Section: ___ **Group Number:** _______ **Score:** ______/10

Names of Group Members PRESENT: LA-16 Key

Credit is only given for group work to those present on all days L&LA is worked in class and who are also present the day it is turned in.

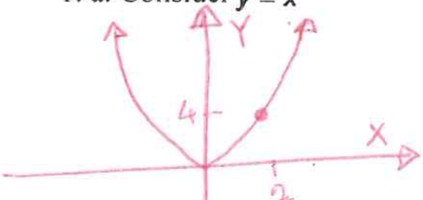
$(2.1)^2 = 4.41$
$(2.01)^2 = 4.0401$
$(2.001)^2 = 4.004001$
$(2.0001)^2 = 4.00040001$

> **1 Definition** Suppose $f(x)$ is defined when x is near the number a. (This means that f is defined on some open interval that contains a, except possibly at a itself.) Then we write
>
> $$\lim_{x \to a} f(x) = L$$
>
> and say "the limit of $f(x)$, as x approaches a, equals L"
>
> if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by taking x to be sufficiently close to a (on either side of a) but not equal to a.

$(1.9)^2 = 3.61$
$(1.99)^2 = 3.9601$
$(1.999)^2 = 3.996001$
$(1.9999)^2 = 3.99960001$

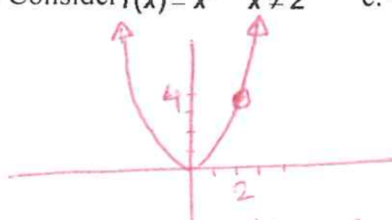
Limits Graphically:

1. a. Consider $y = x^2$



$\lim_{x \to 2} x^2 = 4$

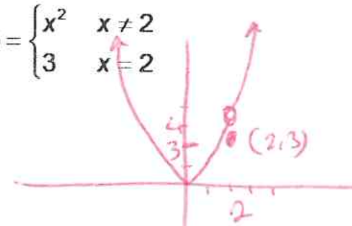
b. Consider $f(x) = x^2 \quad x \neq 2$



$\lim_{x \to 2} f(x) = 4$

Note: $f(2)$ undefined

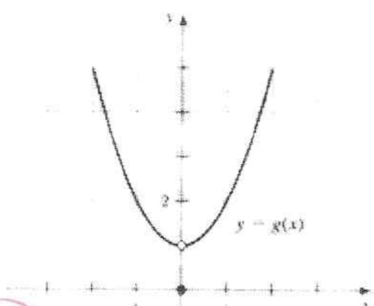
c. Consider $f(x) = \begin{cases} x^2 & x \neq 2 \\ 3 & x = 2 \end{cases}$



$\lim_{x \to 2} f(x) = 4$

$f(2) = 3$

2. Use the graph of $g(x)$ in the figure to find the following values, if they exist.



a. $g(0) = 0$

b. $\lim_{x \to 0} g(x) = 1$

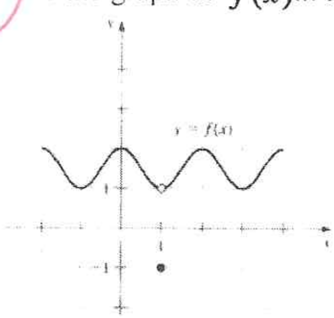
→ Not Same

c. $g(1) = 2$

d. $\lim_{x \to 1} g(x) = 2$

→ same, function is continuous here.

3. Use the graph of $f(x)$ in the figure to find the following values, if they exist.



a. $f(1) = -1$

b. $\lim_{x \to 1} f(x) = 1$

→ Not same

c. $f(0) = 2$

d. $\lim_{x \to 0} f(x) = 2$

→ Same, function is continuous here.

Summary: If $f(x)$ is continuous (more about that later), $\lim_{x\to a} f(x) = f(a)$

Hole at $x = a$ does not affect $\lim_{x\to a} f(x) = L$

Alternate $f(a)$ along with a hole does not affect $\lim_{x\to a} f(x) = L$

Limits Numerically:

4. Let $f(x) = \dfrac{x^3 - 1}{x - 1}$.

a. Calculate $f(x)$ for each value of x in the following table.

x	0.9	0.99	0.999	0.9999
$f(x) = \dfrac{x^3-1}{x-1}$	2.71	2.9701	2.997001	2.9997
x	1.1	1.01	1.001	1.0001
$f(x) = \dfrac{x^3-1}{x-1}$	3.31	3.0301	3.003001	3.0003

b. Make a conjecture about the value of $\lim_{x\to 1} \dfrac{x^3-1}{x-1}$ $= 3$

5. Let $f(x) = (1+x)^{\frac{1}{x}}$

a. Make two tables, one showing the values of f for $x = 0.01,\ 0.001,\ 0.0001,\ 0.00001$ and one showing values of f for $x = -0.01,\ -0.001,\ -0.0001,\ -0.00001$. Round your answers to five digits.

x	$f(x)$
0.01	2.70481
0.001	2.71692
0.0001	2.71815
0.00001	2.71827

x	$f(x)$
-0.01	2.73200
-0.001	2.71964
-0.0001	2.71842
-0.00001	2.71830

I rounded here to six decimal places.

b. Estimate the value of $\lim_{x\to 0} (1+x)^{\frac{1}{x}}$.

≈ 2.71828

c. What mathematical constant does $\lim_{x\to 0} (1+x)^{\frac{1}{x}}$ appear to equal?

$e = 2.718281828459045\ldots$

6. In the electronic text though WebAssign, Video example 3 on page 90 is an important result we will use in the future. Take the time to look at it later.

DEFINITION One-Sided Limits

1. **Right-hand limit** Suppose f is defined for all x near a with $x > a$. If $f(x)$ is arbitrarily close to L for all x sufficiently close to a with $x > a$, we write

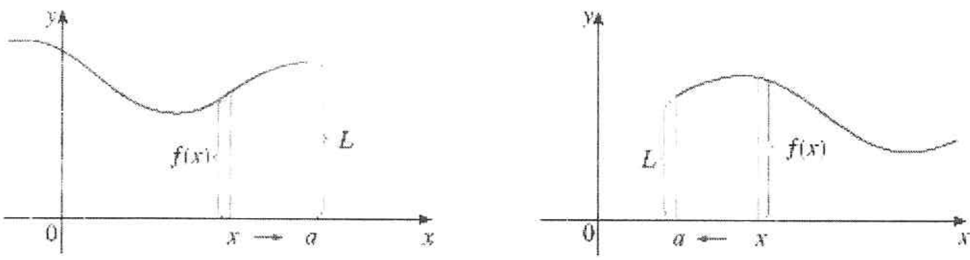
$$\lim_{x \to a^+} f(x) = L$$

and say the limit of $f(x)$ as x approaches a from the right equals L.

2. **Left-hand limit** Suppose f is defined for all x near a with $x < a$. If $f(x)$ is arbitrarily close to L for all x sufficiently close to a with $x < a$, we write

$$\lim_{x \to a^-} f(x) = L$$

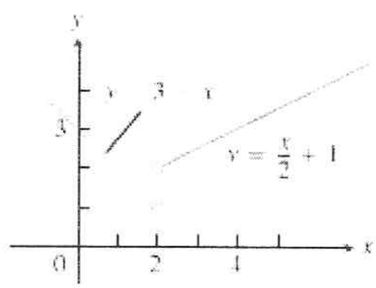
and say the limit of $f(x)$ as x approaches a from the left equals L.



(a) $\lim_{x \to a^-} f(x) = L$ (b) $\lim_{x \to a^+} f(x) = L$

In electronic text, there is a video lecture and video example 7 about one sided limits.

7. Let $f(x) = \begin{cases} 3 - x, & x < 2 \\ \dfrac{x}{2} + 1, & x > 2 \end{cases}$



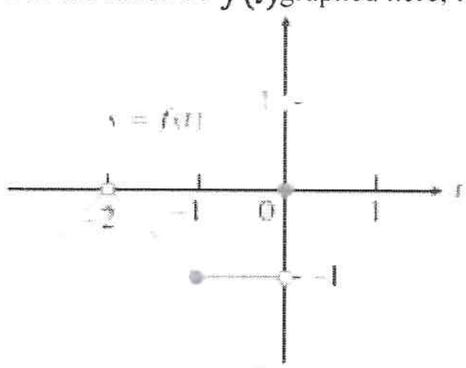
$f(2) =$ Undefined

$\lim_{x \to 2} f(x)$ Does not exist.

But $\lim_{x \to 2^+} f(x) =$ 2

and $\lim_{x \to 2^-} f(x) =$ 1

8. For the function $f(t)$ graphed here, find the following limits.



a) $\lim_{t \to -2} f(t)$ = 0

$\lim_{t \to -2^+} f(t) = 0$

$\lim_{t \to -2^-} f(t) = 0$

b) $\lim_{t \to -1.5} f(t)$ $= -\frac{1}{2}$

c) $\lim_{t \to -1} f(t)$ $= -1$

d) $\lim_{t \to 0} f(t)$ DNE

e) $\lim_{t \to \frac{1}{2}} f(t)$ $= 1$

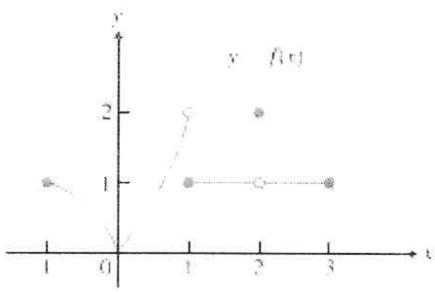
$\boxed{3}$	$\lim\limits_{x \to a} f(x) = L$	if and only if	$\lim\limits_{x \to a^-} f(x) = L$ and $\lim\limits_{x \to a^+} f(x) = L$

Note

As with two-sided limits, the value of a one-sided limit (if it exists) depends on the values of $f(x)$ near $x = a$ but not on the value of $f(a)$.

"Two-sided" limits do not exist when left-hand limit does not equal right hand limits.
"Two-sided" limits do not exist if one of the one-sided limits dne.

9.



$\lim\limits_{x \to 0^-} f(x) = 0$ $\lim\limits_{x \to 0^+} f(x) = 0$ $\lim\limits_{x \to 0} f(x) = 0$

$\lim\limits_{x \to 1^-} f(x) = 2$ $\lim\limits_{x \to 1^+} f(x) = 1$ $\lim\limits_{x \to 1} f(x)$ DNE

$\lim\limits_{x \to 2^-} f(x) = 1$ $\lim\limits_{x \to 2^+} f(x) = 1$ $\lim\limits_{x \to 2} f(x) = 1$

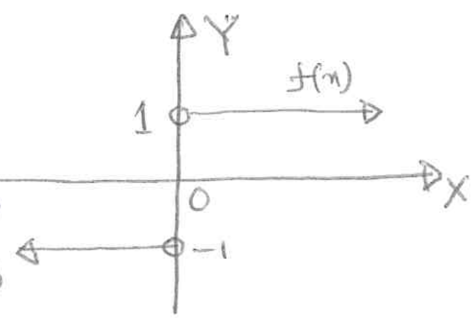
$\lim\limits_{x \to 3^-} f(x) = 1$

$\lim\limits_{x \to -1^+} f(x) = 1$

10. **A step function** Let $f(x) = \dfrac{|x|}{x}$, for $x \neq 0$.

a. Sketch a graph of f on the interval $[-2, 2]$.

$$f(x) = \begin{cases} \dfrac{x}{x}, & x > 0 \\ \dfrac{-x}{x}, & x < 0 \end{cases} = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$$



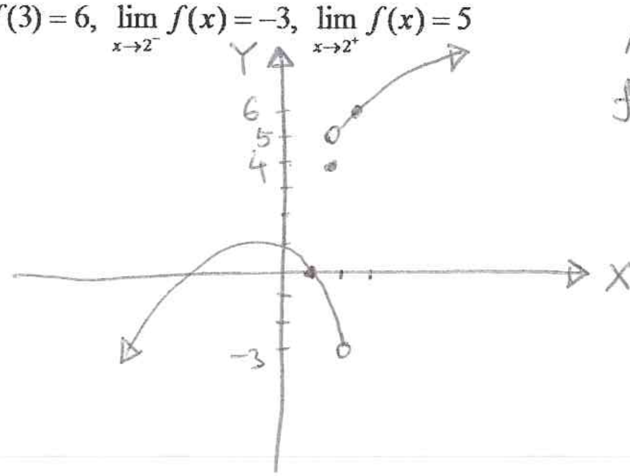
b. Does $\lim\limits_{x \to 0} f(x)$ exist? Explain your reasoning after first examining $\lim\limits_{x \to 0^-} f(x)$ and $\lim\limits_{x \to 0^+} f(x)$.

$\lim\limits_{x \to 0^-} f(x) = -1$ $\lim\limits_{x \to 0^+} f(x) = 1$, $\lim\limits_{x \to 0^-} f(x) \neq \lim\limits_{x \to 0^+} f(x)$ so $\lim\limits_{x \to 0} f(x)$ DNE

11. Electronic text page 91, video example 6 is interesting about the Heaviside function.

12. Sketch the graph of a function with the given properties. You do not need to find a formula for the function.

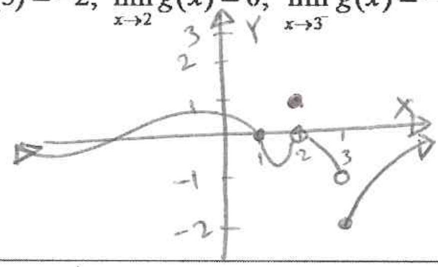
$f(1) = 0, f(2) = 4, f(3) = 6$, $\lim\limits_{x \to 2^-} f(x) = -3$, $\lim\limits_{x \to 2^+} f(x) = 5$

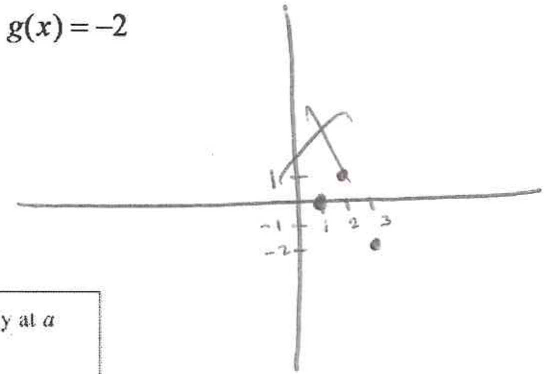


Answer may be different from person to person

13. Sketch the graph of a function with the given properties. You do not need to find a formula for the function.

$g(1)=0,\ g(2)=1,\ g(3)=-2,\ \lim_{x\to2} g(x)=0,\ \lim_{x\to3^-} g(x)=-1,\ \lim_{x\to3^+} g(x)=-2$





Infinite limits

> **4 Definition** Let f be a function defined on both sides of a, except possibly at a itself. Then
>
> $$\lim_{x\to a} f(x)=\infty$$
>
> means that the values of $f(x)$ can be made arbitrarily large (as large as we please) by taking x sufficiently close to a, but not equal to a.

> **5 Definition** Let f be defined on both sides of a, except possibly at a itself. Then
>
> $$\lim_{x\to a} f(x)=-\infty$$
>
> means that the values of $f(x)$ can be made arbitrarily large negative by taking x sufficiently close to a, but not equal to a.

Similar definitions can be given for the one-sided infinite limits

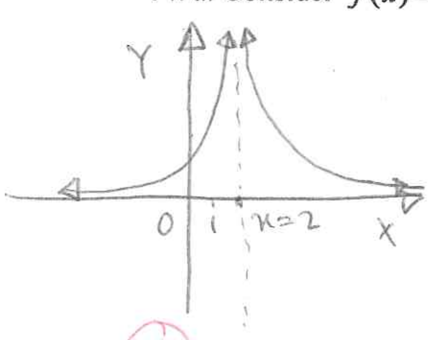
$$\lim_{x\to a^-} f(x)=\infty \qquad \lim_{x\to a^+} f(x)=\infty$$

$$\lim_{x\to a^-} f(x)=-\infty \qquad \lim_{x\to a^+} f(x)=-\infty$$

14. a. Consider $f(x)=\dfrac{1}{(x-2)^2}$

$\lim_{x\to2^+} f(x)=+\infty$

$\lim_{x\to2^-} f(x)=+\infty$

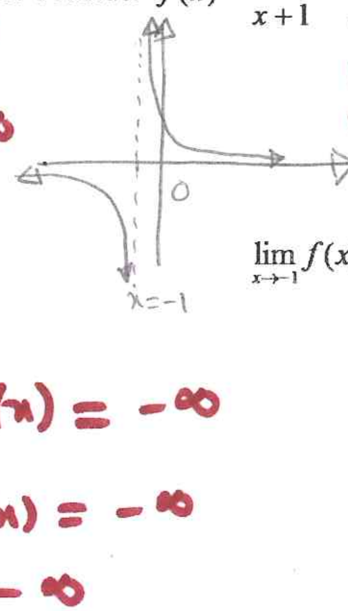


$\lim_{x\to2} f(x)=$ ∞

b. Consider $f(x)=\dfrac{1}{x+1}$

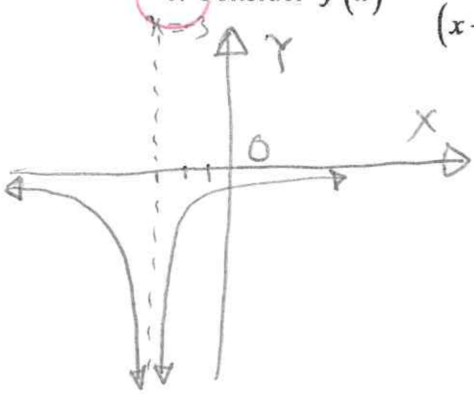
$\lim_{x\to-1^+} f(x)=+\infty$

$\lim_{x\to-1^-} f(x)=-\infty$



$\lim_{x\to-1} f(x)=$ DNE

c. Consider $f(x)=-\dfrac{1}{(x+3)^4}$



$\lim_{x\to-3^+} f(x)=-\infty$

$\lim_{x\to-3^-} f(x)=-\infty$

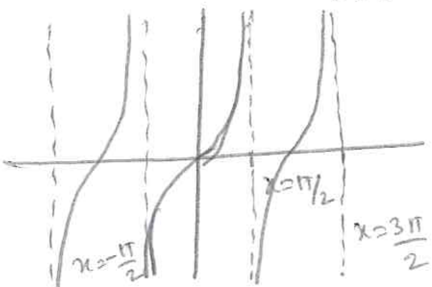
$\lim_{x\to-3} f(x)=$ $-\infty$

6 Definition The line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if at least one of the following statements is true:

$$\lim_{x \to a} f(x) = \infty \qquad \lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = \infty$$

$$\lim_{x \to a} f(x) = -\infty \qquad \lim_{x \to a^-} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = -\infty$$

15a. Consider $f(x) = \tan x$



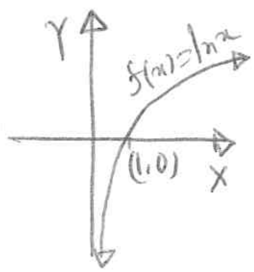
$x=-\frac{3\pi}{2}$

$\lim_{x \to \frac{\pi}{2}^+} \tan x = -\infty$

$\lim_{x \to \frac{\pi}{2}^-} \tan x = +\infty$

$\lim_{x \to \frac{\pi}{2}} \tan x$ DNE

b. Consider $f(x) = \ln x$



$\lim_{x \to 0^+} \ln x = -\infty$

Infinite Limits Analytically

When finding a limit the first thing to try is direct substitution.

If you get $\frac{0}{0}$, you have more work to do. (We will discuss this in section 2.3)

If you get $\frac{\text{constant} \neq 0}{0}$, the limit is infinite. The text says "The fraction $\frac{a}{b}$ grows arbitrarily large in magnitude if b approaches 0 while a remains nonzero and relatively constant."

It must be considered whether the denominator is approaching 0 through negative values or whether the denominator is approaching 0 through positive values.

Then apply the traditional positive/negative rules. $\frac{+}{-} = -$, $\frac{+}{+} = +$, $\frac{-}{+} = -$, $\frac{-}{-} = +$

The text uses writing above and below the function to describe what is happening.

For example:

$$\lim_{x \to 3^+} \frac{\overset{\text{approaches } -13}{\overline{2 - 5x}}}{\underset{\substack{\cancel{\text{negative}}\text{ and} \\ \text{positive} \\ \text{approaches } 0}}{\underline{x - 3}}} = -\infty,$$

To simplify the writing, the 1040/1070 and 1060 instructors have chosen to use a "small –" or "small +" notation or "sp" "sn"

This presentation will be required of you on your HW and test answers to get full credit.

For example:

$$\lim_{x \to 3^+} \frac{2 - 5x}{x - 3} = \frac{-13}{\text{small} +} = -\infty$$

16. Evaluate a. $\lim_{x\to 3^-}\frac{2}{(x-3)^3}$ b. $\lim_{x\to 3^+}\frac{2}{(x-3)^3}$ c. $\lim_{x\to 3}\frac{2}{(x-3)^3}$ dne

$= \frac{2}{\text{Small}-} = -\infty$ | $= \frac{2}{\text{small}+} = +\infty$

17. Evaluate the limits

a. $\lim_{x\to -8^+}\frac{2x}{x+8}$ b. $\lim_{x\to -8^-}\frac{2x}{x+8}$ c. $\lim_{x\to -8}\frac{2x}{x+8}$ DNE

$= \frac{-16}{\text{Small}+} = -\infty$ $= \frac{-16}{\text{small}-} = +\infty$

18. a. Evaluate $\lim_{x\to 7}\frac{4}{(x-7)^2}$ $= \frac{4}{\text{Small}+} = +\infty$

b. Why was $\lim_{x\to 7}\frac{4}{(x-7)^2}$ not presented as a one sided limit?

Square in denominator makes denominator positive no matter which side 7 is approached from

19. (2.2: #30-36 even) Note: #36 looks ahead a bit

30. $\lim_{x\to -3^-}\frac{x+2}{x+3}$ $= \frac{-1}{\text{Small}-} = +\infty$

32. $\lim_{x\to 5^-}\frac{e^x}{(x-5)^3}$ $= \frac{e^5}{(\text{small}-)^3} = -\infty$

34. $\lim_{x\to \pi^-}\cot x$ $= \lim_{x\to\pi^-}\frac{\cos x}{\sin x} = \frac{\cos\pi}{\text{Small}+} = \frac{-1}{\text{Small}+} = -\infty$

36. $\lim_{x\to 2^-}\frac{x^2-2x}{x^2-4x+4}$ $= \lim_{x\to 2^-}\frac{x(x-2)}{(x-2)(x-2)} = \lim_{x\to 2^-}\frac{x}{x-2}$

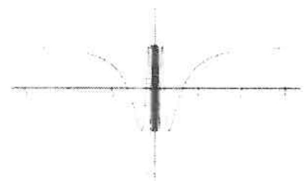
$\lim_{x\to 2} = \frac{4-4}{4-8+4} = \frac{0}{0}$

$= \frac{2}{\text{Small}-} = -\infty$

Oscillations There are some functions with "**Strange Behavior**"

20. Find $\lim_{x \to 0} \cos\left(\frac{1}{x}\right)$ DNE

oscillates too much near $x=0$ to be able to find a limit.

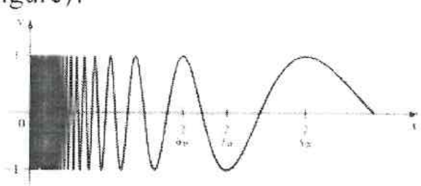


21.a. Create a table of values of $\sin\left(\frac{1}{x}\right)$ for $x = \frac{2}{\pi}, \frac{2}{3\pi}, \frac{2}{5\pi}, \frac{2}{7\pi}, \frac{2}{9\pi}$, and $\frac{2}{11\pi}$. Describe the pattern of values you observe.

x	$\frac{2}{\pi}$	$\frac{2}{3\pi}$	$\frac{2}{5\pi}$	$\frac{2}{7\pi}$	$\frac{2}{9\pi}$	$\frac{2}{11\pi}$
$\sin\frac{1}{x}$	1	-1	1	-1	1	-1

Values of $\sin\frac{1}{x}$ alternates between -1 and 1

b. Why does a graphing utility have difficulty plotting the graph of $y = \sin\left(\frac{1}{x}\right)$ near $x = 0$ (see figure)?



Since $\sin\left(\frac{1}{0}\right)$ is undefined. The graph oscillates too much near $x=0$. There would still be a hole at $x=0$.

c. What do you conclude about $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$?

DNE

Summary examples of ways limits fail to exist:

1. The graph jumps at the specific input.
2. The function approaches positive or negative infinity at the specific input. It grows to large to have a limit. (This only needs to happen on one side for the limit to fail to exist)
3. The function oscillates too much to have a limit. (This only needs to happen on one side for the limit to fail to exist.)

Note on #2: Technically, the "infinite limits" do not exist. But we choose to indicate that they approach infinity in our notation.

HW 2.2: #1, 3, 5, 6, 8, 17, 21, 37