MATB42: Multivariable Calculus II
Lecture Notes

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Pre-reqs are MATB41. Instructor is Eric Moore. If you find any problems in these notes, feel free to contact me at conconjoshua@gmail.com.

Contents

1 Friday, January 5, 2018

1.1 Fourier Expansions

In this section, we will focus on single variable calculus, (so where $f : \mathbb{R} \mapsto \mathbb{R}$)

Let us say that we have a function $f(x)$ and we want to approximate it. We can use an nth degree Taylor Polynomial, but this requires that $f(x)$ has at least n derivatives at some point x_0 and the kth derivative of f $(f^{(k)}(x))$ is determined by properties of f in some neighbourhood of x_0, but what about outside this neighbourhood? How can we be certain of the approximation outside of this neighbour.

Our problem here is that Taylor Polynomial may only approximate "near" x_0

Now, consider the following function:

$$\Delta(x) = \begin{cases} 1, & \lfloor x \rfloor < x, \lfloor x \rfloor \text{ is odd} \\ 0, & \lfloor x \rfloor < x, \lfloor x \rfloor \text{ is even} \end{cases}$$

In this function, Tayler returns either 0 or 1 depending on your choice of x_0 and cannot work for an $x_0 = p \in \mathbb{Z}$. Therefore Taylor polynomials cannot reflect the true nature of this function. Taylor provides a "local" approximation, but we want a "global" approximation. We need an approximation that is more precise over an interval at the cost of being not as precise as precise at any particular x_0.

Note that the example function is **periodic**.

Definition 1.1. A function $y = f(x)$ such that $f(x) = f(x + p), p \neq 0, \forall x$ is said to be **periodic** of period p

Example 1.2. The periodic function $\Delta(x)$ is of period 2.

What we want is a global approximation of a periodic function, and the Fourier Approximation will be periodic, so we can use it for exactly that.

Definition 1.3. A **trigonometric polynomial of degree** N is an expression of the form

$$\frac{a_0}{2} + \sum_{k-1}^{N} a_k cos(kx) + b_k sin(kx)$$

where the a_i, b_i are constants.

We know that $sin(x)$ and $cos(x)$ are the simplest periodic functions and repeat in intervals of 2π, so $cos(kx)$ and $sin(kx)$ have period $\frac{2\pi}{k}$, but the smallest shared period is 2π. If a trigonometric polynomial has period 2π and $f(x)$ has period p, then we must set $x = \frac{pt}{2\pi}$ to fix the period (where t is a variable).

So to approximate $y = f(x)$ by $F_N(x)$ for some N, we use the following equation:

$$F_N(x) = \frac{a_0}{2} + \sum_{k-1}^{N} a_k cos(kx) + b_k sin(kx)$$

Now we need to choose the a_k, b_k. We can define it in the following way:

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) cos(kx) dx, k = 1, 2, 3, ...$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) sin(kx) dx, k = 1, 2, 3, ...$$

When defined in this way, a_i, b_i are called the **Fourier Coefficients** of f over the interval $[-\pi, \pi]$ and we call $F_N(x)$ the **Fourier Polynomial of degree** N.

So why do we add the $\frac{a_0}{2}$? It is the average value of f over $[-\pi, \pi]$.

Note. *sometimes you will see a_0 used instead of $\frac{a_0}{2}$ in the Fourier polynomial where*

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx$$

Example 1.4. Consider $f(x) = \frac{-x}{2}$ over $[-\pi, \pi]$. Use Fourier Approximation.

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)cos(kx)dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(-\frac{x}{2})cos(kx)dx \stackrel{odd}{=} 0$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(-\frac{x}{2})dx \stackrel{odd}{=} 0$$

$$\begin{aligned}
b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)sin(kx)dx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi}(-\frac{x}{2})sin(kx)dx \\
&\stackrel{even}{=} -\frac{1}{\pi}\int_{-\pi}^{\pi} xsin(kx)dx \\
&\underset{u=x,dv=sin(kx)dx}{\stackrel{even}{=}} -\frac{1}{\pi}[-\frac{1}{k}xcos(kx) + \frac{1}{k^2}sin(kx)]_0^{\pi} \\
&= \frac{1}{\pi k}[\pi cos(k\pi)] \\
&= \frac{1}{k}cos(k\pi) \\
&= \frac{(-1)^k}{k}
\end{aligned}$$

Thus we have:

$$F_N(x) = -sin(x) + \frac{1}{2}sin(2x) - \frac{1}{3}sin(3x) + \frac{1}{4}sin(4x) + ...$$

$$F_1(x) = -sin(x)$$

$$F_2(x) = -sin(x) + \frac{1}{2}sin(2x)$$

$$F_3(x) = -sin(x) + \frac{1}{2}sin(2x) - \frac{1}{3}sin(3x)$$

$$...$$

And so on.

2 Monday, January 8, 2017

continuing from the last lecture...

Example 2.1. (continued from example 1.4)
$f(x) = \frac{-x}{2}$
$F_N(x) = -sin(x) + \frac{1}{2}sin(2x) - \frac{1}{3}sin(3x) + \frac{1}{4}sin(4x) + ...$

This can be extended to a Fourier Series:

$$F_N(x) = \sum_{k=1}^{\infty}(-1)^k \frac{sin(kx)}{k}$$

Definition 2.2. For $f : \mathbb{R} \mapsto \mathbb{R}$, the Fourier Series for f is

$$F(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k cos(kx) + b_k sin(kx)$$

where a_i, b_i are Fourier coefficients.

The Nth degree Fourier Polynomial can be regarded as the Nth partial sum of the series.

We haven't talked about convergence yet, but for now, we will assume the series converges ($f(x) = F(x)$)

Definition 2.3. Function $a_k cos(kx) + b_k sin(kx)$ is the kth harmonic of f. The Fourier Series expresses f in terms of its harmonics.

Note. *(Looking at Harmonics in a Musical Sense):*
the 1st harmonic is the fundamental harmonic of f (the fundamental tone).
The 2nd harmonic is the first overtone.

(completely rewrite this amplitude section)

Definition 2.4. The amplitude of the kth harmonic is

$$A_k = \sqrt{(a_k)^2 + (b_k)^2}$$

And note that

$$a_k = A_k sin\alpha, b_k = A_k cos\alpha$$

Definition 2.5. The energy E of a periodic function f of period 2π is

$$E = \frac{1}{\pi}\int_{-\pi}^{\pi}[f(x)]^2 dx$$

So the energy of the kth harmonic is

$$E = \frac{1}{\pi}\int_{-\pi}^{\pi}[a_k cos(kx) + b_k sin(kx)]^2 dx = (a_k)^2 + (b_k)^2 = (A_k)^2$$

And the energy of the constant term is

$$\frac{1}{\pi}\int_{-\pi}^{\pi}[a_0]^2 dx = 2(a_0)^2$$

So we put $A_0 = \frac{1}{\sqrt{2}}a_0$.

For a "nice" periodic function, we have the following equation:

$$E = A_0^2 + A_1^2 + A_2^2 + ...$$

This is known as the Energy Theorem, and comes from the study of periodic waves.

We can draw a graph of this as A_k^2 against k (This graph is known as the Energy Spectrum of f). It shows how the energy of f is distributed among its harmonics.

Note. *Notice that*

$$E = \frac{1}{\pi}\int_{-\pi}^{\pi}[f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{k=1}^{\infty}(a_k^2 + b_k^2) \text{ Parseval's Equation}$$

Assume a function f of period 2π is the sum of a trigonometric series

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k cos(kx) + b_k sin(kx)) \text{ on the interval } [-\pi, \pi]$$

Multiply by $cos(mx)$ and integrate to get

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)cos(mx)dx &= \frac{a_0^2}{2}\int_{-\pi}^{\pi} cos(mx)dx + \int_{-\pi}^{\pi}[\sum_{k=1}^{\infty}(a_k cos(kx) + b_k sin(kx))]dx \\ &= \frac{a_0^2}{2}\int_{-\pi}^{\pi} cos(mx)dx + \sum_{k=1}^{\infty}(a_k \int_{-\pi}^{\pi} cos(kx)dx + b_k \int_{-\pi}^{\pi} sin(kx)dx)\end{aligned}$$

Note. *Recall the following trigonometric identities:*

1. $cosAcosB = \frac{1}{2}[cos(A+B) + cos(A-B)]$
2. $cosAsinB = \frac{1}{2}[sin(A+B) + sin(A-B)]$
3. $sinAsinB = \frac{1}{2}[cos(A-B) - cos(A+B)]$

3 Friday, January 12, 2018

Continuing from where we left off.

$$\int_{-\pi}^{\pi} f(x)cos(mx)dx = \frac{a_0^2}{2}\int_{-\pi}^{\pi} cos(mx)dx + \int_{-\pi}^{\pi}[\sum_{k=1}^{\infty}(a_k cos(kx) + b_k sin(kx))]dx$$
$$= \frac{a_0^2}{2}\int_{-\pi}^{\pi} cos(mx)dx + \sum_{k=1}^{\infty}(a_k \int_{-\pi}^{\pi} cos(kx)dx + b_k \int_{-\pi}^{\pi} sin(kx)dx)$$

We know the following from trigonometric identities

$$\int_{-\pi}^{\pi} cos(kx)cos(mx)dx = \begin{cases} 0, & k \neq m \\ \pi, & k = m \end{cases}$$

As well as the following from odd function properties

$$\int_{-\pi}^{\pi} cos(kx)dx = 0$$

$$\int_{-\pi}^{\pi} sin(kx)cos(mx)dx = 0$$

So now we get

$$\int_{-\pi}^{\pi} f(x)cos(mx)dx = \frac{a_0^2}{2}\int_{-\pi}^{\pi} cos(mx)dx + \sum_{k=1}^{\infty}(a_k \int_{-\pi}^{\pi} cos(kx)dx + b_k \int_{-\pi}^{\pi} sin(kx)dx)$$
$$= a_m\pi$$
$$\implies a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)cos(mx)dx$$

Example 3.1. Lets take

$$f(x) = \begin{cases} 1, & 0 \leq x < \pi \\ -1, & -\pi \leq x < 0 \end{cases}$$

Note that this is an odd function, therefore $a_k = 0, \forall k \geq 0$. So now lets calculate b_k.

$$\begin{aligned}
b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)sin(kx)dx \\
&\overset{even}{=} \frac{2}{\pi}\int_{0}^{\pi} f(x)sin(kx)dx \\
&= \frac{2}{\pi}\int_{0}^{\pi} sin(kx)dx \\
&= \frac{2}{k\pi}[-cos(kx)]_0^{\pi} \\
&= \begin{cases} \frac{4}{k\pi}, & k \text{ is odd} \\ 0, & k \text{ is even} \end{cases}
\end{aligned}$$

And now lets right out the Fourier Polynomial ($F_N(x)$)
if N is odd:

$$F_N(x) = \frac{4}{\pi}sin(x) + \frac{4}{3\pi}sin(3x) + \frac{4}{5\pi}sin(5x) + ...$$

if N is even:

$$F_N(x) = F_{N-1}(x)$$

We can also write it as a Fourier Series

$$F(x) = \frac{4}{\pi}\sum_{l=0}^{\infty}\frac{sin((2l+1)x)}{2l+1}$$

The energy of the function is

$$E = \frac{1}{\pi}\int_{-\pi}^{\pi}[f(x)]^2dx = \frac{2}{\pi}\int_{0}^{\pi}dx = 2$$

The amplitutde of the kth harmonic is

$$A_k = \sqrt{a_k^2 + b_k^2} = \sqrt{0 + \frac{16}{k^2\pi^2}} = \frac{4}{k\pi}$$

The energy of the kth harmonic is

$$A_k^2 = \frac{16}{k^2\pi^2}$$

Note that for this example, both the energy and the amplitude are 0 at an even k.

Lets now evaluate the energy spectrum:

$$k = 1, E = \frac{16}{\pi^2} \approx 1.62, \frac{1.62}{2} = 0.81 = 81\%$$

$$k = 3, E = \frac{16}{9\pi^2} \approx 0.18, \frac{0.18}{2} = 0.09 = 9\%$$

$$k = 5, E = \frac{16}{25\pi^2} \approx 0.06, \frac{0.06}{2} = 0.03 = 3\%$$

$$k = 7, E = \frac{16}{49\pi^2} \approx 0.03, \frac{0.03}{2} = 0.015 = 1.5\%$$

However, we do not need to exclusively work with the interval $[-\pi, \pi]$ we can even work over any interval of length 2π.

3.1 General Fourier Series (interval of length 2π)

$$a_k = \frac{1}{\pi}\int_c^{c+2\pi} f(x)cos(kx)dx, k = 1, 2, 3, ...$$

$$b_k = \frac{1}{\pi}\int_c^{c+2\pi} f(x)sin(kx)dx, k = 1, 2, 3, ...$$

What about for f if f has period p?

$$f(x + p) = f(x), \forall x, \exists p \neq 0$$

We then substitute $x = \frac{pt}{2\pi}$ which gives a new function $f_p(t) = f(\frac{pt}{2\pi})$ with period 2π. So

$$f_p(t + 2\pi) = f(\frac{p}{2\pi}(t + 2\pi)) = f(\frac{pt}{2\pi} + p) = f(\frac{pt}{2\pi}) = f_p(t)$$

So how about the Fourier Expansion for $f_p(t)$? To find this, we must replace t by $\frac{2\pi x}{p}$ giving for $f(x)$.

$$F(x) = \frac{a_0}{2} + \sum_{k-1}^{\infty} a_k cos(\frac{2nx\pi}{p}) + b_k sin(\frac{2nx\pi}{p})$$

$$a_k = \frac{2}{p}\int_c^{c+p} f(x)cos(\frac{2nx\pi}{p}x)dx, k = 1, 2, 3, ...$$
$$b_k = \frac{2}{p}\int_c^{c+p} f(x)sin(\frac{2nx\pi}{p}x)dx, k = 1, 2, 3, ...$$

For any function defined on $[a, b]$, we can extend f to all of $\mathbb{R}$ as a periodic function. Given a periodic function f_E from f of period $p = b - a$, we now have:

$$a_k = \frac{2}{b-a}\int_a^{b} f(x)cos(\frac{2nx\pi}{b-a}x)dx, k = 1, 2, 3, ...$$
$$b_k = \frac{2}{b-a}\int_a^{b} f(x)sin(\frac{2nx\pi}{b-a}x)dx, k = 1, 2, 3, ...$$

Example 3.2. Take the function $f(x) = x, 0 \leq x < 1$ and extend it with period 1. For $k \neq 0$:

$$\begin{aligned} a_k &= 2\int_0^1 xcos(2k\pi x)dx \\ &\overset{parts}{\underset{u=v,dv=cos(2\pi kx)dx}{=}} 2[\frac{xsin(2k\pi x)}{2\pi k}]_0^1 - \frac{2}{2k\pi}\int_0^1 sin(2k\pi x)dx \\ &= 0 \end{aligned}$$

$$a_0 = 2\int_0^1 xdx = [x^2]_0^1$$

$$\begin{aligned} b_k &= 2\int_0^1 xsin(2k\pi x)dx \\ &\overset{parts}{\underset{u=x,dv=sin(2k\pi x)dx}{=}} 2[\frac{-xcos(2k\pi x)}{2\pi k}]_0^1 + \frac{1}{k\pi}\int_0^1 cos(2k\pi x)dx \\ &= \frac{-cos(2k\pi)}{k\pi} \\ &= \frac{-1}{k\pi} \end{aligned}$$

So the Fourier Series will be

$$F(x) = \frac{1}{2} - \frac{1}{\pi}[sin(2\pi x) + \frac{sin(4\pi x)}{2} + \frac{sin(6\pi x)}{3} + ...]$$

Example 3.3. $f(x) = |x|, -\pi < x \leq \pi$. Since $f(x)$ is even, $b_k = 0, \forall k \in \mathbb{N}$.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|dx \overset{even}{=} \frac{2}{\pi}\int_0^{\pi} xdx = \pi$$

$$\begin{aligned}
a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} |x|cos(kx)dx \\
&\overset{even}{=} \frac{2}{pi}\int_0^{\pi} xcos(kx)dx \\
&\underset{u=x,dv=cos(kx)dx}{\overset{parts}{=}} \frac{2}{\pi}[\frac{xsin(kx)}{k} + \frac{cos(kx)}{k^2}]_0^{\pi} \\
&= \frac{2}{\pi k^2}(cos(k\pi) - 1) \\
&= \begin{cases} 0, & k \text{ is even} \\ \frac{-4}{\pi k^2}, & k \text{ is odd} \end{cases}
\end{aligned}$$

So we end up with the Fourier Series

$$F(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{l=0}^{\infty} \frac{cos((2l+1)x)}{(2l+1)^2}$$

Example 3.4. $f(x) = x, -\pi < x \leq \pi$. Note that since $f(x)$ is odd, $a_k = 0, \forall k \geq 0$

$$\begin{aligned}
b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} xsin(kx)dx \\
&\underset{u=x,dv=sin(kx)dx}{\overset{parts}{=}} \frac{1}{\pi}[\frac{-xcos(kx)}{k}]_{-\pi}^{\pi} + \frac{1}{\pi}\int_{-\pi}^{\pi} cos(kx)dx \\
&= \frac{1}{\pi}[\frac{-\pi cos(k\pi)}{k} + \frac{-\pi cos(-k\pi)}{k}] \\
&= \frac{-2}{k}cos(k\pi) \\
&= \frac{(-1)^{k+1}2}{k}
\end{aligned}$$

And the Fourier Series of this is

$$F(x) = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{2}{k}sin(kx)$$

To get a Fourier cosine series or a Fourier sine series, we need

- an f defined on the interval $[0, a]$, and we must extend this interval to also include $[-a, 0)$ to give an even or odd function on $[-a, a]$
- $f(-t) = f(t), -a \leq t < 0$ for the even extension item$f(-t) = -f(t), -a \leq t < 0$ for the odd extension