## 12.3. 'CST' AND 'LST' ELEMENTS:

4

For the stress analysis of two dimensional objects such as plates and sheets, these objects are idealized into surface elements such as triangular, rectangular and quadratic elements. Among them, the triangular elements are considered as the simplest type of two dimensional elements. Depending upon the number of nodes selected for the analysis, these triangular elements are specified as linear elements or non-linear elements. For the linear element, only ***three corner nodes*** (also called as ***primary nodes***) are considered for analysis as shown in **figure 12.2(a)**. On the other hand, for the non-linear element, apart from the corner nodes, some inner nodes are also taken into account as shown in **figure 12.2(b)**.


Fig. 12.2

***In linear triangular element, the displacement is assumed to vary linearly and hence the strain which is the change of displacement per unit length is constant throughout the element and hence this linear triangular element is called as 'constant strain triangle' whereas in non-linear triangular element the displacements vary non-linearly (i.e., quadratically) in such a way that the strain vary linearly within the element and hence it is called as 'Linear strain triangle' or 'Quadratic triangle'.***

If u, v are the displacements developed along X and Y directions due to applied load, for the constant strain triangle, they are expressed as

$$u = a_1 + a_2 x + a_3 y$$

$$\text{and } v = a_4 + a_5 x + a_6 y$$

On the other hand, the displacements for linear strain triangular element are pressed as

$$u = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 xy + a_6 y^2$$

$$v = a_7 + a_8 x + a_9 y + a_{10} x^2 + a_{11} xy + a_{12} y^2$$

## 12.4. DERIVATION OF SHAPE FUNCTIONS FOR TWO DIMENSIONAL LINEAR ELEMENT RELATED TO VECTOR PROBLEMS (i.e., For constant strain triangular element):

Derivation of shape functions for two dimensional vector problem is very similar to scalar problem except that for scalar problem, only one parameter is considered for analysis like temperature at a specific node whereas in vector problem, two parameters such as displacements along global X and Y directions are analysed. Also the vector problems are mostly dealing with stress and strain items. The linear triangular element selected for analysis is specified as constant strain triangular (CST) element because of producing constant strain at the specified triangle. That is, as per hookes law, the stress (σ) is related with strain

(e) as $\sigma = Ee$ where E is the Young's modulus. For any triangle if $\sigma$ and E are constants then automatically the strain e is also constant in that triangle and hence called as constant strain triangle.

Now consider a three noded linear triangular (CST) element whose nodes may be specified as 1, 2, 3 as shown in **figure 12.3.**

Let $\delta_1, \delta_2, \delta_3$ – Displacements at nodes 1, 2, 3

$u_1, u_2, u_3$ – Components of $\delta_1, \delta_2, \delta_3$ along X-axis.

$v_1, v_2, v_3$ – Components of $\delta_1, \delta_2, \delta_3$ along Y-axis.

$(x_1, y_1), (x_2, y_2), (x_3, y_3)$ – Co-ordinates of nodes 1, 2, 3 respectively

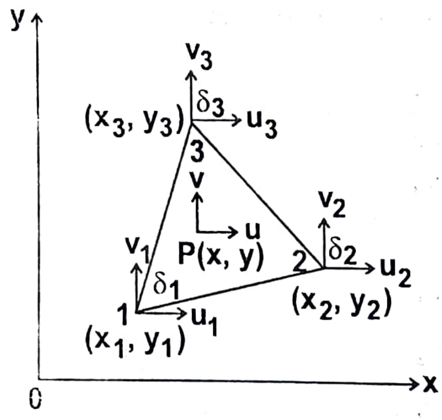



**Fig. 12.3: Nodal displacements of CST element.**

In this vector variable problem, the field variable is $\delta$ and its nodal values are $\delta_1, \delta_2, \delta_3$ or they may be specified by their components as $u_1, v_1, u_2, v_2$ and $u_3, v_3$.

The above nodal displacements can be specified as,

$$\{\delta\} = \begin{Bmatrix} \delta_x \\ \delta_y \end{Bmatrix} = \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{Bmatrix} \quad \text{.... (12.1)}$$

For the linear element, the displacements u and v are linearly varying inside the element and their values at any point (P) inside the element can be expressed by polynomial series as

$$u(x, y) = a_1 + a_2 x + a_3 y \quad \text{.... (12.2)}$$

$$v(x, y) = a_4 + a_5 x + a_6 y \quad \text{.... (12.3)}$$

*(Since the CST element has got two degrees of freedom at each node, there are totally six degrees of freedom at its three nodes and hence we require six polynomial coefficients (also called as **generalised coordinates**, $a_1, a_2 \dots a_6$) for expressing the nodal displacements.)*

Now applying the nodal conditions as

$$u = u_1 \text{ and } v = v_1 \text{ at } x = x_1 \text{ and } y = y_1$$

$$u = u_2 \text{ and } v = v_2 \text{ at } x = x_2 \text{ and } y = y_2$$

$$u = u_3 \text{ and } v = v_3 \text{ at } x = x_3 \text{ and } y = y_3$$

The nodal displacements are expressed as

$$\begin{aligned} u_1 &= a_1 + a_2 x_1 + a_3 y_1 \\ u_2 &= a_1 + a_2 x_2 + a_3 y_2 \\ u_3 &= a_1 + a_2 x_3 + a_3 y_3 \end{aligned} \qquad \text{.... (12.4)}$$

and

$$\begin{aligned} v_1 &= a_4 + a_5 x_1 + a_6 y_1 \\ v_2 &= a_4 + a_5 x_2 + a_6 y_2 \\ v_3 &= a_4 + a_5 x_3 + a_6 y_3 \end{aligned} \qquad \text{.... (12.5)}$$

First we will find the values of polynomial coefficients for u (i.e., $a_1$, $a_2$ and $a_3$) and then for v (i.e., $a_4$, $a_5$ and $a_6$)

Now writing the set of equations (12.4) in matrix form as

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

which can be rewritten as

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}^{-1} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \qquad \text{.... (12.6)}$$

$$\text{(or) } \{a\} = [D]^{-1}\{u\} \qquad \text{.... (12.7)}$$

where,

$$[D] = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \text{ which is } \textit{coordinate matrix.}$$

and

$$[D]^{-1} = \frac{[C]^T}{|D|}$$

where [C] is the ***cofactor matrix*** of [D] and | D | is the ***determinant*** of [D].

Now

$$[C] = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

The values of cofactors $c_{11}$, $c_{12}$ ... etc. can be found out using the relation $c_{ij} = (-1)^{i+j} \, | D_m |$

where $| D_m |$ is the determinant of minor matrix of [D] which can be formed by neglecting $i^{th}$ row and $j^{th}$ column containing the $ij^{th}$ element in matrix D.

The values of cofactors are as follows.

$$c_{11} = (-1)^{(1+1)} \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} = (x_2 y_3 - x_3 y_2)$$

$$c_{12} = (-1)^{(1+2)} \begin{vmatrix} 1 & y_2 \\ 1 & y_3 \end{vmatrix} = -(y_3 - y_2) = y_2 - y_3$$

$$c_{13} = (-1)^{(1+3)} \begin{vmatrix} 1 & x_2 \\ 1 & x_3 \end{vmatrix} = (x_3 - x_2)$$

$$c_{21} = (-1)^{(2+1)} \begin{vmatrix} x_1 & y_1 \\ x_3 & y_3 \end{vmatrix} = -(x_1 y_3 - x_3 y_1) = x_3 y_1 - x_1 y_3$$

$$c_{22} = (-1)^{(2+2)} \begin{vmatrix} 1 & y_1 \\ 1 & y_3 \end{vmatrix} = (y_3 - y_1)$$

$$c_{23} = (-1)^{(2+3)} \begin{vmatrix} 1 & x_1 \\ 1 & x_3 \end{vmatrix} = -(x_3 - x_1) = (x_1 - x_3)$$

$$c_{31} = (-1)^{(3+1)} \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} = (x_1 y_2 - x_2 y_1)$$

$$c_{32} = (-1)^{(3+2)} \begin{vmatrix} 1 & y_1 \\ 1 & y_2 \end{vmatrix} = -(y_2 - y_1) = (y_1 - y_2)$$

$$c_{33} = (-1)^{(3+3)} \begin{vmatrix} 1 & x_1 \\ 1 & x_2 \end{vmatrix} = (x_2 - x_1)$$

The cofactor matrix C is given by

$$[C] = \begin{bmatrix} (x_2 y_3 - x_3 y_2) & (y_2 - y_3) & (x_3 - x_2) \\ (x_3 y_1 - x_1 y_3) & (y_3 - y_1) & (x_1 - x_3) \\ (x_1 y_2 - x_2 y_1) & (y_1 - y_2) & (x_2 - x_1) \end{bmatrix}$$

which implies

$$[C]^T = \begin{bmatrix} (x_2 y_3 - x_3 y_2) & (x_3 y_1 - x_1 y_3) & (x_1 y_2 - x_2 y_1) \\ (y_2 - y_3) & (y_3 - y_1) & (y_1 - y_2) \\ (x_3 - x_2) & (x_1 - x_3) & (x_2 - x_1) \end{bmatrix}$$

The determinant of matrix D is given by

$$|D| = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = 1\,(x_2 y_3 - x_3 y_2) - x_1 (y_3 - y_2) + y_1 (x_3 - x_2)$$

$= 2A$ where A is the area of the triangle calculated through the values of nodal coordinates x and y.

$$\text{i.e., } A = \frac{1}{2} \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

Now the equation 7.7 gives

$$\{a\} = [D]^{-1}\{u\} = \frac{[C]^T}{|D|}\{u\}$$

$$\text{i.e., } \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} (x_2 y_3 - x_3 y_2) & (x_3 y_1 - x_1 y_3) & (x_1 y_2 - x_2 y_1) \\ (y_2 - y_3) & (y_3 - y_1) & (y_1 - y_2) \\ (x_3 - x_2) & (x_1 - x_3) & (x_2 - x_1) \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$= \frac{1}{2A} \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \quad \text{.... (12.8)}$$

where,

$\alpha_1 = x_2 y_3 - x_3 y_2$; $\alpha_2 = x_3 y_1 - x_1 y_3$; $\alpha_3 = x_1 y_2 - x_2 y_1$

$\beta_1 = y_2 - y_3$; $\beta_2 = y_3 - y_1$; $\beta_3 = y_1 - y_2$

$\gamma_1 = x_3 - x_2$; $\gamma_2 = x_1 - x_3$; $\gamma_3 = x_2 - x_1$

Now the equation 12.2 can be rewritten in the matrix form as

$$u(x, y) = [1 \ \ x \ \ y] \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} \qquad \text{.... (12.9)}$$

Substituting the equation 12.8 in equation 12.9, we get

$$u(x, y) = [1 \ \ x \ \ y] \frac{1}{2A} \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$= \frac{1}{2A} \left[ \alpha_1 + \beta_1 x + \gamma_1 y \ \ \ \alpha_2 + \beta_2 x + \gamma_2 y \ \ \ \alpha_3 + \beta_3 x + \gamma_3 y \right] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$= \left[ \frac{\alpha_1 + \beta_1 x + \gamma_1 y}{2A} \ \ \ \frac{\alpha_2 + \beta_2 x + \gamma_2 y}{2A} \ \ \ \frac{\alpha_3 + \beta_3 x + \gamma_3 y}{2A} \right] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$= [N_1 \ \ N_2 \ \ N_3] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

i.e., $u = N_1 u_1 + N_2 u_2 + N_3 u_3$ .... (12.10)

where $N_1$, $N_2$ and $N_3$ are the shape functions for two dimensional triangular element and they are given by

$$N_1 = \frac{\alpha_1 + \beta_1 x + \gamma_1 y}{2A}\ ; \ N_2 = \frac{\alpha_2 + \beta_2 x + \gamma_2 y}{2A}\ ; \ N_3 = \frac{\alpha_3 + \beta_3 x + \gamma_3 y}{2A}$$

If we follow the same procedure for displacement v at the nodes 1, 2, 3, we can get

$$v = N_1 v_1 + N_2 v_2 + N_3 v_3 \qquad \text{.... (12.11)}$$

where $N_1$, $N_2$, $N_3$ are the shape functions and they have the same values as mentioned above.

Combining the equations 12.10 and 12.11, the displacements u and v at any point P inside the element can be expressed as,

$$\text{Displacement } \delta(x, y)_p = \begin{Bmatrix} u(x, y) \\ v(x, y) \end{Bmatrix}_p = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

i.e., $\delta_p = [N]\{\delta\}$ .... (12.12)

**Note:** We can prove that the sum of the shape functions is equal to one.

i.e., $N_1 + N_2 + N_3 = 1$

i.e., $\frac{1}{2A}(\alpha_1 + \beta_1 x + \gamma_1 y) + \frac{1}{2A}(\alpha_2 + \beta_2 x + \gamma_2 y) + \frac{1}{2A}(\alpha_3 + \beta_3 x + \gamma_3 y)$

$$= \frac{1}{2A}\Big[(\alpha_1 + \alpha_2 + \alpha_3) + (\beta_1 + \beta_2 + \beta_3)x + (\gamma_1 + \gamma_2 + \gamma_3)y\Big]$$

$$= \frac{(x_2 y_3 - x_3 y_2 + x_3 y_1 - x_1 y_3 + x_1 y_2 - x_2 y_1) + (y_2 - y_3 + y_3 - y_1 + y_1 - y_2)x + (x_3 - x_2 + x_1 - x_3 + x_2 - x_1)y}{x_2 y_3 - x_3 y_2 - x_1 y_3 + x_1 y_2 + x_3 y_1 - x_2 y_1}$$

$$= \frac{(x_2 y_3 - x_3 y_2 + x_3 y_1 - x_1 y_3 + x_1 y_2 - x_2 y_1) + 0 + 0}{(x_2 y_3 - x_3 y_2 + x_3 y_1 - x_1 y_3 + x_1 y_2 - x_2 y_1)}$$

$= 1$ (Thus proved)

STRAIN RELATIONS: (i.e., Elasticity Equations)

## 12.7. STRAIN-DISPLACEMENT RELATIONSHIP MATRIX (i.e., GRADIENT MATRIX):

From equations 12.31 and 12.33, we have understood the relationship of stresses and strains of two dimensional systems in plane stress and plane strain conditions. Now we will derive the strain-displacement relationship matrix from which we can obtain the stress-displacement relationship also.

From Eqn. 12.12, the displacement produced at any point inside the element is given as

$$\delta(x, y) = \begin{Bmatrix} u(x, y) \\ v(x, y) \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

which can be rewritten as

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3$$

$$v = N_1 v_1 + N_2 v_2 + N_3 v_3$$

The strain which is the ratio of displacement to length can be specified for X and Y axes displacements i.e., for u and v.

The normal strain in X-direction

$$e_x = \frac{\partial u}{\partial x} = \frac{\partial N_1}{\partial x} u_1 + \frac{\partial N_2}{\partial x} u_2 + \frac{\partial N_3}{\partial x} u_3 \quad \text{.... (12.35)}$$

The normal strain in Y-direction,

$$e_y = \frac{\partial v}{\partial y} = \frac{\partial N_1}{\partial y} v_1 + \frac{\partial N_2}{\partial y} v_2 + \frac{\partial N_3}{\partial y} v_3 \quad \text{.... (12.36)}$$

The shear strain in X-Y plane,

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial N_1}{\partial y} u_1 + \frac{\partial N_2}{\partial y} u_2 + \frac{\partial N_3}{\partial y} u_3 + \frac{\partial N_1}{\partial x} v_1 + \frac{\partial N_2}{\partial x} v_2 + \frac{\partial N_3}{\partial x} v_3 \quad \text{.... (12.37)}$$

Writing the equations 12.35, 12.36, 12.37 in matrix form, we get

$$\begin{Bmatrix} e_x \\ e_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial N_1}{\partial x} & 0 & \dfrac{\partial N_2}{\partial x} & 0 & \dfrac{\partial N_3}{\partial x} & 0 \\ 0 & \dfrac{\partial N_1}{\partial y} & 0 & \dfrac{\partial N_2}{\partial y} & 0 & \dfrac{\partial N_3}{\partial y} \\ \dfrac{\partial N_1}{\partial y} & \dfrac{\partial N_1}{\partial x} & \dfrac{\partial N_2}{\partial y} & \dfrac{\partial N_2}{\partial x} & \dfrac{\partial N_3}{\partial y} & \dfrac{\partial N_3}{\partial x} \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} \quad \text{.... (12.38)}$$

(Here $N_1, N_2, N_3$ are the functions of x and y and $u_1, v_1, u_2, v_2, u_3, v_3$ are constant values)

We know that the shape functions are

$$N_1 = \frac{1}{2A}\left(\alpha_1 + \beta_1 x + \gamma_1 y\right)$$

$$N_2 = \frac{1}{2A}\left(\alpha_2 + \beta_2 x + \gamma_2 y\right)$$

$$N_3 = \frac{1}{2A}\left(\alpha_3 + \beta_3 x + \gamma_3 y\right)$$

Now, $\dfrac{\partial N_1}{\partial x} = \dfrac{\beta_1}{2A}$; $\dfrac{\partial N_2}{\partial x} = \dfrac{\beta_2}{2A}$; $\dfrac{\partial N_3}{\partial x} = \dfrac{\beta_3}{2A}$

$\dfrac{\partial N_1}{\partial y} = \dfrac{\gamma_1}{2A}$; $\dfrac{\partial N_2}{\partial y} = \dfrac{\gamma_2}{2A}$; $\dfrac{\partial N_3}{\partial y} = \dfrac{\gamma_3}{2A}$

Substituting the above values in Eqn. 12.38, we get

$$\begin{Bmatrix} e_x \\ e_y \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} \quad \text{.... (12.39)}$$

The above equation can be written in short form as

$$\{e\} = [B]\{\delta\} \quad \text{.... (12.40)}$$

where $[B]$ = Strain-displacement relationship matrix (or) ***Gradient matrix.***

i.e., $$[B] = \frac{1}{2A} \begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix}$$

$\beta_1 = y_2 - y_3$; $\beta_2 = y_3 - y_1$; $\beta_3 = y_1 - y_2$

$\gamma_1 = x_3 - x_2$; $r_2 = x_1 - x_3$; $\gamma_3 = x_2 - x_1$

and $\{\delta\}$ = Nodal displacement vector.

## 12.8. STRESS - DISPLACEMENT RELATIONS:

We know that the stress- strain relations for two dimensional system is

$\{\sigma\} = [D]\{e\}$ (From Eqns. 12.31 & 12.33)

and the strain-displacement relation is

$\{e\} = [B]\{\delta\}$ (From Eqn. 12.40)

Hence the stress-displacement relationship can be obtained as

$$\{\sigma\} = [D][B]\{\delta\} \quad \text{.... (12.41)}$$

With respect to plane stress or plane strain condition, the constitutive matrix [D] can suitably be selected for the above equations 12.41.

That is, for plane stress condition,

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1-\mu^2)} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \left(\frac{1-\mu}{2}\right) \end{bmatrix} \frac{1}{2A} \begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

$$= \frac{E}{(1-\mu^2)\,2A} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \left(\frac{1-\mu}{2}\right) \end{bmatrix} \begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} \quad \text{.... (12.42)}$$

Similarly, for plane strain condition,

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1+\mu)(1-2\mu)} \begin{bmatrix} (1-\mu) & \mu & 0 \\ \mu & (1-\mu) & 0 \\ 0 & 0 & \left(\frac{1-2\mu}{2}\right) \end{bmatrix} \frac{1}{2A} \begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

$$= \frac{E}{(1+\mu)(1-2\mu)(2A)} \underset{(3\times 3)}{\begin{bmatrix} (1-\mu) & \mu & 0 \\ \mu & (1-\mu) & 0 \\ 0 & 0 & \left(\frac{1-2\mu}{2}\right) \end{bmatrix}} \underset{(3\times 6)}{\begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix}} \underset{(6\times 1)}{\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}} \quad .... (12.43)$$

(3 × 3) (3 × 6) → (3 × 6)

(3 × 6) (6 × 1) → (3 × 1) matrix

**Note:**

When any machine component or structure is subjected to two different stresses like normal stress and shear stress, then at some specific planes inside the element, there may be a maximum normal stress, a minimum normal stress and a maximum shear stress. The values of them are,

1. Maximum normal stress,

$$\sigma_1 = \frac{1}{2}\left[(\sigma_x + \sigma_y) + \sqrt{(\sigma_x - \sigma_y)^2 + 4\,\tau_{xy}^2}\right] \quad .... (12.44)$$

2. Minimum normal stress,

$$\sigma_2 = \frac{1}{2}\left[(\sigma_x + \sigma_y) - \sqrt{(\sigma_x - \sigma_y)^2 + 4\,\tau_{xy}^2}\right] \quad .... (12.45)$$

3. Maximum shear stress,

$$\tau_m = \frac{1}{2}(\sigma_1 - \sigma_2) = \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\,\tau_{xy}^2} \quad .... (12.46)$$

4. Principal angle $\theta_p$ can be obtained as

$$\tan 2\theta_p = \frac{2\,\tau_{xy}}{(\sigma_x - \sigma_y)} \text{ (or) } \theta_p = \frac{1}{2}\tan^{-1}\left(\frac{2\,\tau_{xy}}{\sigma_x - \sigma_y}\right) \quad .... (12.47)$$

## 12.9. STIFFNESS MATRIX FOR TWO DIMENSIONAL (CST) ELEMENT:

The stiffness matrix for two dimensional element can be formulated from the general expression (Refer Eqn. 7.21) such as

Stiffness matrix $[K] = \int_V [B]^T [D] [B] \, dV$

$$\text{i.e., } [K] = [B]^T [D] [B] At \quad \text{.... (12.48)}$$

Here $[B]$ = Strain-displacement matrix (or) Gradient matrix for two dimensional element.

$$= \frac{1}{2A}\begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix}$$

$[D]$ = Elasticity matrix or constitutive matrix or stress-strain relationship matrix.

$$= \frac{E}{(1-\mu^2)}\begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \left(\frac{1-\mu}{2}\right) \end{bmatrix} \text{ for plane stress condition}$$

(or)

$$= \frac{E}{(1+\mu)(1-2\mu)}\begin{bmatrix} (1-\mu) & \mu & 0 \\ \mu & (1-\mu) & 0 \\ 0 & 0 & \left(\frac{1-2\mu}{2}\right) \end{bmatrix} \text{ for plane strain condition.}$$

A = Area of triangular element

$$= \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

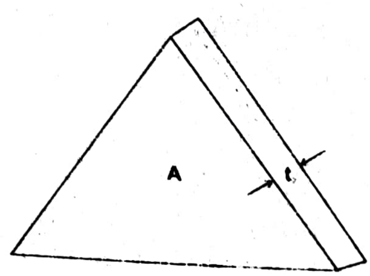



Fig. 12.9(a): CST Element.

t = Thickness of element (Refer Fig. 12.9(a))

E = Modulus of elasticity

$\mu$ = Poisson's ratio.

***The value of t used in Eqn. 12.48 is the actual thickness of the body for plane stress and unity for plane strain.***

## 12.10. TEMPERATURE EFFECTS ON CST ELEMENT:

At the time of function, if the CST element is at a higher temperature or lower temperature than room temperature, the change in temperature $\Delta T$ produces some amount of deformation and the corresponding strain is known as ***thermal strain*** which is considered as initial strain, because this thermal strain is produced before any mechanical loading.

The initial strain (i.e., thermal strain) for plane stress condition is given by

$$\{e_0\} = \begin{Bmatrix} e_{x0} \\ e_{y0} \\ \gamma_{xy0} \end{Bmatrix} = \begin{Bmatrix} \alpha \Delta T \\ \alpha \Delta T \\ 0 \end{Bmatrix} \quad \text{.... (12.49)}$$

and for plane strain condition

$$\{e_0\} = \begin{Bmatrix} e_{x0} \\ e_{y0} \\ \gamma_{xy0} \end{Bmatrix} = (1+\mu) \begin{Bmatrix} \alpha \Delta T \\ \alpha \Delta T \\ 0 \end{Bmatrix} \quad \text{.... (12.50)}$$

where $\alpha$ = Coefficient of thermal expansion

$\Delta T$ = Change in temperature

$\mu$ = Poisson's ratio.

When the CST element is subjected to any mechanical load at the higher or lower temperature, the resultant stress due to mechanical loading and the change of temperature is given by

$$\boxed{\{\sigma_r\} = [D]\left(\{e\} - \{e_0\}\right) = [D]\left([B]\{\delta\} - \{e_0\}\right)} \quad \text{.... (12.51)}$$

where $\{\sigma_r\} = \begin{Bmatrix} \sigma_{x\,r} \\ \sigma_{y\,r} \\ \tau_{xy\,r} \end{Bmatrix}$ ; $\{e\} = \begin{Bmatrix} e_x \\ e_y \\ \gamma_{xy} \end{Bmatrix}$ ; $\{e_0\} = \begin{Bmatrix} e_{x0} \\ e_{y0} \\ \gamma_{xy0} \end{Bmatrix}$

$[D]$ = Constitutive matrix;

$[B]$ = Gradient matrix

$\{\delta\}$ = Nodal displacement vector.

The element thermal load (i.e., thermal force) is given by

$$\boxed{\{F_0\} = [B]^T [D] \{e_0\} At} \quad \text{.... (12.52)}$$

$(6 \times 3)\ (3 \times 3)\ (3 \times 1) \rightarrow (6 \times 1)$ matrix.

where $A$ = Area of CST element

$t$ = Thickness of element

$\{F_0\}$ = Nodal thermal load contribution in X and Y directions.

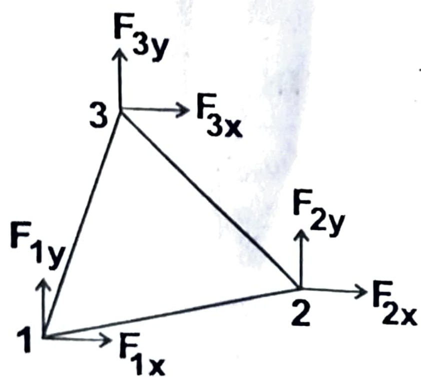
$$= \begin{Bmatrix} F_{01x} \\ F_{01y} \\ F_{02x} \\ F_{02y} \\ F_{03x} \\ F_{03y} \end{Bmatrix} \text{ which is a } (6 \times 1) \text{ matrix.}$$

The above thermal loads must be added to the global mechanical force vector using the connectivity.

The global mechanical force vector is given by

$$\{F\} = \begin{Bmatrix} F_{1x} \\ F_{1y} \\ F_{2x} \\ F_{2y} \\ F_{3x} \\ F_{3y} \end{Bmatrix}$$

The above loading is shown in **figure 12.9(b)**.




1, 2, 3 – Element nodes

**Fig. 12.9(b)**

## 12.11. ISOPARAMETRIC REPRESENTATION:

We know that, for a CST element as shown in fig. 12.10 the displacement at any point P inside the element can be specified as

$u = N_1 u_1 + N_2 u_2 + N_3 u_3$ (From Eqn. 12.10)

and $v = N_1 v_1 + N_2 v_2 + N_3 v_3$ (From Eqn. 12.11)

where $N_1, N_2, N_3$ are shape functions and $u_1, u_2, u_3, v_1, v_2, v_3$ are nodal displacements at X and Y directions.

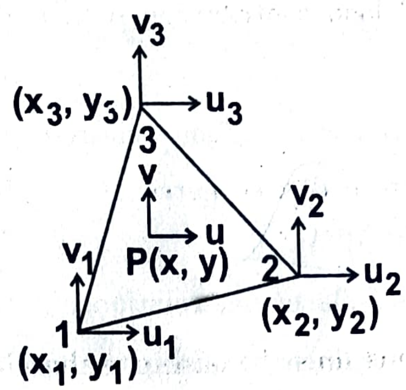



Fig. 12.10

For the triangular element (i.e., CST element), the coordinates x, y of the point P can also be represented in terms of nodal coordinates using the same shape functions. This is referred as ***'Isoparametric representation***.

That is, the coordinates of point P are given by

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3 \qquad \text{.... (12.53)}$$

$$y = N_1 y_1 + N_2 y_2 + N_3 y_3 \qquad \text{.... (12.54)}$$

Also $\quad N_1 + N_2 + N_3 = 1 \Rightarrow N_3 = 1 - N_1 - N_2$

Substituting the above in Eqn. 12.53 we get

$$x = N_1 x_1 + N_2 x_2 + (1 - N_1 - N_2) x_3$$

i.e., $$x = (x_1 - x_3) N_1 + (x_2 - x_3) N_2 + x_3 \qquad \text{.... (12.55)}$$

Similarly $$y = (y_1 - y_3) N_1 + (y_2 - y_3) N_2 + y_3 \qquad \text{.... (12.56)}$$

## 12.12. ANALYSING PROCEDURE FOR THREE NODED TRIANGULAR ELEMENTS:

In finite element analysis, the continuum or system is idealized into number of two dimensional elements of different shapes such as triangle, rectangle or quadratic elements and one of the elements is separated from the continuum and the displacements, stresses developed due to applied load are analysed in this element and this analysis is extended to other finite elements.

In general, the procedure for the stress analysis of different shapes of elements are almost similar. For the stress analysis of three noded triangular element (i.e., linear triangular element or constant strain triangular element), adopt the procedure as follows.

1. From the given problem, first identify that the problem is plane stress type or plane strain type.

2. Define the location of node of the triangular element in terms of X and Y coordinates.

3. Find the stiffness matrix for the selected triangular element using the formula

$$\boxed{[K] = [B]^T [D] [B] At}$$

where [B] – Strain-displacement relationship matrix

[D] – Stress-strain relationship matrix

A – Area of the triangular element

t – Thickness of the triangular element.

For plane stress condition, adopt the thickness as the given value in the problem and for plane strain condition, assume thickness of element as unity.

4. For evaluating the values of [B], [D], A etc. use suitable formulas such as

$$[B] = \frac{1}{2A}\begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix}$$

$$[D] = \frac{E}{1-\mu^2}\begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \left(\frac{1-\mu}{2}\right) \end{bmatrix} \quad \text{for plane stress condition}$$

$$= \frac{E}{(1+\mu)(1-2\mu)}\begin{bmatrix} (1-\mu) & \mu & 0 \\ \mu & (1-\mu) & 0 \\ 0 & 0 & \left(\frac{1-2\mu}{2}\right) \end{bmatrix} \quad \text{for plane strain condition}$$

where E – Modulus of elasticity, $\mu$ – Poisson's ratio.

and $A = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$ where $x_1, x_2, x_3, y_1, y_2, y_3$ are the locations of nodes 1, 2, 3 in X and Y coordinates.

5. Find the nodal displacements by using the finite element equation as

$$[K]\{\delta\} = \{F\}$$

where $\{\delta\} = \{u_1 \quad v_1 \quad u_2 \quad v_2 \quad u_3 \quad v_3\}^T$ – Displacement vector

$\{F\} = \{F_{1x} \quad F_{1y} \quad F_{2x} \quad F_{2y} \quad F_{3x} \quad F_{3y}\}^T$ – Force vector

6. After finding the nodal displacements, by using the relation as $\{\sigma\} = [D]\{e\} = [D][B]\{\delta\}$, find the element stresses. In the above relation

$$\{\sigma\} = \{\sigma_x \quad \sigma_y \quad \tau_{xy}\}^T$$

$$\{e\} = \{e_x \quad e_y \quad \gamma_{xy}\}^T$$

7. Present the findings in a neat form.

## 12.13. SOLVED PROBLEMS FOR LINEAR TRIANGULAR ELEMENTS:

**Problem 12.1:**

**Evaluate the shape functions $N_1$, $N_2$ and $N_3$ at the interior point P for the triangular element shown in figure 12.11.**

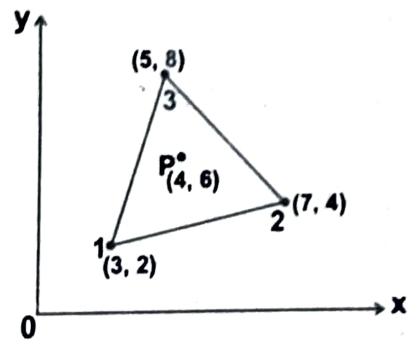



Fig. 12.11

Solution: (Refer Fig. 12.11(a))

For the given triangular element, the coordinates of the nodes are,

$x_1 = 3; \quad y_1 = 2$

$x_2 = 7; \quad y_2 = 4$

$x_3 = 5; \quad y_3 = 8$

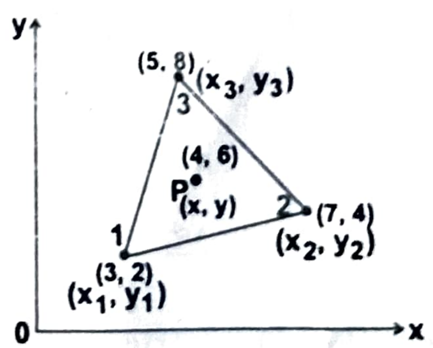



Fig. 12.11(a)

The shape functions $N_1$, $N_2$, $N_3$ at the point P (4, 6) can be evaluated in two ways such as 1. Using the expression of shape functions and 2. by Isoparametric representation method.

**1. Using shape function expression:**

We know that the shape functions for CST element are given by

$$N_1 = \frac{\alpha_1 + \beta_1 x + \gamma_1 y}{2A}; \qquad N_2 = \frac{\alpha_2 + \beta_2 x + \gamma_2 y}{2A};$$

$$N_3 = \frac{\alpha_3 + \beta_3 x + \gamma_3 y}{2A}$$

where,

$$\alpha_1 = x_2 y_3 - x_3 y_2 = (7 \times 8) - (5 \times 4) = 36$$

$$\alpha_2 = x_3 y_1 - x_1 y_3 = (5 \times 2) - (3 \times 8) = -14$$

$$\alpha_3 = x_1 y_2 - x_2 y_1 = (3 \times 4) - (7 \times 2) = -2$$

$$\beta_1 = y_2 - y_3 = 4 - 8 = -4$$

$$\beta_2 = y_3 - y_1 = 8 - 2 = 6$$

$$\beta_3 = y_1 - y_2 = 2 - 4 = -2$$

$$\gamma_1 = x_3 - x_2 = 5 - 7 = -2$$

$$\gamma_2 = x_1 - x_3 = 3 - 5 = -2$$

$$\gamma_3 = x_2 - x_1 = 7 - 3 = 4$$

$$2A = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 2 \\ 1 & 7 & 4 \\ 1 & 5 & 8 \end{vmatrix}$$

$$= 1\,(56 - 20) - 3\,(8 - 4) + 2\,(5 - 7) = 36 - 12 - 4 = 20$$

$$\therefore N_1 = \frac{\alpha_1 + \beta_1 x + \gamma_1 y}{2A} = \frac{36 + (-4)\,4 + (-2)\,6}{20} = \frac{36 - 16 - 12}{20} = \mathbf{0.4}$$

$$N_2 = \frac{\alpha_2 + \beta_2 x + \gamma_2 y}{2A} = \frac{-14 + (6 \times 4) + (-2)\,6}{20} = \frac{-14 + 24 - 12}{20} = \mathbf{-0.1}$$

$$N_3 = \frac{\alpha_3 + \beta_3 x + \gamma_3 y}{2A} = \frac{-2 + (-2)\,4 + (4 \times 6)}{20} = \frac{-2 - 8 + 24}{20} = \mathbf{0.7}$$

**2. Using Isoparametric representation:**

We know that the co-ordinates of any interior point P (x, y) can be represented by the nodal coordinates such as

$$x = (x_1 - x_3)\,N_1 + (x_2 - x_3)\,N_2 + x_3 \qquad \text{.... (1)}$$

$$y = (y_1 - y_3)\,N_1 + (y_2 - y_3)\,N_2 + y_3 \qquad \text{.... (2)}$$

Substituting the coordinate values, we get

Eqn. (1) $\Rightarrow 4 = (3 - 5)\,N_1 + (7 - 5)\,N_2 + 5$

i.e., $4 = -2\,N_1 + 2N_2 + 5$

(or) $2\,N_1 - 2\,N_2 = 1$ .... (3)

Similarly, Eqn. (2) $\Rightarrow 6 = (2 - 8) N_1 + (4 - 8) N_2 + 8$

i.e., $6 = -6 N_1 - 4 N_2 + 8$

(or) $6 N_1 + 4N_2 = 2$ .... (4)

Eqn. (4) + Eqn. (3) $\times 2 \Rightarrow 10 N_1 = 4$

$\therefore N_1 = \frac{4}{10} = \mathbf{0.4}$

Eqn. (3) $\Rightarrow 2N_2 = 2 N_1 - 1 = 2 (0.4) - 1 = 0.8 - 1 = -0.2$

$\therefore N_2 = \frac{-0.2}{2} = \mathbf{-0.1}$

Also $N_1 + N_2 + N_3 = 1$

$\therefore N_3 = 1 - N_1 - N_2 = 1 - 0.4 - (-0.1) = \mathbf{0.7}$

**Result:**

The shape functions at the point P (4, 6) are $\mathbf{N_1 = 0.4}$; $\mathbf{N_2 = -0.1}$ and $\mathbf{N_3 = 0.7}$

**Problem 12.2:**

**The nodal coordinates of a triangular element are shown in figure 12.12. At the point P inside the element, the x-coordinate is 3.3 and the shape function $N_1 = 0.3$. Determine the shape functions $N_2, N_3$ and the y-coordinate of the point P.**

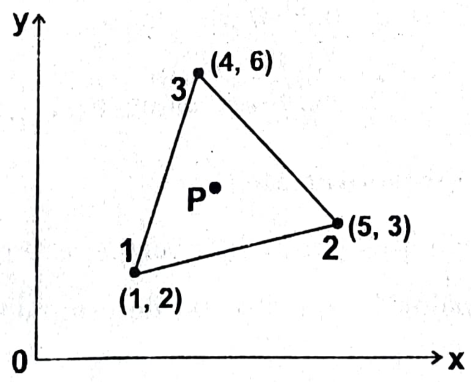



**Fig. 12.12**

**Solution:** (Refer Fig. 12.12(a))

The nodal coordinates are

$x_1 = 1; \quad y_1 = 2$

$x_2 = 5; \quad y_2 = 3$

$x_3 = 4; \quad y_3 = 6$

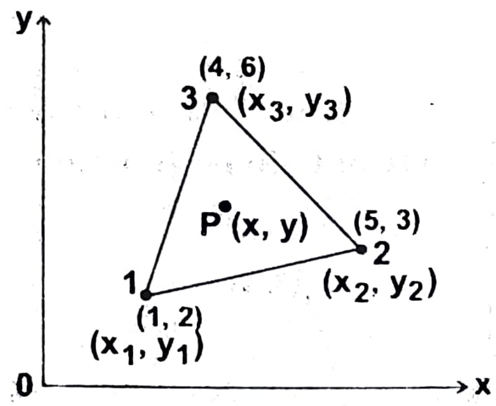



Fig. 12.12(a)

The x-coordinate of the point P is 3.3 and $N_1 = 0.3$.

Using the Isoparametric representation method, we can write the coordinates of any point as

$$x = (x_1 - x_3)\, N_1 + (x_2 - x_3)\, N_2 + x_3 \quad \text{.... (1)}$$

and $$y = (y_1 - y_3)\, N_1 + (y_2 - y_3)\, N_2 + y_3 \quad \text{.... (2)}$$

$(1) \Rightarrow 3.3 = (1 - 4)\, 0.3 + (5 - 4)\, N_2 + 4$

i.e., $3.3 = -0.9 + N_2 + 4$

(or) $N_2 = 3.3 + 0.9 - 4 = \mathbf{0.2}$

Also $N_1 + N_2 + N_3 = 1$

(or) $N_3 = 1 - N_1 - N_2 = 1 - 0.3 - 0.2 = \mathbf{0.5}$

The y-coordinate of point P is obtained using Eqn. (2).

i.e., $y = (2 - 6)\, 0.3 + (3 - 6)\, 0.2 + 6$

$= -1.2 - 0.6 + 6 = \mathbf{4.2}$

**Note:** The y-coordinate of point P can also be determined by another expression such as

$y = N_1\, y_1 + N_2\, y_2 + N_3\, y_3$

$= (0.3 \times 2) + (0.2 \times 3) + (0.5 \times 6) = 0.6 + 0.6 + 3.0 = \mathbf{4.2}$

**Result:** The required shape functions at the point P are

$\mathbf{N_2 = 0.2}$ **and** $\mathbf{N_3 = 0.5}$

and the y-coordinate of point $\mathbf{P = 4.2}$

**Problem 12.3:**

For the point P located inside the triangular element shown in figure 12.13 if the shape functions $N_1$ and $N_2$ are 0.3 and 0.5 respectively, find its x and y-coordinates and the left out shape function.

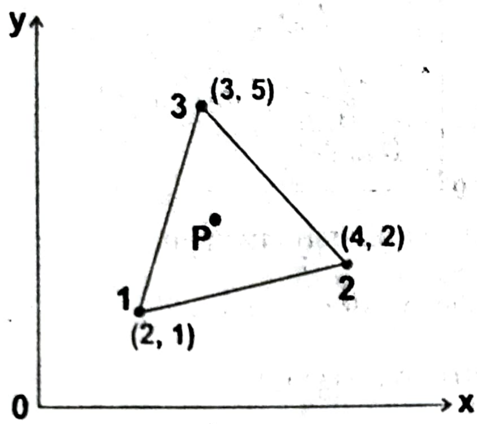



Fig. 12.13

**Solution:** Refer Fig. 12.13(a)

The nodal coordinates are

$x_1 = 2;$ $\quad y_1 = 1$

$x_2 = 4;$ $\quad y_2 = 2$

$x_3 = 3;$ $\quad y_3 = 5$

and $N_1 = 0.3;$ $\quad N_2 = 0.5$

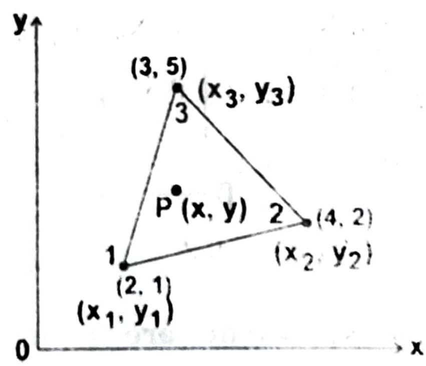



Fig. 12.13(a)

Using Isoparametric representation method the coordinates of the point P inside the element can be written as

$$x = (x_1 - x_3)\, N_1 + (x_2 - x_3)\, N_2 + x_3$$

$$= (2 - 3)\, 0.3 + (4 - 3)\, 0.5 + 3$$

$$= -0.3 + 0.5 + 3 = \mathbf{3.2}$$

and $$y = (y_1 - y_3)\, N_1 + (y_2 - y_3)\, N_2 + y_3$$

$$= (1 - 5)\, 0.3 + (2 - 5)\, 0.5 + 5$$

$$= -1.2 - 1.5 + 5 = \mathbf{2.3}$$

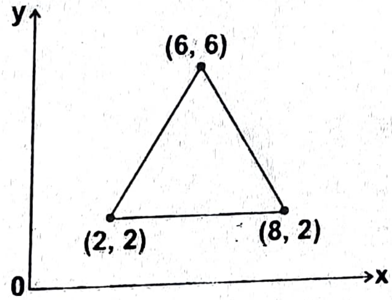
Also, $N_1 + N_2 + N_3 = 1$

$$N_3 = 1 - N_1 - N_2 = 1 - 0.3 - 0.5 = \mathbf{0.2}$$

**Problem 12.6:**

**(i) Distinguish with suitable examples plane stress and plane strain analysis.**

**(ii) For the plane stress element shown in figure 12.16 determine the stiffness matrix. Assume E = 200 GPa. and $\mu$ = 0.3. Thickness = 10 mm.**




**The coordinates are in centimetres**

**Fig. 12.16**

**(Anna University, M.E. Engg. Design, Dec. 2010)**

**Solution:**

(i) For plane stress and plane strain analysis of two dimensional element, refer the article 12.6.2 from 'For two dimensional system ... to... Eqn. 12.34'.

(ii) **Stiffness matrix determination:** (Refer Fig. 12.16(a))

The nodal coordinates of the element are

$x_1 = 2$ cm; $\quad y_1 = 2$ cm

$x_2 = 8$ cm; $\quad y_2 = 2$ cm

$x_3 = 6$ cm; $\quad y_3 = 6$ cm.

and $E = 200 \text{ GPa} = 2 \times 10^{11} \text{ N/m}^2 = 2 \times 10^7 \text{ N/cm}^2$

$\mu = 0.3; \; t = 10 \text{ mm} = 1 \text{ cm}$

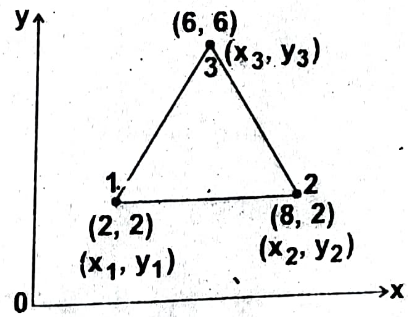



**Fig. 12.16(a)**

For the given triangular element, the stiffness matrix is given by

$$[K] = [B]^T [D] [B] At \qquad \text{.... (1)}$$

where A = Area of the triangle

t = Thickness of the element

[B] = Strain-displacement or Gradient matrix

[D] = Stress-strain or constitutive or ***Hookes law matrix.***

Now

$$\text{Area of the triangle, } A = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 1 & 2 & 2 \\ 1 & 8 & 2 \\ 1 & 6 & 6 \end{vmatrix}$$

$$= \frac{1}{2}[1(48-12) - 2(6-2) + 2(6-8)]$$

$$= \frac{1}{2}(36 - 8 - 4) = 12 \text{ cm}^2 \qquad \text{.... (2)}$$

Strain-displacement matrix,

$$[B] = \frac{1}{2A}\begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix}$$

where $\beta_1 = y_2 - y_3 = 2 - 6 = -4$

$\beta_2 = y_3 - y_1 = 6 - 2 = 4$

$\beta_3 = y_1 - y_2 = 2 - 2 = 0$

$\gamma_1 = x_3 - x_2 = 6 - 8 = -2$

$\gamma_2 = x_1 - x_3 = 2 - 6 = -4$

$\gamma_3 = x_2 - x_1 = 8 - 2 = 6$

Substituting the above values is the gradient matrix, we get

$$[B] = \frac{1}{2\times 12}\begin{bmatrix} -4 & 0 & 4 & 0 & 0 & 0 \\ 0 & -2 & 0 & -4 & 0 & 6 \\ -2 & -4 & -4 & 4 & 6 & 0 \end{bmatrix} = \frac{2}{2\times 12}\begin{bmatrix} -2 & 0 & 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & -2 & 0 & 3 \\ -1 & -2 & -2 & 2 & 3 & 0 \end{bmatrix}$$

$$= \frac{1}{12}\begin{bmatrix} -2 & 0 & 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & -2 & 0 & 3 \\ -1 & -2 & -2 & 2 & 3 & 0 \end{bmatrix}$$

$$\therefore\ [B]^T = \frac{1}{12}\begin{bmatrix} -2 & 0 & -1 \\ 0 & -1 & -2 \\ 2 & 0 & -2 \\ 0 & -2 & 2 \\ 0 & 0 & 3 \\ 0 & 3 & 0 \end{bmatrix}$$

For plane stress element, the stress-strain relationship matrix is given by

$$[D] = \frac{E}{1-\mu^2}\begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \left(\frac{1-\mu}{2}\right) \end{bmatrix} = \frac{2\times 10^7}{1-0.3^2}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & \frac{1-0.3}{2} \end{bmatrix}$$

$$= \frac{2\times 10^7}{0.91}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}$$

Eqn. (1) $\Rightarrow$ $[K] = [B]^T\,[D]\,[B]\,At$

Now,

$$[B]^T\,[D] = \frac{1}{12}\begin{bmatrix} -2 & 0 & -1 \\ 0 & -1 & -2 \\ 2 & 0 & -2 \\ 0 & -2 & 2 \\ 0 & 0 & 3 \\ 0 & 3 & 0 \end{bmatrix}\frac{2\times 10^7}{0.91}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}$$

$$= \frac{1\times 10^7}{6\times 0.91}\begin{bmatrix} -2 & -0.6 & -0.35 \\ -0.3 & -1 & -0.7 \\ 2 & 0.6 & -0.7 \\ -0.6 & -2 & 0.7 \\ 0 & 0 & 1.05 \\ 0.9 & 3 & 0 \end{bmatrix}$$

$\therefore$ $[K] = [B]^T [D] [B] At$

$$= \frac{1\times10^7}{5.46}\begin{bmatrix} -2 & -0.6 & -0.35 \\ -0.3 & -1 & -0.7 \\ 2 & 0.6 & -0.7 \\ -0.6 & -2 & 0.7 \\ 0 & 0 & 1.05 \\ 0.9 & 3 & 0 \end{bmatrix} \frac{1}{12}\begin{bmatrix} -2 & 0 & 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & -2 & 0 & 3 \\ -1 & -2 & -2 & 2 & 3 & 0 \end{bmatrix} (12\times1)$$

$$= \frac{1\times10^7\times12\times1}{5.46\times12}\begin{bmatrix} 4.35 & 1.3 & -3.3 & 0.5 & -1.05 & -1.8 \\ 1.3 & 2.4 & 0.8 & 0.6 & -2.1 & -3.0 \\ -3.3 & 0.8 & 5.4 & -2.6 & -2.1 & 1.8 \\ 0.5 & 0.6 & -2.6 & 5.4 & 2.1 & -6.0 \\ -1.05 & -2.1 & -2.1 & 2.1 & 3.15 & 0 \\ -1.8 & -3.0 & 1.8 & -6.0 & 0 & 9 \end{bmatrix}$$

$$\text{i.e., } [K] = 18.315\times10^5\begin{bmatrix} 4.35 & 1.3 & -3.3 & 0.5 & -1.05 & -1.8 \\ 1.3 & 2.4 & 0.8 & 0.6 & -2.1 & -3.0 \\ -3.3 & 0.8 & 5.4 & -2.6 & -2.1 & 1.8 \\ 0.5 & 0.6 & -2.6 & 5.4 & 2.1 & -6.0 \\ -1.05 & -2.1 & -2.1 & 2.1 & 3.15 & 0 \\ -1.8 & -3.0 & 1.8 & -6.0 & 0 & 9 \end{bmatrix} \text{N/cm}$$

The above stiffness matrix satisfies the conditions that 1) the matrix is symmetric and 2) the sum of values in any row or column is zero. **(Answer)**

**Result:** The thermal force vector is,

$$\{F_0\} = \begin{Bmatrix} F_{01x} \\ F_{01y} \\ F_{02x} \\ F_{02y} \\ F_{03x} \\ F_{03y} \end{Bmatrix} = \begin{Bmatrix} -900 \\ -300 \\ 900 \\ -300 \\ 0 \\ 600 \end{Bmatrix} \text{N}$$

and the element stresses are,

$$\{\sigma_r\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{Bmatrix} -2133.2 \\ -7932.8 \\ 2500.4 \end{Bmatrix} \text{N/cm}^2$$

**Problem 12.13:**

4

**For the two dimensional plate shown in figure 12.23, determine the deflection at the point of load application**

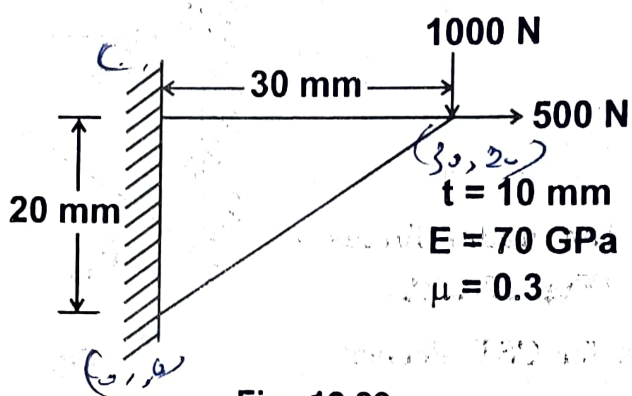



Fig. 12.23

**Solution:** Refer Fig. 12.23(a)

Let the given plate has $u_1, v_1, u_2, v_2$ and $u_3, v_3$ as nodal displacements produced in global X and Y directions as shown.

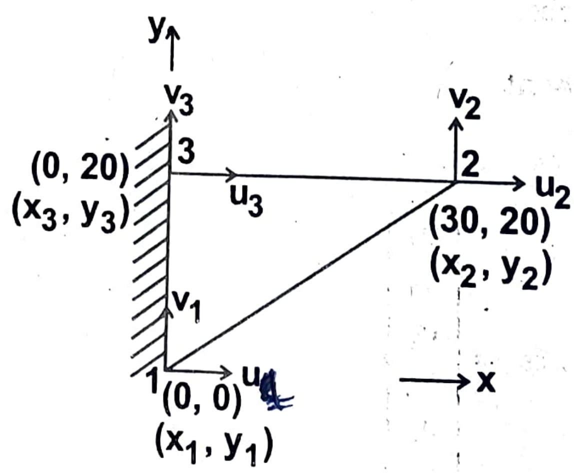



Fig. 12.23(a)

The nodal coordinates of the triangular plate are

$x_1 = 0$ mm; $\quad y_1 = 0$ mm $\quad t = 10$ mm

$x_2 = 30$ mm; $\quad y_2 = 20$ mm $\quad E = 70 \times 10^9 \text{ N/m}^2$

$x_3 = 0$ mm; $\quad y_3 = 20$ mm $\quad = 7 \times 10^4 \text{ N/mm}^2$

$\mu = 0.3$

We know that, for any loaded member, the force applied is related with member displacement through its stiffness property.

i.e., $F = k\,\delta$ (Refer Eqn. 7.19)

Similarly for this CST element, the point load is related with its nodal displacements as

$$\{F\} = [K]\{\delta\} \quad \text{.... (1)}$$

where $\{F\}$ = Nodal force vector (i.e., Point load vector)

$$= \begin{Bmatrix} F_{1x} \\ F_{1y} \\ F_{2x} \\ F_{2y} \\ F_{3x} \\ F_{3y} \end{Bmatrix} \quad \text{.... (2)}$$

and $F_{1x}, F_{1y} \ldots F_{3y}$ are the nodal forces in global X and Y-directions (**Fig. 12.23(b)**)

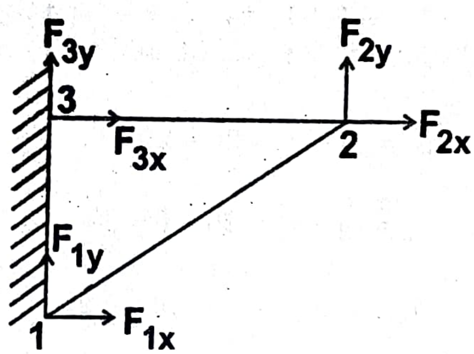



**Fig. 12.23(b)**

$[K]$ = Stiffness matrix for CST element

$$= [B]^T [D] [B] At \quad \text{.... (3)}$$

where

$[B]$ = Strain displacement matrix

$[D]$ = Stress-strain matrix

$A$ = Area of CST element

$t$ = Thickness of the element.

$$\text{and } \{\delta\} = \text{Nodal displacements} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} \quad \text{.... (4)}$$

To find the nodal displacements due to the given nodal forces, first the stiffness matrix must be evaluated

Now,

The strain-displacement matrix is given by

$$[B] = \frac{1}{2A}\begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix}$$

where A = Area of triangle

$$= \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 1 & 0 & 0 \\ 1 & 30 & 20 \\ 1 & 0 & 20 \end{vmatrix} = 300\text{ mm}^2$$

$$\beta_1 = y_2 - y_3 = 20 - 20 = 0$$

$$\beta_2 = y_3 - y_1 = 20 - 0 = 20$$

$$\beta_3 = y_1 - y_2 = 0 - 20 = -20$$

$$\gamma_1 = x_3 - x_2 = 0 - 30 = -30$$

$$\gamma_2 = x_1 - x_3 = 0 - 0 = 0$$

$$\gamma_3 = x_2 - x_1 = 30 - 0 = 30$$

substituting the above values, [B] implies,

$$[B] = \frac{1}{600}\begin{bmatrix} 0 & 0 & 20 & 0 & -20 & 0 \\ 0 & -30 & 0 & 0 & 0 & 30 \\ -30 & 0 & 0 & 20 & 30 & -20 \end{bmatrix} = \frac{1}{60}\begin{bmatrix} 0 & 0 & 2 & 0 & -2 & 0 \\ 0 & -3 & 0 & 0 & 0 & 3 \\ -3 & 0 & 0 & 2 & 3 & -2 \end{bmatrix} \quad \ldots.(5)$$

$$\therefore [B]^T = \frac{1}{60}\begin{bmatrix} 0 & 0 & -3 \\ 0 & -3 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 2 \\ -2 & 0 & 3 \\ 0 & 3 & -2 \end{bmatrix} \quad \ldots.(6)$$

Assuming plane stress condition, the stress-strain displacement matrix,

$$[D] = \frac{E}{(1-\mu^2)}\begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \left(\frac{1-\mu}{2}\right) \end{bmatrix} = \frac{7\times10^4}{1-0.3^2}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & \left(\frac{1-0.3}{2}\right) \end{bmatrix}$$

$$= \frac{7\times10^4}{0.91}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix} = \frac{7\times10^5}{91}\begin{bmatrix} 10 & 3 & 0 \\ 3 & 10 & 0 \\ 0 & 0 & 3.5 \end{bmatrix} \quad \text{.... (7)}$$

Then, the stiffness matrix, $[K] = [B]^T [D] [B] At$

i.e.,

$$[K] = \frac{1}{60}\begin{bmatrix} 0 & 0 & -3 \\ 0 & -3 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 2 \\ -2 & 0 & 3 \\ 0 & 3 & -2 \end{bmatrix} \frac{7\times10^5}{91}\begin{bmatrix} 10 & 3 & 0 \\ 3 & 10 & 0 \\ 0 & 0 & 3.5 \end{bmatrix} \frac{1}{60}\begin{bmatrix} 0 & 0 & 2 & 0 & -2 & 0 \\ 0 & -3 & 0 & 0 & 0 & 3 \\ -3 & 0 & 0 & 2 & 3 & -2 \end{bmatrix}(300\times10)$$

$$(6\times3) \qquad (3\times3) \qquad (3\times6)$$

$$= \frac{7\times10^5\times300\times10}{60\times60\times91}\begin{bmatrix} 0 & 0 & -3 \\ 0 & -3 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 2 \\ -2 & 0 & 3 \\ 0 & 3 & -2 \end{bmatrix}\begin{bmatrix} 0 & -9 & 20 & 0 & -20 & 9 \\ 0 & -30 & 6 & 0 & -6 & 30 \\ -10.5 & 0 & 0 & 7 & 10.5 & -7 \end{bmatrix}$$

$$(6\times3) \qquad (3\times6)$$

$$= \frac{10^6}{156}\begin{bmatrix} 31.5 & 0 & 0 & -21 & -31.5 & 21 \\ 0 & 90 & -18 & 0 & 18 & -90 \\ 0 & -18 & 40 & 0 & -40 & 18 \\ -21 & 0 & 0 & 14 & 21 & -14 \\ -31.5 & 18 & -40 & 21 & 71.5 & -39 \\ 21 & -90 & 18 & -14 & -39 & 104 \end{bmatrix} \quad \text{.... (8)}$$

The above stiffness matrix is symmetric and the sum of all elements in any row or column is zero and also the size of matrix $(6\times6)$ is equal to the number of degrees of freedom (i.e., 6). Hence the above stiffness matrix is correct.

Now, using the equation (1), we get

$$[K]\ \{\delta\} = \{F\}$$

$$\text{i.e.,}\quad \frac{10^6}{156}\begin{bmatrix} 31.5 & 0 & 0 & -21 & -31.5 & 21 \\ 0 & 90 & -18 & 0 & 18 & -90 \\ 0 & -18 & 40 & 0 & -40 & 18 \\ -21 & 0 & 0 & 14 & 21 & -14 \\ -31.5 & 18 & -40 & 21 & 71.5 & -39 \\ 21 & -90 & 18 & -14 & -39 & 104 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} F_{1x} \\ F_{1y} \\ F_{2x} \\ F_{2y} \\ F_{3x} \\ F_{3y} \end{Bmatrix} \qquad \text{.... (9)}$$

Applying the nodal conditions such as

$u_1 = v_1 = u_3 = v_3 = 0$; $F_{2x} = 500$ N; $F_{2y} = -1000$ N in equation (9), we get

$$\frac{10^6}{156}\begin{bmatrix} 31.5 & 0 & 0 & -21 & -31.5 & 21 \\ 0 & 90 & -18 & 0 & 18 & -90 \\ 0 & -18 & 40 & 0 & -40 & 18 \\ -21 & 0 & 0 & 14 & 21 & -14 \\ -31.5 & 18 & -40 & 21 & 71.5 & -39 \\ 21 & -90 & 18 & -14 & -39 & 104 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ u_2 \\ v_2 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} F_{1x} \\ F_{1y} \\ 500 \\ -1000 \\ F_{3x} \\ F_{3y} \end{Bmatrix} \qquad \text{.... (10)}$$

Since $u_1 = v_1 = u_3 = v_3 = 0$; neglecting 1st, 2nd, 5th, 6th rows and columns in stiffness matrix, the remaining terms imply that

$$\frac{10^6}{156}\begin{bmatrix} 40 & 0 \\ 0 & 14 \end{bmatrix} \begin{Bmatrix} u_2 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} 500 \\ -1000 \end{Bmatrix}$$

$$\text{i.e.,}\quad \frac{10^6}{156}(40\ u_2) = 500$$

$$\therefore\ u_2 = \frac{500 \times 156}{10^6 \times 40} = \mathbf{1.95 \times 10^{-3}\ mm}$$

$$\text{Similarly}\quad \frac{10^6}{156}(14\ v_2) = -1000$$

$$\therefore\ v_2 = \frac{-1000 \times 156}{10^6 \times 14} = \mathbf{-11.14 \times 10^{-3}\ mm.}$$

The forces $F_{1x}$, $F_{1y}$, $F_{3x}$, $F_{3y}$ are acting as reaction forces in order to keep the system in equilibrium.

**Result:** The deflections at the point of load application are,

$$\mathbf{u_2 = 1.95 \times 10^{-3}\ mm}$$

$$\mathbf{v_2 = -11.14 \times 10^{-3}\ mm}$$

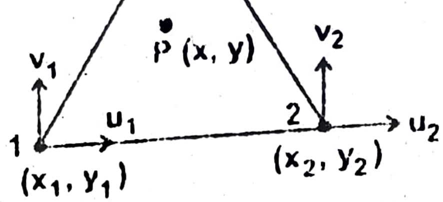



Fig. 12.27

$$\{e\} = [B]\{\delta\}$$

i.e.,
$$\begin{Bmatrix} e_x \\ e_y \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{2A}\begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

where $\beta_1 = y_2 - y_3$; $\beta_2 = y_3 - y_1$; $\beta_3 = y_1 - y_2$

$\gamma_1 = x_3 - x_2$; $\gamma_2 = x_1 - x_3$; $\gamma_3 = x_2 - x_1$

Since the strain displacement matrix [B] is independent of any inside point displacement and also depends only on the nodal points coordinates, the strain is constant throughout the element and hence called as CST element.

**3. Differentiate CST and LST elements:**

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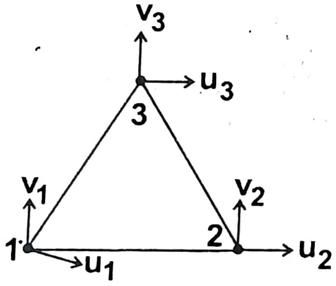
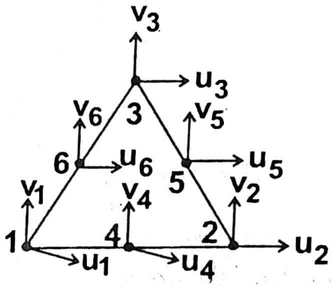
| | CST element | LST element |
|---|---|---|
| 1. | It is a two dimensional linear element (i.e., simplex element) | It is a two-dimensional non-linear element (i.e., complex element) |
| 2. | It has only three primary nodes at the corners (Refer Fig. 12.28(a)) | It has three primary nodes at the corners and three secondary nodes at the midsides. (Refer Fig. 12.28(b)) |
| 3. | The strain is constant throughout the element | The strain is varying linearly inside the element. |

| | CST element | LST element |
|---|---|---|
| 4. | The polynomial functions for CST element are<br>$u = a_1 + a_2 x + a_3 y$<br>and $v = a_4 + a_5 x + a_6 y$<br><br><br>Fig. 12.28(a): CST element | The polynomial functions for LST element are<br>$u = a_1 + a_2x + a_3 y + a_4 x^2 + a_5 xy + a_6 y^2$<br>$v = a_7 + a_8 x + a_9 y + a_{10} x^2 + a_{11} xy + a_{12} y^2$<br><br><br>Fig. 12.28(b): LST element |

where $N_1$, $N_2$, $N_3$ are shape functions for axisymmetric triangular element and they are

$$N_1 = \frac{\alpha_1 + \beta_1 r + \gamma_1 z}{2A} \; ; \; N_2 = \frac{\alpha_2 + \beta_2 r + \gamma_2 z}{2A} \; ; \; N_3 = \frac{\alpha_3 + \beta_3 r + \gamma_3 z}{2A}$$

Assembling the equations 13.17 and 13.18 in matrix form, we get the displacements u and w at any point P inside the element as

$$\delta(r, z)_p = \begin{Bmatrix} u(r, z) \\ w(r, z) \end{Bmatrix}_p = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ w_1 \\ u_2 \\ w_2 \\ u_3 \\ w_3 \end{Bmatrix} \quad \ldots (13.19)$$

.... (13.20)

i.e., $\delta_p = [N] \{ \delta \}$

where $\delta_p$ = Displacements at the point $P = \begin{Bmatrix} u \\ w \end{Bmatrix}_p$

$\{ \delta \}$ = Nodal displacements vector

$[N]$ = Shape functions matrix.

## 13.5. STRESS-STRAIN RELATIONSHIP MATRIX (i.e., CONSTITUTIVE MATRIX) FOR AXISYMMETRIC TRIANGULAR ELEMENT:

Since the axisymmetric solid is the circular three dimensional solid, we can adopt the stress-strain relationship expressions of three dimensional solid for axisymmetric element by replacing x by r, y by θ and leaving z as it is.

Now, for three dimensional solid, the developed strains are given by,

$$e_x = \frac{\sigma_x}{E} - \mu \frac{\sigma_y}{E} - \mu \frac{\sigma_z}{E}$$

$$e_y = -\mu \frac{\sigma_x}{E} + \frac{\sigma_y}{E} - \mu \frac{\sigma_z}{E}$$

$$e_z = -\mu \frac{\sigma_x}{E} - \mu \frac{\sigma_y}{E} + \frac{\sigma_z}{E}$$

$$\gamma_{xz} = \frac{2(1+\mu)}{E} \tau_{xz}$$

Replace x by r; y by θ in the above equations we can get the expressions for axisymmetric solid.

That is, for axisymmetric solid,

$$e_r = \frac{\sigma_r}{E} - \mu \frac{\sigma_\theta}{E} - \mu \frac{\sigma_z}{E}$$

$$e_\theta = -\mu \frac{\sigma_r}{E} + \frac{\sigma_\theta}{E} - \mu \frac{\sigma_z}{E}$$

$$e_z = -\mu \frac{\sigma_r}{E} - \mu \frac{\sigma_\theta}{E} + \frac{\sigma_z}{E}$$

$$\gamma_{rz} = \frac{2(1+\mu)}{E} \tau_{rz} \qquad \text{.... (13.21)}$$

Solving the above set of equations (13.21) we get the stresses for axisymmetric solid as

Radial stress, $\sigma_r = \frac{E}{(1+\mu)(1-2\mu)} \left[ (1-\mu) e_r + \mu e_\theta + \mu e_z \right]$

Circumferential stress, $\sigma_\theta = \frac{E}{(1+\mu)(1-2\mu)} \left[ \mu e_r + (1-\mu) e_\theta + \mu e_z \right]$

Axial stress, $\sigma_z = \frac{E}{(1+\mu)(1-2\mu)} \left[ \mu e_r + \mu e_\theta + (1-\mu) e_z \right]$

Shear stress, $\tau_{rz} = \frac{E}{(1+\mu)(1-2\mu)} \left( \frac{1-2\mu}{2} \right) \gamma_{rz}$ .... (13.22)

Assembling the above equations in matrix form we have

$$\begin{Bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \\ \tau_{rz} \end{Bmatrix} = \frac{E}{(1+\mu)(1-2\mu)} \begin{bmatrix} (1-\mu) & \mu & \mu & 0 \\ \mu & (1-\mu) & \mu & 0 \\ \mu & \mu & (1-\mu) & 0 \\ 0 & 0 & 0 & \left(\frac{1-2\mu}{2}\right) \end{bmatrix} \begin{Bmatrix} e_r \\ e_\theta \\ e_z \\ \gamma_{rz} \end{Bmatrix} \qquad \text{.... (13.23)}$$

(or) $\{\sigma\} = [D]\{e\}$ .... (13.24)

where $\{\sigma\}$ = Stress vector

$\{e\}$ = Strain vector

**[D] = Stress-strain relationship matrix or constitutive matrix.**

And

$$[D] = \frac{E}{(1+\mu)(1-2\mu)} \begin{bmatrix} (1-\mu) & \mu & \mu & 0 \\ \mu & (1-\mu) & \mu & 0 \\ \mu & \mu & (1-\mu) & 0 \\ 0 & 0 & 0 & \left(\frac{1-2\mu}{2}\right) \end{bmatrix} \quad \text{.... (13.25)}$$

## 13.6. STRAIN-DISPLACEMENT MATRIX (i.e., GRADIENT MATRIX):

In the previous article, we derived the stress-strain relationship matrix. Now, we will derive strain-displacement matrix for axisymmetric triangular element.

The displacement functions at any point inside the axisymmetric triangular element are specified as

$$\delta(r, z) = \begin{Bmatrix} u(r, z) \\ w(r, z) \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ w_1 \\ u_2 \\ w_2 \\ u_3 \\ w_3 \end{Bmatrix} \quad \text{(Refer Eqn. 13.19)}$$

which can be rewritten as

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3$$

$$w = N_1 w_1 + N_2 w_2 + N_3 w_3$$

The strains for axisymmetric element are given by

$$\{e\} = \begin{Bmatrix} e_r \\ e_\theta \\ e_z \\ \gamma_{rz} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial r} \\ \frac{u}{r} \\ \frac{\partial w}{\partial z} \\ \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \end{Bmatrix} \quad \text{(Refer Eqn. 13.5)}$$

Now,

$$\text{Radial strain, } e_r = \frac{\partial u}{\partial r} = \frac{\partial N_1}{\partial r} u_1 + \frac{\partial N_2}{\partial r} u_2 + \frac{\partial N_3}{\partial r} u_3 \quad \text{.... (13.26)}$$

$$\text{Circumferential strain, } e_\theta = \frac{u}{r} = \frac{N_1}{r} u_1 + \frac{N_2}{r} u_2 + \frac{N_3}{r} u_3 \quad \text{.... (13.27)}$$

$= \begin{bmatrix} u_1 & w_1 & u_2 & w_2 & u_3 & w_3 \end{bmatrix}^T$.

## 13.8. STIFFNESS MATRIX FOR AXISYMMETRIC TRIANGULAR ELEMENT:

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The stiffness matrix for the axisymmetric triangular element is given by

$$[K] = \int_V [B]^T [D] [B] \, dV$$

$$= [B]^T [D] [B] V$$

$$= [B]^T [D] [B] \, 2\pi rA = 2\pi rA \, [B]^T [D] [B] \qquad \ldots\ldots (13.35)$$

where $r = \dfrac{r_1 + r_2 + r_3}{3}$ (Refer Fig. 13.7)

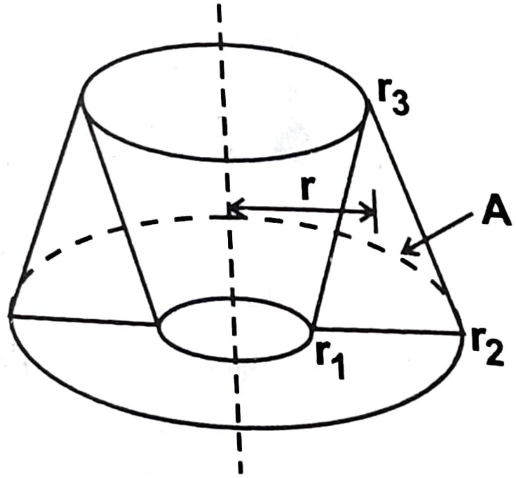


Fig. 13.7

$r_1$, $r_2$, $r_3$ are radial distances of nodes 1, 2, 3.

$$A = \text{Area of the triangle} = \frac{1}{2} \begin{vmatrix} 1 & r_1 & z_1 \\ 1 & r_2 & z_2 \\ 1 & r_3 & z_3 \end{vmatrix}$$

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## .9. SOLVED PROBLEMS:

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### oblem 13.1:

nodal coordinates for an axisymmetric ngular element shown in figure 13.8 are n below.

$r_1 = 20$ mm ; $z_1 = 10$ mm

$r_2 = 40$ mm ; $z_2 = 10$ mm

$r_3 = 30$ mm ; $z_3 = 50$ mm

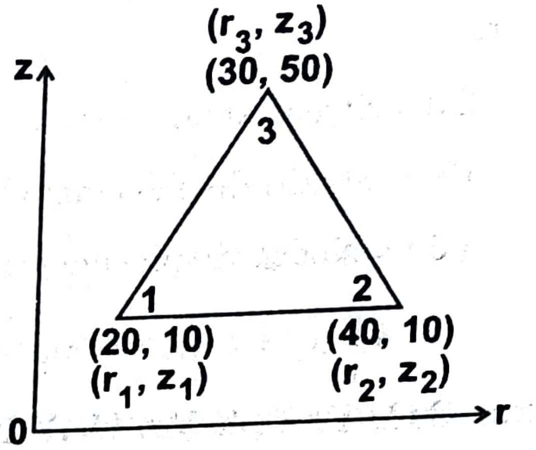


Fig. 13.8

Determine the strain-displacement matrix for that element.

**Solution:** The nodal coordinates are

$r_1 = 20$ mm; $z_1 = 10$ mm

$r_2 = 40$ mm; $z_2 = 10$ mm

$r_3 = 30$ mm; $z_3 = 50$ mm.

The strain-displacement matrix for the axisymmetric triangular element is given by

$$[B] = \frac{1}{2A}\begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ \left(\frac{\alpha_1 + \beta_1 r + \gamma_1 z}{r}\right) & 0 & \left(\frac{\alpha_2 + \beta_2 r + \gamma_2 z}{r}\right) & 0 & \left(\frac{\alpha_3 + \beta_3 r + \gamma_3 z}{r}\right) & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix}$$

where A = Area of the triangular element

$$= \frac{1}{2}\begin{vmatrix} 1 & r_1 & z_1 \\ 1 & r_2 & z_2 \\ 1 & r_3 & z_3 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 1 & 20 & 10 \\ 1 & 40 & 10 \\ 1 & 30 & 50 \end{vmatrix}$$

$$= \frac{1}{2}[\,1\,(2000 - 300) - 20\,(50 - 10) + 10\,(30 - 40)\,]$$

$$= \frac{1}{2}(1700 - 800 - 100) = 400 \text{ mm}^2$$

Now $\alpha_1 = r_2 z_3 - r_3 z_2 = (40 \times 50) - (30 \times 10) = 1700$

$\alpha_2 = r_3 z_1 - r_1 z_3 = (30 \times 10) - (20 \times 50) = -700$

$\alpha_3 = r_1 z_2 - r_2 z_1 = (20 \times 10) - (40 \times 10) = -200$

$\beta_1 = z_2 - z_3 = 10 - 50 = -40$

$\beta_2 = z_3 - z_1 = 50 - 10 = 40$

$\beta_3 = z_1 - z_2 = 10 - 10 = 0$

$\gamma_1 = r_3 - r_2 = 30 - 40 = -10$

$\gamma_2 = r_1 - r_3 = 20 - 30 = -10$

$\gamma_3 = r_2 - r_1 = 40 - 20 = 20$

The coordinates,

$$r = \frac{r_1 + r_2 + r_3}{3} = \frac{20 + 40 + 30}{3} = \frac{90}{3} = 30 \text{ mm.}$$

$$z = \frac{z_1 + z_2 + z_3}{3} = \frac{10 + 10 + 50}{3} = \frac{70}{3} = 23.3 \text{ mm.}$$

Now
$$\frac{\alpha_1 + \beta_1 r + \gamma_1 z}{r} = \frac{1700 + (-40)\,30 + (-10)\,23.3}{30} = 8.9$$

$$\frac{\alpha_2 + \beta_2 r + \gamma_2 z}{r} = \frac{-700 + (40 \times 30) + (-10)\,23.3}{30} = 8.9$$

$$\frac{\alpha_3 + \beta_3 r + \gamma_3 z}{r} = \frac{-200 + 0 + 20\,(23.3)}{30} = 8.9$$

Hence, the strain-displacement matrix

$$[B] = \frac{1}{800}\begin{bmatrix} -40 & 0 & 40 & 0 & 0 & 0 \\ 8.9 & 0 & 8.9 & 0 & 8.9 & 0 \\ 0 & -10 & 0 & -10 & 0 & 20 \\ -10 & -40 & -10 & 40 & 20 & 0 \end{bmatrix}$$ **Answer**

**Problem 13.2:**

[8]

**Compute the strain-displacement matrix for the axisymmetric triangular element shown in figure 13.9. Also determine the element strains. The nodal displacements are found out as**

[e]

$u_1 = 0.002;$ $\quad w_1 = 0.001$

$u_2 = 0.001;$ $\quad w_2 = -0.004$