{S_n\}$.
Let $A_N = \sup\{S_n | n > N\}$ A_N forms a decreasing Sequence and converges, define

$$\lim_{n\to\infty} \sup S_n = \lim_{N\to\infty} A_N$$

Similarly, $B_N = \inf\{S_n | n > N\}$, B_N is an increasing sequence bounded above, Let

$$\lim_{n\to\infty} \inf S_n = \lim_{N\to\infty} B_N$$

Example 18.1.

$$S_n = (-1)^n, \lim_{n\to\infty} \sup S_n = 1, \lim_{n\to\infty} \inf S_n = -1$$

Example 18.2.

$$S_n = n(1 + (-1)^n) = \begin{cases} 2n, & \text{if } n \text{ even} \\ 0, & \text{if } n \text{ odd} \end{cases}, \lim_{n\to\infty} \sup S_n = \infty, \lim_{n\to\infty} \inf S_n = 0$$

Example 18.3. Let $S_n = \sin(\frac{n\pi}{3})$. The only possible solutions are $\pm\frac{\sqrt{3}}{2}, 0$, so

$$\lim_{n\to\infty} \sup S_n = \frac{\sqrt{3}}{2}, \lim_{n\to\infty} \inf S_n = -\frac{\sqrt{3}}{2}$$

Proposition 18.4. For any sequence S_n, there exists a subsequence, whose limit is $lim_{n\to\infty} \sup S_n$ and one whose limit is $lim_{n\to\infty} \inf S_n$

Proof. Assume $\lim_{n\to\infty} \sup S_n \neq \infty$. Let $a = \lim_{n\to\infty} \sup S_n$, so $a = \lim_{N\to\infty} A_N$
So given K, there exists M, such that for $N \geq M, |a_k - a| < \frac{1}{k}$

Now, $A_m = \sup_{n>M}\{s_n\} \implies$ There exists $m_k > M_k$ such that $|A_m - S_{m_k}| < \frac{1}{2k}$

$$|S_{n_k} - a| \leq |S_{n_k} - A_m| + |A_m - a| < \frac{1}{k}, S_{n_k} \to a \text{ as } k \to \infty$$

Analogous construction for $\liminf S_n$ ■

Remark. *For any N, $B_N \leq A_N$*

$$\implies \lim_{n\to\infty} \inf S_n = \lim_{n\to\infty} B_n \leq \lim_{n\to\infty} A_n = \lim_{n\to\infty} \sup S_n$$

Proposition 18.5. Suppose $\{S_n\}$ converges to a, then

$$\lim_{n\to\infty} \sup S_n = a = \lim_{n\to\infty} \inf S_n$$

Proof. Since $\{S_n\} \to a$, every subsequence to a in particular, a substance which converges to $\lim_{n\to\infty} \sup S_n$, converges to $a, \implies a = \lim_{n\to\infty} \sup S_n$ ■

There's something here I'm missing

Proof. let $\{S_{n_k}\}$ be a convergent subsequence, then

$$\lim_{k\to\infty} S_{n_k} = \lim_{k\to\infty} \sup(S_{n_k}) \leq \lim_{n\to\infty} \sup(S_n) \implies \sup(S) \leq \lim_{n\to\infty} \sup(S)$$

But there exists a convergent subsequence whose limit is

$$\lim_{n\to\infty} \sup(S) \implies \sup(S) = \lim_{n\to\infty} \sup(S_n)$$

Similar argument for inf case. ■

18.2 Root Test Revisited

Given a series $\sum_{n=1}^{\infty} a_n$

Proposition 18.6. If

$$A = \lim_{n\to\infty} \sup |a_n|^{\frac{1}{n}} < 1$$

then $\sum_n a_n$ converges. If $A > 1$ then$\sum_n a_n$ diverges.

Proof. Let $A_n = \inf_{n\geq N}\{|a_n|^{\frac{1}{n}}\}$ so

$$\lim_{n\to\infty}\inf |a_n|^{\frac{1}{n}} = \lim_{N\to\infty} A_N$$

Suppose

$$\lim_{n\to\infty}\inf |a_n|^{\frac{1}{n}} < 1$$

Pick a c such that

$$\lim_{n\to\infty}\inf |a_n|^{\frac{1}{n}} < c < 1$$

then there exists an M such that for $N \geq M, A_n < c, since \mathrm{A}_M = \sup_{n\geq M}\{|a_n|^{\frac{1}{n}}$

$$\implies |a_n|^{\frac{1}{n}} < c \text{ for } n \geq M$$

$$\implies |a_n| < c^n \text{ for } n \geq M$$

since $\sum_{n\geq M} c^n$ converges, so does $\sum_{n\geq M} a_n \implies \sum_{n=0}^{\infty} a_n$ converges.

Just do

$$\lim_{n\to\infty}\sup |a_n|^{\frac{1}{n}} > c > 1$$

∎

18.3 Radius of convergent power series

$$\sum_{n=0}^{\infty} a_n x^n, c = \lim_{n\to\infty}\sup |a_n|^{\frac{1}{n}}$$

$R = \frac{1}{c}$ if $c \neq 0$ is the radius of convergence, if $c = 0$, then $R = (-\infty, \infty)$