PROOF OF MACLAURIN INEQUALITY

1. Introduction

Given n real numbers $y = (y_1, \dots, y_n) \in \mathbb{R}^n$ and $0 \le k \le n$, let $s_k(y)$ denote the elementary symmetric means

$$s_k(y) := \frac{1}{\binom{n}{k}} \sum_{1 \le i_1 < \cdots < i_k \le n} y_{i_1} \dots y_{i_k}$$

(thus for instance $s_0(y) = 1$).

Lemma 1.1 (Symmetries of attainable tuples)**.** *Let $(s_0, \dots, s_n)$ be an attainable tuple.*

(i) *(Scaling) For any real λ, $(s_0, \lambda s_1, \dots, \lambda^n s_n)$ is attainable.*
(ii) *(Reflection) If $s_n \neq 0$, then $(1, s_{n-1}/s_n, \dots, s_0/s_n)$ is attainable. (In particular, if $|s_n| = 1$, then $\pm(s_n, \dots, s_0)$ is attainable with $\pm$ the sign of s_n.)*
(iii) *(Truncation) If $1 \le \ell \le n$, then $(s_0, \dots, s_\ell)$ is attainable.*

Proof. We can write $s_k = s_k(y_1, \dots, y_n)$ for some real $y_1, \dots, y_n$. The claims (i), (ii) are immediate from the homogeneity identity

$$s_k(\lambda y_1, \dots, \lambda y_n) = \lambda^k s_k(y_1, \dots, y_n)$$

and the reflection identity

$$s_k(1/y_1, \dots, 1/y_n) = s_{n-k}(y_1, \dots, y_n)/s_n(y_1, \dots, y_n)$$

respectively for all $0 \le k \le n$ (note that the non-vanishing of $s_n(y_1, \dots, y_n)$ implies that all the $y_1, \dots, y_n$ are non-zero). To prove (iii), observe from $n - \ell$ applications of Rolle's theorem that the degree ℓ polynomial

$$\frac{\ell!}{n!} \frac{d^{n-\ell}}{dx^{n-\ell}} \prod_{i=1}^{n} (z - y_i) = \sum_{k=0}^{\ell} (-1)^k \binom{\ell}{k} s_k(y_1, \dots, y_n) z^{\ell-k}$$

is monic with all roots real, and hence the tuple $(s_0(y_1, \dots, y_n), \dots, s_\ell(y_1, \dots, y_n))$ is attainable. $\square$

2020 *Mathematics Subject Classification.* 26D15.