GRAPH THEORY

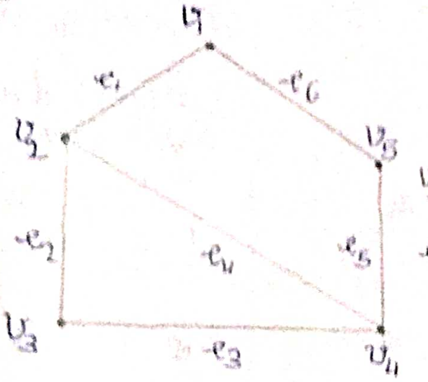


Fig.1

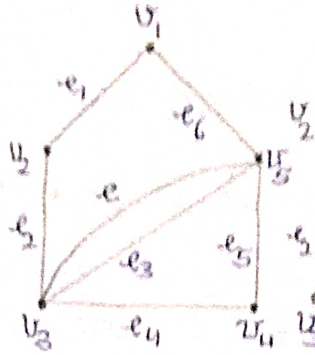


Fig.2

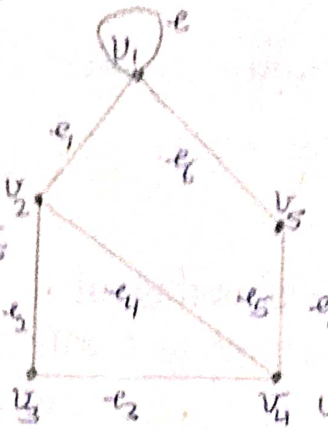


Fig.3

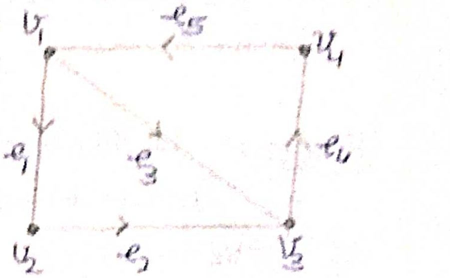


Fig.4

$V = \{v_1, v_2, v_3, v_4, v_5\}$

$E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$

$S = \{\{v_1, v_2\} \{v_2, v_3\} \{v_2, v_4\}, \{v_3, v_4\} \{v_4, v_5\} \{v_1, v_5\}\}$

: a set/collection of 2-element sets of elements in V.

$\psi : E \to S$ by

$\psi(e_1) = \{v_1, v_2\} = \{v_2, v_1\}$

$\psi(e_2) = \{v_2, v_3\} = \{v_3, v_2\}$

etc.

A triplet (V, E, ψ): Undirected Graph.

An undirected graph (V, E) without multiple edges and self-loops
: SIMPLE GRAPH

An undirected graph (V, E) with parallel/multiple edges
: MULTI GRAPH

An undirected graph (V, E) with self-loop(s) (with or without parallel edges)
: PSEUDO GRAPH.

Let $G = (V, E)$

If $|V|$: finite & $|E|$: finite, then G FINITE Graph.

If $E = \phi$, then G is null graph

$V = \{v_1, v_2, v_3, v_4\}$

$E = \{e_1, e_2, e_3, e_4, e_5\}$

$S = \{\langle v_1, v_2\rangle, \langle v_1, v_3\rangle, \langle v_2, v_3\rangle, \langle v_3, v_4\rangle, \langle v_1, v_4\rangle\}$
$\subseteq V \times V$

$\psi : E \to S$ by

$\psi(e_1) = \langle v_1, v_2\rangle$

$\psi(e_2) = \langle v_2, v_3\rangle$

$\psi(e_3) = \langle v_1, v_3\rangle$

$\psi(e_4) = \langle v_3, v_4\rangle$

$\psi(e_5) = \langle v_4, v_1\rangle$

A triplet (V, E, ψ): Directed Graph.

If $\psi(e) = \{v_i, v_j\}$, then
v_i & v_j : end vertices of e
: adjacent vertices

e is incident with vertices v_i and v_j

If $\psi(e) = \{v_i, v_i\}$, then e is called a loop

A vertex v that has no incident edges is called an isolated vertex.

If $\psi(e) = \{v_i, v_j\} = \psi(e')$, e and e' are called parallel/multiple edges.

If $\psi(e) = \{v_i, v_j\}$ & $\psi(e') = \{v_j, v_k\}$ then e and e' are adjacent edges.

Degree(v) = deg(v)
= Number of edges incident with v

If deg(v) = 1 then v : pendant vertex

If deg(v) = 0, v : isolated vertex

If $\psi(e) = \langle v_i, v_j\rangle$

v_i & v_j : adjacent vertices

v_i : initial vertex,
v_j : terminating vertex

Edge e is incident with vertices v_i, v_j

v_i is adjacent to v_j whereas v_j is adjacent to v_i.

Out-degree

$d^+(v)$ = Number of edges incident out of v

In-degree

$d^-(v)$ = Number of edges incident into v.

The Handshaking Theorem (Lemma):

$$\sum_{v \in V} \deg(v) = 2e = 2|E|$$

Theorum: The number of vertices of odd degree in a graph is always EVEN.

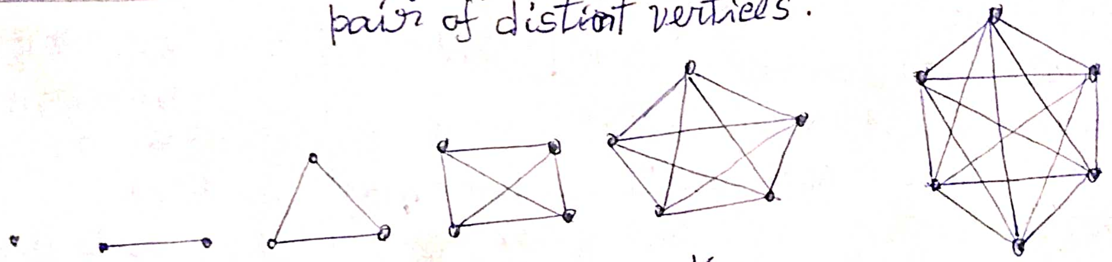
Theorum: digraph G=(

$$\sum_{v \in V} d^+(v) = \sum_{v \in V} d^-(v) =$$

Walk: L graph

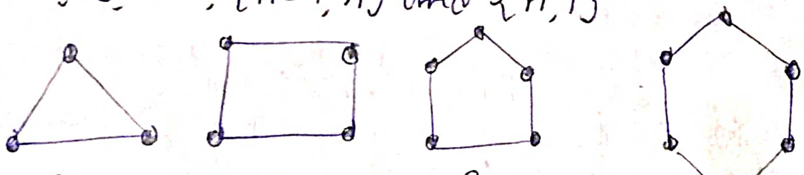
Regular Graph: If $\deg(v) = n, \forall v \in V$, then $G = (V, E)$ is called n-regular graph.

Complete Graph, K_n: simple graph that contains exactly one edge between each pair of distinct vertices.



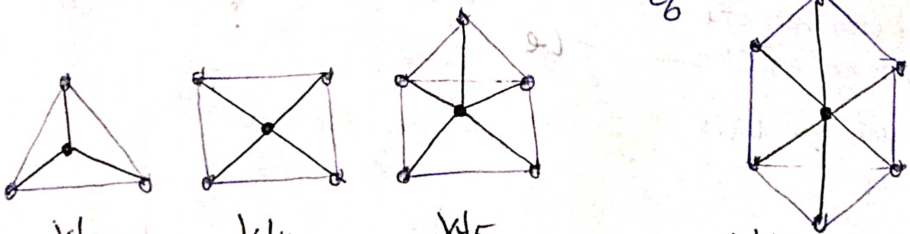
K_1 K_2 K_3 K_4 K_5 K_6

Cycle: The cycle $C_n, n \geq 3$, consists of n vertices $1, 2, \ldots, n$ and edges $\{1,2\}$, $\{2,3\}, \ldots, \{n-1, n\}$ and $\{n, 1\}$



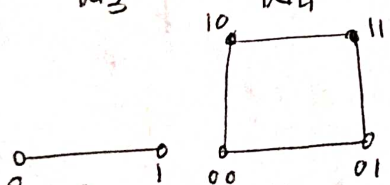
C_3 C_4 C_5 C_6

Wheel, W_n



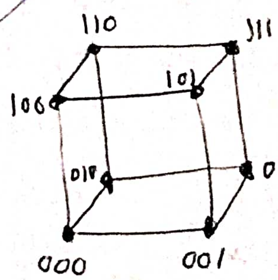
W_3 W_4 W_5 W_6

n-Cube : Q_n

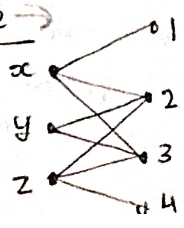


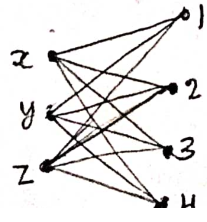
Q_1 Q_2

Q_3 :



Bipartite →

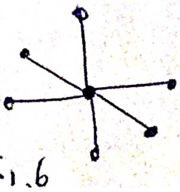




: $K_{3,4}$

← Complete Bipartite $K_{m,n}$

star



$K_{1,6}$

Walk: Let x, y be (not necessarily distinct) vertices in an undirected graph $G = (V, E)$.

An x-y walk in G is a (loop-free) finite alternating sequence:

$$x = x_0, e_1, x_1, e_2, x_2, e_3, \dots, e_{n-1}, x_{n-1}, e_n, x_n = y$$

of vertices and edges from G, starting at vertex x and ending at vertex y and involving n edges $e_i = \{x_{i-1}, x_i\}$, where $1 \le i \le n$.

Length of the walk = the number of edges in the walk = n.

Any x-y walk where $x = y$ (and $n > 1$) is called a closed walk. Otherwise, the walk is called open.

Note that a walk may repeat both vertices and edges.

If no edge in the x-y walk is repeated, then the walk is called
$e_i \neq e_j, i \neq j$
an x-y trail. A closed x-x trail is called a circuit.

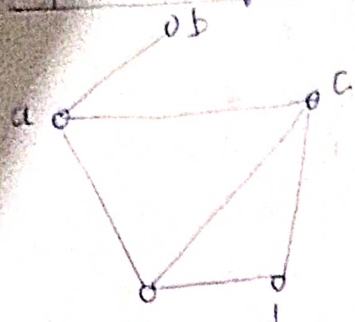
If no vertex of the x-y walk occurs more than once, then the
$x_i \neq x_j, i \neq j, i, j = 0, n$
walk is called an x-y path. A closed x-x path is called the cycle.

Theorem: Let $G = (V, E)$ be an undirected graph with $a, b \in V$, $a \neq b$. If there exists a trail (in G) from a to b, then there is a path (in G) from a to b.

DEF Let $G = (V, E)$ be an undirected graph.
If there is a path between any two distinct vertices of G, then G is called connected.
A graph that is not connected is called disconnected.

Theorem: An undirected graph $G = (V, E)$ is disconnected if and only if V can be partitioned into at least two subsets V_1, V_2 such that there is no edge in E of the form $\{x, y\}$ where $x \in V_1$ any $y \in V_2$.

Representing Graphs:



A simple Graph

:

Adjacency List:

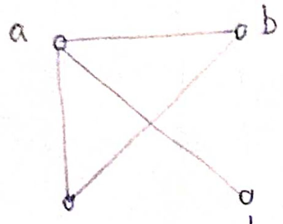
Vertex	Adjacent vertices
a	b, c, e
b	a
c	a, d, e
d	c, e
e	a, c, d

&

Adjacency Matrix:

$$\begin{array}{c|c} & \begin{matrix} a & b & c & d & e \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \\ e \end{matrix} & \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

$a_{ij} = a_{ji}$



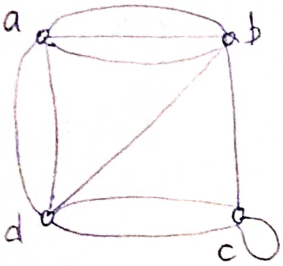
Simple Graph

:

Vertex	adjacent vertices
a	b, c, d
b	a, c
c	a, b
d	a

&

$$\begin{array}{c|c} & \begin{matrix} a & b & c & d \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \end{matrix} & \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$



A Pseudograph

:

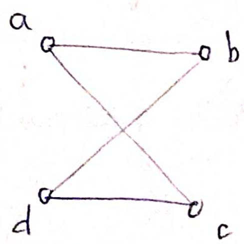
$$\begin{array}{c|c} & \begin{matrix} a & b & c & d \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \end{matrix} & \begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix} \end{array}$$

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

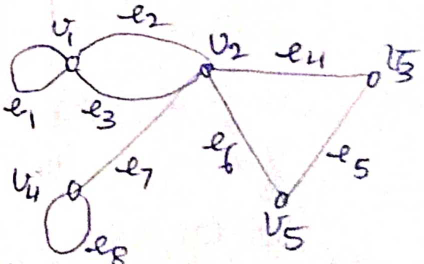
Adjacency matrix

Ordering of vertices: a, b, c, d

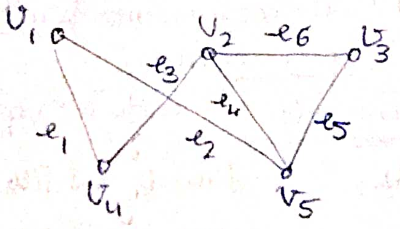
:



Simple Graph



Pseudograph: ↓



Simple Graph

:

Incidence matrix:

$$\begin{array}{c|c} & \begin{matrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{matrix} \\ \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{matrix} & \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Incidence matrix

$$\begin{array}{c|c} & \begin{matrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{matrix} \\ \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{matrix} & \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

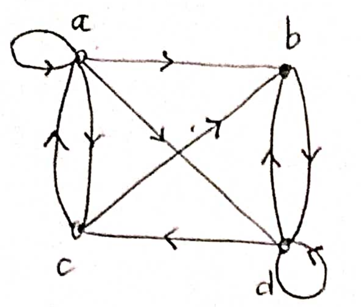
Q. Find an adjacency matrix for each of the graphs:

a) K_n b) C_n c) W_n d) $K_{m,n}$ e) Q_n

: Parallel edges:

Self-loops:

i) Determine



Directed graph

:

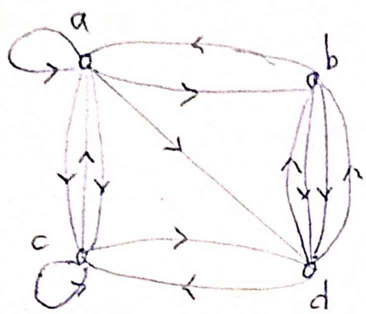
Adjacency List:

Vertex	Terminal vertices
a	a, b, c, d
b	d
c	a, b
d	b, c, d

&

Adjacency Matrix:

$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$



:

Vertex	Terminal vertices

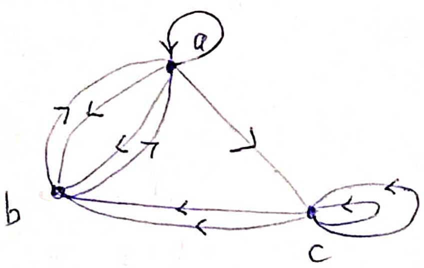
&

$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 0 & 0 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix} \end{array}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 0 \\ 0 & 2 & 2 \end{bmatrix}$$

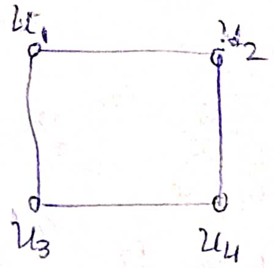
Adjacency matrix

:

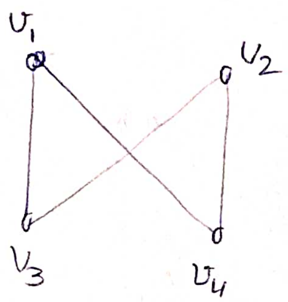


Digraph.

EX:



$G=(V,E)$



$H=(W,F)$

: Define a function $f: V \longrightarrow W$ by

$f(u_1) = v_1$
$f(u_2) = v_4$
$f(u_3) = v_3$
$f(u_4) = v_2$

(i) one-to-one correspondence between V and W

(ii) ? correspondence preserves adjacency.

$$A(G) = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{array}\begin{array}{c} \begin{array}{cccc} u_1 & u_2 & u_3 & u_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

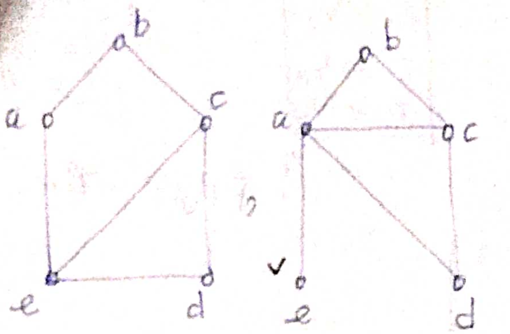
$$A(H) = \begin{array}{c} \\ v_1 \\ v_4 \\ v_3 \\ v_2 \end{array}\begin{array}{c} \begin{array}{cccc} v_1 & v_4 & v_3 & v_2 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Identical

$G=(V,E)$ and $H=(W,F)$ are ISOMORPHIC.

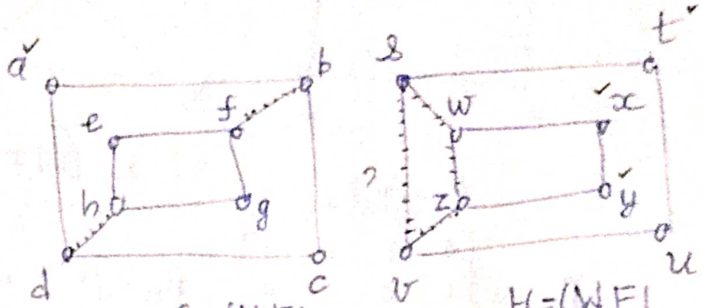
adjacent vertices in G	adjacent vertices in H
u_1 and u_2	$f(u_1)=v_1$ and $f(u_2)=v_4$
u_1 and u_3	$f(u_1)=v_1$ and $f(u_3)=v_3$
u_2 and u_4	$f(u_2)=v_4$ and $f(u_4)=v_2$
u_3 and u_4	$f(u_3)=v_3$ and $f(u_4)=v_2$

EX Determine whether the graphs shown in Fig. 1 and Fig. 2 are isomorphic.



$G=(V,E)$ $H=(W,F)$

Fig. 1



$G=(V,E)$ $H=(W,F)$

Fig. 2

Solution :

(i) $|V|=$ $|W|=$

(ii) $|E|=$ $|F|=$

(iii) Degree sequence of G :

Degree sequence of H :

Solution :

(i) $|V|=$ $|W|=$

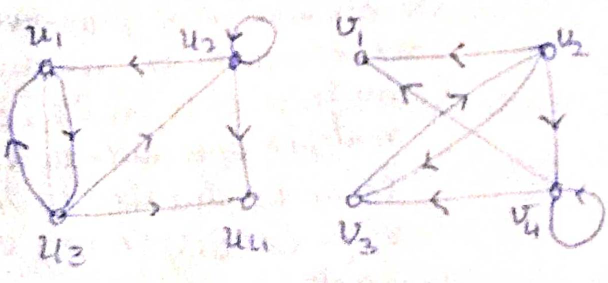
(ii) $|E|=$ $|F|=$

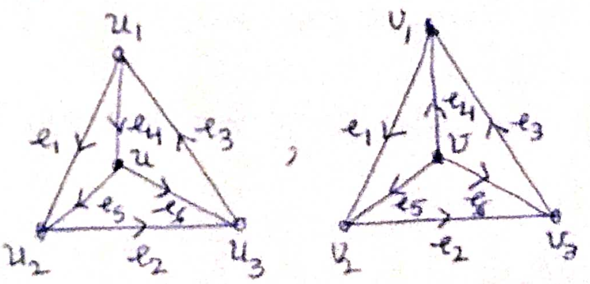
(iii) Degree sequence of G :

Degree sequence of H :

Subgraphs formed by/made up of vertices of degree 3 and the edges connecting are not ISOMORPHIC.

EX : Determine whether the given pair of directed graphs are isomorphic.





$d^+(u)=2$; $d^+(v)=3$

$d^-(u)=1$; $d^-(v)=0$

Sol : To be isomorphic

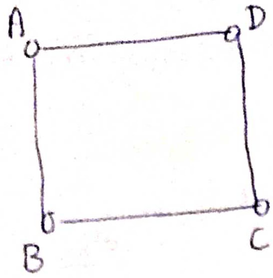
① Corresponding undirected graphs must be isomorphic

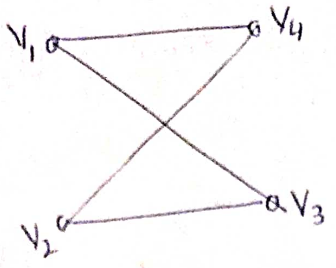
② The directions of the corresponding edges must also agree.

ISOMORPHIC

NOT ISOMORPHIC

EX:





$G = (V, E)$ $\quad$ $H = (W, F)$

MAPPING

$A \to V_1$
$B \to V_3$
$C \to V_2$
$D \to V_4$

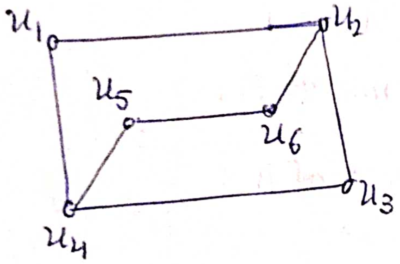
Assume

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

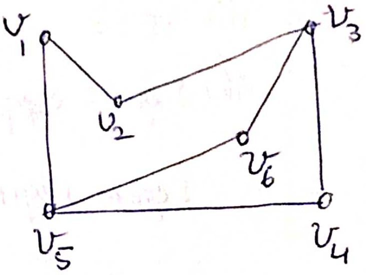
$$? \quad P A_G P^T = A_H$$

$$A_G = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \quad A_H = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

EX: Determine whether the following pair of graphs are isomorphic.



$G = (V, E)$



$H = (W, F)$

(1) $|V| =$ $\quad$ $|W| =$

(2) $|E| =$ $\quad$ $|F| =$

(3) Degree sequence of G:

Degree sequence of H:

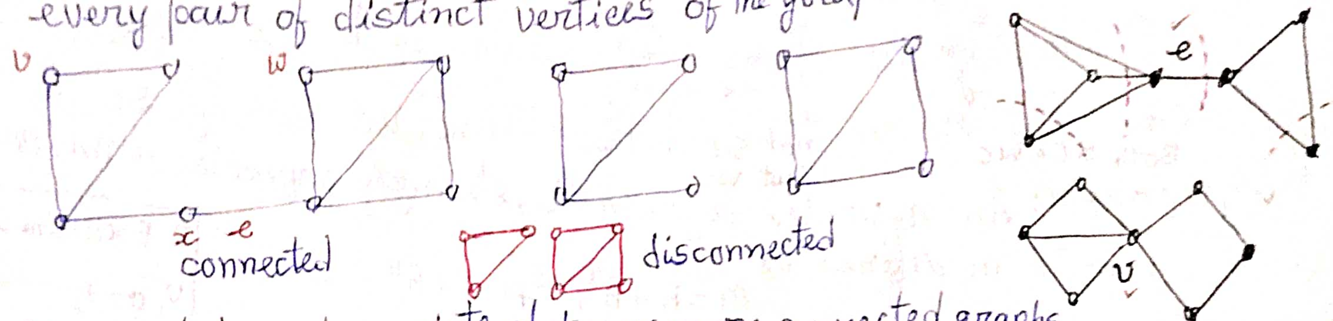
(4) $f : V \to W$ by

$f(u_1) = v_6$
$f(u_2) = v_3$
$f(u_3) = v_4$
$f(u_4) = v_5$
$f(u_5) = v_1$
$f(u_6) = v_2$

To see whether f preserves edges, we examine the adjacency matrix of G, and the adjacency matrix of H with rows and columns labeled by the images of the corresponding vertices in G.

Connectedness in Undirected Graphs:

DEF. An undirected graph is called connected if there is a path between every pair of distinct vertices of the graph. Otherwise disconnected.



A disconnected graph consists of two or more connected graphs. Each of these connected subgraphs is called a component.

Theorem: A graph is disconnected if and only its vertex set V can be partitioned into two nonempty subsets V_1 and V_2 such that there exists no edge in G whose one end vertex is in V_1 and the other in V_2

Theorem: If a graph (connected or disconnected) has exactly two vertices of odd degree, there must be a path joining these two vertices.

Theorem: If G is a bipartite graph, then each cycle of G has even length.

Theorem: Let G be a simple graph on n vertices. If G has k components, then the number m of edges of graph satisfies

$$(n-k) \le m \le (n-k)(n-k+1)/2$$

Corollary: Any simple graph with n vertices and more than $(n-1)(n-2)/2$ edges is connected.

Q. Determine $k(G)$ and $\lambda(G)$ for each of : (i) C_6 (ii) W_6 (iii) $K_{4,7}$ (iv) Q_4.

A disconnecting set in a connected graph is a set of edges whose removal disconnects G. A disconnecting set, no proper subset of which is a disconnecting set is called a cutset. A cutset with only one edge is called a bridge.

Edge connectivity. $\lambda(G)$ = size of the smallest cutset in G. G is k-edge connected if $\lambda(G) \ge k$.

A separating set in a connected graph G is a set of vertices whose deletion disconnects G. A separating set with only one vertex is called a CUT-VERTEX.

(Vertex) connectivity $k(G)$ = size of small separating set in G. G is k-connected if $k(G) \ge k$