In track reconstruction applications, this can lead to a reduction of the track reconstruction efficiency and an underestimation of the track parameter uncertainties for triplet with weak bending.

This problem can be solved to some extend by using a regularised χ^2-function for the triplet quality:

$$\chi^2_{\rm reg}(c_{3D}) = \frac{1}{\sigma_{\rm MS}^{(2-r)}\ b_{\rm MS}^r} \left[\frac{\Theta_{\rm MS}(c_{3D})^2 + \sin^2\hat{\vartheta}\ \Phi_{\rm MS}(c_{3D})^2}{|c_{3D}|^r} \right] \tag{60}$$

with r being a regularisation parameter, which is zero for no regularisation or otherwise positive. The solutions and the validity ranges for $r = 1$ and $r = 2$ are given in the appendix. For $r = 0$ (constrained fit) the fit gives very reliable results[5] for $b_{\rm MS} \lesssim d_{02}/8$, as also derived in [5].

3.3 Simple Hit Uncertainties Fit (case B)

In case of dominating hit uncertainties (negligile multiple scattering uncertaintes) and assuming a solenoidal magentic field, the triplet trajectory is described by a helix. From equation 3 a simple expression for the triplet quality is derived:

$$\chi^2(c_{3D}; \vec{\delta}_k) = \sum_{k=0}^{2} \sum_{i=1}^{3} \frac{\delta_k^{u_i\,2}}{\sigma_k^{u_i\,2}} \qquad \text{with} \quad \Theta_{\rm MS} \stackrel{!}{=} 0 \ \wedge \ \Phi_{\rm MS} \stackrel{!}{=} 0 \tag{61}$$

First, we consider the problem in the bending and non-bending planes separately. From equations 18 and 19 we obtain the conditions:

$$0 \stackrel{!}{=} (\varphi_{12} - \varphi_{01}) - \frac{\Phi_1 + \Phi_2}{2} \quad , \tag{62}$$

$$0 \stackrel{!}{=} \vartheta_2 - \vartheta_1 \quad . \tag{63}$$

The first equation (62) can always be fullfilled if the transverse curvature is determined as:

$$c_\perp = 2\, \frac{(\vec{d}'_{12} \times \vec{d}'_{01})\ \vec{e}_z}{d'_{01}\ d'_{12}\ d'_{02}} \tag{64}$$

with

$$\vec{d}'_{kl} = (\vec{x}_l + \vec{\delta}_l)_\perp - (\vec{x}_k + \vec{\delta}_k)_\perp \tag{65}$$

being the hit position vectors projected to the bending plane, see also figure 1. Since three points in the bending plane can always be connected by a circle with curvature $c_\perp$, the only real constraint comes from equation 63, which corresponds to the conservation of longitudinal momentum. In other words, the hit position shifts $\boldsymbol{\vec{\delta}} = \{\vec{\delta}_1, \vec{\delta}_2, \vec{\delta}_3\}$ need to be determined by the fit such that the polar angles in the two arcs are the same: $\vartheta_{\rm fit} = \vartheta'_1 = \vartheta'_2$.

For sake of simplicity, we assume that the spatial hit position uncertainties are uncorrelated between the bending and non-bending plane (a general solution to this problem is

[5] We consider a fit beeing relaible if the average χ^2 value does not deviate from 1 by more than 10%.

discussed in the appendix ??). This is the case, for example, for a pixel detector with an ideal cylindrical detector geometry or an strip detector with the strips oriented either along the z-axis or transverse to it.

Even if the hit uncertainties are assumed to be uncorrelated between bending and non-bending plane, there is a small correlation between the transverse and longitudinal hit position shifts, as we will see in the following. For the here discussed simple fit, we ignore these correlations and assume that the hit positions shifts in the bending plane then vanish $\vec{\boldsymbol{\delta}}_\perp = 0$.

With the $2D$ curvature given by equation 26, the bending angles can be calculated:

$$\Phi_1 = 2 \arcsin\left(\frac{c_\perp\, d_{01}}{2}\right) \qquad \text{and} \qquad \Phi_2 = 2 \arcsin\left(\frac{c_\perp\, d_{12}}{2}\right) \quad , \tag{66}$$

and correspondingly the arc lengths[6]:

$$s_{01} = \frac{\Phi_1}{c_\perp} \qquad \text{and} \qquad s_{12} = \frac{\Phi_2}{c_\perp} \quad . \tag{67}$$

Instead of fitting the hit shifts, we can directly fit the polar angle using a regression:

$$\chi^2_{\rm hit} \quad = \quad \sum_{k=0,2} \frac{(z_k - z_0 - s_k \cot\vartheta)^2}{{\sigma^z_k}^2} \tag{68}$$

with s_i being the arc length position, z_0 a free parameter of the fit and σ^z_k the hit uncertainties in z-direction. Minimisation of equation 68 gives:

$$\cot\vartheta_{\rm min,hit} \quad = \quad \frac{\left(\sum_k w_k s_k z_k\right)\left(\sum_k w_k\right) \;-\; \left(\sum_k w_k z_k\right)\left(\sum_k w_k s_k\right)}{\left(\sum_k w_k s_k^2\right)\left(\sum_k w_k\right) \;-\; \left(\sum_k w_k s_k\right)^2} \tag{69}$$

where all sums run over the triplet hits and weights are introduced, defined as

$$w_k \quad = \quad \frac{1}{{\sigma^z_k}^2} \quad . \tag{70}$$

Evaluation of equation 69 yields for the triplet:

$$\cot\vartheta_{\rm min,hit} \quad = \quad \frac{s_{12} z_{12}/w_0 + s_{01} z_{01}/w_2 + s_{02} z_{02}/w_1}{s_{12}^2/w_0 + s_{01}^2/w_2 + s_{02}^2/w_1} \quad . \tag{71}$$

The fit quality is given by

$$\begin{aligned} \chi^2_{\rm min,hit} \quad &= \quad \frac{(s_{12} z_{01} - s_{01} z_{12})^2}{s_{12}^2/w_0 + s_{01}^2/w_2 + s_{02}^2/w_1} \\ &= \quad \frac{s_{01}^2 s_{12}^2}{s_{12}^2/w_0 + s_{01}^2/w_2 + s_{02}^2/w_1} \,(\cot\vartheta_2 - \cot\vartheta_1)^2 \end{aligned} \tag{72}$$

[6]Note that for small bending radii $|\Phi_{1,2}| \ll 1$ (high particle momenta) the arc length can be approximated by $s_{01} \approx d_{01}$ and $s_{12} \approx d_{12}$.

and the uncertainty on the slope is:

$$\sigma(\cot\vartheta)^2 \quad = \quad \frac{w_0 + w_1 + w_2}{s_{12}^2 w_1 w_2 + s_{01}^2 w_0 w_1 + s_{02}^2 w_0 w_2} \tag{73}$$

Finally, the $3D$ curvature is calculated using the relation::

$$c_{3\mathrm{D}} \quad = \quad c_\perp \sin\vartheta \tag{74}$$

The curvature uncertainty has in general two contributions, one from the bending plane affecting the measurement of $c_\perp$, and one from the non-bending plane affecting the polar angle measurement (equation 73):

$$\begin{aligned} \sigma(c_{3\mathrm{D}})^2 \quad &= \quad c_{3\mathrm{D}} \left(\frac{\sigma(c_\perp)^2}{c_\perp^2} + \frac{\sigma(\sin\theta)^2}{\sin^2\theta} \right) \\ &= \quad c_{3\mathrm{D}} \left(\frac{\sigma(c_\perp)^2}{c_\perp^2} + \frac{\sin^2(2\theta)}{4}\,\sigma(\cot\theta)^2 \right) \end{aligned} \tag{75}$$

3.3.1 Alternative Method (to be validated)

The fit quality can be calculated using a very simple expression:

$$\chi^2_{\mathrm{min,hit}} \;=\; \frac{(\vartheta_2 - \vartheta_1)^2}{{\sigma_\vartheta^{\mathrm{hit}}}^2} \quad , \tag{76}$$

with the pre-fit polar angles given by

$$\cot\vartheta_1 = \frac{z_{01}}{s_{01}} \qquad \text{and} \qquad \cot\vartheta_2 = \frac{z_{12}}{s_{12}} \quad , \tag{77}$$

and the polar angle uncertainty

$$\sigma_\vartheta^{\mathrm{hit}} \quad = \quad \left(\sum_{k=0}^{2} (\zeta_k\, \sigma_k^z)^2 \right)^{1/2} \quad . \tag{78}$$

The coefficients ζ_k propagate the spatial hit uncertainties and are given by the derivatives:

$$\zeta_0 \;:=\; \frac{\mathrm{d}(\vartheta_2 - \vartheta_1)}{\mathrm{d}z_0} \quad = \quad -\frac{c_\perp\,\Phi_1}{\Phi_1^2 + z_{01}^2\,c_\perp^2} \tag{79}$$

$$\zeta_1 \;:=\; \frac{\mathrm{d}(\vartheta_2 - \vartheta_1)}{\mathrm{d}z_1} \quad = \quad \frac{c_\perp\,\Phi_1}{\Phi_1^2 + z_{01}^2\,c_\perp^2} + \frac{c_\perp\,\Phi_2}{\Phi_2^2 + z_{12}^2\,c_\perp^2} \tag{80}$$

$$\zeta_2 \;:=\; \frac{\mathrm{d}(\vartheta_2 - \vartheta_1)}{\mathrm{d}z_2} \quad = \quad -\frac{c_\perp\,\Phi_2}{\Phi_2^2 + z_{12}^2\,c_\perp^2} \tag{81}$$

In the limit of small bending angles above equations simplify to

$$\zeta_0 \quad \approx \quad -\frac{d_{01}}{d_{01}^2 + z_{01}^2} \quad = \quad -\frac{d_{01}}{\ell_{01}^2} \tag{82}$$

$$\zeta_1 \quad \approx \quad \frac{d_{01}}{d_{01}^2 + z_{01}^2} + \frac{d_{12}}{d_{12}^2 + z_{12}^2} \quad \approx \quad \frac{d_{02}}{\ell_{02}^2} \tag{83}$$

$$\zeta_2 \quad \approx \quad -\frac{d_{12}}{d_{12}^2 + z_{12}^2} \quad = \quad -\frac{d_{12}}{\ell_{12}^2} \quad . \tag{84}$$