Numerical Methods 2
Task-2 Report

9. Computing the condition number of a symmetric and positive definite matrix A ($\mathrm{cond}_2(A) = \lambda_{max}/\lambda_{min}$).
Use the power method to calculate λ_{max} and the inverse power method to calculate λ_{min}. Use Gaussian elimination to obtain the LU decomposition of A (to solve the system of linear equations in the inverse power iteration)

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1 Introduction

If a dominating eigenvalue (one with the biggest absolute value) and matching eigenvector exist, the Power Method is used to find them.
Begin by guessing the eigenvector of the dominating eigenvalue for a square matrix A to use the Power Method. Repeat the process multiplying the most recently acquired vector on the left by A, the responses converge to the required eigenvector, normalizing the result, and repeating the process until the answers converge to the desired eigenvector (or until it is clear the results are not converging).
The absolute value of the dominant eigenvalue is the norm of the final vector if convergence happens.
Similarly, the power method is applied to the inverse of a matrix A is the inverse power method. The inverse power method, in general, converges to the smallest eigenvalue in A's absolute value.

2 Task

In this task problem, we are given a symmetric and positive definite matrix, and our goal is to calculate the conditional value with is calculated as the quotient of the dominant eigenvalue and the smallest eigenvalue of the given matrix. The dominant eigenvalue is calculated using the Power method and the smallest eigenvalue is calculated using the inverse power method. The Gaussian elimination approach is used to obtain the LU decomposition of A to solve the linear equations in the inverse power method iterations.

The results for some of the matrices using this method have been displayed in the Results section of this report.

3 About the code

Attached to this report, are two matlab files which are named as conditionVar.m and LUfact.m. The second file contains the LU factorization of the matrix A, with partial pivoting. This function is used in the main file named, conditionVar.m for the inverse power method calculations.
The main file is divided in 4 sections. The first section is assigned to check if the matrix entered is symmetric and positive definite. If not, the function returns. The second and third section are the power and inverse power methods. And the final section is the result section which contains the output of the function. The parameters of the function are the matrix A, a variable named iter, which contains the number of iterations required for inverse power method, and the last parameter is named tol, which is the tolerance for the power method.
Also, a testcase.m file is attached to the submission, which has command lines readied to be copied for testing for the initialized matrices in it with various sizes. The matrices are not same as the ones in the project.

A generateSPDmatrix.m function is also included that can generate a nxn symmetric and positive definite matrix, where n is a parameter.

4 Result

4.1 $\mathbf{A} = \begin{bmatrix} 34 & 12 & 0 & 0 \\ 12 & 41 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

In this above matrix, through calculation **by hand**, we get that the eigenvalues of the above matrix is equal to 1,25, and 50. Hence, $\lambda_{min} = 1$ and $\lambda_{max} = 50$. Thus, $\text{cond}_2(A) = 50$.

Iteration	Tolerance	λ_{max}	λ_{min}	$\text{cond}_2(A)$
100	0.01	50.00523	1	50.00523
200	0.001	50.00065	1	50.00065
300	0.1	50.08399	1	50.08399
400	0.01	50.00523	1	50.00523
500	0.0001	50.00008	1	50.00008

From the above table, we can gather that on average, the absolute error = 0.019036 , and the relative error percentage = 0.0381%.

4.2 $\mathbf{A} = \begin{bmatrix} 8.31 & 1.1876 & 1.7664 & 1.9263 & 1.1145 & 1.7291 \\ 1.1876 & 6.8676 & 0.7309 & 1.2123 & 0.6991 & 1.2092 \\ 1.7664 & 0.7309 & 7.8699 & 1.7054 & 1.2178 & 1.0433 \\ 1.9263 & 1.2123 & 1.7054 & 8.4640 & 1.7825 & 1.7353 \\ 1.1145 & 0.6991 & 1.2178 & 1.7825 & 7.4938 & 1.0093 \\ 1.7291 & 1.2092 & 1.0433 & 1.7353 & 1.0093 & 7.7632 \end{bmatrix}$

In this above matrix, through the **eig function in matlab**, we get that the eigenvalues are equal to 6.0175, 6.0283, 6.0456, 6.8286, 7.0484, and 14.8000. Hence, $\lambda_{min} = 6.1075$ and $\lambda_{max} = 14.8000$. Thus, $cond_2(A) = 2.4595$.

Iteration	Tolerance	λ_{max}	λ_{min}	$\text{cond}_2(A)$
100	0.05	14.81689	6.02781	2.45809
200	0.001	14.80040	6.02494	2.45652
300	0.7	15.07359	6.02263	2.50282
400	0.45	15.07359	6.02091	2.50354
500	0.01	14.80264	6.01971	2.45903

From the above table, we can gather that on average, the absolute error = 0.0165 , and the relative error percentage = 0.671%.

4.3 A =

$$\begin{bmatrix}
12.2833 & 2.1944 & 0.8454 & 2.1755 & 2.4978 & 1.2296 & 2.7206 & 1.6619 & 2.5598 & 2.4322 \\
2.1944 & 13.2599 & 1.2399 & 3.0234 & 3.2451 & 2.1212 & 2.9971 & 2.4200 & 3.0508 & 2.6329 \\
0.8454 & 1.2399 & 11.4795 & 1.2654 & 1.4279 & 0.8914 & 1.3675 & 1.2401 & 1.4102 & 0.9567 \\
2.1755 & 3.0234 & 1.2654 & 13.6672 & 2.9376 & 2.1877 & 3.3605 & 2.9097 & 3.1824 & 2.7939 \\
2.4978 & 3.2451 & 1.4279 & 2.9376 & 13.7995 & 2.1415 & 3.2717 & 2.6379 & 3.1771 & 2.8721 \\
1.2296 & 2.1212 & 0.8914 & 2.1877 & 2.1415 & 12.1247 & 1.7811 & 1.6923 & 1.8598 & 1.6879 \\
2.7206 & 2.9971 & 1.3675 & 3.3605 & 3.2717 & 1.7811 & 14.5343 & 3.4170 & 3.5747 & 2.9053 \\
1.6619 & 2.4200 & 1.2401 & 2.9097 & 2.6379 & 1.6923 & 3.4170 & 13.2292 & 2.8026 & 2.0518 \\
2.5598 & 3.0508 & 1.4102 & 3.1824 & 3.1771 & 1.8598 & 3.5747 & 2.8026 & 13.9465 & 3.1561 \\
2.4322 & 2.6329 & 0.9567 & 2.7939 & 2.8721 & 1.6879 & 2.9053 & 2.0518 & 3.1561 & 13.1229
\end{bmatrix}$$

In this above matrix, through the **eig function in matlab**, we get that the eigenvalues are equal to 10.0041, 10.0843, 10.2775, 10.3344, 10.4297, 10.7629, 11.0032, 11.6336, 11.7514 and, 35.1659. Hence, $\lambda_{min} = 10.0041$ and $\lambda_{max} = 35.1659$. Thus, $cond_2(A) = 3.5151$.

Iteration	Tolerance	λ_{max}	λ_{min}	cond$_2(A)$
100	0.001	35.16642	10.00722	3.51410
200	0.01	35.17035	10.00343	3.51583
300	0.005	35.16738	10.00376	3.51542
400	0.05	35.17950	10.00394	3.51656
500	0.1	35.20784	10.00403	3.51937

From the above table, we can gather that on average, the absolute error = 0.001156 , and the relative error percentage = 0.0329%.

4.4 A = $\begin{bmatrix} 3.6280 & 0.1537 & 0.2689 \\ 0.1537 & 3.9841 & 0.2698 \\ 0.2689 & 0.2698 & 3.1981 \end{bmatrix}$

In this above matrix, through calculation **eig function in matlab**, we get that the eigenvalues of the above matrix is equal to 3.03261, 3.60722, and 4.17035. Hence, $\lambda_{min} = 3.03261$ and $\lambda_{max} = 4.17035$. Thus, $cond_2(A) = 1.3752$.

Iteration	Tolerance	λ_{max}	λ_{min}	$\text{cond}_2(A)$
1000	0.01	4.21455	3.03262	1.38974
2000	0.01	4.21455	3.03262	1.38974
3000	0.01	4.21455	3.03262	1.38974
4000	0.01	4.21455	3.03262	1.38974
5000	0.01	4.21455	3.03262	1.38974

From the above table, we can gather that on average, the absolute error = 0.01454 , and the relative error percentage = 1.057%.

4.5 $$\mathbf{A} = \begin{bmatrix} 7.4897 & 0.3812 & 0.8090 & 0.5457 & 0.2567 & 0.5831 & 0.2605 \\ 0.3812 & 7.5479 & 0.7409 & 0.2252 & 0.9823 & 0.3917 & 0.4958 \\ 0.8090 & 0.7409 & 7.6981 & 0.6139 & 0.1673 & 0.1243 & 0.8081 \\ 0.5457 & 0.2252 & 0.6139 & 7.8819 & 0.7623 & 0.3900 & 0.3171 \\ 0.2567 & 0.9823 & 0.1673 & 0.7623 & 7.6448 & 0.3012 & 0.5076 \\ 0.5831 & 0.3917 & 0.1243 & 0.3900 & 0.3012 & 7.3846 & 0.7828 \\ 0.2605 & 0.4958 & 0.8081 & 0.3171 & 0.5076 & 0.7828 & 7.7302 \end{bmatrix}$$

In this above matrix, through calculation **eig function in matlab**, we get that the eigenvalues of the above matrix is equal to 6.0884, 6.5130, 7.1637, 7.4767, 7.6923, 7.8008, and 10.6423. Hence, $\lambda_{min} = 6.0884$ and $\lambda_{max} = 10.6423$. Thus, $cond_2(A) = 1.7480$.

Iteration	Tolerance	λ_{max}	λ_{min}	$\text{cond}_2(A)$
100	0.002	10.64666	6.08846	1.74866
200	0.05	10.75409	6.08840	1.76632
300	0.0001	10.64242	6.08840	1.74798
400	0.07	10.80122	6.08840	1.77406
500	0.001	10.64438	6.08840	1.74830

From the above table, we can gather that on average, the absolute error = 0.009064 , and the relative error percentage = 0.519%.

4.6 Analytical Result

$$\mathbf{A} = \begin{bmatrix} 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

(Analytical Result using the tri-diagonal method of calculating the eigenvalues of a symmetric and positive definite matrix)

$\lambda_k = d + 2 * a * cos(\frac{k\pi}{m+1})$
where, d = middle diagonal value, a = adjacent diagonal value, m = size of matrix, and k = 1,2,....,m.
Using this above formula, we get that the eigenvalues of the above matrix are 5.9190, 5.6825, 5.3097, 4.8308, 4.2846, 3.7154, 3.1692, 2.6903, 2.3175, and 2.0810. Hence, $\lambda_{min} = 2.0810$ and $\lambda_{max} = 5.9190$. Thus, $cond_2(A) = 2.8443$.

Iteration	Tolerance	λ_{max}	λ_{min}	cond$_2(A)$
100	0.02	6	2.31749	2.58900
200	0.05	6	2.31749	2.58900
300	0.0001	6	2.31273	2.59434
400	0.07	6	2.08107	2.88314
500	0.001	6	2.08101	2.88321

From the above table, we can gather that on average, the absolute error = 0.13656 , and the relative error percentage = 4.8%.

5 Conclusion

As we can see from the results section of this report, the power method is still very primitive and still has absolute and relative errors present in its calculations, which in turn also affects the inverse power method to calculate the smallest eigenvalue of a matrix. The conditional variable is calculated accordingly for each of the matrices and is displayed in the tables above.