Section: _9__ LA-20 **Score: ______/20**

Names of Group Members PRESENT: ______________________________________

Credit is only given for group work to those present on all days L&LA is worked in class and who are also present the day it is turned in. ______________________________________

____

Limits at infinity

Limits at infinity – as opposed to infinite limits – occur when the independent variable becomes large in magnitude. Limits at infinity determine what is called the *end behavior* of a function.

1 Definition Let f be a function defined on some interval (a, ∞). Then

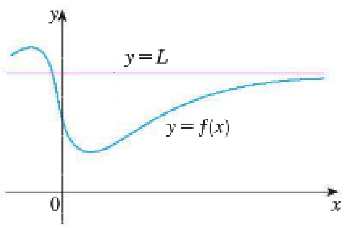
$$\lim_{x \to \infty} f(x) = L$$

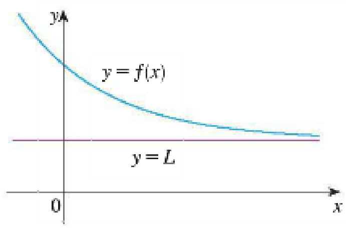
means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large.

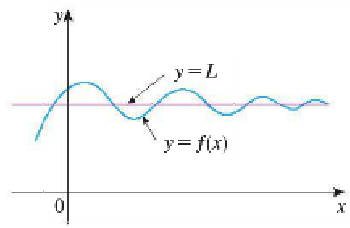
2 Definition Let f be a function defined on some interval $(-\infty, a)$. Then

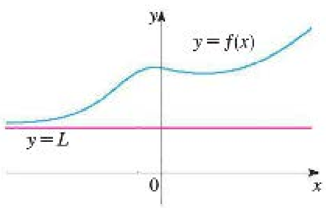
$$\lim_{x \to -\infty} f(x) = L$$

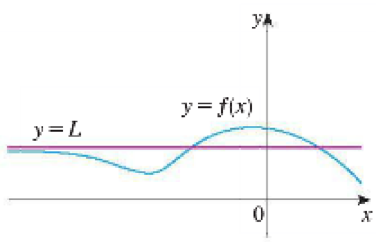
means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large negative.









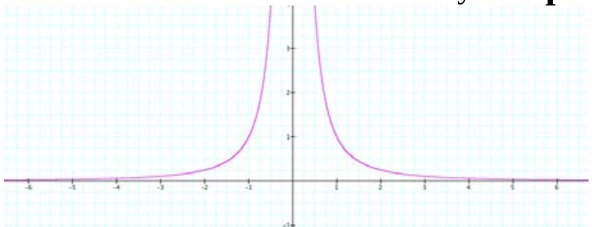


3 Definition The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = L \qquad \text{or} \qquad \lim_{x \to -\infty} f(x) = L$$

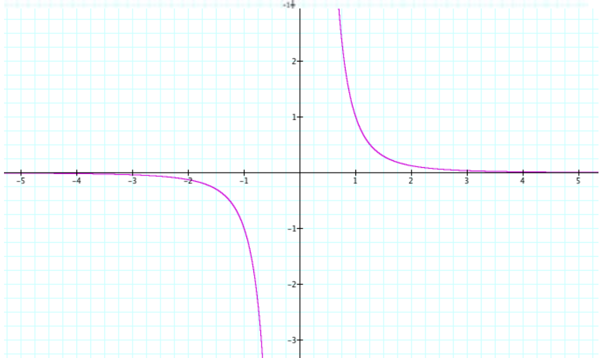
1. Lets consider several functions and their limits at infinity **Graphically**:

a. $f(x)=\dfrac{1}{x^2}$



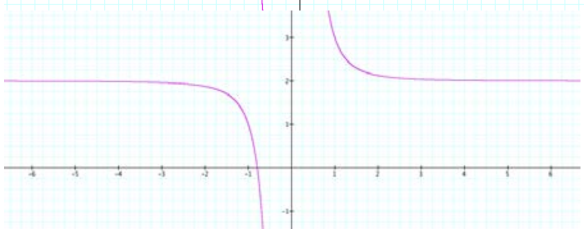
$\lim\limits_{x\to\infty} f(x)$ $\lim\limits_{x\to-\infty} f(x)$

b. $f(x)=\dfrac{1}{x^3}$



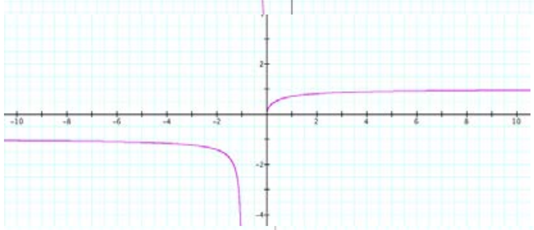
$\lim\limits_{x\to\infty} f(x)$ $\lim\limits_{x\to-\infty} f(x)$

c. $f(x)=\dfrac{1}{x^3}+2$



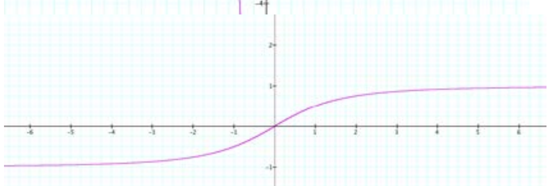
$\lim\limits_{x\to\infty} f(x)$ $\lim\limits_{x\to-\infty} f(x)$

d. $f(x)=\dfrac{x}{\sqrt{x^2+x}}$



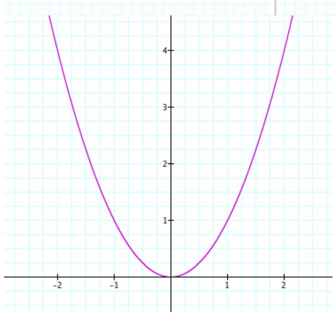
$\lim\limits_{x\to\infty} f(x)$ $\lim\limits_{x\to-\infty} f(x)$

e. $f(x)=\dfrac{x}{\sqrt{x^2+3}}$



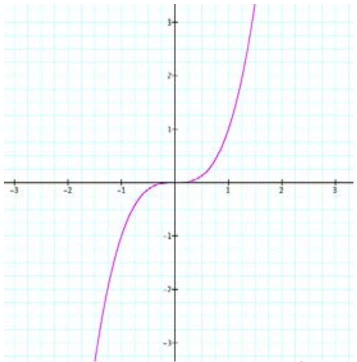
$\lim\limits_{x\to\infty} f(x)$ $\lim\limits_{x\to-\infty} f(x)$

f. $f(x)=x^2$



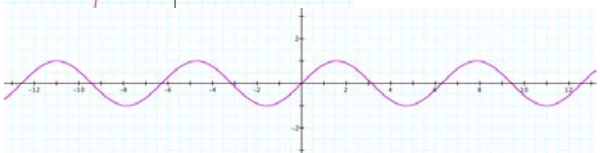
$\lim\limits_{x\to\infty} f(x)$ $\lim\limits_{x\to-\infty} f(x)$

g. $f(x)=x^3$



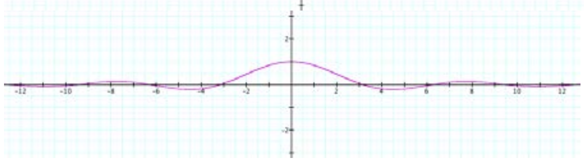
$\lim_{x\to\infty} f(x)$ $\lim_{x\to-\infty} f(x)$

h. $f(x)=\sin x$



$\lim_{x\to\infty} f(x)$ $\lim_{x\to-\infty} f(x)$

i. $f(x)=\dfrac{\sin x}{x}$



$\lim_{x\to\infty} f(x)$ $\lim_{x\to-\infty} f(x)$

2. Summary questions:

How many horizontal asymptotes can a function have?

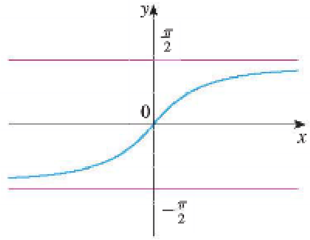
Can limits at infinity be infinite?

What is the end behavior of $f(x)=\sin x$ and $f(x)=\cos x$?

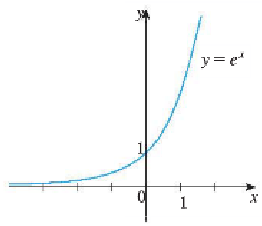
Can a function cross its horizontal asymptote(s)?

Before we can work on limits at infinity Analytically we need to consider a couple of special functions and develop graphical support for an important theorem.

Inverse Tangent Function:

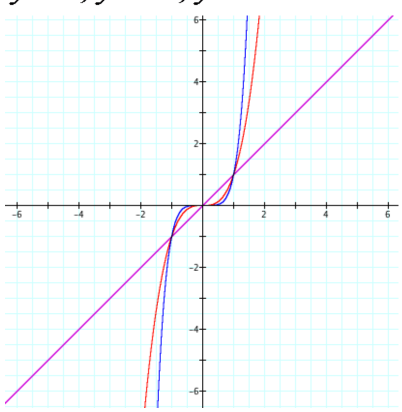


the Natural Exponential Function:



Basic Power Functions

$y = x, y = x^3, y = x^5$

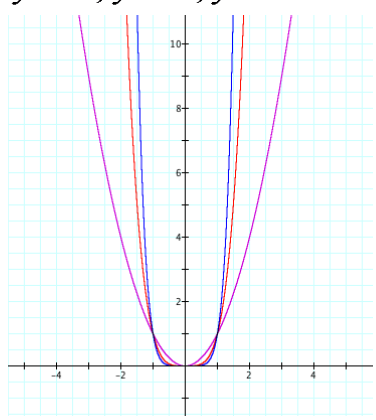


If n is an odd positive integer,

$\lim_{x \to \infty} x^n$

$\lim_{x \to -\infty} x^n$

$y = x^2, y = x^4, y = x^6$



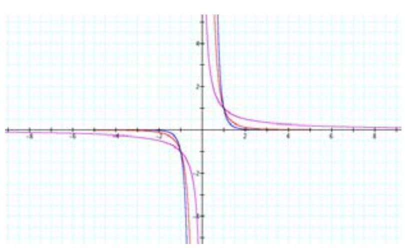
If n is an even positive integer,

$\lim_{x \to \infty} x^n$

$\lim_{x \to -\infty} x^n$

Basic Rational Functions

$y = \frac{1}{x},\ y = \frac{1}{x^3},\ y = \frac{1}{x^5}$

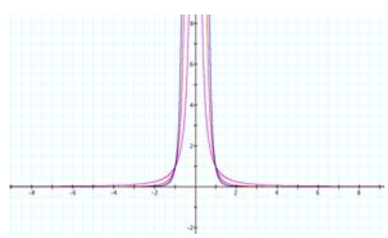


If n is an odd positive integer,

$\lim_{x \to \infty} x^{-n} = \lim_{x \to \infty} \frac{1}{x^n}$

$\lim_{x \to -\infty} x^{-n} = \lim_{x \to -\infty} \frac{1}{x^n}$

$y = \frac{1}{x^2},\ y = \frac{1}{x^4},\ y = \frac{1}{x^6}$



If n is an even positive integer,

$\lim_{x \to \infty} x^{-n} = \lim_{x \to \infty} \frac{1}{x^n}$

$\lim_{x \to -\infty} x^{-n} = \lim_{x \to -\infty} \frac{1}{x^n}$

5 Theorem If $r > 0$ is a rational number, then

$$\lim_{x \to \infty} \frac{1}{x^r} = 0$$

If $r > 0$ is a rational number such that x^r is defined for all x, then

$$\lim_{x \to -\infty} \frac{1}{x^r} = 0$$

Also notice $\lim_{x \to \pm\infty} \frac{k}{cx^n} = \frac{k}{c} \lim_{x \to \pm\infty} \frac{1}{x^n} = \frac{k}{c} \cdot 0 = 0$.

Now we are read to work **on** limits at infinity **Algebraically:**

3. Evaluate $\lim_{x \to \infty} \left(5 + \frac{1}{x} + \frac{10}{x^2} \right)$

4. a. $\lim_{x \to -\infty} 2x^{-8}$

b. $\lim_{x \to \infty} (-12x^{-5})$

When finding a limit as $x \to \infty$ or $x \to -\infty$ for a rational function $\frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials, we use a technique of multiplying by one in the form on $\frac{\frac{1}{x^n}}{\frac{1}{x^n}}$ where n is the highest power of the denominator $q(x)$.

WARNING: Do NOT use this technique on limits as $x \to a$ where a is finite. You are very likely to get the wrong answer.

5. Evaluate and state the equation for the horizontal asymptote(s) if any.

a. $\lim_{x \to \infty} \frac{1 - x}{2x}$

b. $\lim_{x \to \infty} \frac{1 - x}{x^2}$

c. $\lim_{x \to \infty} \frac{1 - x^2}{2x}$

Yes, you must show all the fractions in your work.

6. Evaluate $\lim_{x\to\infty}\frac{2x^3+7}{x^3-x^2+x+7}$ and $\lim_{x\to-\infty}\frac{2x^3+7}{x^3-x^2+x+7}$ and state the equation for the horizontal asymptote(s) if any.

7. Evaluate $\lim_{x\to\infty}\frac{1}{x^3-4x+1}$ and $\lim_{x\to-\infty}\frac{1}{x^3-4x+1}$ and state the equation for the horizontal asymptote(s) if any.

8. Evaluate $\lim_{x\to\infty}\frac{3x^5+2x^2-2}{4x^4-3x}$ and $\lim_{x\to-\infty}\frac{3x^5+2x^2-2}{4x^4-3x}$ and state the equation for the horizontal asymptote(s) if any.

9. Evaluate $\lim_{x\to\infty}\frac{4x^2-2x+3}{7x^2-1}$ and $\lim_{x\to-\infty}\frac{4x^2-2x+3}{7x^2-1}$ and state the equation for the horizontal asymptote(s) if any.

End Behavior and Asymptotes of Rational Functions

Suppose $f(x)=\frac{p(x)}{q(x)}$ is a rational function, where

$$p(x)=a_m x^m + a_{m-1}\, x^{m-1}+\cdots+a_2 x^2+a_1 x+a_0 \quad \text{and}$$
$$q(x)=b_n x^n + b_{n-1} x^{n-1}+\cdots+b_2 x^2+b_1 x+b_0$$

and $a_m \neq 0$ and $b_n \neq 0$.

a. If $m<n$, then $\lim_{x\to\pm\infty} f(x)=0$, and $y=0$ is a horizontal asymptote of f.

b. If $m=n$, then $\lim_{x\to\pm\infty} f(x)=a_m/b_n$, and $y=a_m/b_n$ is a horizontal asymptote of f.

c. If $m>n$, then $\lim_{x\to\pm\infty} f(x)=\infty$ or $-\infty$, and f has no horizontal asymptote.

d. Assuming that $f(x)$ is in reduced form (p and q share no common factors), vertical asymptotes occur at the zeros of q.

If we have a function $\dfrac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$are NOT both polynomials (not an rational function, but an algebraic function), we have more work to do alongside the "highest power of the denominator" technique.

10. Evaluate $\lim\limits_{x\to\infty}\dfrac{7x^3-2}{-x^3+\sqrt{25x^6+4}}$and $\lim\limits_{x\to-\infty}\dfrac{7x^3-2}{-x^3+\sqrt{25x^6+4}}$. State the equation(s) of the horizontal asymptotes.

11. Evaluate $\lim\limits_{x\to\infty}\dfrac{\sqrt[3]{x^6+8}}{4x^2+\sqrt{3x^4+1}}$and $\lim\limits_{x\to-\infty}\dfrac{\sqrt[3]{x^6+8}}{4x^2+\sqrt{3x^4+1}}$. State the equation(s) of the horizontal asymptotes.

12. (2.6:#26) Evaluate $\lim\limits_{x\to-\infty}\left(x+\sqrt{x^2+2x}\right)$.

13. Find the limit or show that it does not exist.

a. (2.6:#30) $\lim_{x\to\infty}\left(e^{-x}+2\cos 3x\right)$

b. (2.6:#31) $\lim_{x\to -\infty}\left(x^4+x^5\right)$

c. (2.6:#33) $\lim_{x\to\infty}\tan^{-1}\left(e^x\right)$

d. (2.6:#34) $\lim_{x\to\infty}\frac{e^{3x}-e^{-3x}}{e^{3x}+e^{-3x}}$

Let's look at the horizontal and vertical asymptotes of functions together:

$\boxed{3}$ **Definition** The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = L \qquad \text{or} \qquad \lim_{x\to-\infty} f(x) = L$$

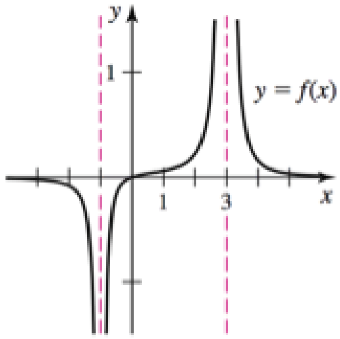
$\boxed{6}$ **Definition** The line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if at least one of the following statements is true:

$$\lim_{x\to a} f(x) = \infty \qquad \lim_{x\to a^-} f(x) = \infty \qquad \lim_{x\to a^+} f(x) = \infty$$

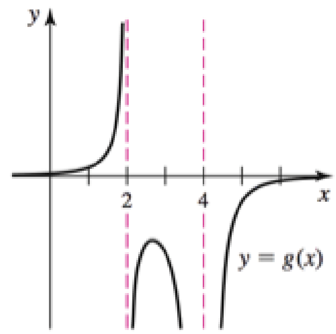
$$\lim_{x\to a} f(x) = -\infty \qquad \lim_{x\to a^-} f(x) = -\infty \qquad \lim_{x\to a^+} f(x) = -\infty$$

14. State the horizontal and vertical asymptotes of these functions.

a.



b.



Analytically, possible candidates for vertical asymptotes for a rational function $\frac{p(x)}{q(x)}$ are the x-values where $q(x)=0$. But $q(x)=0$, does not guarantee a vertical asymptote. Again, we have more work to do to decide what is happening at these x-values.

15. Find the vertical and horizontal asymptotes of $f(x)=\frac{2x}{x^2+x}$. Prove the asymptotes with limit(s). If there is no asymptote, say so.

16. Find the vertical and horizontal asymptotes of $f(x)=\frac{x^2-1}{2x+4}$. Prove the asymptotes with limit(s). If there is no asymptote, say so.

17. Find the vertical and horizontal asymptotes of $f(x)=\dfrac{x^2-3x+2}{x^3-2x^2}$. Prove the asymptotes with limit(s). If there is no asymptote, say so.

18. Find the vertical and horizontal asymptotes of $f(x)=\dfrac{x^3-10x^2+16x}{x^2-8x}$. Prove the asymptotes with limit(s). If there is no asymptote, say so.

19. Find the vertical and horizontal asymptotes of $f(x)=\dfrac{2x}{\sqrt{x^2-x-2}}$

Prove the asymptotes with limit(s).

20. Sketch a graph of a function that satisfies the given conditions. You need not come up with an equation for the function.

$f(1)=2,\ f(-1)=3, f(0)$ is undefined, $\lim_{x\to 1} f(x)=4,\ \lim_{x\to 3^-} f(x)=\infty,\ \lim_{x\to 3^+} f(x)=-\infty$ and there are no other infinite limits.

21. Sketch a graph of polynomials $p(x)$ and $q(x)$ such that $\frac{p(x)}{q(x)}$ is undefined at $x=1$ and $x=2$, but $\frac{p(x)}{q(x)}$ has a vertical asymptote only at $x=2$.

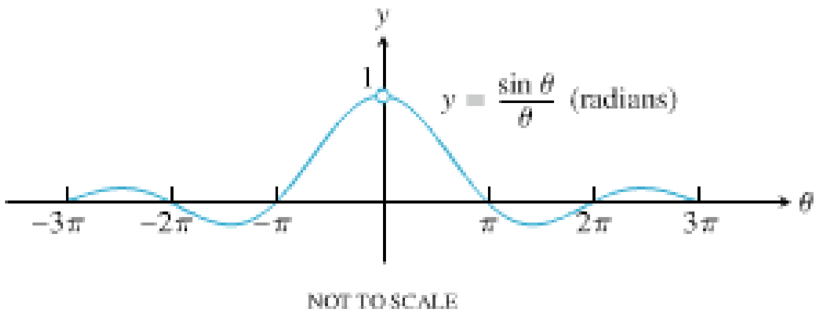
22. Sketch a possible graph of a function $f(x)$ that satisfies all of the given conditions. Be sure to identify all vertical and horizontal asymptotes.

$\lim_{x\to 0} f(x)=-\infty,\ \lim_{x\to 2} f(x)=\frac{5}{4},\ \lim_{x\to\pm\infty} f(x)=1,\ f(2)$ is undefined, $f(1)=1,\ f(-1)=-1$

23. Sketch a possible graph of a function $f(x)$ that satisfies all of the given conditions. Be sure to identify all vertical and horizontal asymptotes.

$\lim_{x\to -1^-} f(x)=\infty,\ \lim_{x\to -1^+} f(x)=-\infty\ \lim_{x\to 0} f(x)=0,\ \lim_{x\to\pm\infty} f(x)=2,\ f(0)=-2,\ f(1)=1,\ f(-2)=4$

Some functions need the Squeeze Theorem as $x \to \pm\infty$, particularly when trig functions are involved.
Here is a graph from another text that is not drawn to scale to emphasize some properties of the function $f(x) = \dfrac{\sin x}{x}$.



24. Consider $f(x) = \dfrac{\sin\theta}{\theta}$. Answer the following based on this graph.

a. $f(0)$ b. $\lim\limits_{x\to 0} \dfrac{\sin x}{x}$ c. $\lim\limits_{x\to\infty} \dfrac{\sin x}{x}$ d. $\lim\limits_{x\to -\infty} \dfrac{\sin x}{x}$

25. (2.6: #57) Use the Squeeze Theorem to prove that $\lim\limits_{x\to\infty} \dfrac{\sin x}{x} = 0$

26. Find $\lim\limits_{\theta\to -\infty} \dfrac{\cos\theta}{3\theta}$

27. (2.6: #37) Evaluate $\lim\limits_{x\to\infty} e^{-2x} \cos x$.

HW 2.6: #6, 9, 16, 20, 21, 24, 25, 29, 32, 36, 38, 41, 44, 45