# MATB42: Multivariable Calculus II
# Lecture Notes

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Pre-reqs are MATB41. Instructor is Eric Moore. If you find any problems in these notes, feel free to contact me at conconjoshua@gmail.com.

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## 1.1 Fourier Expansions

In this section, we will focus on single variable calculus, (so where $f : \mathbb{R} \mapsto \mathbb{R}$)

Let us say that we have a function $f(x)$ and we want to approximate it. We can use an $n$th degree Taylor Polynomial, but this requires that $f(x)$ has at least $n$ derivatives at some point $x_0$ and the $k$th derivative of $f$ ($f^{(k)}(x)$) is determined by properties of $f$ in some neighbourhood of $x_0$, but what about outside this neighbourhood? How can we be certain of the approximation outside of this neighbour.

Our problem here is that Taylor Polynomial may only approximate "near" $x_0$

Now, consider the following function:

$$\Delta(x) = \begin{cases} 1, & \lfloor x \rfloor < x, \lfloor x \rfloor \text{ is odd} \\ 0, & \lfloor x \rfloor < x, \lfloor x \rfloor \text{ is even} \end{cases}$$

In this function, Tayler returns either 0 or 1 depending on your choice of $x_0$ and cannot work for an $x_0 = p \in \mathbb{Z}$. Therefore Taylor polynomials cannot reflect the true nature of this function. Taylor provides a "local" approximation, but we want a "global" approximation. We need an approximation that is more precise over an interval at the cost of being not as precise as precise at any particular $x_0$.

Note that the example function is **periodic**.

**Definition 1.1.** A function $y = f(x)$ such that $f(x) = f(x + p), p \neq 0, \forall x$ is said to be **periodic** of period $p$

**Example 1.2.** The periodic function $\Delta(x)$ is of period 2.

What we want is a global approximation of a periodic function, and the Fourier Approximation will be periodic, so we can use it for exactly that.

**Definition 1.3.** A **trigonometric polynomial of degree** $N$ is an expression of the form

$$\frac{a_0}{2} + \sum_{k-1}^{N} a_k cos(kx) + b_k sin(kx)$$

where the $a_i, b_i$ are constants.

We know that $sin(x)$ and $cos(x)$ are the simplest periodic functions and repeat in intervals of $2\pi$, so $cos(kx)$ and $sin(kx)$ have period $\frac{2\pi}{k}$, but the smallest shared period is $2\pi$. If a trigonometric polynomial has period $2\pi$ and $f(x)$ has period $p$, then we must set $x = \frac{pt}{2\pi}$ to fix the period (where $t$ is a variable).

So to approximate $y = f(x)$ by $F_N(x)$ for some $N$, we use the following equation:

$$F_N(x) = \frac{a_0}{2} + \sum_{k-1}^{N} a_k cos(kx) + b_k sin(kx)$$

Now we need to choose the $a_k, b_k$. We can define it in the following way:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)cos(kx)dx, k = 1, 2, 3, ...$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)sin(kx)dx, k = 1, 2, 3, ...$$

When defined in this way, $a_i, b_i$ are called the **Fourier Coefficients** of $f$ over the interval $[-\pi, \pi]$ and we call $F_N(x)$ the **Fourier Polynomial of degree** $N$.

So why do we add the $\frac{a_0}{2}$? It is the average value of $f$ over $[-\pi, \pi]$.

**Note.** *sometimes you will see* $a_0$ *used instead of* $\frac{a_0}{2}$ *in the Fourier polynomial where*

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$$

**Example 1.4.** Consider $f(x) = \frac{-x}{2}$ over $[-\pi, \pi]$. Use Fourier Approximation.

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)cos(kx)dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(-\frac{x}{2})cos(kx)dx \overset{odd}{=} 0$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(-\frac{x}{2})dx \overset{odd}{=} 0$$

$$\begin{aligned}
b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)sin(kx)dx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi}(-\frac{x}{2})sin(kx)dx \\
&\overset{even}{=} -\frac{1}{\pi}\int_{-\pi}^{\pi} xsin(kx)dx \\
&\underset{u=x,dv=sin(kx)dx}{\overset{even}{=}} -\frac{1}{\pi}[-\frac{1}{k}xcos(kx) + \frac{1}{k^2}sin(kx)]_0^{\pi} \\
&= \frac{1}{\pi k}[\pi cos(k\pi)] \\
&= \frac{1}{k}cos(k\pi) \\
&= \frac{(-1)^k}{k}
\end{aligned}$$

Thus we have:

$$F_N(x) = -sin(x) + \frac{1}{2}sin(2x) - \frac{1}{3}sin(3x) + \frac{1}{4}sin(4x) + ...$$

$$F_1(x) = -sin(x)$$

$$F_2(x) = -sin(x) + \frac{1}{2}sin(2x)$$

$$F_3(x) = -sin(x) + \frac{1}{2}sin(2x) - \frac{1}{3}sin(3x)$$

$$...$$

And so on.