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# Numerical Study of MHD Forced Convective Flow of a Micropolar Fluid Past a Non-linear Stretching Sheet with Variable Viscosity

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**Abstract**

Magnetohydrodynamic forced convective flow of a micropolar fluid past a non-linear stretching sheet with various parameters in presence of magnetic field has been studied numerically. The governing partial differential equations are transformed into non-linear ordinary differential equations with the help of suitable transformations and then solved numerically. The results obtained are compared with a previously published work on special case of the model and found excellent agreement. The numerical results in the form of velocity, microrotation and temperature profiles are shown graphically and discussed. The effect of the pertinent parameters on the local skin-friction coefficient and the local Nusselt number are also studied numerically and presented graphically.

**Key words:** MHD, forced convection, micropolar fluid, variable suction, variable viscosity.

## I. Introduction

In the recent years the study of MHD Flow of an electrically conducting fluid on stretching sheet becomes an important matter due to its many engineering and physical applications in modern metallurgical and metal-working processes. Hot rolling, drawing of plastic films and artificial fibers, glass fiber production, metal extrusion are examples of such practical applications. Sakiadis [1] initiated the study of boundary layer flow of a continuous stretched surface at constant speed for viscous fluid and developed a different class of numerical solution from the classical Blasius problem. Erickson et al. [2] extended the problem of Sakiadis taking wall suction to investigate its effects on heat and mass transfer over a stretched sheet surface. Gupta and Gupta [3] worked on the same fluid with suction/injection in an extended view for linearly moving sheet. Tsou et al. [4] reported both analytical and experimental results for the flow and heat transfer in the boundary layer on a continuously moving surface. Ali [5] investigated the effects of temperature dependent viscosity with suction/injection of thermal boundary layer flow on a power-law stretched vertical surface.

Micropolar fluids are fluids which exhibit microrotational effects and microrotational inertia. Micropolar fluids are used for analyzing the behavior of exotic lubricants, the flow of colloidal suspensions, liquid crystals, animal blood and in many other cases. The study of MHD flow of micropolar fluid has been a great interest to the researchers due to the effect of magnetic field on the boundary layer flow control. Gorla and Takhar [6], Yucel [7], Gorla et al. [8] are the worth mentioning researchers who have studied micropolar fluids. Desseaux and Kelson [9] investigated the flow of a micropolar fluid by a stretching sheet. Recently, Rahman and Sattar [10] studied MHD convective flow of a micropolar fluid past a continuously moving vertical porous plate in the presence of heat generation/absorption for free convection case.

In the present study, the work of Rahman and Sattar [10] has been extended on a non-linear stretching sheet to a steady forced convective flow of a micropolar fluid taking into account the variable suction, variable surface velocity, variable wall temperature, variable viscosity and viscous dissipation.

## II. Mathematical Analysis

A steady two dimensional viscous incompressible MHD convective flow of a micropolar fluid of temperature $T$ past a non-linear stretching sheet with variable wall temperature $T_w = T_\infty + Ax^p$ where $A$ is constant and $p$ is exponent has been considered. It is also assumed that the surface is moving with a surface velocity $u = cx^m$ where $c$ is constant and $m$ is exponent. The $x$ axis is measured along the surface and there is a suction velocity $v_0(x)$ at the surface. A magnetic field of uniform strength $B_0$ is applied along the $y$-axis normal to the flow direction. Considering also the variable viscosity and viscous dissipation term, with the usual boundary layer the problem is governed by the following equations (Rahman and Sattar [10]):

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{1}{\rho_\infty}\frac{\partial}{\partial y}\left[(\mu+s)\frac{\partial u}{\partial y}\right]+\frac{S}{\rho_\infty}\frac{\partial \sigma}{\partial y}-\frac{\sigma' B_0^2 u}{\rho_\infty} \tag{2}$$

$$u\frac{\partial \sigma}{\partial x}+v\frac{\partial \sigma}{\partial y}=\frac{\upsilon_s}{\rho_\infty j}\frac{\partial^2 \sigma}{\partial y^2}-\frac{S}{\rho_\infty j}(2\sigma+\frac{\partial u}{\partial y}) \tag{3}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{k}{\rho_\infty c_p}\frac{\partial^2 T}{\partial y^2}+\frac{Q_0}{\rho_\infty c_p}(T-T_\infty)+\frac{\mu}{c_p\rho_\infty}\left(\frac{\partial u}{\partial y}\right)^2 \tag{4}$$

where $u, v$ are the velocity components along $x, y$ co-ordinates respectively, $\mu$ is the coefficient of dynamic viscosity, $S$ is the microrotation coupling coefficient (also known as the vortex viscosity), $\rho_\infty$ is the mass density of the fluid, $\sigma$ is the microrotation component normal to the $xy$-plane, $\sigma'$ is the electric conductivity, $\upsilon_s=(\mu+\frac{S}{2})j$ is the microrotation viscosity or spin-gradient viscosity, $j$ is the micro-inertia density, $T$ is the temperature of the fluid in the boundary layer, $T_\infty$ is the temperature of the fluid outside the boundary layer, $c_p$ is the specific heat of the fluid at constant pressure, $k$ is the thermal conductivity, $Q_0$ is the heat generation/absorption constant.

The corresponding boundary conditions of the above problem are given by,

$$\left.\begin{aligned} &u = cx^m, \quad v = v_0(x), \quad \sigma = -n\frac{\partial u}{\partial y}, \\ &T_w(x) = T_\infty + Ax^p \qquad \text{at } y = 0 \\ &u = 0, \quad \sigma = 0, \; T = T_\infty \quad \text{as } y \to \infty \end{aligned}\right\} \tag{5}$$

Positive and negative values for $v_0$ indicate injection and suction of fluid at the sheet respectively, while $v_0 = 0$ corresponds to an impermeable sheet. When microrotation parameter $n=0$ we obtain $\sigma = 0$, which represents that the microelements in a concentrated particle flow-close to the wall are not able to rotate. The case corresponding to $n = 0.5$ results in the vanishing of the anti-symmetric part of the stress tensor and weak concentration and the case corresponding to $n = 1$ represents turbulent boundary layer flow.

## III. Non-dimensionalisation

To acquire similarity solution we introduce the following non-dimensional transformations:

$$\left.\begin{aligned} &\eta = y\sqrt{\frac{c}{\upsilon_\infty}}x^{\frac{m-1}{2}}, \\ &\psi = \sqrt{c\upsilon_\infty x^{m+1}}\, f(\eta), \\ &\sigma = \sqrt{\frac{c^3}{\upsilon_\infty}}x^{\frac{3m-1}{2}} g(\eta), \\ &\theta = \frac{T - T_\infty}{T_w - T_\infty}, \end{aligned}\right\} \tag{6}$$

where $\psi$ is the stream function.

Since $u = \dfrac{\partial \psi}{\partial y}$ and $v = -\dfrac{\partial \psi}{\partial x}$ satisfy the continuity equation (1) we have,

$u = cx^m f'$ and

$$v = -\sqrt{c\upsilon_\infty}\left\{ f'y\sqrt{\frac{c}{\upsilon_\infty}}\frac{m-1}{2}x^{m-1} + f\frac{m+1}{2}x^{\frac{m-1}{2}} \right\} \tag{7}$$

where prime denotes differentiation with respect to $\eta$ .Now for viscous fluid we have (see Ling and Dybbs[11]),

$$\mu = \frac{\mu_\infty}{1 + \gamma_*(T - T_\infty)} \tag{8}$$

where $\gamma_*$ is the thermal property. The dimensionless temperature $\theta$ can be written as,

$$\theta = \frac{T - T_r}{T_w - T_\infty} + \theta_r \tag{9}$$

where $T_r$ is a constant whose value depends on the thermal property of the fluid. Hence we may have the relation,

$$\mu = \mu_\infty\left(\frac{\theta_r}{\theta_r - \theta}\right) \tag{10}$$

which indicates variable viscosity.

Now introducing the relations (6)-(7) and (10) into the equations (2)-(4) the ordinary differential equations representing the flow field are:

$$\left(\frac{\theta_r}{\theta_r - \theta} + \Delta\right)f''' + \frac{m+1}{2}ff'' - mf'^2 + \frac{\theta_r}{(\theta_r - \theta)^2}\theta' f'' + \Delta g' - Mf' = 0 \tag{11}$$

$$\left(\frac{\theta_r}{\theta_r - \theta} + \frac{1}{2}\Delta\right)\xi g'' - \Delta(2g + f'') - \xi\left(\frac{3m-1}{2}f'g - \frac{m+1}{2}g'f\right) = 0 \tag{12}$$

$$\theta'' + \Pr_\infty\left(\frac{m+1}{2}f\theta' - pf'\theta\right) + \Pr_\infty Q\theta + \Pr_\infty \frac{\theta_r}{\theta_r - \theta}Ecf''^2 = 0 \tag{13}$$

where $\Delta = \dfrac{S}{\mu_\infty}$ is the vortex viscosity parameter, $M = \dfrac{\sigma' B_0^2}{\rho_\infty c}x^{1-m}$ is the magnetic field parameter, $\xi = \dfrac{jc}{\upsilon_\infty}x^{m-1}$ is the spin-gradient viscosity parameter, $E_C = \dfrac{(cx^m)^2}{c_p(T_W - T_\infty)}$ is the Eckert number, $\Pr_\infty = \dfrac{\mu_\infty c_p}{k}$ is the Prandtl number and $Q = \dfrac{Q_0 x^{1-m}}{\rho_\infty c_p c}$ is the heat source / sink parameter.

The transformed boundary conditions become,

$$\left.\begin{aligned} &f = F_w,\; f' = 1,\; g = -nf'',\; \theta = 1 \quad \text{at } \eta = 0 \\ &f' = 0,\; g = 0,\; \theta = 0 \quad \text{as } \eta \to \infty \end{aligned}\right\} \tag{14}$$

where $Fw = \pm\dfrac{v_0(x)}{\sqrt{c\upsilon_\infty}\,\dfrac{m+1}{2}x^{\frac{m-1}{2}}}$ is the suction velocity at the sheet for $F_w > 0$.

## IV. Skin-friction Coefficient and Nusselt Number

The local skin-friction coefficient and local Nusselt number are the important physical parameters for the problem. The local skin-friction coefficient is defined as

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho_\infty (cx^m)^2}$$

$$C_f = (2\mathrm{Re}_x^{-1})^{\frac{1}{2}}\left[\frac{\theta_r}{\theta_r - \theta} + \Delta(1 - n)\right]f''(0) \tag{15}$$

where $\mathrm{Re}_x = \dfrac{cx^{m+1}}{\upsilon_\infty}$ is the Reynolds number and $\tau_w$ refers to the shear stress at the sheet. The local Nusselt number may be defined as

$$Nu_x = \frac{x\,h(x)}{k} = -(\mathrm{Re}_x)^{\frac{1}{2}}\theta'(0) \tag{16}$$

where $h(x)$is the local heat transfer coefficient.

Thus from equation (15) and (16) we see that $C_f$ and $Nu_x$ are proportional to $f''(0)$ and $-\theta'(0)$ respectively.

## V. Method of Numerical Solution

The set of transformed governing equations (11)-(13) are solved using Nachtsheim-Swigert [12] shooting iteration technique together with sixth order Runge-Kutta Butcher initial value solver where $\Delta, M, \xi, Ec, \Pr_{\infty}, Q, F_w, n, \theta_r, m$ and $p$ are as prescribed parameters. Throughout the whole calculation we have fixed the value of step size $\Delta\eta = 0.001$ for sufficient accuracy of the solutions as the solutions are independent of step size. Velocity profiles for three distinct step sizes are shown in Fig. 1. For a convergence of the solution, error $10^{-6}$ was considered in all cases.

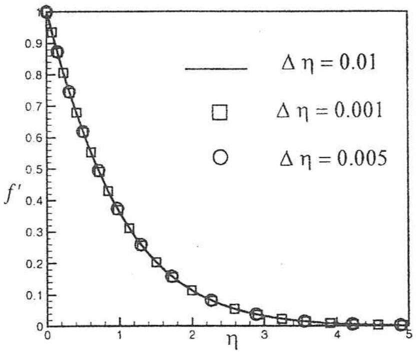



**Fig. 1.** Velocity profiles for different value of $\Delta\eta$

The value of $\eta_{\infty}$ has been obtained by $\eta_{\infty} = \eta_{\infty} + \Delta\eta$ to each iteration loop. The maximum value of $\eta_{\infty}$ to each group of the prescribed parameters is determined, when the value of unknown boundary conditions at $\eta = 0$ not change to successful loop with error less than $10^{-6}$.

**Table. 1.** Comparison of $f''(0)$ and for different values of $\theta_r$ at $Pr = 0.7$

| $\theta_r$ | | Pop et al. [13] | M.E. Ali [14] | Present results |
|---|---|---|---|---|
| -8.0 | $f''(0)$<br>$\theta'(0)$ | -0.477357<br>-0.349318 | -0.476323<br>-0.343233 | -0.476347<br>-0.343665 |
| -0.1 | $f''(0)$<br>$\theta'(0)$ | -1.506173<br>-0.219139 | -1.496515<br>-0.165239 | -1.496523<br>-0.166148 |
| -<br>0.01 | $f''(0)$<br>$\theta'(0)$ | -4.485664<br>-0.154491 | -4.468356<br>-0.056184 | -4.457745<br>-0.076409 |
| 8.0 | $f''(0)$<br>$\theta'(0)$ | -0.408915<br>-0.360522 | -0.408347<br>-0.355582 | -0.408351<br>-0.356002 |

To assess the accuracy of our code, we calculated the values of $f''(0)$ and $-\theta'(0)$. We have compared these physical quantities with that of Pop et al. [13] and with that of M.E. Ali [14] for their $\lambda = 0$ case presented in **Table-1.** and found excellent agreement among them.

## VI. Results and Discussion

The obtained numerical results in the form of non-dimensional velocity $(f')$, microrotation $(g)$ and temperature gradient $(\theta)$ are plotted versus $\eta$ for various micro polar fluid parameters in the following figures and discussed them from the physical point of view.

Figs. 2(a)-(c), respectively, show the velocity, microrotation and temperature profiles for different values of $\theta_r$. Fig. 2(a) shows velocity profile decreases with the increasing values of variable viscosity parameter $\theta_r$. From Fig. 2(b) we see that microrotation profile increase within the region $0 \le \eta \le 1$ with the increase of $\theta_r$. Fig. 2(c) reveals that there is no effect in temperature profile for the increase of $\theta_r$.

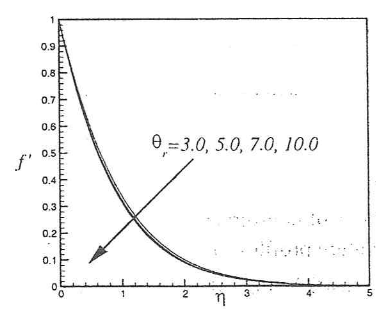
Fig. 3(a) illustrates that the velocity profile decrease with the increase of suction parameter $Fw$ indicating the usual fact that suction stabilizes the boundary layer growth. From Fig. 3(b) we see that microrotation profile increases for $0 \le \eta \le 0.5$ and then decreases for $\eta > 0.5$ with the increase of suction parameter $Fw$. From Fig. 3(c) we observe the usual effect of suction on the temperature profile that is suction parameter $Fw$ has a rapid decreasing effect on the temperature profile.

It is observed from Fig. 4(a) that the fluid velocity decreases with the increase of magnetic field parameter $M$ indicating that magnetic field tends to retard the motion of the fluid. From Fig. 4(b) we see that microrotation profile increases within the domain $0 \le \eta \le 1.5$ and then it decreases for $\eta > 1.5$ with the increase of magnetic field parameter $M$. Fig. 4(c) reveals that temperature profile increases slowly with the increase of $M$.

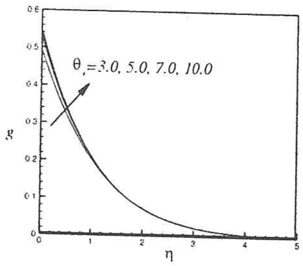
From Fig. 5(a) it is clear that velocity profile decreases with a large effect with the increase of microrotation parameter $n$. Fig. 5(b) shows increasing effect of $n$ on the microrotation profile. From Fig. 5(c) we see that temperature profile increases with the increase of microrotation parameter $n$.

From Fig. 6(a) it is clear that there is a decreasing effect on the velocity field with the increase of variable surface velocity parameter $m$. Fig. 6(b) shows that microrotatrion profile increases for the region $0 \le \eta \le 0.5$ and then decreases for $\eta > 0.5$ with the increase of variable surface velocity parameter $m$. From Fig. 6(c) we see that temperature profile decreases rapidly with the increase of variable surface velocity parameter $m$.

Fig. 7(a) shows the local skin-friction coefficient for different values of variable wall temperature parameter $p$ and magnetic field parameter $M$ keeping all other parameters values fixed. From this Fig. we see that for fixed $p$, local skin-friction $C_f$ decreases as $M$ increases and local skin-friction increases as $p$ increases. Fig. 7(b) shows that $Nu_x$ increases as $p$ increases whereas $Nu_x$ decreases as $M$ increases. Fig. 8(a) shows that for fixed $\Delta$ local skin-friction coefficient $C_f$ increases as $\Pr_{\infty}$ increases. Fig. 8(b) shows that increasing values of $\Delta$ and $\Pr_{\infty}$, increase the local Nusselt number $Nu_x$.




(a)

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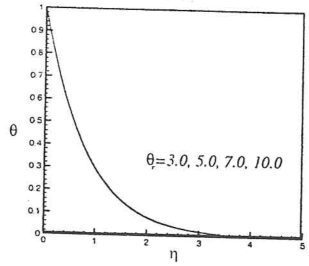
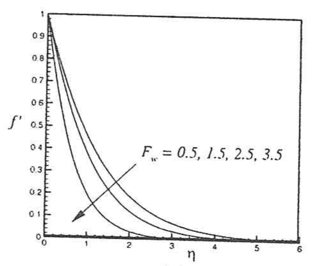
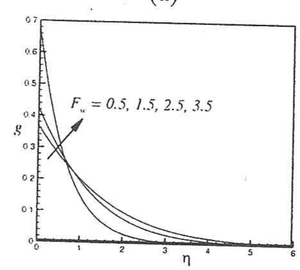



(c)

**Fig. 2.** Variations of non-dimensional (a) velocity(b) microrotation and (c) temperature profiles for different values of $\theta_r$ and $\Delta=3.0$, $\xi=2.0, n=0.5, \mathrm{Pr}_\infty=0.73, M=1.0$, $Q=0.5$, $Ec=0.02$, $F_w=2.0, m=0.5, p=0.5$.




(a)

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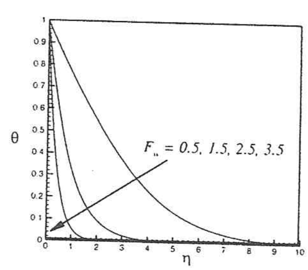
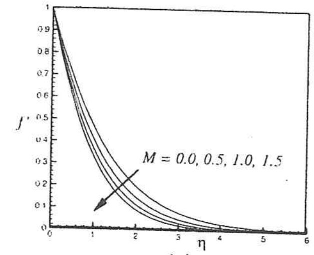
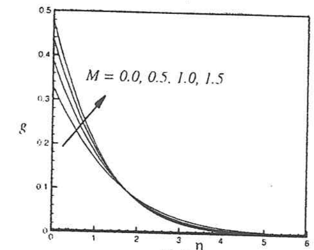



(c)

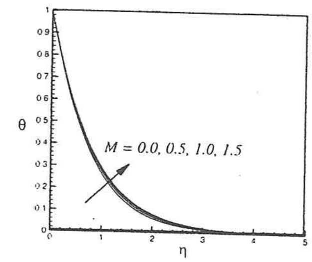
**Fig. 3.** Variations of non-dimensional (a) velocity (b) microrotation and (c) temperature profiles for different values of $F_w$ and $p=0.5$, $,\Delta=3.0, \xi=2.0, n=0.5$, $\mathrm{Pr}_\infty=0.73, M=1.0$, $Q=0.5, Ec=0.02$, $\theta_r=2.0, m=0.5$,




(a)

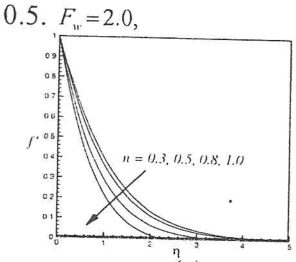



(b)

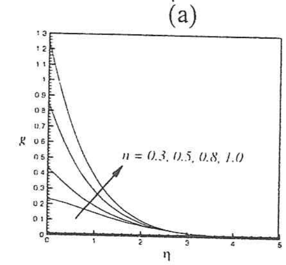



(c)

**Fig. 4.** Variations of non-dimensional (a) velocity (b) microrotation and (c) temperature profiles for different values of $M$ and $\mathrm{Pr}_\infty=0.73$, $\Delta=3.0, \xi=2.0, n=0.5$, $Q=0.5, Ec=0.02, \theta_r=2.0$, $m=0.5, p=0.5$. $F_w=2.0$,




(a)

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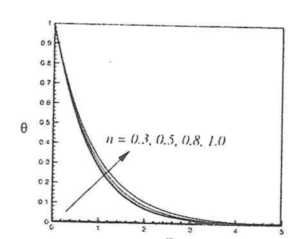
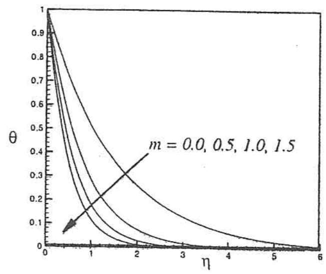



(c)

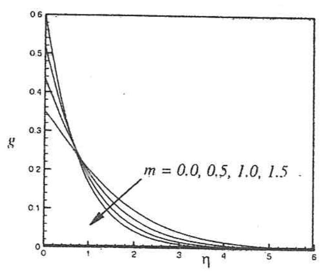
**Fig. 5.** Variations of non-dimensional (a) velocity (b) microrota-tion and (c) temperature profiles for different values of $n$ and $\Delta=3.0, \xi=2.0$, $Q=0.5$, $F_w=2.0$, $\mathrm{Pr}_\infty=0.73$, $Ec=0.02, \theta_r=2.0$, $m=0.5, p=0.5$, $M=1.0$.

In Figs. 7(a)-(b) the effect of suction parameter $F_w$ and variable velocity parameter $m$ on the local skin-friction coefficient $C_f$ and rate of heat transfer $Nu_x$ are shown respectively. In these Figs. we found that local skin-friction coefficient decreases and Nusselt number increases with the increase of $F_w$ and $m$.

From Figs. 8(a)-(b) we see that both local skin-friction as well as Nusselt number decrease with the increase of variable viscosity parameter $\theta_r$ and heat generation/ absorption parameter $Q$.




(a)

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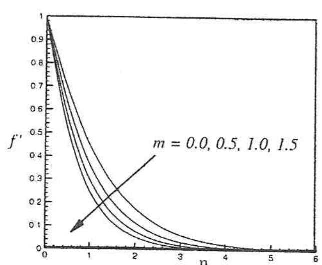
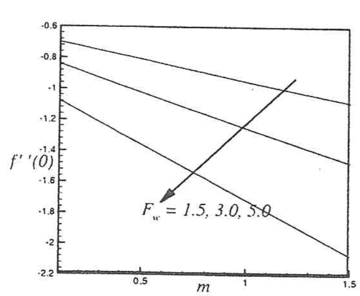
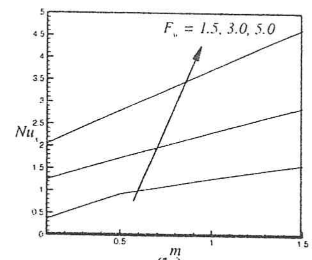



(c)

**Fig. 6.** Variations of non-dimensional (a) velocity (b) micro rotation and (c) temperature profiles for different values of $m$ and $\Delta$=3.0, $\xi$=2.0, $M$=1.0, $n$=0.5, $F_w$=2.0, $Q$=0.5, $Ec$=0.02, $\theta_r$=2.0, $\Pr_\infty$=0.73, $p$=0.5.

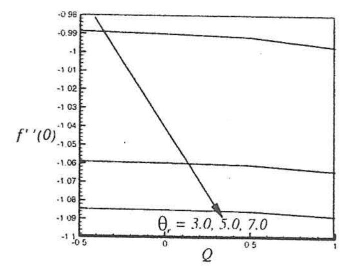



(a)

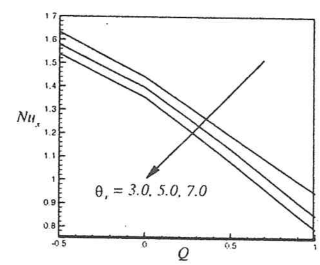



(b)

**Fig.7.** (a) Local skin-friction coefficient and (b) Nusselt number for different values of $m$ and $F_w$.




(a)




(b)

**Fig. 8.** (a) Local skin-friction coefficient and (b) Nusselt number for different values of $Q$ and $\theta_r$.

## Conclusions

1. An increase in the values of suction parameter $F_w$, leads to decrease in the values of local skin-friction coefficient $C_f$ but local Nusselt number $Nu_x$ increases with the increasing values $F_w$.

2. An increase in the values of variable wall temperature $p$, vortex viscosity parameter $\Delta$ and Prandtl number $\Pr_\infty$ increases the local skin-friction and local Nusselt number.

3. Local skin-friction coefficient $C_f$ as well as local Nusselt number $Nu_x$ decrease with the increase of variable viscosity parameter $\theta_r$, heat generation/absorption parameter $Q$ and magnetic field parameter $M$.

4. Variable surface velocity parameter $m$ has decreasing effects on velocity and temperature profiles while it has increasing effect on the microrotation profile for particular region.

5. Magnetic field parameter $M$ tends to retard the motion of the fluid and thus it can be applied in boundary layer control.

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