MATB43: Introduction to Analysis Lecture Notes

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1.1 Countability

Countability is all about the cardinality of sets

Definition 1.1. For sets X, Y, they have the same cardinality if there exists a bijection between them ($f : X \mapsto Y$ has an inverse as well). Equality of cardinality of 2 sets is written as $card(X) = card(Y)$

Example 1.2. $card(\{1, 2, 3, 4, 5\}) = card(\{a, b, c, d, e\})$

Example 1.3. $card(\mathbb{N}) = card(2\mathbb{N})$

This example is correct because $f : \mathbb{N} \mapsto 2\mathbb{N}, f(n) = 2n$ is a bijection.

Example 1.4. $card(\{\text{set of odd numbers}\}) = card(\mathbb{N})$

This example is correct because $g : \mathbb{N} \mapsto \{\text{set of odd numbers}\}$
$g(n) = 2n + 1, n \in \mathbb{N}$

Definition 1.5. X is **finite** if for some $n \in \mathbb{N}$ there exists a bijection where $A = \{1, 2, ..., n\}$ $f : A \mapsto X$. So essentially $card(A) = card(X)$

Intuitively, we can label the elements of X as $X = \{x_1, x_2, ..., x_n\}$

Definition 1.6. X is **inifinite** if X is not finite.

Examples of this include: $\mathbb{N}, 2\mathbb{N}, \mathbb{Q}, \mathbb{R}$
However, not all infinite sets have the same cardinality.

Definition 1.7. X is **countable** if $card(X) = card(A)$ where $A \subseteq \mathbb{N}$

Again intuitively, we can label the elements of X as $X = \{x_1, x_2, ..., x_n, ...\}$ (using elements of $\mathbb{N}$)

Example 1.8. $2\mathbb{N}$

Example 1.9. The set of odd numbers

Example 1.10. $\mathbb{Z}$

Theorem 1.11. *In general, if $Z = X \cup Y$ and Y, X are both countable then Z is countable as well*

Proof. label elements of $X = \{x_1, x_2, ..., x_n, ...\}$
label elements of $Y = \{y_1, y_2, ..., y_n, ...\}$
define $h : \mathbb{N} \mapsto \mathbb{Z}$ as, for $n \in \mathbb{N}$:

$$h(2n-1) = x_n$$

$$h(2n) = y_n$$

Then h is a bijection, therefore Z is also countable ■

Example 1.12. $\mathbb{N} \times \mathbb{N}$

This example is countable as we can label its elements in the following pattern.

$$(1,1) \mapsto 1$$

$$(2,1) \mapsto 2$$

$$(1,2) \mapsto 3$$

$$(1,3) \mapsto 4$$

$$(2,2) \mapsto 5$$

$$(3,1) \mapsto 6$$

$$(4,1) \mapsto 7$$

And so on, in this pattern. Intuitively, if you list out all the pairs of $\mathbb{N} \times \mathbb{N}$ like a matrix, this would create a sort of zig zag pattern.

Proposition 1.13. Suppose X is countable and $Y \subset X$, then Y is either finite (which also means that it is countable) or just countable

Proof. write $X = \{x_1, x_2, ..., x_n, ...\}$
let j_1 be the smallest index such that $x_{j_1} \in Y$
let j_2 be the smallest index such that $x_{j_2} \in Y, j_2 > j_1$
let j_3 be the smallest index such that $x_{j_3} \in Y, j_3 > j_2$
...
and so on.

Now either:
This process terminates, which means that $Y = \{x_{j_1}, ..., x_{j_n}\}$ for some

$j_1, ..., j_n$ which Y is finite.

or:
Y is countable, since $Y = \{x_{j_1}, x_{j_2}, ..., x_{j_n}, ...\}$ ■

Proposition 1.14. $\mathbb{Q}$ is countable

Proof. Note that $\mathbb{Q} = \mathbb{Q}^- \cup \{0\} \cup \mathbb{Q}^+$
so we must check that $\mathbb{Q}^+$ is countable, note that

$$\mathbb{Q}^+ = \{\frac{m}{n} | m, n \in \mathbb{N}, n \neq 0\}$$

Since it is a pairing of two natural numbers (m, n), this is a subset of $\mathbb{N} \times \mathbb{N}$, which means that $\mathbb{Q}^+$ is countable, we can say that same thing for $\mathbb{Q}^-$ as well, note that it is

$$\mathbb{Q}^+ = \{\frac{-m}{n} | m, n \in \mathbb{N}, n \neq 0\}$$

and since $\mathbb{Q}^+$ and $\mathbb{Q}^-$ are both countable and $\{0\}$ is a set of cardinality 1, therefore $\mathbb{Q}$ is countable ■

Theorem 1.15. *Let $S = \{s = (s_1, s_2, ...) | s_j = 0 \text{ or } 1 \forall j\}$ (note here that s is a sequence). Then S is not countable*

Proof. For proof by contradiction, suppose S is countable.
We can then label the elements of S as $S^1, S^2, ..., S^n, ... \in S$.
So an example of this would be:

$$S^1 = (s_1^1, s_2^1, s_3^1, ..., s_n^1, ...)$$
$$S^2 = (s_1^2, s_2^2, s_3^2, ..., s_n^2, ...)$$
$$...$$
$$S^m = (s_1^m, s_2^m, s_3^m, ..., s_n^m, ...)$$
$$...$$

And we will define a $t \in S$ as follows, let $t = (t_1, t_2, ..., t_m, ...)$

$$t_1 = \begin{cases} 0, & \text{if } S_1^1 = 1 \\ 1, & \text{if } S_1^1 = 0 \end{cases}$$

$$t_2 = \begin{cases} 0, & \text{if } S_2^2 = 1 \\ 1, & \text{if } S_2^2 = 0 \end{cases}$$
$$\dots$$
$$t_m = \begin{cases} 0, & \text{if } S_m^m = 1 \\ 1, & \text{if } S_m^m = 0 \end{cases}$$
$$\dots$$

Therefore $t \neq S^m, \forall m$. Therefore this is a contradiction as t was not in the listed elements of S, therefore S is not countable. ■

Corollary 1.16. $\mathbb{R}$ is not countable

Proof. Regard $\mathbb{R}$ as a set of infinite decimal fractions.
Identify $s \in S$ with the decimal number $0.s_1s_2...s_n...$ so S can be regarded as a subset of $\mathbb{R}$. If $\mathbb{R}$ were countable, then S would be countable, therefore $\mathbb{R}$ must not be countable. ■

1.2 Real Numbers

Examples of real numbers that are not rational include: π, e (which are algebraic numbers), $\sqrt{2}, \sqrt{3}, \frac{\sqrt{5}+1}{2}$ (which are roots of polynomial equations with integer coefficients.)

1.3 Algebraic Numbers

The set of Algebraic Numbers is a set $\overline{\mathbb{Q}}$ such that

$$\overline{\mathbb{Q}} = \{x \in \mathbb{R} | x^n + a_{n-1}x^{n-1} + ... + a_0 = 0 \text{ for some } a_0, ..., a_{n-1} \in \mathbb{Z}\}$$

Note that $\mathbb{Q} \subset \overline{\mathbb{Q}}$, and in fact, $\overline{\mathbb{Q}}$ is countable.

if $x \in (\mathbb{R} \setminus \overline{\mathbb{Q}})$ then. x is transcendental.

1.4 Bounds

Definition 1.17. $X \subset \mathbb{R}, X$ is **bounded above** if $\exists a \in \mathbb{R}$ such that $a \geq x, \forall x \in X$

Definition 1.18. $X \subset \mathbb{R}, X$ is **bounded below** if $\exists a \in \mathbb{R}$ such that $a \leq x, \forall x \in X$

Example 1.19. $X = \{x \in \mathbb{R} | x^2 \leq 2\}$ is bounded above by $\frac{3}{2}$ since $(\frac{3}{2})^2 \geq 2$

Definition 1.20. a is the **least upper bound** of X if a is an upper bound of X and if $b < a$, then there exists $x \in X$ such that $x > b$.

We write $a = lub(X)$ or $a = sup(X)$ if a is the least upper bound of X

Definition 1.21. a is the **greatest lower bound** of X if a is an lower bound of X and if $b > a$, then there exists $x \in X$ such that $x < b$.

Example 1.22. if $X = \{x | x^2 \leq 2\}$, then $sup(X) = \sqrt{2} \notin \mathbb{Q}$

Property 1.23. if $X \subset \mathbb{R}$ is bounded above then there exists $a \in \mathbb{R}$, a least upper bound of X

Theorem 1.24. *given $a, b \in \mathbb{R}$ $a, b > 0$ $\exists n \in \mathbb{N}$ such that $na > b$ (This is known as the **archemedian property**)*

Proof. Suppose that $\forall n \in \mathbb{N}$, $na \leq b$ implies that b is an upper bound for $X = \{na | n \in \mathbb{N}\}$.

Since X is bounded above, let $c = sup(X)$, this implies that $c - a$ is not an upper bound for X, which further implies that there exists an $a \in \mathbb{N}$ such that $na > c - a$, and that

$$na > c - a$$
$$na + a > c$$
$$(n+1)a > c$$
$$c < (n+1)a \in X$$

This is impossible since we've previously stated that c is the upper bound. Therefore X is not bounded above. ∎

Theorem 1.25. *given $c, d \in \mathbb{R}$, there exists $q = \frac{m}{n} \in \mathbb{Q}$ such that $c < q < d$*

Proof. We want $c < \frac{m}{n} < d$ iff $nc < m < nd$.
let $\epsilon = d - c > 0$
pick $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$
pick m, such that $m > nc$
(since we also need $m < nd$, choose m to be as small as possible such that $m - 1 \leq nc < m$)

$$m - 1 \leq nc \leq m$$
$$\frac{m}{n} - \frac{1}{n} = \frac{m-1}{n} \leq c \leq \frac{m}{n}$$

or

$$\frac{m}{n} \leq c + \frac{1}{n} < c + \epsilon = d$$

So now we have $c < \frac{m}{n} < d$ as desired.

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