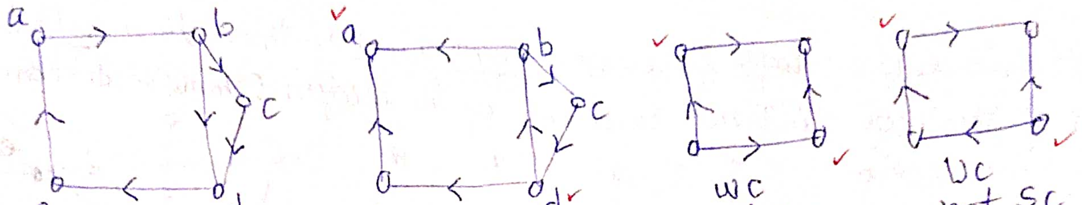
Connectedness in Directed Graphs:

DEF. A directed graph is strongly connected if there is a path from a to b and from b to a whenever a and b are vertices in the graph.

DEF. A directed graph is weakly connected if there is a path between every two vertices in the underlying undirected graph.

DEF. A digraph is unilaterally connected if for every pair of vertices one is reachable from the other



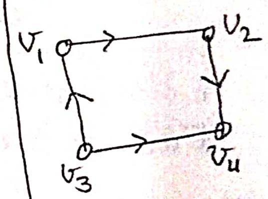
Both SC & WC — not SC, not UC but WC — WC not UC — UC not SC

✓ **THEOREM:** An n-vertex digraph is strongly connected if and only if the matrix M defined by

$$M = A + A^2 + A^3 + \cdots + A^n$$

has no zero entry, A is the adjacency matrix.

Q. Examine whether....

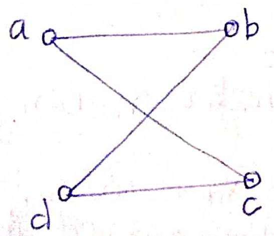


is strongly connected.

Counting Paths Between Vertices:

✓ **Theorem:** Let G be a graph with adjacency matrix A with respect to the ordering 1, 2, 3, ---, n (with directed or undirected edges, with multiple edges and loops allowed). The number of different paths of length r from i to j, where r is a positive integer, equals the (i, j)th entry of A^r.

EX. Determine the number of paths of length four from a to d in the simple graph:



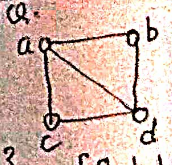
$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix};$$

The number of paths of length four from a to d is the (1, 4)th entry of A^4:

$$A^2 = \begin{bmatrix} \quad \end{bmatrix}; A^3 = \begin{bmatrix} \quad \end{bmatrix}; A^4 = \begin{bmatrix} 8 & 0 & 0 & 8 \\ 0 & 8 & 8 & 0 \\ 0 & 8 & 8 & 0 \\ 8 & 0 & 0 & 8 \end{bmatrix}$$

? $X = A + A^2 + A^3 = \begin{bmatrix} \quad \end{bmatrix}$

Q.



?

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

✓ **THEOREM:** If A is the adjacency matrix of an undirected graph G with n vertices, and

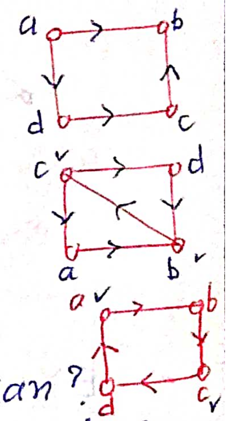
$$X = A + A^2 + A^3 + \cdots + A^{n-1}$$

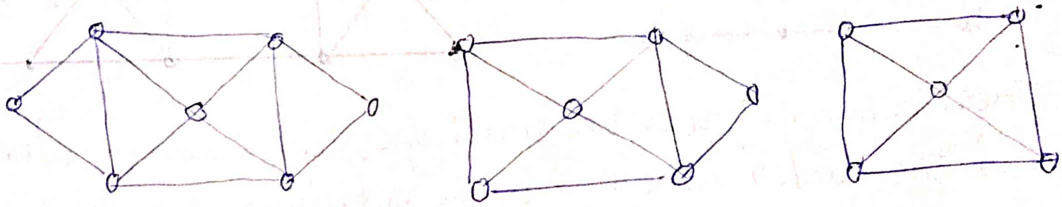
Then, G is disconnected iff and only if there exists at least one entry in matrix that is zero.

EULER GRAPHS

- A closed walk in a graph that contains every edge of the graph exactly once is called an Euler line/Euler circuit, and a graph that consists of an Euler line is called an Euler graph.
- A open walk in a graph that includes (or traces or covers) all edges of the graph without retracing any edge is called a unicursal line or an open Euler or an Euler path.

  A (connected) graph that has a unicursal line will be termed/called a unicursal graph or semi-Euler graph.
- A graph that has neither Euler line nor unicursal line is called non-Euler graph.





Q. Discuss Königsberg bridges problem.

Q. Which of the following graphs are Eulerian? semi-Eulerian?

(i) K_5 (ii) $K_{2,3}$ (iii) the graph of the cube (iv) the graph of the octahedron (v) the Petersen graph.

Q. Examine each of the following for an Euler graph.

(i) K_n (ii) $K_{m,n}$ (iii) W_n (iv) Q_k (v) Platonic graphs.

Theorem: A connected graph G an Euler graph if and only if all vertices of G are of even degree.

Theorem. A connected graph G is an Euler graph if and only if it can be decomposed into circuits/its set of edges can be split up into disjoint cycles.

Corollary: A connected graph is semi-Eulerian if and only if it has exactly two vertices of odd degree.

Theorem: In a connected graph G with exactly $2k$ odd vertices, there exist k edge-disjoint subgraphs such that they together contain all edges of G and that each is a unicursal graph.

Fleury's Algorithm :

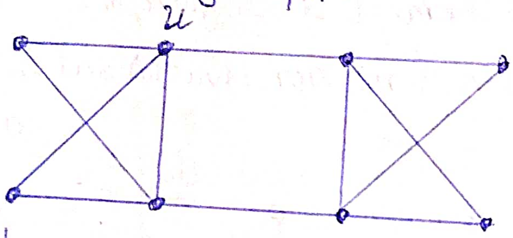
Theorem: Let G be an Eulerian graph. Then the following construction is always possible, and produces an Eulerian line of G.

Start at any vertex u and traverse the edges in an arbitrary manner, subject only to the following rules:

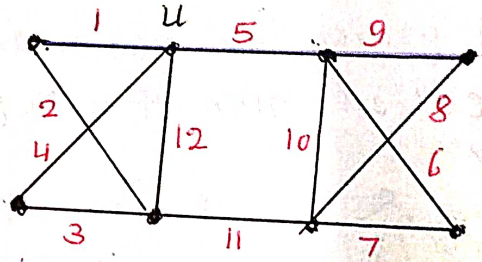
(i) erase the edges as they are traversed, and if any isolated vertices result, erase them too;

(ii) at each stage, use a bridge only if there is no alternative.

EX: Use Fleury's algorithm to produce an Eulerian trail/line/circuit for the graph.

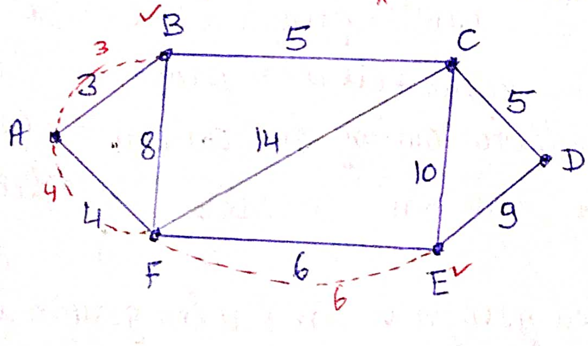


Solution →

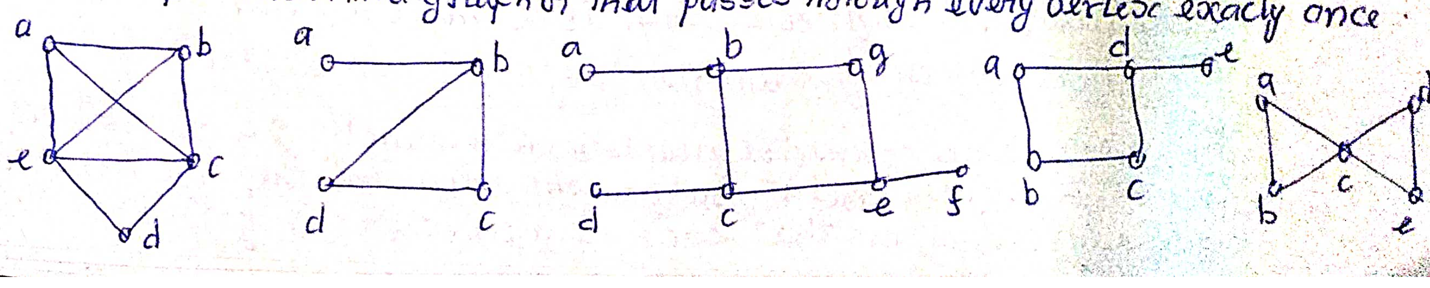


Solution: There are many possible solutions; for example, traverse the edges in the order indicated by the adjoining diagram.

EX: Solve the Chinese postman problem for the weighted graph:



EX. Find a closed walk in a graph G that passes through every vertex exactly once.
Find an open walk in a graph G that passes through every vertex exactly once.

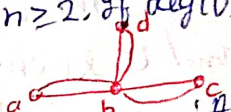


HAMILTONIAN CIRCUITS AND PATHS:

- A Hamiltonian circuit in a connected graph G is defined as a closed walk that traverses/visits every vertex of G exactly once, except of course, the starting vertex, at which the walk also terminates. A graph with a Hamiltonian circuit is called Hamiltonian graph.
- An open walk in a connected graph G (without self-loop & parallel edges) that traverses every vertex of G exactly once is called a Hamiltonian path. A graph which contains a Hamiltonian path is called a semi-Hamiltonian graph.
- A connected graph which is neither Hamiltonian nor semi-Hamiltonian is called non-Hamiltonian.

Theorem: Let $G=(V,E)$ be a loop-free graph with $|V|=n\geq 2$. If $\deg(x)+\deg(y)\geq n-1$ for all $x,y\in V$, $x\neq y$ (non-adjacent), then G has a Hamilton path.

Theorem: Let $G=(V,E)$ be a loop-free graph with $|V|=n\geq 2$. If $\deg(v)\geq (n-1)/2$ for all $v\in V$, then G has a Hamilton path.



no Hamilton path — multigraph.

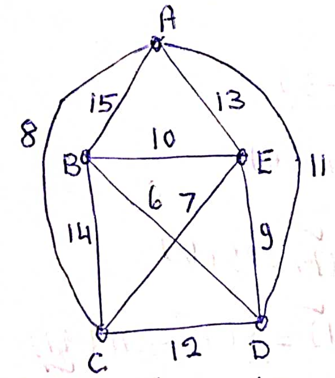
$\deg(x)+\deg(y)\geq 4>3=4-1$

→ Q. Which of the following graphs are : Hamiltonian? semi-Hamiltonian?
(i) K_5 (ii) $K_{2,3}$ (iii) the graph of the octahedron, (iv) K_6 (v) the 4-cube, Q_4.

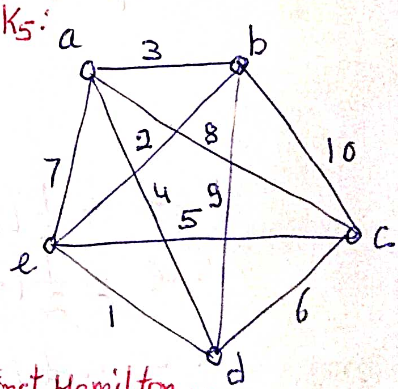
→ Q. Examine the following for Hamiltonian graph?
(i) K_n, (ii) $K_{m,n}$ (iii) Platonic graphs, (iv) W_n, (v) the k-cube, Q_k.

EX. Solve the TSP for the weighted graph. | No. of different Hamilton circuits in K_5 or permutations $=\frac{(5-1)!}{2}=12$

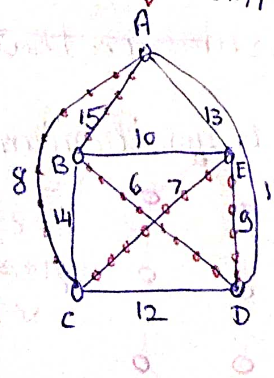
Weighted K_5

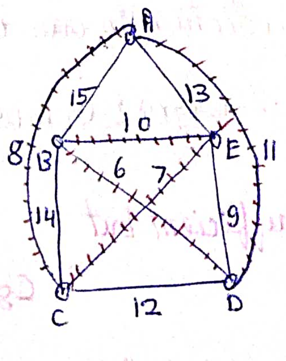


Weighted K_5:



Method of Nearest-Neighbourhood: (Approx. Solution)





Total distance = 115 Approx.   Total distance = 42 Correct Sol.

List all distinct Hamilton circuits and their weights:

Permutation	Weight	Permutation	Weight
A, B, C, D, E, A	63		

ORE'S THEOREM. Let G be a simple connected graph with $n \geq 3$ ver
then G is Hamiltonian if

$$\boxed{\deg(u) + \deg(v) \geq n}$$

for every pair of non-adjacent vertices u and v.

DIRAC'S THEOREM. A simple connected graph with $n \geq 3$ vertices is Hamiltonian if

$$\boxed{\deg(v) \geq \frac{n}{2}}, \; \cancel{\forall v} \text{ for every } v \text{ in } G.$$

Corollarly: The connected graph G with $n \geq 3$ vertices has a Hamiltonian circuit provided the number of edges in G

$$\boxed{e \geq \frac{1}{2}(n^2 - 3n + 6)}$$

Proof. If possible, let the graph G be non-Hamiltonian. Then by Dirac's theorem, there will exist a pair of non-adjacent vertices u and v such that

$$\deg(u) + \deg(v) \leq n - 1$$

Let H be the subgraph of G obtained by deleting the u and v from G. The graph H will have $(n-2)$ vertices and $e - \deg(u) - \deg(v)$ edges. the maximum number of edges in H can be ${}^{n-2}C_2$.

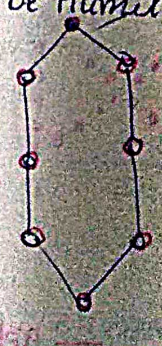
$$\therefore [e - \deg(u) - \deg(v)] \leq {}^{n-2}C_2$$
$$= \frac{(n-2)(n-3)}{2}$$
$$= \frac{1}{2}(n^2 - 5n + 6)$$

$$\therefore e \leq \frac{1}{2}(n^2 - 5n + 6) + \deg(u) + \deg(v)$$
$$\therefore e \leq \frac{1}{2}(n^2 - 5n + 6) + (n-1) = \frac{1}{2}(n^2 - 3n + 4)$$
$$\therefore e < \frac{1}{2}(n^2 - 3n + 6)$$

which is a contradiction to ~~our assumption that~~
$\therefore$ Our assumption is wrong. Therefore G must be Hamiltonian.

Illustration (Given conditions are sufficient but not necessary)

C8:



$n = 8$
$e = 8$
$\deg(v) = 2$
$\forall v$.

THEOREM: Let D be a strongly connected digraph with n vertices. If $\text{outdeg}(v) \geq n/2$ and $\text{indeg}(v) \geq n/2$ for each vertex v, then G is Hamiltonian.