# High order algebraic splitting for magnetohydrodynamics simulation

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October 27, 2016

**Abstract**

This paper proposes, analyzes and tests high order algebraic splitting methods for magnetohydrodynamic (MHD) flows. The main idea is to apply, at each time step, Yosida-type algebraic splitting to a block saddle point problem that arises from a particular incremental formulation of MHD. By doing so, we dramatically reduce the complexity of the nonsymmetric block Schur complement by decoupling it into two Stokes-type Schur complements, each of which is symmetric positive definite and also is the same at each time step. We prove the splitting is $O(\Delta t^3)$ accurate, and if used together with (block-)pressure correction, is fourth order. A full analysis of the solver is given, both as a linear algebraic approximation, but also in a finite element context that uses the natural spatial norms. Numerical tests are given to illustrate the theory and show the effectiveness of the method.

## 1 Introduction

The flow of electrically conducting fluids in the presence of a magnetic field is called magnetohydrodynamics (MHD) flow. Such flows arise, for example, in astrophysics and geophysics [14, 20, 9, 8, 2, 3], liquid metal cooling of nuclear reactors [1, 12, 24], and process metallurgy [7]. However, simulation of these flows is known to be quite difficult, and one major reason for this is the difficulties that arise because of the large, nonsymmetric, ill-conditioned block saddle point linear systems that arise at each time step. It is the purpose of this paper to propose, analyze and test an accurate and efficient linear solver for these systems, by extending some recent work of the authors on saddle point linear systems for Navier-Stokes (NS) [22] to the block saddle point systems that arise in MHD. The key ideas are combining the Yosida algebraic splitting with a particular incremental formulation of the MHD system at each time step, which leads to a Schur complement (the main difficulty of the linear solve) that decouples into two Stokes-type Schur complements, each of which are symmetric positive definite, and are also the same at each time step. We will fully analyze the splitting error and show it is third order (fourth order if block pressure-correction is applied), and to our knowledge, this is the first higher order algebraic splitting method studied for the block saddle point systems in MHD.

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The governing equations of MHD consist of a non-linear system of partial differential equations that nonlinearly couple the NS equations for fluid flow to Maxwell's equations for electromagnetics, which in the case of smooth boundaries can be written as

$$\underline{u}_t - \nu \Delta \underline{u} + (\underline{u} \cdot \nabla)\underline{u} - s(\underline{B} \cdot \nabla)\underline{B} + \nabla \underline{P} = \underline{f}, \tag{1.1}$$

$$\nabla \cdot u = 0, \tag{1.2}$$

$$\underline{B}_t - \nu_m \Delta \underline{B} + (\underline{u} \cdot \nabla)\underline{B} - (\underline{B} \cdot \nabla)\underline{u} - \nabla \underline{\lambda} = \nabla \times \underline{g}, \tag{1.3}$$

$$\nabla \cdot \underline{B} = 0, \tag{1.4}$$

where $\underline{u}$ is the velocity, $\underline{P} := \underline{p} + \frac{s}{2}|\underline{B}|^2$ is the modified pressure and $p$ the pressure, $\underline{f}$ is the body force, $\nabla \times \underline{g}$ is a force acting on the magnetic field $\underline{B}$, $\nu$ is the kinematic viscosity (inversely proportional to the Reynolds number), $\nu_m$ is the magnetic diffusivity (inversely proportional to the magnetic Reynolds number), $s$ is the coupling number, and $\underline{\lambda} := \nu_m \nabla \cdot \underline{B}$ is a dummy variable acting as a Lagrange multiplier corresponding to the solenoidal constraint of the magnetic field. If $\nabla \cdot \underline{B}_0 = 0$ in the continuous case, then (1.4) is enforced by (1.3), and $\underline{\lambda} = 0$. However, in the discrete case, non-zero $\underline{\lambda}$ will allow for the divergence-free condition on $\underline{B}$ to be enforced explicitly [6].

Applying a temporal discretization to the MHD system above, we obtain the problem at each time step: find a velocity $u$, a magnetic field $B$ and Lagrange multipliers $P, \lambda$ satisfying

$$\frac{\alpha}{\Delta t} u - \nu \Delta u + \mathcal{U} \cdot \nabla u - \mathcal{B} \cdot \nabla B + \nabla P = \tilde{f}, \tag{1.5}$$

$$\nabla \cdot u = 0, \tag{1.6}$$

$$\frac{\alpha}{\Delta t} B - \nu_m \Delta B + \mathcal{U} \cdot \nabla B - \mathcal{B} \cdot \nabla u - \nabla \lambda = \tilde{g}, \tag{1.7}$$

$$\nabla \cdot B = 0, \tag{1.8}$$

where $\Delta t$ is a time-step size, $\mathcal{U}$ and $\mathcal{B}$ are given solenoidal velocity and magnetic fields (e.g. extrapolated from previous time steps), and $\tilde{f}$ and $\tilde{g}$ are the forcing terms combined with left hand side terms that are known from previous time steps. For a BDF2 time-stepping scheme, for example, $\alpha = \frac{3}{2}$, $\mathcal{U} = 2u^n - u^{n-1}$, $\mathcal{B} = 2B^n - B^{n-1}$, $\tilde{f} = f + \frac{2}{\Delta t}u^n - \frac{1}{\Delta t}u^{n-1}$, and $\tilde{g} = \nabla \times g + \frac{2}{\Delta t}B^n - \frac{1}{\Delta t}B^{n-1}$.

Applying a finite element discretization to (1.5)-(1.8), where we search for $\bar{u}, \bar{B} \in X_h$ and $\bar{P}, \bar{\lambda} \in Q_h$, with $(X_h, Q_h)$ satisfying the LBB stability property [5] (details of the finite element discretization are given in section 3), a block linear system arises of the form

$$\begin{pmatrix} A_1 & N_1 & C_1 & 0 \\ N_2 & A_2 & 0 & C_1 \\ C_1^T & 0 & 0 & 0 \\ 0 & C_1^T & 0 & 0 \end{pmatrix} \begin{pmatrix} \bar{u} \\ \bar{B} \\ \bar{P} \\ \bar{\lambda} \end{pmatrix} = \begin{pmatrix} \bar{F}_1 \\ \bar{F}_2 \\ 0 \\ 0 \end{pmatrix}, \tag{1.9}$$

where $A_1 := \frac{\alpha}{\Delta t}M + \nu S + \tilde{N}_1$, $A_2 := \frac{\alpha}{\Delta t}M + \nu_m S + \tilde{N}_2$, with $M$ denoting the $X_h$ mass matrix, $S$ the $X_h$ stiffness matrix, $C_1$ the rectangular matrix representing the gradient operator acting on $Q_h$ and tested with $X_h$, $\tilde{N}_1$ the nonlinear contributions from the momentum equation, and $\tilde{N}_2$ the nonlinear contributions from Maxwell's equation. Denoting

$$A = \begin{pmatrix} A_1 & N_1 \\ N_2 & A_2 \end{pmatrix}, C = \begin{pmatrix} C_1 & 0 \\ 0 & C_1 \end{pmatrix}, \bar{X} = \begin{pmatrix} \bar{u} \\ \bar{B} \end{pmatrix}, \bar{Y} = \begin{pmatrix} \bar{P} \\ \bar{\lambda} \end{pmatrix}, \bar{F} = \begin{pmatrix} \bar{F}_1 \\ \bar{F}_2 \end{pmatrix},$$

the equation (1.9) can now be written as a block saddle point linear system:

$$\begin{pmatrix} A & C \\ C^T & 0 \end{pmatrix} \begin{pmatrix} \bar{X} \\ \bar{Y} \end{pmatrix} = \begin{pmatrix} \bar{F} \\ 0 \end{pmatrix}. \tag{1.10}$$

A common approach to solving saddle point systems that arise in Navier-Stokes saddle point systems is to apply algebraic splitting methods, which reduces the difficulty of the solves, but introduces error due to the approximations that are made. Yosida-type splitting methods work by making an approximation of the Schur complement. Setting

$$\tilde{A} = \begin{pmatrix} \frac{\alpha}{\Delta t} M + \nu S & 0 \\ 0 & \frac{\alpha}{\Delta t} M + \nu_m S \end{pmatrix},$$

or possibly without viscous contributions, the following approximation of the block LU decomposition is made:

$$\begin{pmatrix} A & C \\ C^T & 0 \end{pmatrix} \approx \begin{pmatrix} A & 0 \\ C^T & -C^T \tilde{A}^{-1} C \end{pmatrix} \begin{pmatrix} I & A^{-1}C \\ 0 & Q \end{pmatrix} = \begin{pmatrix} A & C \\ C^T & C^T A^{-1} C - C^T \tilde{A}^{-1} C Q \end{pmatrix}, \quad (1.11)$$

where $Q = I$ yields the classical Yosida method, and $Q = (C^T \tilde{A}^{-1} A \tilde{A}^{-1} C)^{-1} \left( C^T \tilde{A}^{-1} C \right)$ yields the pressure corrected Yosida method developed by Veneziani et. al [23, 11]. We note that this clever choice of $Q$ increases the accuracy of the Yosida splitting from $O(\Delta t^2)$ to $O(\Delta t^3)$, and requires two more solves with SPD matrix $\tilde{A}$, and one more SPD Schur complement solve. Additionally, this approach has the advantage of easily allowing for adaptive time stepping. One potential disadvantage was shown in [22] with both analysis and numerical tests, which is that the pressure correction step can have error that scales negatively with respect to the spatial mesh width, and thus seems best suited for problems where the temporal scales are smaller than the spatial scales (e.g. for blood flow problems Veneziani et. al has shown it works very well).

Taking $Q = I$ for simplicity, the Yosida approximation requires solving

$$\begin{pmatrix} A & 0 \\ C^T & -C^T \tilde{A}^{-1} C \end{pmatrix} \begin{pmatrix} I & A^{-1}C \\ 0 & I \end{pmatrix} \begin{pmatrix} \hat{X} \\ \hat{Y} \end{pmatrix} = \begin{pmatrix} F \\ 0 \end{pmatrix},$$

which is equivalent to the following three steps:

1. Solve $A\hat{z} = \hat{F}$ for $\hat{z}$.
2. Solve $C^T \widetilde{A}^{-1} C \hat{Y} = C^T \hat{z}$ for $\hat{Y}$.
3. Solve $A\hat{X} = A\hat{z} - C\hat{Y}$ for $\hat{X}$.

The only difference between the linear systems arising from the Yosida method and unaltered linear system is that the Yosida method uses the matrix $\tilde{A}$ instead of $A$ in step 2, but this small change leads to a dramatic reduction of complexity. Since $\tilde{A}$ is block diagonal, the Schur complement $C^T \widetilde{A}^{-1} C$ reduces to

$$C^T \widetilde{A}^{-1} C = \begin{pmatrix} C_1^T \left( \frac{\alpha}{\Delta t} M + \nu S \right)^{-1} C_1 & 0 \\ 0 & C_1^T \left( \frac{\alpha}{\Delta t} M + \nu_m S \right)^{-1} C_1, \end{pmatrix}$$

and thus is decoupled into two smaller SPD time-dependent-Stokes-type Schur complements. It is well known how to solve such systems (see [22] and references therein). Hence the Yosida splitting creates linear systems that are much easier to solve.

Of course, with approximations comes error, and from (1.11) we observe the error created is in the 2,2 block of the recombined matrix, as the term $C^T(A^{-1} - \tilde{A}^{-1})C$ appears instead of 0. An expansion of $A^{-1} - \tilde{A}^{-1}$ from [23] reveals that

$$A^{-1} - \tilde{A}^{-1} = \tilde{A}^{-1} N \tilde{A}^{-1} + (\tilde{A}^{-1} N)^2 \tilde{A}^{-1} + ...,$$

which implies a splitting error of $\mathcal{O}(\Delta t^2)$, since $N = O(1)$ and $\tilde{A} = O(\Delta t^{-1})$. We note that if pressure correction is used, the first term of the expansion is cancelled, producing $\mathcal{O}(\Delta t^3)$ error.

Our goal in this paper is to apply our ideas of [22] for NSE saddle point linear systems to the block MHD systems. In particular, before applying the Yosida approximation, we will rewrite the system (1.5)-(1.8) in terms of increments of the pressure variables $\delta_P = P - P^n$, $\delta_\lambda = \lambda - \lambda^n$, where we seek $(u, \delta_P, B, \delta_\lambda)$ satisfying

$$\frac{\alpha}{\Delta t}u - \nu\Delta u + \mathcal{U}\cdot\nabla u - \mathcal{B}\cdot\nabla B + \nabla\delta_P = \tilde{\tilde{f}}, \tag{1.12}$$

$$\nabla\cdot u = 0, \tag{1.13}$$

$$\frac{\alpha}{\Delta t}B - \nu_m\Delta B + \mathcal{U}\cdot\nabla B - \mathcal{B}\cdot\nabla u - \nabla\delta_\lambda = \tilde{\tilde{g}}, \tag{1.14}$$

$$\nabla\cdot B = 0, \tag{1.15}$$

with appropriately defined right hand sides. Since the problem is linearized at each time step, this change of variables produces the exact same matrix, but with an altered right hand side. The general idea is that the Yosida appproximation creates $O(\Delta t^2)$ error in the primitive variables, so if approximation is made to $O(\Delta t)$ increments instead of the $O(1)$ original variables, then the total error will be reduced to $O(\Delta t^3)$ (and if pressure correction is used, then accuracy will be $O(\Delta t^4)$). Analysis and testing of this idea showed that it works quite well, and these higher order rates were found to hold. Interestingly, our finite element analysis in the NS-case revealed that the same asymptotic rates could be found if only the pressure increment was used [22], and velocity was solved for as usual; this leads back to a method proposed for the NSE in [13]. Since using only pressure increments is a simpler approach, we will apply this approach herein. Another nice feature of using only the pressure increments is that grad-div stabilization can be immediately applied, which is well known to provide for reduction in divergence errors, improvements in overall accuracy, improvements in accuracy in Yosida methods, and aids in effectively preconditioning Schur complement solvers [21, 19, 17, 15, 4].

The purpose of this paper will be to analyze and test the Yosida method for the system (1.12)-(1.15), and we call this method the 'Yosida-updates' (YU) method for MHD. The linear algebraic splitting analysis is identical to the NSE case, and is discussed above. However, such analysis is not attractive mathematically, since the linear algebra vector norms are not in the natural spaces of the variables, and any negative scaling of the error with respect to the spatial mesh width would be neglected. Hence, we apply a finite element-type error analysis to quantify the difference between solutions found by solving the linear system using an exact linear solver and the YU splitting approximation by casting the algebraic systems back into finite element problems. Our analysis considers the basic YU case, with no grad-div stabilization and without pressure correction. However, the ideas of [21, 22] can be applied to the block systems herein to extend our results further. In our computational tests, however, both grad-div stabilization and pressure correction are used. The paper is organized as follows. Section 2 gives notation and preliminary results used throughout the paper. Section 3 performs the analysis for YU applied to MHD while Section 4 the analysis for YUPC applied to MHD, and in Section 5 we give numerical experiments to illustrate the theory and show the effectiveness of the method on a benchmark test problem.

# 2 Mathematical preliminaries

In this study, we consider a domain $\Omega \subset \mathbb{R}^d$, $(d = 2, 3)$, that is either a convex polygon or has a smooth boundary. Denote the $L^2$-inner product and induced norm by $(\cdot,\cdot)$ and $\|\cdot\|$, respectively. All other norms will be clearly labeled.

The natural function spaces for analysis of the MHD system above are

$$X := H_0^1(\Omega) = \{v \in H^1(\Omega),\ v = 0 \text{ on } \partial\Omega\},$$
$$Q := L_0^2(\Omega) = \{q \in L^2(\Omega),\ \int_{\partial\Omega} q\, dx = 0\ \}.$$

Use of these boundary conditions are mainly for analysis purposes in the finite element context. Extension to periodic boundary conditions or inhomogeneous Dirichlet conditions can be performed. Other types of boundary conditions could require additional analysis in the finite element context, although the linear algebra analysis above will hold for any commonly used boundary conditions.

Recall that the Poincaré-Friedrich's inequality holds in $X$: there exists a constant $C_{PF} > 0$, dependent only on the size of the domain $\Omega$, such that for every $\phi \in X$, $\|\phi\| \leq C_{PF}\|\nabla\phi\|$.

We assume finite element spaces $(X_h, Q_h) \subset (X, Q)$ satisfy the LBB-condition,

$$\inf_{q_h \in Q_h} \sup_{v_h \in X_h} \frac{(\nabla \cdot v_h, q_h)}{\|\nabla v_h\|} \geq \beta > 0,$$

with $\beta$ independent of $h$. The discretely divergence-free subspace of $X_h$ is denoted by

$$V_h := \{v_h \in X_h :\ \ (\nabla \cdot v_h, q_h) = 0\ \ \forall q_h \in Q_h\}.$$

We also assume the mesh is sufficiently regular for the inverse inequality to hold in $X_h$: there exists a constant $C_i$, independent of the mesh width $h$, such that

$$\|\nabla\phi_h\| \leq C_i h^{-1}\|\phi_h\|,\ \ \forall\phi_h \in X_h.$$

# 3 Analysis of the Yosida updates method

The linear algebra analysis of the splitting methods is in terms of the vector norms of the coefficients of the variables, and not in terms of the natural norms of the finite element spaces. Thus, important constants and potential negative scalings with respect to the mesh width could be left out of the linear algebra results. We begin by presenting the numerical method related to the usual finite element discretization of the MHD equations. For simplicity of analysis, we only consider one step time discretization, and take the coupling number $s = 1$.

The exact (unapproximated) single step discrete MHD scheme is defined by: find $(\hat{u}_h, \hat{P}_h, \hat{B}_h, \hat{\lambda}_h) \in (X_h, Q_h, X_h, Q_h)$ satisfying, $\forall(v_h, q_h, \psi_h, r_h) \in (X_h, Q_h, X_h, Q_h)$,

$$\frac{\alpha}{\Delta t}(\hat{u}_h, v_h) + \nu(\nabla\hat{u}_h, \nabla v_h) + (\mathcal{U} \cdot \nabla\hat{u}_h, v_h) - (\mathcal{B} \cdot \nabla\hat{B}_h, v_h) - (\hat{P}_h, \nabla \cdot v_h) = (\widetilde{f}, v_h), \quad (3.1)$$

$$(\nabla \cdot \hat{u}_h, q_h) = 0, \quad (3.2)$$

$$\frac{\alpha}{\Delta t}\left(\hat{B}_h, \psi_h\right) + \nu_m(\nabla\hat{B}_h, \nabla\psi_h) + (\mathcal{U} \cdot \nabla\hat{B}_h, \psi_h) - (\mathcal{B} \cdot \nabla\hat{u}_h, \psi_h) + (\hat{\lambda}_h, \nabla \cdot \psi_h) = (\widetilde{g}, \psi_h), \quad (3.3)$$

$$(\nabla \cdot \hat{B}_h, r_h) = 0. \quad (3.4)$$

Given $(P^n, \lambda^n) \in (Q_h, Q_h)$, which are the solutions from the previous step, denote $\hat{\delta}_P := \hat{P}_h - P^n$, $\hat{\delta}_\lambda := \hat{\lambda}_h - \lambda^n$. Then the scheme (3.1)-(3.4) is equivalently written in terms of velocity, magnetic field, and updates of the Lagrange multipliers: find $(\hat{u}_h, \hat{\delta_P}, \hat{B}_h, \hat{\delta}_\lambda) \in (X_h, Q_h, X_h, Q_h)$ such that $\forall(v_h, q_h, \psi_h, r_h) \in (X_h, Q_h, X_h, Q_h)$,

$$\frac{\alpha}{\Delta t}(\hat{u}_h, v_h) + \nu(\nabla\hat{u}_h, \nabla v_h) + (\mathcal{U}\cdot\nabla\hat{u}_h, v_h) - (\mathcal{B}\cdot\nabla\hat{B}_h, v_h) - (\hat{\delta}_P, \nabla\cdot v_h) = (\tilde{\tilde{f}}, v_h), \quad (3.5)$$

$$(\nabla\cdot\hat{u}_h, q_h) = 0, \quad (3.6)$$

$$\frac{\alpha}{\Delta t}\left(\hat{B}_h, \psi_h\right) + \nu_m(\nabla\hat{B}_h, \nabla\psi_h) + (\mathcal{U}\cdot\nabla\hat{B}_h, \psi_h) - (\mathcal{B}\cdot\nabla\hat{u}_h, \psi_h) + (\hat{\delta}_\lambda, \nabla\cdot\psi_h) = (\tilde{\tilde{g}}, \psi_h), \quad (3.7)$$

$$(\nabla\cdot\hat{B}_h, r_h) = 0, \quad (3.8)$$

where $\tilde{\tilde{f}} := \tilde{f} - \nabla P^n$, $\tilde{\tilde{g}} := \tilde{g} + \nabla\lambda^n$. To recover $\hat{P}_h$ and $\hat{\lambda}_h$, add the increments to the previous time step solutions: $\hat{P}_h = \hat{\delta}_P + P^n$, $\hat{\lambda}_h = \hat{\delta}_\lambda + \lambda^n$. The system (3.5)-(3.8) produces the block saddle point linear system:

$$\begin{pmatrix} A & C \\ C^T & 0 \end{pmatrix}\begin{pmatrix} \hat{X} \\ \hat{\delta}_Y \end{pmatrix} = \begin{pmatrix} F \\ 0 \end{pmatrix}, \quad (3.9)$$

where $\hat{X} = \begin{pmatrix} \hat{u} \\ \hat{B} \end{pmatrix}$, $\hat{\delta}_Y = \begin{pmatrix} \hat{\delta}_P \\ \hat{\delta}_\lambda \end{pmatrix}$, $F = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$. Applying the Yosida splitting to the updates formulation algebraic system (3.5)-(3.8): first write the matrix $A$ as

$$A = \begin{pmatrix} \tilde{A}_1 & 0 \\ 0 & \tilde{A}_2 \end{pmatrix} + \begin{pmatrix} \tilde{N}_1 & N_1 \\ N_2 & \tilde{N}_2 \end{pmatrix} =: \tilde{A} + \tilde{N}, \quad (3.10)$$

where $\tilde{A}$ is SPD. Now use the approximation $C^T\tilde{A}^{-1}C$ for the Schur complement matrix $C^TA^{-1}C$, which leads to the approximation:

$$\begin{pmatrix} A & C \\ C^T & 0 \end{pmatrix} \approx \begin{pmatrix} A & 0 \\ C^T & -C^T\tilde{A}^{-1}C \end{pmatrix}\begin{pmatrix} I & A^{-1}C \\ 0 & I \end{pmatrix}. \quad (3.11)$$

With this approximation, we solve the linear system

$$\begin{pmatrix} A & 0 \\ C^T & -C^T\tilde{A}^{-1}C \end{pmatrix}\begin{pmatrix} I & A^{-1}C \\ 0 & I \end{pmatrix}\begin{pmatrix} X \\ \delta_Y \end{pmatrix} = \begin{pmatrix} F \\ 0 \end{pmatrix}, \quad (3.12)$$

with $X = \begin{pmatrix} u \\ B \end{pmatrix}$, $\delta_Y = \begin{pmatrix} \delta_P \\ \delta_\lambda \end{pmatrix}$. Solving the linear system (3.12) is equivalent to the three steps:

1. Solve $\begin{pmatrix} A_1 & N_1 \\ N_2 & A_2 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$ for $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$,

2a. Solve $(C_1^T\tilde{A}_1^{-1}C_1)\delta_P = C_1^Tz_1$ for $\delta_P$,

2b. Solve $(C_1^T\tilde{A}_2^{-1}C_1)\delta_\lambda = C_1^Tz_2$ for $\delta_\lambda$,

3. Solve $\begin{pmatrix} A_1 & N_1 \\ N_2 & A_2 \end{pmatrix}\begin{pmatrix} u \\ B \end{pmatrix} = \begin{pmatrix} A_1 & N_1 \\ N_2 & A_2 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} - \begin{pmatrix} C_1^T & 0 \\ 0 & C_1^T \end{pmatrix}\begin{pmatrix} \delta_P \\ \delta_\lambda \end{pmatrix}$ for $\begin{pmatrix} u \\ B \end{pmatrix}$.

Finally, set $P_h = \delta_P + P^n$ and $\lambda_h = \delta_\lambda + \lambda^n$.

For analysis purposes, we define a finite element formulation that is equivalent to 3-step YU linear algebraic system above. We note that YU implementation should not be computed in this way, but as the simple linear algebraic implementation presented in the introduction.

**Algorithm 3.1 (YU finite element formulation)** *Given $(P^n, \lambda^n) \in (Q_h, Q_h)$, find $(u_h, P_h, B_h, \lambda_h) \in (X_h, Q_h, X_h, Q_h)$ via the following steps:*

1. *Find $(z_h, \omega_h) \in (X_h, X_h)$ satisfying, $\forall (v_h, \psi_h) \in (X_h, X_h)$,*

$$\frac{\alpha}{\Delta t}(z_h, v_h) + \nu(\nabla z_h, \nabla v_h) + (\mathcal{U} \cdot \nabla z_h, v_h) - (\mathcal{B} \cdot \nabla \omega_h, v_h) = (\tilde{\tilde{f}}, v_h), \tag{3.13}$$

$$\frac{\alpha}{\Delta t}(\omega_h, \psi_h) + \nu_m(\nabla \omega_h, \nabla \psi_h) + (\mathcal{U} \cdot \nabla \omega_h, \psi_h) - (\mathcal{B} \cdot \nabla z_h, \psi_h) = (\tilde{\tilde{g}}, \psi_h). \tag{3.14}$$

2. *Find $(\chi_h, \delta_P, \mu_h, \delta_\lambda) \in (X_h, Q_h, X_h, Q_h)$ satisfying, $\forall (v_h, q_h, \psi_h, r_h) \in (X_h, Q_h, X_h, Q_h)$,*

$$\frac{\alpha}{\Delta t}(\chi_h, v_h) + \nu(\nabla \chi_h, \nabla v_h) - (\delta_P, \nabla \cdot v_h) = 0, \tag{3.15}$$

$$(\nabla \cdot \chi_h, q_h) = -(\nabla \cdot z_h, q_h), \tag{3.16}$$

$$\frac{\alpha}{\Delta t}(\mu_h, \psi_h) + \nu_m(\nabla \mu_h, \nabla \psi_h) + (\delta_\lambda, \nabla \cdot \psi_h) = 0, \tag{3.17}$$

$$(\nabla \cdot \mu_h, r_h) = -(\nabla \cdot \omega_h, r_h). \tag{3.18}$$

3. *Find $(u_h, B_h) \in (X_h, X_h)$ satisfying, $\forall (v_h, \psi_h) \in (X_h, X_h)$,*

$$\begin{aligned} &\frac{\alpha}{\Delta t}(u_h, v_h) + \nu(\nabla u_h, \nabla v_h) + (\mathcal{U} \cdot \nabla u_h, v_h) - (\mathcal{B} \cdot \nabla B_h, v_h) \\ &\quad = \frac{\alpha}{\Delta t}(z_h, v_h) + \nu(\nabla z_h, \nabla v_h) + (\mathcal{U} \cdot \nabla z_h, v_h) - (\mathcal{B} \cdot \nabla \omega_h, v_h) + (\delta_P, \nabla \cdot v_h), \end{aligned} \tag{3.19}$$

$$\begin{aligned} &\frac{\alpha}{\Delta t}(B_h, \psi_h) + \nu_m(\nabla B_h, \nabla \psi_h) + (\mathcal{U} \cdot \nabla B_h, \psi_h) - (\mathcal{B} \cdot \nabla u_h, \psi_h) \\ &\quad = \frac{\alpha}{\Delta t}(\omega_h, \psi_h) + \nu_m(\nabla \omega_h, \nabla \psi_h) + (\mathcal{U} \cdot \nabla \omega_h, \psi_h) - (\mathcal{B} \cdot \nabla z_h, \psi_h) - (\delta_\lambda, \nabla \cdot \psi_h). \end{aligned} \tag{3.20}$$

4. *Recover $P_h, \lambda_h$ by setting $P_h = \delta_P + P^n$ and $\lambda_h = \delta_\lambda + \lambda^n$.*

**Remark 3.1** *Combining Step 1 and Step 3 of Algorithm 3.1 produces*

$$\frac{\alpha}{\Delta t}(u_h, v_h) + \nu(\nabla u_h, \nabla v_h) + (\mathcal{U} \cdot \nabla u_h, v_h) - (\mathcal{B} \cdot \nabla B_h, v_h) - (\delta_P, \nabla \cdot v_h,) = (\tilde{\tilde{f}}, v_h), \tag{3.21}$$

$$\frac{\alpha}{\Delta t}(B_h, \psi_h) + \nu_m(\nabla B_h, \nabla \psi_h) + (\mathcal{U} \cdot \nabla B_h, \psi_h) - (\mathcal{B} \cdot \nabla u_h, \psi_h) + (\delta_\lambda, \nabla \cdot \psi_h) = (\tilde{\tilde{g}}, \psi_h). \tag{3.22}$$

*Replacing $\tilde{\tilde{f}}$, $\tilde{\tilde{g}}$, $\delta_P$ and $\delta_\lambda$ by their definitions and recombining gives, for any $v_h$ or $\psi_h \in X_h$,*

$$\frac{\alpha}{\Delta t}(u_h, v_h) + \nu(\nabla u_h, \nabla v_h) + (\mathcal{U} \cdot \nabla u_h, v_h) - (\mathcal{B} \cdot \nabla B_h, v_h) - (P_h, \nabla \cdot v_h) = (\tilde{f}, v_h), \tag{3.23}$$

$$\frac{\alpha}{\Delta t}(B_h, \psi_h) + \nu_m(\nabla B_h, \nabla \psi_h) + (\mathcal{U} \cdot \nabla B_h, \psi_h) - (\mathcal{B} \cdot \nabla u_h, \psi_h) + (\lambda_h, \nabla \cdot \psi_h) = (\tilde{g}, \psi_h), \tag{3.24}$$

*which shows that the YU method preserves the momentum and magnetic field evolution equations.*

We now prove that the solutions of the Yosida updates method converge to the solutions of the unaltered discrete scheme (3.5)-(3.8). For simplicity of the analysis, we assume that $\alpha = 1$ and the convective velocity $\mathcal{U}$ from (1.5) and the magnetic field $\mathcal{B}$ from (1.7) satisfy $\nabla \cdot \mathcal{U} = 0$ with $\|\mathcal{U}\|_{L^\infty(\Omega)} \leq C_U < \infty$ and $\nabla \cdot \mathcal{B} = 0$ with $\|\mathcal{B}\|_{L^\infty(\Omega)} \leq C_B < \infty$. Extension to the case of only weakly divergence-free $\mathcal{U}$ and $\mathcal{B}$ can be done by skew-symmetrizing the nonlinear terms, and if these variables have only $H^1(\Omega)$ regularity then different Hölder and Sobolev bounds could be used. Neither of these changes would affect convergence rates in $\Delta t$, but could affect the scaling with the mesh width $h$.

**Theorem 3.1** *Let $\hat{u}_h, \hat{P}_h, \hat{B}_h$ and $\hat{\lambda}_h$ be the solution of (3.1)-(3.4) (the unapproximated linear system solution), and $u_h, P_h, B_h, \lambda_h$ the solutions to Algorithm 3.1 (the YU solution). Further, assume that the pressure solutions of the unapproximated solution satisfy $\|\hat{P}_h - P^n\| \leq C_P\Delta t, \quad \|\hat{\lambda}_h - \lambda^n\| \leq C_\lambda \Delta t$. Then the error in YU satisfies*

$$\|\hat{u}_h - u_h\| + \|\hat{B}_h - B_h\| \leq \frac{2C_i^3\Delta t^3}{\beta h^3}\Big(C_U + C_B\Big)\Big(C_P + C_\lambda\Big)\Big(2C_{PF} + \frac{C_* C_i \Delta t}{h}\Big) = O(\Delta t^3),$$

*where $C_* := \min\{\nu, \nu_m\}$.*

**Remark 3.2** *The negative scaling with respect to $h$ arises due to use of the inverse inequality, as to find the $O(\Delta t^3)$, it was necessary to bound $H^1$ terms from the right hand side, in $L^2$ terms on the left hand side. The negative dependence on $h$ could be reduced or even eliminated in the analysis, but this would in turn lower the order of convergence with respect to $\Delta t$. However, it was observed for the YU applied to the NSE that the negative scaling of $h$ on several test problems was much milder, $O(h^{-1/2})$, instead of $O(h^{-3})$. In our convergence rate tests, we observed no negative scaling with $h$, although we did observe some slight deterioration of the third order convergence with respect to $\Delta t$.*

**Proof:** The proof is rather long, and we split it up into three major steps. Denote $e_u := \hat{u}_h - u_h, \quad e_B := \hat{B}_h - B_h$, and note that

$$\hat{P}_h - P_h = (\hat{P}_h - P^n) - (P_h - P^n) =: \hat{\delta}_P - \delta_P,$$
$$\hat{\lambda}_h - \lambda_h = (\hat{\lambda}_h - \lambda^n) - (\lambda_h - \lambda^n) =: \hat{\delta}_\lambda - \delta_\lambda.$$

**Step 1: Claim:**

$$\|e_u\| + \|e_B\| \leq \frac{\sqrt{2}C_i\Delta t}{h}\left(\|\hat{\delta}_P - \delta_P\| + \|\hat{\delta}_\lambda - \delta_\lambda\|\right). \tag{3.25}$$

Begin by subtracting (3.21) from (3.5) and (3.22) from (3.7), which gives

$$\frac{1}{\Delta t}(e_u, v_h) + \nu(\nabla e_u, \nabla v_h) + (\mathcal{U}\cdot\nabla e_u, v_h) - (\mathcal{B}\cdot\nabla e_B, v_h) - (\hat{\delta}_P - \delta_P, \nabla\cdot v_h) = 0,$$
$$\frac{1}{\Delta t}(e_B, \psi_h) + \nu_m(\nabla e_B, \nabla \psi_h) + (\mathcal{U}\cdot\nabla e_B, \psi_h) - (\mathcal{B}\cdot\nabla e_u, \psi_h) + (\hat{\delta}_\lambda - \delta_\lambda, \nabla\cdot \psi_h) = 0.$$

Setting $v_h = e_u$ and $\psi_h = e_B$ yields $(\mathcal{U}\cdot\nabla e_u, e_u) = 0, \ (\mathcal{U}\cdot\nabla e_B, e_B) = 0$ since $\nabla\cdot\mathcal{U} = 0$ [16]. Then using Cauchy-Schwarz and Young's inequalities along with the inverse inequality provides

$$\begin{aligned}
\frac{1}{\Delta t}\|e_u\|^2 + \nu\|\nabla e_u\|^2 &\leq (\mathcal{B}\cdot\nabla e_B, e_u) + \|\hat{\delta}_P - \delta_P\|\|\nabla\cdot e_u\| \\
&\leq (\mathcal{B}\cdot\nabla e_B, e_u) + \|\hat{\delta}_P - \delta_P\|\|\nabla e_u\| \\
&\leq (\mathcal{B}\cdot\nabla e_B, e_u) + \frac{C_i}{h}\|\hat{\delta}_P - \delta_P\|\|e_u\| \\
&\leq (\mathcal{B}\cdot\nabla e_B, e_u) + \frac{1}{2\Delta t}\|e_u\|^2 + \frac{C_i^2\Delta t}{2h^2}\|\hat{\delta}_P - \delta_P\|^2, && (3.26)\\
\frac{1}{\Delta t}\|e_B\|^2 + \nu_m\|\nabla e_B\|^2 &\leq (\mathcal{B}\cdot\nabla e_u, e_B) + \|\hat{\delta}_\lambda - \delta_\lambda\|\|\nabla\cdot e_B\| \\
&\leq (\mathcal{B}\cdot\nabla e_u, e_B) + \frac{C_i}{h}\|\hat{\delta}_\lambda - \delta_\lambda\|\|e_B\| \\
&\leq (\mathcal{B}\cdot\nabla e_u, e_B) + \frac{1}{2\Delta t}\|e_B\|^2 + \frac{C_i^2\Delta t}{2h^2}\|\hat{\delta}_\lambda - \delta_\lambda\|^2. && (3.27)
\end{aligned}$$

Next, sum (3.26) and (3.27). Notice that $(\mathcal{B}\cdot\nabla e_B, e_u) = -(\mathcal{B}\cdot\nabla e_u, e_B)$, and after dropping the viscous terms on the left hand side, we find that

$$\|e_u\|^2 + \|e_B\|^2 \le \frac{C_i^2\Delta t^2}{h^2}\left(\|\hat{\delta}_P - \delta_P\|^2 + \|\hat{\delta}_\lambda - \delta_\lambda\|^2\right),$$

and taking square root of both sides gives the claimed bound.

**Step 2: Claim:**

$$\begin{aligned}
&\|\hat{\delta}_P - \delta_P\| + \|\hat{\delta}_\lambda - \delta_\lambda\| \\
&\qquad \le \beta^{-1}\Big(C_U + C_B\Big)\Big(2C_{PF} + \frac{C_* C_i \Delta t}{h}\Big)\Big(\|\nabla(\hat{u}_h - z_h)\| + \|\nabla(\hat{B}_h - \omega_h)\|\Big) \\
&\qquad\qquad \le \frac{C_i}{\beta h}\Big(C_U + C_B\Big)\Big(2C_{PF} + \frac{C_* C_i \Delta t}{h}\Big)\Big(\|\hat{u}_h - z_h\| + \|\hat{B}_h - \omega_h\|\Big). \qquad (3.28)
\end{aligned}$$

For this step of the proof, begin by adding Step 1 and Step 2 from the Yosida updates algorithm to obtain

$$\begin{aligned}
\frac{1}{\Delta t}(\chi_h + z_h, v_h) + \nu(\nabla(\chi_h + z_h), \nabla v_h) + (\mathcal{U}\cdot\nabla z_h, v_h) - (\mathcal{B}\cdot\nabla\omega_h, v_h) - (\delta_P, \nabla\cdot v_h) &= (\tilde{\tilde{f}}, v_h), \\
(\nabla\cdot(\chi_h + z_h), q_h) &= 0, \\
\frac{1}{\Delta t}(\mu_h + \omega_h, \psi_h) + \nu_m(\nabla(\mu_h + \omega_h), \nabla\psi_h) + (\mathcal{U}\cdot\nabla\omega_h, \psi_h) - (\mathcal{B}\cdot\nabla z_h, \psi_h) + (\delta_\lambda, \nabla\cdot\psi_h) &= (\tilde{\tilde{g}}, \psi_h), \\
(\nabla\cdot(\mu_h + \omega_h), r_h) &= 0.
\end{aligned}$$

Subtracting this system from the unapproximated MHD system (3.5)-(3.8) yields

$$\begin{aligned}
\frac{1}{\Delta t}(\hat{u}_h - (\chi_h + z_h), v_h) + \nu(\nabla(\hat{u}_h - (\chi_h + z_h)), \nabla v_h) + (\mathcal{U}\cdot\nabla(\hat{u}_h - z_h), v_h)& \\
-(\mathcal{B}\cdot\nabla(\hat{B}_h - \omega_h), v_h) - (\hat{\delta}_P - \delta_P, \nabla\cdot v_h) &= 0, \qquad (3.29)\\
(\nabla\cdot(\hat{u}_h - (\chi_h + z_h)), q_h) &= 0, \qquad (3.30)\\
\frac{1}{\Delta t}(\hat{B}_h - (\mu_h + \omega_h), \psi_h) + \nu_m(\nabla(\hat{B}_h - (\mu_h + \omega_h)), \nabla\psi_h) + (\mathcal{U}\cdot\nabla(\hat{B}_h - \omega_h), \psi_h)& \\
-(\mathcal{B}\cdot\nabla(\hat{u}_h - z_h), \psi_h) + (\hat{\delta}_\lambda - \delta_\lambda, \nabla\cdot\psi_h) &= 0, \qquad (3.31)\\
(\nabla\cdot(\hat{B}_h - (\mu_h + \omega_h)), r_h) &= 0. \qquad (3.32)
\end{aligned}$$

Now isolate the pressure error in (3.29), and divide both sides by $\|\nabla v_h\|$. Then using the Cauchy-Schwarz and Hölder's inequalities produces

$$\begin{aligned}
\frac{\left(\hat{\delta}_P - \delta_P, \nabla\cdot v_h\right)}{\|\nabla v_h\|} \le \frac{1}{\Delta t}\frac{\|\hat{u}_h - (\chi_h + z_h)\|\|v_h\|}{\|\nabla v_h\|} + \frac{\nu\|\nabla(\hat{u}_h - (\chi_h + z_h))\|\|\nabla v_h\|}{\|\nabla v_h\|}& \\
+\frac{\|\mathcal{U}\|_{L^\infty}\|\nabla(\hat{u}_h - z_h)\|\|v_h\|}{\|\nabla v_h\|} + \frac{\|\mathcal{B}\|_{L^\infty}\|\nabla(\hat{B}_h - \omega_h)\|\|v_h\|}{\|\nabla v_h\|}&. \qquad (3.33)
\end{aligned}$$

Similarly, we can get the following from (3.31):

$$\begin{aligned}
\frac{\left(\hat{\delta}_\lambda - \delta_\lambda, \nabla\cdot\psi_h\right)}{\|\nabla\psi_h\|} \le \frac{1}{\Delta t}\frac{\|\hat{B}_h - (\mu_h + \omega_h)\|\|\psi_h\|}{\|\nabla\psi_h\|} + \frac{\nu_m\|\nabla(\hat{B}_h - (\mu_h + \omega_h))\|\|\nabla\psi_h\|}{\|\nabla\psi_h\|}& \\
+\frac{\|\mathcal{U}\|_{L^\infty}\|\nabla(\hat{B}_h - \omega_h)\|\|\psi_h\|}{\|\nabla\psi_h\|} + \frac{\|\mathcal{B}\|_{L^\infty}\|\nabla(\hat{u}_h - z_h)\|\|\psi_h\|}{\|\nabla\psi_h\|}&. \qquad (3.34)
\end{aligned}$$

Using the LBB condition together with the Poincaré-Friedrich's inequality and reducing gives the estimates

$$\beta\|\hat{\delta}_P - \delta_P\| \leq \frac{C_{PF}}{\Delta t}\|\hat{u}_h - (\chi_h + z_h)\| + \nu\|\nabla(\hat{u}_h - (\chi_h + z_h))\| \\ + C_{PF}C_U\|\nabla(\hat{u}_h - z_h)\| + C_{PF}C_B\|\nabla(\hat{B}_h - \omega_h)\|, \quad (3.35)$$

and

$$\beta\|\hat{\delta}_\lambda - \delta_\lambda\| \leq \frac{C_{PF}}{\Delta t}\|\hat{B}_h - (\mu_h + \omega_h)\| + \nu_m\|\nabla(\hat{B}_h - (\mu_h + \omega_h))\| \\ + C_{PF}C_U\|\nabla(\hat{B}_h - \omega_h)\| + C_{PF}C_B\|\nabla(\hat{u}_h - z_h)\|. \quad (3.36)$$

After applying the inverse inequality to the second right hand side terms of (3.35)-(3.36), sum them to get

$$\|\hat{\delta}_P - \delta_P\| + \|\hat{\delta}_\lambda - \delta_\lambda\| \\ \leq \beta^{-1}\Bigg[\left(\frac{C_{PF}}{\Delta t} + \frac{C_*C_i}{h}\right)\Big(\|\hat{u}_h - (\chi_h + z_h)\| + \|\hat{B}_h - (\mu_h + \omega_h)\|\Big) \\ + C_{PF}(C_U + C_B)\Big(\|\nabla(\hat{u}_h - z_h)\| + \|\nabla(\hat{B}_h - \omega_h)\|\Big)\Bigg], \quad (3.37)$$

where $C_* = \min\{\nu, \nu_m\}$.

Next, set $v_h = \hat{u}_h - (\chi_h + z_h)$ in (3.29), $q_h = \hat{\delta}_P - \delta_P$ in (3.30), $\psi_h = \hat{B}_h - (\mu_h + \omega_h)$ in (3.31), and $r_h = \hat{\delta}_\lambda - \delta_\lambda$ in (3.32). Apply Hölder's inequality to produce

$$\frac{1}{\Delta t}\|\hat{u}_h - (\chi_h + z_h)\|^2 + \nu\|\nabla(\hat{u}_h - (\chi_h + z_h))\|^2 \\ \leq \|\mathcal{U}\|_{L^\infty}\|\nabla(\hat{u}_h - z_h)\|\|\hat{u}_h - (\chi_h + z_h)\| + \|\mathcal{B}\|_{L^\infty}\|\nabla(\hat{B}_h - \omega_h)\|\|\hat{u}_h - (\chi_h + z_h)\| \\ = \Big(C_U\|\nabla(\hat{u}_h - z_h)\| + C_B\|\nabla(\hat{B}_h - \omega_h)\|\Big)\|\hat{u}_h - (\chi_h + z_h)\|,$$

and

$$\frac{1}{\Delta t}\|\hat{B}_h - (\mu_h + \omega_h)\|^2 + \nu_m\|\nabla(\hat{B}_h - (\mu_h + \omega_h))\|^2 \\ \leq \|\mathcal{U}\|_{L^\infty}\|\nabla(\hat{B}_h - \omega_h)\|\|\hat{B}_h - (\mu_h + \omega_h)\| + \|\mathcal{B}\|_{L^\infty}\|\nabla(\hat{u}_h - z_h)\|\|\hat{B}_h - (\mu_h + \omega_h)\| \\ = \Big(C_U\|\nabla(\hat{B}_h - \omega_h)\| + C_B\|\nabla(\hat{u}_h - z_h)\|\Big)\|\hat{B}_h - (\mu_h + \omega_h)\|.$$

Reducing the terms produces

$$\|\hat{u}_h - (\chi_h + z_h)\| \leq \Delta t\left(C_U\|\nabla(\hat{u}_h - z_h)\| + C_B\|\nabla(\hat{B}_h - \omega_h)\|\right), \quad (3.38)$$

and

$$\|\hat{B}_h - (\mu_h + \omega_h)\| \leq \Delta t\left(C_U\|\nabla(\hat{B}_h - \omega_h)\| + C_B\|\nabla(\hat{u}_h - z_h)\|\right). \quad (3.39)$$

Now sum (3.38) and (3.39) to get the estimate

$$\|\hat{u}_h - (\chi_h + z_h)\| + \|\hat{B}_h - (\mu_h + \omega_h)\| \\ \leq \Delta t(C_U + C_B)\Big(\|\nabla(\hat{u}_h - z_h)\| + \|\nabla(\hat{B}_h - \omega_h)\|\Big). \quad (3.40)$$

Finally, using (3.40) in (3.37) together with the inverse inequality yields

$$
\begin{aligned}
&\|\hat{\delta}_P - \delta_P\| + \|\hat{\delta}_\lambda - \delta_\lambda\| \\
&\qquad \leq \beta^{-1}\Bigg[\left(C_{PF} + \frac{C_* C_i \Delta t}{h}\right)(C_U + C_B)\Big(\|\nabla(\hat{u}_h - z_h)\| + \|\nabla(\hat{B}_h - \omega_h)\|\Big) \\
&\qquad\qquad + C_{PF}(C_U + C_B)\Big(\|\nabla(\hat{u}_h - z_h)\| + \|\nabla(\hat{B}_h - \omega_h)\|\Big)\Bigg] \\
&\qquad = \beta^{-1}\Big(C_U + C_B\Big)\left(2C_{PF} + \frac{C_* C_i \Delta t}{h}\right)\Big(\|\nabla(\hat{u}_h - z_h)\| + \|\nabla(\hat{B}_h - \omega_h)\|\Big) \\
&\qquad\qquad \leq \frac{C_i}{\beta h}\Big(C_U + C_B\Big)\left(2C_{PF} + \frac{C_* C_i \Delta t}{h}\right)\Big(\|\hat{u}_h - z_h\| + \|\hat{B}_h - \omega_h\|\Big), \quad (3.41)
\end{aligned}
$$

which proves the stated second claim.

**Step 3: Completion of the proof**

It remains to bound the terms $\|\hat{u}_h - z_h\|$ and $\|\hat{B}_h - \omega_h\|$ to get our stated result in the theorem. Subtract (3.13) from (3.5), and (3.14) from (3.7) to obtain

$$
\begin{aligned}
\frac{1}{\Delta t}(\hat{u}_h - z_h, v_h) + \nu(\nabla(\hat{u}_h - z_h), \nabla v_h) + (\mathcal{U}\cdot\nabla(\hat{u}_h - z_h), v_h) &= (\mathcal{B}\cdot\nabla(\hat{B}_h - \omega_h), v_h) \\
&\quad + (\hat{\delta}_P, \nabla\cdot v_h), \qquad (3.42)
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{\Delta t}(\hat{B}_h - \omega_h, \psi_h) + \nu_m(\nabla(\hat{B}_h - \omega_h), \nabla \psi_h) + (\mathcal{U}\cdot\nabla(\hat{B}_h - \omega_h), \psi_h) &= (\mathcal{B}\cdot\nabla(\hat{u}_h - z_h), \psi_h) \\
&\quad - (\hat{\delta}_\lambda, \nabla\cdot \psi_h). \qquad (3.43)
\end{aligned}
$$

Setting $v_h = \hat{u}_h - z_h$ in (3.42) and $\psi_h = \hat{B}_h - \omega_h$ (3.43) vanishes the nonlinear terms $(\mathcal{U}\cdot\nabla(\hat{u}_h - z_h), \hat{u}_h - z_h)$ and $(\mathcal{U}\cdot\nabla(\hat{B}_h - \omega_h), \hat{B}_h - \omega_h)$ since $\nabla\cdot\mathcal{U} = 0$. The equations (3.42) and (3.43) are then reduced to

$$
\frac{1}{\Delta t}\|\hat{u}_h - z_h\|^2 + \nu\|\nabla(\hat{u}_h - z_h)\|^2 = (\mathcal{B}\cdot\nabla(\hat{u}_h - \omega_h), \hat{u}_h - z_h) + (\hat{\delta}_P, \nabla\cdot(\hat{u}_h - z_h)), \quad (3.44)
$$

$$
\frac{1}{\Delta t}\|\hat{B}_h - \omega_h\|^2 + \nu_m\|\nabla(\hat{B}_h - \omega_h)\|^2 = (\mathcal{B}\cdot\nabla(\hat{u}_h - z_h), \hat{B}_h - \omega_h) - (\hat{\delta}_\lambda, \nabla\cdot(\hat{B}_h - \omega_h)). \ (3.45)
$$

Now apply the Cauchy-Schwarz, Young's and the inverse inequalities on the second right hand side term of (3.44) and (3.45) to get

$$
\frac{1}{\Delta t}\|\hat{u}_h - z_h\|^2 + \nu\|\nabla(\hat{u}_h - z_h)\|^2 \leq (\mathcal{B}\cdot\nabla(\hat{B}_h - \omega_h), \hat{u}_h - z_h) + \frac{1}{2\Delta t}\|\hat{u}_h - z_h\|^2 + \frac{C_i^2\Delta t}{2h^2}\|\hat{\delta}_P\|^2,
$$

$$
\frac{1}{\Delta t}\|\hat{B}_h - \omega_h\|^2 + \nu_m\|\nabla(\hat{B}_h - \omega_h)\|^2 \leq (\mathcal{B}\cdot\nabla(\hat{u}_h - z_h), \hat{B}_h - \omega_h) + \frac{1}{2\Delta t}\|\hat{B}_h - \omega_h\|^2 + \frac{C_i^2\Delta t}{2h^2}\|\hat{\delta}_\lambda\|^2.
$$

Rearranging terms now yields

$$
\|\hat{u}_h - z_h\|^2 + 2\nu\Delta t\|\nabla(\hat{u}_h - z_h)\|^2 \leq 2\Delta t(\mathcal{B}\cdot\nabla(\hat{B}_h - \omega_h), \hat{u}_h - z_h) + \frac{C_i^2\Delta t^2}{h^2}\|\hat{\delta}_P\|^2, \quad (3.46)
$$

$$
\|\hat{B}_h - \omega_h\|^2 + 2\nu_m\Delta t\|\nabla(\hat{B}_h - \omega_h)\|^2 \leq 2\Delta t(\mathcal{B}\cdot\nabla(\hat{u}_h - z_h), \hat{B}_h - \omega_h) + \frac{C_i^2\Delta t^2}{h^2}\|\hat{\delta}_\lambda\|^2. \quad (3.47)
$$

Notice that $(\mathcal{B}\cdot\nabla(\hat{B}_h - \omega_h), \hat{u}_h - z_h) = -(\mathcal{B}\cdot\nabla(\hat{u}_h - z_h), \hat{B}_h - \omega_h)$ since $\nabla\cdot\mathcal{B} = 0$, so by summing (3.46), (3.47), and using the assumptions $\|\hat{\delta}_P\| \leq C_P\Delta t$, $\|\hat{\delta}_\lambda\| \leq C_\lambda\Delta t$, we find that

$$
\|\hat{u}_h - z_h\|^2 + \|\hat{B}_h - \omega_h\|^2 \leq \frac{C_i^2\Delta t^2}{h^2}\Big(\|\hat{\delta}_P\|^2 + \|\hat{\delta}_\lambda\|^2\Big) \leq \frac{C_i^2\Delta t^4}{h^2}\Big(C_P^2 + C_\lambda^2\Big),
$$

which after taking square root of both sides produces

$$\|\hat{u}_h - z_h\| + \|\hat{B}_h - \omega_h\| \le \frac{\sqrt{2}C_i\Delta t^2}{h}\Big(C_P + C_\lambda\Big). \tag{3.48}$$

To finish the proof, use (3.48) in (3.41), which gives the bound

$$\|\hat{\delta}_P - \delta_P\| + \|\hat{\delta}_\lambda - \delta_\lambda\| \ \le \ \frac{\sqrt{2}C_i^2\Delta t^2}{\beta h^2}\Big(C_U + C_B\Big)\Big(C_P + C_\lambda\Big)\Big(2C_{PF} + \frac{C_* C_i \Delta t}{h}\Big), \tag{3.49}$$

and finally use (3.49) in (3.25). □

**Lemma 3.1 (Stability of YU)** *Assume $\tilde{f} = f + \frac{1}{\Delta t}u^n$ and $\tilde{g} = g + \frac{1}{\Delta t}B^n$ (i.e. the backward Euler case). Then if $\Delta t \le O(h^{4/3}) \le O(h)$, the Yosida updates method is stable, and solutions satisfies*

$$\frac{1}{\Delta t}\left(\|u_h\|^2 - \|u^n\|^2 + \|u_h - u^n\|^2\right) + \frac{1}{\Delta t}\left(\|B_h\|^2 - \|B^n\|^2 + \|B_h - B^n\|^2\right) \\ + \nu\|\nabla u_h\|^2 + \nu_m\|\nabla B_h\|^2 \le C(data).$$

**Proof:** Setting $v_h = u_h$ in (3.23), $\psi_h = B_h$ in (3.24) and using the definition of $\tilde{f}$ and $\tilde{g}$ gives the following:

$$\frac{1}{2\Delta t}\left(\|u_h\|^2 - \|u^n\|^2 + \|u_h - u^n\|^2\right) + \nu\|\nabla u_h\|^2 = (\mathcal{B}\cdot\nabla B_h, u_h) + (P_h, \nabla\cdot u_h) + (f, u_h), \tag{3.50}$$

$$\frac{1}{2\Delta t}\left(\|B_h\|^2 - \|B^n\|^2 + \|B_h - B^n\|^2\right) + \nu_m\|\nabla B_h\|^2 = (\mathcal{B}\cdot\nabla u_h, B_h) - (\lambda_h, \nabla\cdot B_h) + (\nabla\times g, B_h). \tag{3.51}$$

Notice that $(P_h, \nabla\cdot\hat{u}_h) = 0$ and $(\lambda_h, \nabla\cdot\hat{B}_h) = 0$ since $\hat{u}_h, \hat{B}_h \in V_h$. Now add these terms to the right hand side of (3.50) and (3.51), respectively. Then apply the Cauchy-Schwarz, Young's inequalities on the forcing terms and Cauchy-Schwarz, Young's inequality together with the inverse inequality to the second right hand side terms, which produces

$$\begin{aligned}
&\frac{1}{2\Delta t}(\|u_h\|^2 - \|u^n\|^2 + \|u_h - u^n\|^2) + \nu\|\nabla u_h\|^2 \\
&\qquad = (\mathcal{B}\cdot\nabla B_h, u_h) - (P_h, \nabla\cdot(\hat{u}_h - u_h)) + (f, u_h) \\
&\qquad \le (\mathcal{B}\cdot\nabla B_h, u_h) + \|P_h\|\|\nabla\cdot(\hat{u}_h - u_h)\| + \|f\|_{-1}\|\nabla u_h\| \\
&\qquad \le (\mathcal{B}\cdot\nabla B_h, u_h) + \frac{C_i}{h}\|P_h\|\|\hat{u}_h - u_h\| + \frac{\nu^{-1}}{2}\|f\|_{-1}^2 + \frac{\nu}{2}\|\nabla u_h\|^2 \\
&\qquad \le (\mathcal{B}\cdot\nabla B_h, u_h) + \frac{C\Delta t^3}{h^4}\|P_h\| + \frac{\nu^{-1}}{2}\|f\|_{-1}^2 + \frac{\nu}{2}\|\nabla u_h\|^2,
\end{aligned} \tag{3.52}$$

and similarly

$$\begin{aligned}
&\frac{1}{2\Delta t}(\|B_h\|^2 - \|B^n\|^2 + \|B_h - B^n\|^2) + \nu_m\|\nabla B_h\|^2 \\
&\qquad \le (\mathcal{B}\cdot\nabla u_h, B_h) + \frac{C\Delta t^3}{h^4}\|\lambda_h\| + \frac{\nu_m^{-1}}{2}\|\nabla\times g\|_{-1}^2 + \frac{\nu_m}{2}\|\nabla B_h\|^2,
\end{aligned} \tag{3.53}$$

where $C$ is a constant independent of $h$ and $\Delta t$. Sum (3.52) and (3.53) and notice that $(\mathcal{B}\cdot\nabla u_h, B_h) = -(\mathcal{B}\cdot\nabla B_h, u_h)$, Then multiplying by 2 produces

$$\frac{1}{\Delta t}(\|u_h\|^2 - \|u^n\|^2 + \|u_h - u^n\|^2) + \frac{1}{\Delta t}\left(\|B_h\|^2 - \|B^n\|^2 + \|B_h - B^n\|^2\right)$$
$$+\nu\|\nabla u_h\|^2 + \nu_m\|\nabla B_h\|^2 \le \frac{C\Delta t^3}{h^4}\left(\|P_h\| + \|\lambda_h\|\right) + \nu^{-1}\|f\|_{-1}^2 + \nu_m^{-1}\|\nabla\times g\|_{-1}^2.$$

We now bound the term $(\|P_h\|+\|\lambda_h\|)$. Adding $\pm\hat{P}^n$, $\pm\hat{\lambda}^n$, and applying the triangle inequality along with (3.41) provides

$$\begin{aligned}\|P_h\| + \|\lambda_h\| &= \|\hat{P}_h + P_h - \hat{P}_h\| + \|\hat{\lambda}_h + \lambda_h - \hat{\lambda}_h\| \\ &\le \|\hat{P}_h\| + \|\hat{\lambda}_h\| + \|\hat{\delta}_P - \delta_P\| + \|\hat{\delta}_\lambda - \delta_\lambda\| \le C\left(1 + \frac{\Delta t^2}{h^2}\right),\end{aligned} \tag{3.54}$$

where $C$ is a constant independent of $\Delta t$ and $h$. Then using the assumption $\Delta t \le O(h^{4/3}) \le O(h)$ in (3.54) gives the desired stability bound. $\square$

# 4 The Yosida updates pressure correction (YUPC) method

The addition of pressure correction to Yosida algorithms has been shown by Veneziani et. al [11, 23] to increase the order of accuracy of the solver. We now consider pressure correction applied to the YU method, which we call the Yosida-updates pressure correction (YUPC) method, and show it is $O(\Delta t^4)$ accurate.

YUPC is defined by applying the pressure-corrected Yosida method to (3.5)-(3.8): Find approximations $u_h, \delta_P, B_h, \delta_\lambda$ satisfying

$$\begin{pmatrix} A & 0 \\ C^T & -C^T\tilde{A}^{-1}C \end{pmatrix}\begin{pmatrix} I & A^{-1}C \\ 0 & Q \end{pmatrix}\begin{pmatrix} X \\ \delta_Y \end{pmatrix} = \begin{pmatrix} F \\ 0 \end{pmatrix}, \tag{4.1}$$

where $X = \begin{pmatrix} u \\ B \end{pmatrix}$, $\delta_Y = \begin{pmatrix} \delta_P \\ \delta_\lambda \end{pmatrix}$, with $Q := (C^T\tilde{A}^{-1}A\tilde{A}^{-1}C)^{-1}(C^T\tilde{A}^{-1}C)$. Finding approximations for $u, \delta_P, B, \delta_\lambda$ is equivalent to 6-steps:

1. Solve $\begin{pmatrix} A_1 & N_1 \\ N_2 & A_2 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$ for $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$,
2. Solve $\begin{pmatrix} C_1^T\tilde{A}_1^{-1}C_1 & 0 \\ 0 & C_1^T\tilde{A}_2^{-1}C_1 \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} C_1^T & 0 \\ 0 & C_1^T \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ for $\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$,
3. Solve $\begin{pmatrix} \tilde{A}_1 & 0 \\ 0 & \tilde{A}_2 \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$ for $\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$.
4. Solve $\begin{pmatrix} \tilde{A}_1 & 0 \\ 0 & \tilde{A}_2 \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} A_1 & N_1 \\ N_2 & A_2 \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ for $\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$.
5. Solve $\begin{pmatrix} C_1^T\tilde{A}_1^{-1}C_1 & 0 \\ 0 & C_1^T\tilde{A}_2^{-1}C_1 \end{pmatrix}\begin{pmatrix} \delta_P \\ \delta_\lambda \end{pmatrix} = \begin{pmatrix} C_1^T & 0 \\ 0 & C_1^T \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ for $\begin{pmatrix} \delta_P \\ \delta_\lambda \end{pmatrix}$,
6. Solve $\begin{pmatrix} A_1 & N_1 \\ N_2 & A_2 \end{pmatrix}\begin{pmatrix} u \\ B \end{pmatrix} = \begin{pmatrix} A_1 & N_1 \\ N_2 & A_2 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} - \begin{pmatrix} C_1 & 0 \\ 0 & C_1 \end{pmatrix}\begin{pmatrix} \delta_P \\ \delta_\lambda \end{pmatrix}$ for $\begin{pmatrix} u \\ B \end{pmatrix}$.

Then set $P_h = \delta_P + P^n$ and $\lambda_h = \delta_\lambda + \lambda^n$.

**Remark 4.1** *The additional work resulting form the YUPC method is to solve a second pressure-update solution with $Q$. This produces three additional steps: one step is solved with the SPD Yosida Schur complement and the other two steps with $\tilde{A}$. Thus it is not a major expense to apply pressure correction.*

In order to analyze the method, we now cast it into a finite element framework. We note that this is for analysis purposed only, and implementation of YUPC should only be considered from a linear algebraic viewpoint.

**Algorithm 4.1 (YUPC)** .

*Given $(P^n, \lambda^n) \in (Q_h, Q_h)$, find $(u_h, P_h, B_h, \lambda_h) \in (X_h, Q_h, X_h, Q_h)$ via the following steps:*

1. *Find $(z_h, \omega_h) \in (X_h, X_h)$ satisfying, $\forall (v_h, \psi_h) \in (X_h, X_h)$,*

$$\frac{\alpha}{\Delta t}(z_h, v_h) + \nu(\nabla z_h, \nabla v_h) + (\mathcal{U} \cdot \nabla z_h, v_h) - (\mathcal{B} \cdot \nabla \omega_h, v_h) = (\tilde{\tilde{f}}, v_h), \tag{4.2}$$

$$\frac{\alpha}{\Delta t}(\omega_h, \psi_h) + \nu_m(\nabla \omega_h, \nabla \psi_h) + (\mathcal{U} \cdot \nabla \omega_h, \psi_h) - (\mathcal{B} \cdot \nabla z_h, \psi_h) = (\tilde{\tilde{g}}, \psi_h). \tag{4.3}$$

2. *Find $(\chi_h, p_h, \mu_h, \pi_h) \in (X_h, Q_h, X_h, Q_h)$ satisfying, $\forall (v_h, q_h, \psi_h, r_h) \in (X_h, Q_h, X_h, Q_h)$,*

$$\frac{\alpha}{\Delta t}(\chi_h, v_h) + \nu(\nabla \chi_h, \nabla v_h) - (p_h, \nabla \cdot v_h) = 0, \tag{4.4}$$

$$(\nabla \cdot \chi_h, q_h) = -(\nabla \cdot z_h, q_h), \tag{4.5}$$

$$\frac{\alpha}{\Delta t}(\mu_h, \psi_h) + \nu_m(\nabla \mu_h, \nabla \psi_h) + (\pi_h, \nabla \cdot \psi_h) = 0, \tag{4.6}$$

$$(\nabla \cdot \mu_h, r_h) = -(\nabla \cdot \omega_h, r_h). \tag{4.7}$$

3. *Find $(\varphi_h, \theta_h) \in (X_h, X_h)$ satisfying, $\forall (v_h, \psi_h) \in (X_h, X_h)$,*

$$\frac{\alpha}{\Delta t}(\varphi_h, v_h) + \nu(\nabla \varphi_h, \nabla v_h) = -(p_h, \nabla \cdot v_h), \tag{4.8}$$

$$\frac{\alpha}{\Delta t}(\theta_h, \psi_h) + \nu_m(\nabla \theta_h, \nabla \psi_h) = -(\pi_h, \nabla \cdot \psi_h). \tag{4.9}$$

4. *Find $(\gamma_h, \sigma_h) \in (X_h, X_h)$ satisfying, $\forall (v_h, \psi_h) \in (X_h, X_h)$,*

$$\begin{aligned}\frac{\alpha}{\Delta t}(\gamma_h, v_h) &+ \nu(\nabla \gamma_h, \nabla v_h) \\ &= \frac{\alpha}{\Delta t}(\varphi_h, v_h) + \nu(\nabla \varphi_h, \nabla v_h) + (\mathcal{U} \cdot \nabla \varphi_h, v_h) - (\mathcal{B} \cdot \nabla \theta_h, v_h),\end{aligned} \tag{4.10}$$

$$\begin{aligned}\frac{\alpha}{\Delta t}(\sigma_h, \psi_h) &+ \nu_m(\nabla \sigma_h, \nabla \psi_h) \\ &= \frac{\alpha}{\Delta t}(\theta_h, \psi_h) + \nu_m(\nabla \theta_h, \nabla \psi_h) + (\mathcal{U} \cdot \nabla \theta_h, \psi_h) - (\mathcal{B} \cdot \nabla \varphi_h, \psi_h).\end{aligned} \tag{4.11}$$

5. *Find $(\phi_h, \delta_P, \kappa_h, \delta_\lambda) \in (X_h, Q_h, X_h, Q_h)$ satisfying, $\forall (v_h, q_h, \psi_h, r_h) \in (X_h, Q_h, X_h, Q_h)$,*

$$\frac{\alpha}{\Delta t}(\phi_h, v_h) + \nu(\nabla \phi_h, \nabla v_h) - (\delta_P, \nabla \cdot v_h) = 0, \tag{4.12}$$

$$(\nabla \cdot \phi_h, q_h) = -(\nabla \cdot \gamma_h, q_h), \tag{4.13}$$

$$\frac{\alpha}{\Delta t}(\kappa_h, \psi_h) + \nu_m(\nabla \kappa_h, \nabla \psi_h) + (\delta_\lambda, \nabla \cdot \psi_h) = 0, \tag{4.14}$$

$$(\nabla \cdot \kappa_h, r_h) = -(\nabla \cdot \sigma_h, r_h). \tag{4.15}$$

6. *Find $(u_h, B_h) \in (X_h, X_h)$ satisfying, $\forall (v_h, \psi_h) \in (X_h, X_h)$,*

$$\frac{\alpha}{\Delta t}(u_h, v_h) + \nu(\nabla u_h, \nabla v_h) + (\mathcal{U} \cdot \nabla u_h, v_h) - (\mathcal{B} \cdot \nabla B_h, v_h) \\ = \frac{\alpha}{\Delta t}(z_h, v_h) + \nu(\nabla z_h, \nabla v_h) + (\mathcal{U} \cdot \nabla z_h, v_h) - (\mathcal{B} \cdot \nabla \omega_h, v_h) + (\delta_P, \nabla \cdot v_h), \quad (4.16)$$

$$\frac{\alpha}{\Delta t}(B_h, \psi_h) + \nu_m(\nabla B_h, \nabla \psi_h) + (\mathcal{U} \cdot \nabla B_h, \psi_h) - (\mathcal{B} \cdot \nabla u_h, \psi_h) \\ = \frac{\alpha}{\Delta t}(\omega_h, \psi_h) + \nu_m(\nabla \omega_h, \nabla \psi_h) + (\mathcal{U} \cdot \nabla \omega_h, \psi_h) - (\mathcal{B} \cdot \nabla z_h, \psi_h) - (\delta_\lambda, \nabla \cdot \psi_h). \quad (4.17)$$

7. *Recover $P_h, \lambda_h$ by setting $P_h = \delta_P + P^n$ and $\lambda_h = \delta_\lambda + \lambda^n$.*

We now present a theorem for the error in YUPC scheme.

**Theorem 4.1** *Let $\hat{u}_h, \hat{P}_h, \hat{B}_h$ and $\hat{\lambda}_h$ be the unapproximated solutions to (3.1)-(3.4), with $\|\hat{P}_h - P^n\| \leq C_P \Delta t$, $\|\hat{\lambda}_h - \lambda^n\| \leq C_\lambda \Delta t$. Besides, let $u_h, P_h, B_h, \lambda_h$ be the solutions to Algorithm 4.1, i.e., the Yosida-updates pressure corrected solutions. The error satisfies the following bound:*

$$\|\hat{u}_h - u_h\| + \|\hat{B}_h - B_h\| \leq \frac{2C_i^4 \Delta t^4}{\beta h^4}\Big(C_U + C_B\Big)^2 \Big(C_P + C_\lambda\Big)\Big(2C_{PF} + \frac{C_* C_i \Delta t}{h}\Big),$$

*where $C_* := \min\{\nu, \nu_m\}$.*

**Proof:** The proof of this theorem is long and technical. However, it follows very closely the structure of the NSE proof, but handling the Maxwell equation just as is done above for the YU analysis. Thus, we omit the proof. □

Just as in the YU case, the stability can be proven using the convergence estimate.

**Corollary 4.1 (Stability of the YUPC scheme)** . *Suppose $\tilde{f} = f + \frac{1}{\Delta t}u^n$, $\tilde{g} = g + \frac{1}{\Delta t}B^n$, and $\Delta t \leq O(h^{5/4}) \leq O(h)$. Then the YUPC method is stable, and solutions satisfies*

$$\frac{1}{\Delta t}\left(\|u_h\|^2 - \|u^n\|^2 + \|u_h - u^n\|^2\right) + \frac{1}{\Delta t}\left(\|B_h\|^2 - \|B^n\|^2 + \|B_h - B^n\|^2\right) \\ + \nu\|\nabla u_h\|^2 + \nu_m\|\nabla B_h\|^2 \leq C(data).$$

# 5 Numerical experiments

We now present two numerical tests, the first to test the predicted convergence rates of the theory above, and second to test the feasibility of the method on a larger scale problem. For both tests, we use the viscous Orszag-Tang problem, which is a benchmark MHD test problem studied in [18, 10]. The domain is the periodic box $(0, 2\pi)^2$, and the setup is as follows. We take as the initial conditions

$$u_0 = \langle -\sin(y+2), \sin(x+1.4)\rangle^T, \quad B_0 = \langle -\frac{1}{3}\sin(y+2), \frac{2}{3}\sin(2x+3)\rangle^T,$$

add no external forcing, $f = \nabla \times g = 0$, and allow the flow to evolve. We choose $\nu = \nu_m = 0.01$, and in the second test we also choose $\nu = \nu_m = 0.001$.

In all of our computations below, we use $(P_2, P_1^{disc})$ elements for $(u_h, p_h)$ and $(B_h, \lambda_h)$.

### 5.1 Numerical experiment 1: Convergence rates

To compute the predicted convergence rates, we compute the unapproximated method for four time steps, the YU method for four time steps, and also for comparison the classical Yosida method (Y), and then compare solutions in the $L^2(\Omega)$ norm. For the first time step of all methods, we use Crank-Nicolson time stepping with an exact linear solver. The subsequent steps use BDF2, and the various solvers. To test the negative scaling with respect to $h$, we also compute on three different meshes: barycenter refinements of uniform meshes with $h = \frac{L}{8}, \frac{L}{16}, \frac{L}{32}$, with $L = 2\pi$.

The table below shows the errors in the Y and YU approximations, and the corresponding convergence rates with respect to $\Delta t$. Convergence of the usual Y method is clearly observed to be second order. For YU, we observe essentially third order, but with a slight deterioration in the rate as $\Delta t$ gets smaller. However, this deterioration does not occur until errors approach $10^{-8}$, which is at the level where linear solver error can be a factor ($10^{-10}$ is the tolerance for the CG Schur complement solver). We note that YU is clearly much more accurate than Y, and we stress that these two methods require the same amount of work to solve and have exactly the same system matrices.

Regarding the negative scaling with respect to $h$ from the analysis, we do not see a deterioration in the errors as $h$ decreases. However, we do observe a reduced scaling with respect to $\Delta t$, which could be related to this issue since the analysis does allow for a tradeoff between better scaling with respect to $h$ and a reduced scaling with respect to $\Delta t$. However, the reduced order of convergence is only observed when errors are near .

$h = \frac{L}{8}$

| $\Delta t$ | $\|u_{YU} - u_h\|$ | Rate | $\|B_{YU} - B_h\|$ | Rate | $\|u_Y - u_h\|$ | Rate | $\|B_Y - B_h\|$ | Rate |
|---|---|---|---|---|---|---|---|---|
| $\frac{0.2}{4}$ | 2.128e-4 | | 1.232e-4 | | 3.348e-3 | | 1.530e-3 | |
| $\frac{0.2}{8}$ | 3.137e-5 | 2.76 | 1.860e-5 | 2.73 | 9.098e-4 | 1.88 | 4.218e-4 | 1.86 |
| $\frac{0.2}{16}$ | 4.473e-6 | 2.81 | 3.103e-6 | 2.58 | 2.381e-4 | 1.93 | 1.144e-4 | 1.88 |
| $\frac{0.2}{32}$ | 6.368e-7 | 2.81 | 5.542e-7 | 2.48 | 6.108e-5 | 1.96 | 3.059e-5 | 1.90 |
| $\frac{0.2}{64}$ | 8.918e-8 | 2.84 | 9.229e-8 | 2.57 | 1.550e-5 | 1.98 | 8.102e-6 | 1.92 |
| $\frac{0.2}{128}$ | 1.218e-8 | 2.87 | 1.400e-8 | 2.72 | 3.910e-6 | 1.99 | 2.119e-6 | 1.93 |
| $\frac{0.2}{256}$ | 1.622e-9 | 2.91 | 1.972e-9 | 2.83 | 9.826e-7 | 1.99 | 5.464e-7 | 1.96 |

$h = \frac{L}{16}$

| $\Delta t$ | $\|u_{YU} - u_h\|$ | Rate | $\|B_{YU} - B_h\|$ | Rate | $\|u_Y - u_h\|$ | Rate | $\|B_Y - B_h\|$ | Rate |
|---|---|---|---|---|---|---|---|---|
| $\frac{0.2}{4}$ | 2.075e-4 | | 2.075e-4 | | 3.336e-3 | | 1.501e-3 | |
| $\frac{0.2}{8}$ | 3.100e-5 | 2.74 | 1.698e-5 | 3.61 | 9.088e-4 | 1.88 | 4.138e-4 | 1.86 |
| $\frac{0.2}{16}$ | 4.114e-6 | 2.91 | 1.975e-6 | 3.10 | 2.377e-4 | 1.93 | 1.106e-4 | 1.90 |
| $\frac{0.2}{32}$ | 5.597e-7 | 2.88 | 3.215e-7 | 2.62 | 6.087e-5 | 1.96 | 2.880e-5 | 1.94 |
| $\frac{0.2}{64}$ | 7.817e-8 | 2.84 | 6.055e-8 | 2.41 | 1.541e-5 | 1.98 | 7.368e-6 | 1.97 |
| $\frac{0.2}{128}$ | 1.110e-8 | 2.82 | 1.077e-8 | 2.49 | 3.877e-6 | 1.99 | 1.869e-6 | 1.98 |
| $\frac{0.2}{256}$ | 1.562e-9 | 2.83 | 1.732e-9 | 2.64 | 9.726e-7 | 1.99 | 4.721e-7 | 1.98 |

$h = \frac{L}{32}$

| $\Delta t$ | $\|u_{YU} - u_h\|$ | Rate | $\|B_{YU} - B_h\|$ | Rate | $\|u_Y - u_h\|$ | Rate | $\|B_Y - B_h\|$ | Rate |
|---|---|---|---|---|---|---|---|---|
| $\frac{0.2}{4}$ | 1.857e-5 | | 7.683e-5 | | 3.328e-3 | | 1.484e-3 | |
| $\frac{0.2}{8}$ | 2.878e-5 | -0.63 | 1.301e-5 | 2.56 | 9.080e-4 | 1.87 | 4.122e-4 | 1.85 |
| $\frac{0.2}{16}$ | 3.942e-6 | 2.87 | 1.516e-6 | 3.10 | 2.377e-4 | 1.93 | 1.104e-4 | 1.90 |
| $\frac{0.2}{32}$ | 5.213e-7 | 2.92 | 1.815e-7 | 3.06 | 6.086e-5 | 1.97 | 2.873e-5 | 1.94 |
| $\frac{0.2}{64}$ | 6.822e-8 | 2.93 | 2.762e-8 | 2.72 | 1.540e-5 | 1.98 | 7.337e-6 | 1.97 |
| $\frac{0.2}{128}$ | 9.027e-9 | 2.92 | 5.105e-9 | 2.44 | 3.875e-6 | 1.99 | 1.854e-6 | 1.98 |
| $\frac{0.2}{256}$ | 1.234e-9 | 2.87 | 9.695e-10 | 2.40 | 9.718e-7 | 2.00 | 4.662e-7 | 1.99 |

### 5.2 Numerical experiment 2: viscous Orszag-Tang problem

For this same test problem, we now run the problem for longer times on finer meshes, as a feasibility test. Here, we consider two cases of viscosity: $\nu = \nu_m = 0.01$ and $\nu = \nu_m = 0.001$, and compute to an end time of $T = 2.7$, using time step size $\Delta t = 0.01$ and third order linearized BDF3 time stepping. For the spatial discretization, we used a barycenter refinement of a 128x128 uniform mesh for $\nu = \nu_m = 0.01$, and 196x196 for $\nu = \nu_m = 0.001$. This provided 1,378,308 total degrees of freedom for $\nu = \nu_m = 0.01$, and 3,230,084 for $\nu = \nu_m = 0.001$. Grad-div stabilization was added to both the momentum and Maxwell equations, with stabilization parameter 100. For the linear solves, we used the YU method studied above, and with pressure correction - since the time stepping method is third order, the accuracy order of the linear solver at each time step should be $O(\Delta t^4)$ to match.

The block Schur complements were decomposed into Stokes-type Schur complements (a key feature of the proposed method), each of which was solved using PCG as the outer solver (preconditioned with the pressure mass matrix), and the inner solves were performed using PCG (with ILU preconditioning, using drop tolerance $10^{-6}$). For the solves with the nonsymmetric block matrix $A$, ILU-0 preconditioned BICGSTAB was used and was very effective - the preconditioner was updated whenever 20 BICGSTAB iterations were needed to converge the solver, and this happened about every 10 time steps.

Results for the $\nu = \nu_m = 0.01$ test are shown in figure 1, and for $\nu = \nu_m = 0.001$ in figure 2, as contours of the magnitude of the magnetic field and velocity field, and the (scalar) curl of the magnetic field. Results for $\nu = \nu_m = 0.001$ are consistent with the ideal MHD (i.e. $\nu = \nu_m = 0$) results from [18, 10]. We note we also ran these test problems using an exact (unapproximated) linear solver and the resulting plots were visually indistinguishable.

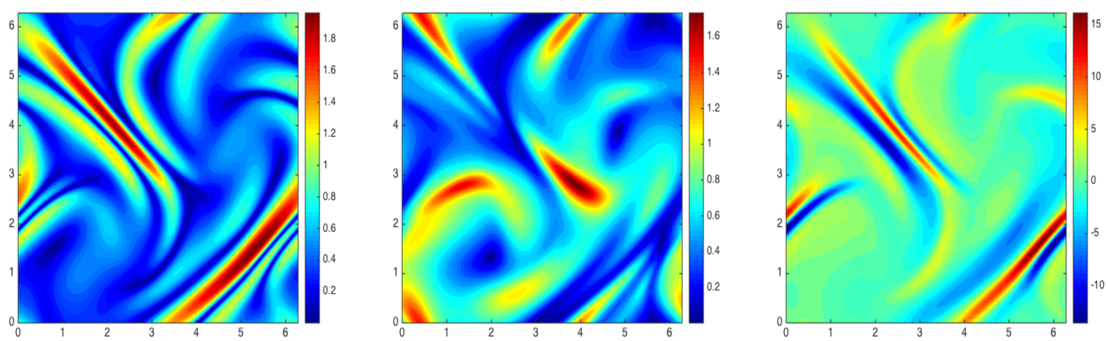

Figure 1: Shown above are the T=2.7 solution for $|B|$ (left), $|u|$ (center), and $\nabla \times B$ (right) for $\nu = \nu_m = 0.01$

# 6 Conclusions and future directions

In this paper, we have proposed, analyzed and tested the YU method (and with pressure correction) for MHD. The method provides for very efficient solves of the block saddle point linear systems that arise in MHD, as they decompose the nonsymmetric block Schur complement into 2 Stokes-type Schur complements that are the same at each time step. The method is proven to be third order (fourth order with pressure correction) with respect to $\Delta t$, and the analysis is done using the natural norms of the problem. Numerical tests were given that show the

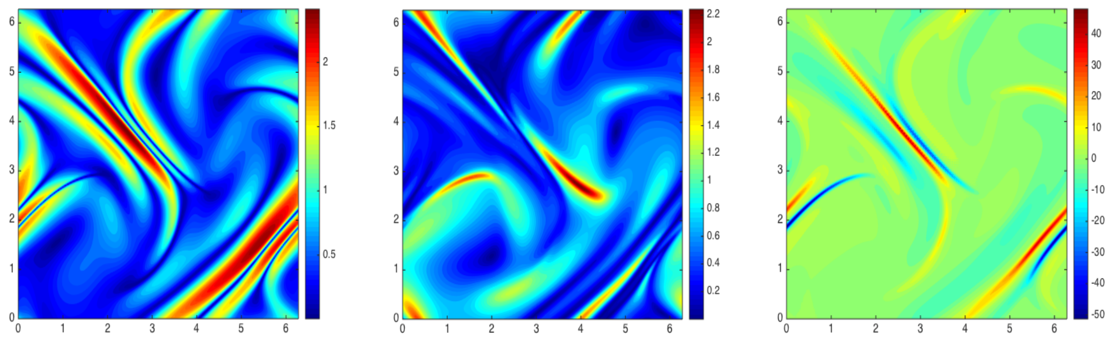

Figure 2: Shown above are the T=2.7 solution for $|B|$ (left), $|u|$ (center), and $\nabla \times B$ (right) for $\nu = \nu_m = 0.001$

effectiveness of the method.

There are at least two potential future directions for research that comes from this study. The first is testing the YU and YUPC methods efficiency and accuracy against various solvers of MHD systems that do not approximate, but solve with preconditioned iterative solvers. Second, the analysis predicts a potential negative scaling of $h^{-3}$ in the convergence, however the numerical test showed no negative scaling in $h$. Even for the YU method applied to the incremental NSE, the same $h^{-3}$ is predicted by the analysis, but only a $h^{-1/2}$ is observed in numerical tests [22]. Further study should be done here to see if a sharper analysis with respect to $h$ can be discovered.

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