PROOF OF MACLAURIN INEQUALITY

1. Introduction

Given n real numbers $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ and $0 \leq k \leq n$, let $s_k(y)$ denote the elementary symmetric means

$$s_k(y) := \frac{1}{\binom{n}{k}} \sum_{1 \leq i_1 < \cdots < i_k \leq n} y_{i_1} \ldots y_{i_k}$$

(thus for instance $s_0(y) = 1$). We call a $n+1$-tuple $(s_0, \ldots, s_n)$ of real numbers *attainable* if it is of the form $(s_0(y), \ldots, s_n(y))$ for some y, or equivalently if the polynomial $\sum_{k=0}^n (-1)^k \binom{n}{k} s_k z^{n-k}$ is monic with all roots real. We have previously proven

Proposition 1.1 (Newton's inequality)**.** *If $(s_0, \ldots, s_n)$ is attainable, then $s_k s_{k+2} \leq s_{k+1}^2$ for all $0 \leq k \leq n$.*

We now prove

Theorem 1.2 (Maclaurin inequality)**.** *Suppose that $(s_0, \ldots, s_n)$ is an attainable tuple with all s_i non-negative. Then $s_\ell^{1/\ell} \leq s_k^{1/k}$ for all $1 \leq k \leq \ell \leq n$.*

Proof. By induction on ℓ it suffices to verify the case $\ell = k + 1$.

Suppose that $s_i = 0$ for some $1 \leq i \leq k$. From the Newton inequality $s_{i-1} s_{i+1} \leq s_i^2$ and non-negativity we conclude that $s_{i+1} = 0$. By induction we conclude that $s_{k+1} = 0$ and the claim is true in this case. Thus we may assume that $s_i \neq 0$ for all $0 \leq i \leq k+1$ (the case $i = 0$ is easy since $s_0 = 1$).

Now write $d_i = s_i / s_{i-1}$ for $1 \leq i \leq k + 1$, then the d_i are positive. The Newton inequality can then be rewritten as

$$d_{i+1} \leq d_i$$

for all $1 \leq i \leq k$, while the Maclaurin inequality

$$s_{k+1}^{\frac{1}{k+1}} \leq s_k^{\frac{1}{k}}$$

is equivalent to

$$s_{k+1}^k \leq s_k^{k+1}$$

which expands to

$$(d_1 \ldots d_{k+1})^k \leq (d_1 \ldots d_k)^{k+1}$$

which simplifies to

$$d_{k+1}^k \leq d_1 \ldots d_k$$

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so it suffices to show that $d_{k+1} \leq d_i$ for all $1 \leq i \leq k$. But this follows from the monotone decreasing nature of the d_i. $\square$