HIGH ORDER EFFICIENT ALGORITHM FOR COMPUTATION OF MHD FLOW ENSEMBLE

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Abstract: In [38, 39], we proposed, analyzed and tested a fully discrete efficient algorithm for computing magnetohydrodynamic (MHD) flow ensemble dealing with uncertainties in initial conditions and body forces. The algorithm was first order accurate in time, decoupled and based on Elsässer variable formulation. In this paper, we propose, analyze and test a new fully discrete and efficient algorithm for computing MHD flow ensemble under uncertainties in the initial conditions, which aims at improving the temporal convergence rate of the first order algorithm. Similar to the previous algorithm, the new algorithm is decoupled, based on Elsässer variable formulation and uses the breakthrough idea of Jiang and Layton, 2014 to approximate the ensemble average of J realizations. That is, at each time step, each of the J realization shares the same coefficient matrix for different right-hand side matrices. Thus, storage requirements and computational time are reduced by building preconditioners once per time step and reuse them. We prove stability and optimal convergence with respect to the time step restriction. On some manufactured solutions, numerical experiments are given to verify the predicted convergence rates of our analysis. Finally, we test the scheme on a benchmark channel flow over a step problem and it performs well.

Keywords: Magnetohydrodynamics; uncertainty quantification; fast ensemble calculation; finite element method; elsässer variables; second order scheme

1 Introduction:

When an electrically conducting fluid, e.g. plasmas, salt water and liquid metals, moves in presence of a magnetic field, the dynamics of the magnetic field is studied in magnetohydrodynamics (MHD) and the flow is called MHD flow. Recently, the study of MHD flows has become important due to applications in e.g. engineering, physical science, geophysics and astrophysics [5, 7, 13, 15, 22, 43], liquid metal cooling of nuclear reactors [4, 19, 45], process metallurgy [12, 44], and MHD propulsion[32, 37]. The physical principle governing such flows is that the magnetic field induces currents in the moving conductive fluid, which in turn create forces on the fluid and also changes the magnetic field. The time dependent, viscous and incompressible MHD flow is governed by a system of non-linear partial differential equations (PDEs) which nonlinearly couple the Navier-Stokes equations (NSEs) of fluid dynamics to the Maxwell's equations of electromagnetism. In a convex domain $\Omega \subset \mathbb{R}^d (d = 2 \text{ or } 3)$, these nonlinear PDEs can be represented as follows [6, 12, 29]

$$\begin{aligned} u_t + u \cdot \nabla u - B \cdot \nabla B - \nu \Delta u + \nabla p &= f, \\ B_t + u \cdot \nabla B - B \cdot \nabla u - \nu_m \Delta B + \nabla \lambda &= \nabla \times g, \\ \nabla \cdot u = \nabla \cdot B &= 0, \end{aligned}$$

in $\Omega \times (0, T)$. Where Ω is the domain of the fluid, u is the velocity vector, p is a modified pressure, ν is the kinematic viscosity, ν_m is the magnetic resistivity, f is body forces, $\nabla \times g$ is the forcing on the magnetic field B, T is the time period. The artificial magnetic pressure λ is a Lagrange

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multiplier introduced in the induction equation to enforce divergence free constraint on the Maxwell equation in the discrete case but in continuous case $\lambda = 0$. Here all the variables are dimensionless. Assuming the domain is smooth enough, which is a common assumption in, e.g. applications in geophysics and astrophysics, we can avoid the curl formulation of the induction equation.

Numerical simulations of fluid flows are greatly affected by input data like initial condition, the boundary condition, body forces, viscosity, geometry etc, which involve uncertainties. As a result uncertainty quantification (UQ) plays an important role in the validation of simulation methodologies and helps in developing rigorous methods to characterize the effect of the uncertainties on the final quantities of interest. Moreover, many fluid dynamics applications e.g. ensemble Kalman filter approach, weather forecasting, and sensitivity analyses of solutions [10, 31, 34, 35, 36, 42], require multiple numerical simulations of a flow subject to J different input conditions (realizations), are then used to compute means and sensitivities. For MHD simulations, this leads to solve the following J separate nonlinearly coupled systems of PDEs:

$$\begin{aligned}
u_{j,t} + u_j \cdot \nabla u_j - B_j \cdot \nabla B_j - \nu \Delta u_j + \nabla p_j &= f_j(x,t), \text{ in } \Omega \times (0,T), && (1.1)\\
B_{j,t} + u_j \cdot \nabla B_j - B_j \cdot \nabla u_j - \nu_m \Delta B_j + \nabla \lambda_j &= \nabla \times g_j(x,t) \text{ in } \Omega \times (0,T), && (1.2)\\
\nabla \cdot u_j &= 0, \text{ in } \Omega \times (0,T), && (1.3)\\
\nabla \cdot B_j &= 0, \text{ in } \Omega \times (0,T), && (1.4)\\
u_j(x,0) &= u_j^0(x) \text{ in } \Omega, && (1.5)\\
B_j(x,0) &= B_j^0(x) \text{ in } \Omega, && (1.6)
\end{aligned}$$

where u_j, B_j, and p_j denote the solution of the j-th member of the ensemble with initial condition data u_j^0 and B_j^0, and body forces f_j and $\nabla \times g_j$ and $j = 1, 2, \cdots, J$. For the sake of simplicity of our analysis, we consider homogeneous Dirichlet boundary conditions for both velocity and magnetic fields. For periodic boundary conditions or inhomogeneous Dirichlet boundary conditions, our analyses and results will still work after a minor modifications. To obtain an accurate numerical NSE simulation for a single member of the ensemble, the required number of degrees of freedom (dof) are very high, which is known from Kolmogorovs 1941 results [30]. Thus, even for a single member of MHD ensemble simulation, where velocity and magnetic fields are nonlinearly coupled together, is computationally very expensive with respect to time and memory. As a result, the computational cost of the above giant system (1.1)-(1.6) will be approximately equal to $J\times$(cost of one MHD simulation) and will generally be computationally be infeasible. Our objective in this paper is to build and study an efficient and accurate algorithm for solving the above ensemble systems. It has been shown in recent works [1, 20, 39, 46] that using Elsässer variables formulation, efficient MHD simulation algorithms can be created. Since they can be decoupled stable way so that at each time step, in lieu of solving a fully coupled linear system, two separate Oseen-type problems need to be solved.

Defining $v_j := u_j + B_j$, $w_j := u_j - B_j$, $f_{1,j} := f_j + \nabla \times g_j$, $f_{2,j} := f_j - \nabla \times g_j$, $q_j := p_j + \lambda_j$ and $r_j := p_j - \lambda_j$ produces the Elsässer variable formulation of the ensemble systems:

$$\begin{aligned}
v_{j,t} + w_j \cdot \nabla v_j + \nabla q_j - \frac{\nu + \nu_m}{2} \Delta v_j - \frac{\nu - \nu_m}{2} \Delta w_j &= f_{1,j}, && (1.7)\\
w_{j,t} + v_j \cdot \nabla w_j + \nabla r_j - \frac{\nu + \nu_m}{2} \Delta w_j - \frac{\nu - \nu_m}{2} \Delta v_j &= f_{2,j}, && (1.8)\\
\nabla \cdot v_j = \nabla \cdot w_j &= 0. && (1.9)
\end{aligned}$$

together with initial and boundary conditions.

To reduce the ensemble simulation cost, an excellent idea was presented in [27] to find a set of J solutions of the NSEs for different initial conditions and body forces. The fundamental idea is that, at each time step, each of the J systems shares a common coefficient matrix but the right-hand vectors are different. Thus, the preconditioners need to build only once per time step and can reuse for all J systems, also the algorithm can save storage requirement and take advantage of block linear solvers. This breakthrough idea has been implemented in heat condution[14], Navier-Stokes simulations [24, 25, 28, 41], magnetohydrodynamics [39], parameterized flow problems [18, 33], and turbulence modeling [26]. We use the same idea for a second oder time stepping scheme for MHD flow ensemble simulation subject to different initial conditions. The author proposed a first order scheme to compute MHD flow ensemble subject to different initial conditions [39] and body forces [38].

We consider a uniform timestep size Δt and let $t_n = n\Delta t$ for $n = 0, 1, \cdots$., for simplicity, we suppress the spatial discretization momentarily. Then computing the J solutions independently, takes the following form:
Step 1: for j=1,...,J,

$$\begin{aligned}\frac{3v_j^{n+1}-4v_j^n+v_j^{n-1}}{2\Delta t}+\nabla q_j^{n+1}-\frac{\nu+\nu_m}{2}\Delta v_j^{n+1}-\frac{\nu-\nu_m}{2}\Delta(2w_j^n-w_j^{n-1})&\\ +<w>^n\cdot\nabla v_j^{n+1}+w_j^{'n}\cdot\nabla(2v_j^n-v_j^{n-1})=f_{1,j}(t^{n+1}),&\\ \nabla\cdot v_j^{n+1}=0.&\end{aligned}\tag{1.10}$$

Step 2: for j=1,...,J,

$$\begin{aligned}\frac{3w_j^{n+1}-4w_j^n+w_j^{n-1}}{2\Delta t}+\nabla r_j^{n+1}-\frac{\nu+\nu_m}{2}\Delta w_j^{n+1}-\frac{\nu-\nu_m}{2}\Delta(2v_j^n-v_j^{n-1})&\\ +<v>^n\cdot\nabla w_j^{n+1}+v_j^{'n}\cdot\nabla(2w_j^n-w_j^{n-1})=f_{2,j}(t^{n+1}),&\\ \nabla\cdot w_j^{n+1}=0.&\end{aligned}\tag{1.11}$$

Where v_j^n, w_j^n, q_j^n, and r_j^n denote approximations of $v_j(\cdot,t^n), w_j(\cdot,t^n), q_j(\cdot,t^n)$, and $r_j(\cdot,t^n)$. The ensemble mean and fluctuation about the mean are denoted by $<u>$, $u_j^{'}$ respectively and these are defined as follows:

$$<u>^n:=\frac{1}{J}\sum_{j=1}^{J}(2u_j^n-u_j^{n-1}),\ \ u_j^{'n}:=2u_j^n-u_j^{n-1}-<u>^n\ . \tag{1.12}$$

The key to the efficiencies of the above algorithm are that (1) the MHD system is decoupled into two Oseen problems and can be solved simultaneously if the computational resources are available, (2) the coefficient matrices of (1.10) and (1.11) at each time step are independent of j, thus all the J members for each sub-problems in the ensemble share a same coefficient matrix. That is, at every time step, we do not need to solve J individual systems of equations for each sub-problem instead a single linear system with J different right-hand-side constant vectors.

We give a rigorous proof that the decoupled scheme is stable and the ensemble average of the J different computed solutions converges to the ensemble average of the J different true solutions, as the timestep size and the spatial mesh width tend to zero.

This paper is organized as follows. Section 2 presents notation and mathematical preliminaries those are necessary for a smooth presentation and analysis to follow. In section 3, we present and analyze a fully discrete and decoupled algorithm corresponding to (1.10)-(1.11), and prove it is stable and convergent. Numerical tests are presented in section 4, and finally, conclusions are drawn in section 5.

2 Notation and Preliminaries:

Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be a convex polygonal or polyhedral domain in $\mathbb{R}^d(d = 2, 3)$ with boundary $\partial\Omega$. The usual $L^2(\Omega)$ norm and inner product are denoted by $\|.\|$ and $(.,.)$ respectively. Similarly, the $L^p(\Omega)$ norms and the Sobolev $W^k_p(\Omega)$ norms are $\|.\|_{L^p}$ and $\|.\|_{W^k_p}$ respectively for $k \in \mathbb{N}$, $1 \le p \le \infty$. Sobolev space $W^k_2(\Omega)$ is represented by $H^k(\Omega)$ with norm $\|.\|_k$. The natural function spaces for our problem are

$$X := H^1_0(\Omega) = \{v \in (L^p(\Omega))^d : \nabla v \in L^2(\Omega)^{d\times d}, v = 0 \text{ on } \partial\Omega\},$$

$$Q := L^2_0(\Omega) = \{q \in L^2(\Omega) : \int_\Omega q \ dx = 0\}.$$

Recall the Poincare inequality holds in X: there exists C depending only on the size of Ω satisfying for all $\phi \in X$,

$$\|\phi\| \le C\|\nabla\phi\|.$$

The divergence free velocity space is given by

$$V := \{v \in X : (\nabla \cdot v, q) = 0, \forall q \in Q\}.$$

We define the trilinear form $b : X \times X \times X \to \mathbb{R}$ by

$$b(u, v, w) := (u \cdot \nabla v, w),$$

and recall from [16] that $b(u, v, v) = 0$ if $u \in V$, and

$$|b(u, v, w)| \le C(\Omega)\|\nabla u\|\|\nabla v\|\|\nabla w\|, \text{ for any } u, v, w \in X. \tag{2.1}$$

The conforming finite element spaces are denoted by $X_h \subset X$ and $Q_h \subset Q$, and we assume a regular triangulation $\tau_h(\Omega)$, where h is the maximum triangle diameter. We assume that (X_h, Q_h) satisfies the usual discrete inf-sup condition

$$\inf_{q_h\in Q_h} \sup_{v_h\in X_h} \frac{(q_h, \nabla \cdot v_h)}{\|q_h\|\|\nabla v_h\|} \ge \beta > 0, \tag{2.2}$$

where β is independent of h.

The space of discretely divergence free functions is defined as

$$V_h := \{v_h \in X_h : (\nabla \cdot v_h, q_h) = 0, \ \forall q_h \in Q_h\}.$$

For simplicity of our analysis, we will use Scott-Vogelius $(X_h, Q_h) = ((P_k)^d, P^{disc}_{k-1})$ finite element pair, which satisfies the *inf-sup* condition when the mesh is created as a barycenter refinement of a regular mesh, and the polynomial degree $k \ge d$ [3, 48]. Our analysis can be extended without difficulty to any inf-sup stable element choice, however, there will be additional terms that appear in the convergence analysis if non-divergence-free elements are chosen.

We have the following approximation properties in (X_h, Q_h): [9]

$$\inf_{v_h\in X_h} \|u - v_h\| \leq Ch^{k+1}|u|_{k+1}, \quad u \in H^{k+1}(\Omega), \tag{2.3}$$

$$\inf_{v_h\in X_h} \|\nabla(u - v_h)\| \leq Ch^{k}|u|_{k+1}, \quad u \in H^{k+1}(\Omega), \tag{2.4}$$

$$\inf_{q_h\in Q_h} \|p - q_h\| \leq Ch^{k}|p|_{k}, \quad p \in H^{k}(\Omega), \tag{2.5}$$

where $|\cdot|_r$ denotes the H^r seminorm.

We will assume the mesh is sufficiently regular for the inverse inequality to hold, and with this and the LBB assumption, we have approximation properties

$$\|\nabla(u - P_{L^2}^{V_h}(u))\| \leq Ch^{k}|u|_{k+1}, \quad u \in H^{k+1}(\Omega), \tag{2.6}$$

$$\inf_{v_h\in V_h} \|\nabla(u - v_h)\| \leq Ch^{k}|u|_{k+1}, \quad u \in H^{k+1}(\Omega), \tag{2.7}$$

where $P_{L^2}^{V_h}(u)$ is the L^2 projection of u into V_h.

The following lemma for the discrete Gronwall inequality was given in [21].

Lemma 2.1. *Let Δt, H, a_n, b_n, c_n, d_n be non-negative numbers for $n = 1, \cdots, M$ such that*

$$a_M + \Delta t\sum_{n=1}^{M} b_n \leq \Delta t\sum_{n=1}^{M-1} d_n a_n + \Delta t\sum_{n=1}^{M} c_n + H \quad \textit{for } M \in \mathbb{N},$$

then for all $\Delta t > 0$,

$$a_M + \Delta t\sum_{n=1}^{M} b_n \leq exp\left(\Delta t\sum_{n=1}^{M-1} d_n\right)\left(\Delta t\sum_{n=1}^{M} c_n + H\right) \quad \textit{for } M \in \mathbb{N}.$$

3 Fully discrete scheme and analysis:

Now we present and analyze an efficient, fully discrete, decoupled, and second-order scheme for computing MHD flow ensemble. At first, we define the scheme then analyze its stability and convergence. The scheme is derived by applying the linearized BDF2 scheme to the Elsas̈er MHD system (1.7)-(1.9) with a finite element spatial discretization. Similar to the other BDF2 schemes, two initial conditions should be known; if the first initial condition is known, using the linearized backward Euler scheme in [39] without the ensemble eddy viscosity terms on the first time step, we can get the second initial condition without affecting stability or accuracy. The scheme is defined below.

Algorithm 3.1. *Given time step $\Delta t > 0$, end time $T > 0$, initial conditions $v_j^0, w_j^0, v_j^1, w_j^1 \in V_h$ and $f_{1,j}, f_{2,j} \in L^\infty(0, T; H^{-1}(\Omega)^d)$ for $j = 1, 2, \cdots, J$. Set $M = T/\Delta t$ and for $n = 1, \cdots, M-1$, compute:*

Find $v_{j,h}^{n+1} \in V_h$ satisfying, for all $\chi_h \in V_h$:

$$\left(\frac{3v_{j,h}^{n+1} - 4v_{j,h}^{n} + v_{j,h}^{n-1}}{2\Delta t}, \chi_h\right) + \frac{\nu+\nu_m}{2}(\nabla v_{j,h}^{n+1}, \nabla\chi_h) + \frac{\nu-\nu_m}{2}(\nabla(2w_{j,h}^{n} - w_{j,h}^{n-1}), \nabla\chi_h)$$
$$+(< w_h >^n \cdot\nabla v_{j,h}^{n+1}, \chi_h) + (w_{j,h}^{'n} \cdot \nabla(2v_{j,h}^{n} - v_{j,h}^{n-1}), \chi_h) = (f_{1,j}(t^{n+1}), \chi_h), \tag{3.1}$$

Find $w_{j,h}^{n+1} \in V_h$ satisfying, for all $l_h \in V_h$:

$$\left(\frac{3w_{j,h}^{n+1} - 4w_{j,h}^n + w_{j,h}^{n-1}}{2\Delta t}, l_h\right) + \frac{\nu + \nu_m}{2}(\nabla w_{j,h}^{n+1}, \nabla l_h) + \frac{\nu - \nu_m}{2}(\nabla(2v_{j,h}^n - v_{j,h}^{n-1}), \nabla l_h)$$
$$+(< v_h >^n \cdot\nabla w_{j,h}^{n+1}, l_h) + (v_{j,h}'^n \cdot \nabla(2w_{j,h}^n - w_{j,h}^{n-1}), l_h) = (f_{2,j}(t^{n+1}), l_h). \qquad (3.2)$$

3.1 Stability Analysis: We now prove stability and well-posedness for the Algorithm (3.1). To simplify our calculation, we define $\alpha := \nu + \nu_m - |\nu - \nu_m| > 0$.

Lemma 3.1. *Consider the Algorithm 3.1. If the mesh is sufficiently regular so that the inverse inequality holds (with constant C_i) and the time step is chosen to satisfy*

$$\Delta t \leq \frac{\alpha h^2}{3(\nu - \nu_m)^2 C_i + 12C^2C_i^2 \max\limits_{1\leq j\leq J}\{\|\nabla v_{j,h}'^n\|, \|\nabla v_{j,h}'^n\|\}}$$

then the method is stable and solutions to (3.1)-(3.2) satisfy

$$\|v_{j,h}^M\|^2 + \|w_{j,h}^M\|^2 + \|2v_{j,h}^M - v_{j,h}^{M-1}\|^2 + \|2w_{j,h}^M - w_{j,h}^{M-1}\|^2 + \alpha\Delta t \sum_{n=1}^{M-1}(\|\nabla v_{j,h}^{n+1}\|^2 + \|\nabla w_{j,h}^{n+1}\|^2)$$
$$\leq \|v_{j,h}^1\|^2 + \|w_{j,h}^1\|^2 + \|2v_{j,h}^1 - v_{j,h}^0\|^2 + \|2w_{j,h}^1 - w_{j,h}^0\|^2$$
$$+ \frac{12\Delta t}{\alpha}\sum_{n=1}^{M-1}(\|f_{1,j}(t^{n+1})\|_{-1} + \|f_{2,j}(t^{n+1})\|_{-1}). \qquad (3.3)$$

Proof. Choos $\chi_h = v_{j,h}^{n+1}$ in (3.1), using the following identity

$$(3a - 4b + c, a) = \frac{a^2 + (2a-b)^2}{2} - \frac{b^2 + (2b-c)^2}{2} + \frac{(a - 2b + c)^2}{2}, \qquad (3.4)$$

we obtain

$$\frac{1}{4\Delta t}\left(\|v_{j,h}^{n+1}\|^2 - \|v_{j,h}^n\|^2 + \|2v_{j,h}^{n+1} - v_{j,h}^n\|^2 - \|2v_{j,h}^n - v_{j,h}^{n-1}\|^2 + \|v_{j,h}^{n+1} - 2v_{j,h}^n + v_{j,h}^{n-1}\|^2\right)$$
$$+\frac{\nu + \nu_m}{2}\|\nabla v_{j,h}^{n+1}\|^2 + \frac{\nu - \nu_m}{2}(\nabla(2w_{j,h}^n - w_{j,h}^{n-1}), \nabla v_{j,h}^{n+1}) + (w_{j,h}'^n \cdot \nabla(2v_{j,h}^n - v_{j,h}^{n-1}), v_{j,h}^{n+1})$$
$$= (f_{1,j}(t^{n+1}), v_{j,h}^{n+1}). \qquad (3.5)$$

Similarly, choose $l_h = w_{j,h}^{n+1}$ in (3.2), we have

$$\frac{1}{4\Delta t}\left(\|w_{j,h}^{n+1}\|^2 - \|w_{j,h}^n\|^2 + \|2w_{j,h}^{n+1} - w_{j,h}^n\|^2 - \|2w_{j,h}^n - w_{j,h}^{n-1}\|^2 + \|w_{j,h}^{n+1} - 2w_{j,h}^n + w_{j,h}^{n-1}\|^2\right)$$
$$+ \frac{\nu + \nu_m}{2}\|\nabla w_{j,h}^{n+1}\|^2 + \frac{\nu - \nu_m}{2}(\nabla(2v_{j,h}^n - v_{j,h}^{n-1}), \nabla w_{j,h}^{n+1}) + (v_{j,h}'^n \cdot \nabla(2w_{j,h}^n - w_{j,h}^{n-1}), w_{j,h}^{n+1})$$
$$= (f_{2,j}(t^{n+1}), w_{j,h}^{n+1}). \qquad (3.6)$$

Next, using

$$
\begin{aligned}
(w_{j,h}^{\prime n}\cdot\nabla(2v_{j,h}^{n}-v_{j,h}^{n-1}),v_{j,h}^{n+1}) &= (w_{j,h}^{\prime n}\cdot\nabla v_{j,h}^{n+1},v_{j,h}^{n+1}-2v_{j,h}^{n}+v_{j,h}^{n-1})\\
&\leq C\|\nabla w_{j,h}^{\prime n}\|\|\nabla v_{j,h}^{n+1}\|\ \|\nabla(v_{j,h}^{n+1}-2v_{j,h}^{n}+v_{j,h}^{n-1})\|\\
&\leq \frac{CC_i}{h}\|\nabla w_{j,h}^{\prime n}\|\|\nabla v_{j,h}^{n+1}\|\ \|v_{j,h}^{n+1}-2v_{j,h}^{n}+v_{j,h}^{n-1}\|,
\end{aligned}
$$

adding equations (3.5) and (3.6) and applying Cauchy-Schwarz inequality, yields

$$
\begin{aligned}
&\frac{1}{4\Delta t}\Big(\|v_{j,h}^{n+1}\|^2-\|v_{j,h}^{n}\|^2+\|2v_{j,h}^{n+1}-v_{j,h}^{n}\|^2-\|2v_{j,h}^{n}-v_{j,h}^{n-1}\|^2+\|v_{j,h}^{n+1}-2v_{j,h}^{n}+v_{j,h}^{n-1}\|^2\\
&+\|w_{j,h}^{n+1}\|^2-\|w_{j,h}^{n}\|^2+\|2w_{j,h}^{n+1}-w_{j,h}^{n}\|^2-\|2w_{j,h}^{n}-w_{j,h}^{n-1}\|^2+\|w_{j,h}^{n+1}-2w_{j,h}^{n}+w_{j,h}^{n-1}\|^2\Big)\\
&+\frac{\nu+\nu_m}{2}(\|\nabla v_{j,h}^{n+1}\|^2+\|\nabla w_{j,h}^{n+1}\|^2)\\
&+\frac{\nu-\nu_m}{2}\Big\{\big(\nabla(2v_{j,h}^{n}-v_{j,h}^{n-1}),\nabla w_{j,h}^{n+1}\big)+\big(\nabla(2w_{j,h}^{n}-w_{j,h}^{n-1}),\nabla v_{j,h}^{n+1}\big)\Big\}\\
\leq{}&\frac{CC_i}{h}\|\nabla w_{j,h}^{\prime n}\|\|\nabla v_{j,h}^{n+1}\|\ \|v_{j,h}^{n+1}-2v_{j,h}^{n}+v_{j,h}^{n-1}\|+\frac{CC_i}{h}\|\nabla v_{j,h}^{\prime n}\|\|\nabla w_{j,h}^{n+1}\|\ \|w_{j,h}^{n+1}-2w_{j,h}^{n}+w_{j,h}^{n-1}\|\\
&+\|f_{1,j}(t^{n+1})\|_{-1}\|\nabla v_{j,h}^{n+1}\|+\|f_{2,j}(t^{n+1})\|_{-1}\|\nabla w_{j,h}^{n+1}\|.
\end{aligned}
$$

Adding and subtracting the term $\frac{\nu-\nu_m}{2}\left(\nabla v_{j,h}^{n+1},\nabla w_{j,h}^{n+1}\right)$ twice provides

$$
\begin{aligned}
&\frac{1}{4\Delta t}\Big(\|v_{j,h}^{n+1}\|^2-\|v_{j,h}^{n}\|^2+\|2v_{j,h}^{n+1}-v_{j,h}^{n}\|^2-\|2v_{j,h}^{n}-v_{j,h}^{n-1}\|^2+\|v_{j,h}^{n+1}-2v_{j,h}^{n}+v_{j,h}^{n-1}\|^2\\
&+\|w_{j,h}^{n+1}\|^2-\|w_{j,h}^{n}\|^2+\|2w_{j,h}^{n+1}-w_{j,h}^{n}\|^2-\|2w_{j,h}^{n}-w_{j,h}^{n-1}\|^2+\|w_{j,h}^{n+1}-2w_{j,h}^{n}+w_{j,h}^{n-1}\|^2\Big)\\
&+\frac{\nu+\nu_m}{2}(\|\nabla v_{j,h}^{n+1}\|^2+\|\nabla w_{j,h}^{n+1}\|^2)\\
&-\frac{\nu-\nu_m}{2}\Big\{\big(\nabla(v_{j,h}^{n+1}-2v_{j,h}^{n}+v_{j,h}^{n-1}),\nabla w_{j,h}^{n+1}\big)+\big(\nabla(w_{j,h}^{n+1}-2w_{j,h}^{n}+w_{j,h}^{n-1}),\nabla v_{j,h}^{n+1}\big)\Big\}\\
&+\frac{\nu-\nu_m}{2}(\nabla v_{j,h}^{n+1},\nabla w_{j,h}^{n+1})+\frac{\nu-\nu_m}{2}(\nabla w_{j,h}^{n+1},\nabla v_{j,h}^{n+1})\\
\leq{}&\frac{CC_i}{h}\|\nabla w_{j,h}^{\prime n}\|\|\nabla v_{j,h}^{n+1}\|\ \|v_{j,h}^{n+1}-2v_{j,h}^{n}+v_{j,h}^{n-1}\|+\frac{CC_i}{h}\|\nabla v_{j,h}^{\prime n}\|\|\nabla w_{j,h}^{n+1}\|\ \|w_{j,h}^{n+1}-2w_{j,h}^{n}+w_{j,h}^{n-1}\|\\
&+\|f_{1,j}(t^{n+1})\|_{-1}\|\nabla v_{j,h}^{n+1}\|+\|f_{2,j}(t^{n+1})\|_{-1}\|\nabla w_{j,h}^{n+1}\|.
\end{aligned}
$$

Using Cauchy-Schwarz inequality we have that

$$
\begin{aligned}
&\frac{1}{4\Delta t}\Big(\|v_{j,h}^{n+1}\|^2 - \|v_{j,h}^{n}\|^2 + \|2v_{j,h}^{n+1} - v_{j,h}^{n}\|^2 - \|2v_{j,h}^{n} - v_{j,h}^{n-1}\|^2 + \|v_{j,h}^{n+1} - 2v_{j,h}^{n} + v_{j,h}^{n-1}\|^2 \\
&+ \|w_{j,h}^{n+1}\|^2 - \|w_{j,h}^{n}\|^2 + \|2w_{j,h}^{n+1} - w_{j,h}^{n}\|^2 - \|2w_{j,h}^{n} - w_{j,h}^{n-1}\|^2 + \|w_{j,h}^{n+1} - 2w_{j,h}^{n} + w_{j,h}^{n-1}\|^2\Big) \\
&+ \frac{\nu+\nu_m}{2}\big(\|\nabla v_{j,h}^{n+1}\|^2 + \|\nabla w_{j,h}^{n+1}\|^2\big) \le |\nu-\nu_m|\|\nabla v_{j,h}^{n+1}\|\|\nabla w_{j,h}^{n+1}\| \\
&+ \frac{|\nu-\nu_m|}{2}\|\nabla(v_{j,h}^{n+1} - 2v_{j,h}^{n} + v_{j,h}^{n-1})\|\|\nabla w_{j,h}^{n+1}\| + \frac{|\nu-\nu_m|}{2}\|\nabla(w_{j,h}^{n+1} - 2w_{j,h}^{n} + w_{j,h}^{n-1})\|\|\nabla v_{j,h}^{n+1}\| \\
&+ \frac{CC_i}{h}\|\nabla w_{j,h}'^{n}\|\|\nabla v_{j,h}^{n+1}\|\,\|v_{j,h}^{n+1} - 2v_{j,h}^{n} + v_{j,h}^{n-1}\| + \frac{CC_i}{h}\|\nabla v_{j,h}'^{n}\|\|\nabla w_{j,h}^{n+1}\|\,\|w_{j,h}^{n+1} - 2w_{j,h}^{n} + w_{j,h}^{n-1}\| \\
&+ \|f_{1,j}(t^{n+1})\|_{-1}\|\nabla v_{j,h}^{n+1}\| + \|f_{2,j}(t^{n+1})\|_{-1}\|\nabla w_{j,h}^{n+1}\|. \qquad (3.7)
\end{aligned}
$$

Young's inequality provides the following bounds on the last seven terms in (3.7):

$$
\begin{aligned}
|\nu-\nu_m|\,\|\nabla v_{j,h}^{n+1}\|\|\nabla w_{j,h}^{n+1}\| &\le \frac{|\nu-\nu_m|}{2}\big(\|\nabla v_{j,h}^{n+1}\|^2 + \|\nabla w_{j,h}^{n+1}\|^2\big), \\
\frac{|\nu-\nu_m|}{2}\|\nabla(v_{j,h}^{n+1} - 2v_{j,h}^{n} + v_{j,h}^{n-1})\|\|\nabla w_{j,h}^{n+1}\| & \\
&\le \frac{\alpha}{12}\|\nabla w_{j,h}^{n+1}\|^2 + \frac{3(\nu-\nu_m)^2}{4\alpha}\|\nabla(v_{j,h}^{n+1} - 2v_{j,h}^{n} + v_{j,h}^{n-1})\|^2, \\
\frac{|\nu-\nu_m|}{2}\|\nabla(w_{j,h}^{n+1} - 2w_{j,h}^{n} + w_{j,h}^{n-1})\|\|\nabla v_{j,h}^{n+1}\| & \\
&\le \frac{\alpha}{12}\|\nabla v_{j,h}^{n+1}\|^2 + \frac{3(\nu-\nu_m)^2}{4\alpha}\|\nabla(w_{j,h}^{n+1} - 2w_{j,h}^{n} + w_{j,h}^{n-1})\|^2, \\
\frac{CC_i}{h}\|\nabla w_{j,h}'^{n}\|\|\nabla v_{j,h}^{n+1}\|\,\|v_{j,h}^{n+1} - 2v_{j,h}^{n} + v_{j,h}^{n-1}\| & \\
&\le \frac{\alpha}{12}\|\nabla v_{j,h}^{n+1}\|^2 + \frac{3C^2C_i^2}{\alpha h^2}\|\nabla w_{j,h}'^{n}\|^2\|v_{j,h}^{n+1} - 2v_{j,h}^{n} + v_{j,h}^{n-1}\|^2, \\
\frac{CC_i}{h}\|\nabla v_{j,h}'^{n}\|\|\nabla w_{j,h}^{n+1}\|\,\|w_{j,h}^{n+1} - 2w_{j,h}^{n} + w_{j,h}^{n-1}\| & \\
&\le \frac{\alpha}{12}\|\nabla w_{j,h}^{n+1}\|^2 + \frac{3C^2C_i^2}{\alpha h^2}\|\nabla v_{j,h}'^{n}\|^2\|w_{j,h}^{n+1} - 2w_{j,h}^{n} + w_{j,h}^{n-1}\|^2, \\
\|f_{1,j}(t^{n+1})\|_{-1}\|\nabla v_{j,h}^{n+1}\| &\le \frac{\alpha}{12}\|\nabla v_{j,h}^{n+1}\|^2 + \frac{3}{\alpha}\|f_{1,j}(t^{n+1})\|_{-1}^2, \\
\|f_{2,j}(t^{n+1})\|_{-1}\|\nabla w_{j,h}^{n+1}\| &\le \frac{\alpha}{12}\|\nabla w_{j,h}^{n+1}\|^2 + \frac{3}{\alpha}\|f_{2,j}(t^{n+1})\|_{-1}^2.
\end{aligned}
$$

Using these estimates and the following inverse inequality in (3.7)

$$
\|\nabla(u_{j,h}^{n+1} - 2u_{j,h}^{n} + u_{j,h}^{n-1})\|^2 \le \frac{C_i^2}{h^2}\|u_{j,h}^{n+1} - 2u_{j,h}^{n} + u_{j,h}^{n-1}\|^2
$$

produces

$$
\begin{aligned}
&\frac{1}{4\Delta t}\Big(\|v_{j,h}^{n+1}\|^2-\|v_{j,h}^{n}\|^2+\|2v_{j,h}^{n+1}-v_{j,h}^{n}\|^2-\|2v_{j,h}^{n}-v_{j,h}^{n-1}\|^2\\
&\quad+\|w_{j,h}^{n+1}\|^2-\|w_{j,h}^{n}\|^2+\|2w_{j,h}^{n+1}-w_{j,h}^{n}\|^2-\|2w_{j,h}^{n}-w_{j,h}^{n-1}\|^2\Big)\\
&+\left\{\frac{1}{4\Delta t}-\frac{3(\nu-\nu_m)^2C_i+12C^2C_i^2\|\nabla w_{j,h}'^{n}\|^2}{4\alpha h^2}\right\}\|v_{j,h}^{n+1}-2v_{j,h}^{n}+v_{j,h}^{n-1}\|^2\\
&+\left\{\frac{1}{4\Delta t}-\frac{3(\nu-\nu_m)^2C_i+12C^2C_i^2\|\nabla v_{j,h}'^{n}\|^2}{4\alpha h^2}\right\}\|w_{j,h}^{n+1}-2w_{j,h}^{n}+w_{j,h}^{n-1}\|^2\\
&+\frac{\alpha}{4}\left(\|\nabla v_{j,h}^{n+1}\|^2+\|\nabla w_{j,h}^{n+1}\|^2\right)\leq\frac{3}{\alpha}\left(\|f_{1,j}(t^{n+1})\|_{-1}+\|f_{2,j}(t^{n+1})\|_{-1}\right).
\end{aligned}\tag{3.8}
$$

Now if we choose $\Delta t\leq\frac{\alpha h^2}{3(\nu-\nu_m)^2C_i+12C^2C_i^2\max\limits_{1\leq j\leq J}\left\{\|\nabla v_{j,h}'^{n}\|,\|\nabla v_{j,h}'^{n}\|\right\}}$, dropping the non-negative terms on left, multiplying both sides by $4\Delta t$ and summing over time steps from $n=1$ to $n=M-1$ results in (3.3). □

3.2 Error Analysis:

Now we consider the convergence of the proposed decoupled scheme.

Theorem 3.1. *Suppose (v_j, w_j, q_j, r_j) satisfies (1.7)-(1.9) with the regularity assumptions v_j,w_j $\in L^\infty(0,T;\ H^m(\Omega)^d)$ for $m=\max\{2,k+1\}$, $v_{j,tt}, w_{j,tt}\in L^\infty(0,T;H^1(\Omega)^d)$, and $v_{j,ttt}, w_{j,ttt}\in L^\infty(0,T;L^2(\Omega)^d)$. Then the ensemble average solution $(<v_h>,<w_h>)$ to Algorithm (3.1) converges to the true ensemble average solution: for any $\Delta t\leq\frac{\alpha h^2}{9C_i^2(\nu-\nu_m)^2+9C_i^2C^2\max\limits_{1\leq j\leq J}\left\{\|\nabla v_{j,h}'^{n}\|,\|\nabla w_{j,h}'^{n}\|\right\}}$, one has*

$$
\begin{aligned}
\|<v>^T-<v_h>^M\|^2&+\alpha\Delta t\sum_{n=2}^{M}\|\nabla(<v>(t^n)-<v_h>^n)\|^2\\
&\leq\frac{2^{J+2}C}{J^2\alpha}e^{\frac{9TC}{\alpha}}\left((\nu^2+\nu_m^2+1)h^{2k}+((\nu-\nu_m)^2+1)\Delta t^4\right)
\end{aligned}\tag{3.9}
$$

Proof. We start our proof by obtaining the error equations. Testing (1.7) and (1.8) with $\chi_h, l_h\in V_h$ at the time level t^{n+1}, the continuous variational formulations can be written as

$$
\begin{aligned}
&\left(\frac{3v_j(t^{n+1})-4v_j(t^n)+v_j(t^{n-1})}{2\Delta t},\chi_h\right)+(w_j(t^{n+1})\cdot\nabla v_j(t^{n+1}),\chi_h)\\
&\quad+\frac{\nu+\nu_m}{2}(\nabla v_j(t^{n+1}),\nabla\chi_h)+\frac{\nu-\nu_m}{2}(\nabla(2w_j(t^n)-w_j(t^{n-1})),\chi_h)\\
&=(f_{1,j}(t^{n+1}),\chi_h)-\frac{\nu-\nu_m}{2}(\nabla(w_j(t^{n+1})-2w_j(t^n)+w_j(t^{n-1})),\chi_h)\\
&\quad-\left(v_{j,t}(t^{n+1})-\frac{3v_j(t^{n+1})-4v_j(t^n)+v_j(t^{n-1})}{2\Delta t},\chi_h\right),
\end{aligned}\tag{3.10}
$$

and

$$
\begin{aligned}
&\left(\frac{3w_j(t^{n+1})-4w_j(t^n)+w_j(t^{n-1})}{2\Delta t}, l_h\right) + (v_j(t^{n+1})\cdot\nabla w_j(t^{n+1}), l_h)\\
&\quad+\frac{\nu+\nu_m}{2}(\nabla w_j(t^{n+1}),\nabla l_h)+\frac{\nu-\nu_m}{2}(\nabla(2v_j(t^n)-v_j(t^{n-1})), l_h)\\
&\quad=(f_{2,j}(t^{n+1}), l_h)-\frac{\nu-\nu_m}{2}(\nabla(v_j(t^{n+1})-2v_j(t^n)+v_j(t^{n-1})), l_h)\\
&\qquad-\left(w_{j,t}(t^{n+1})-\frac{3w_j(t^{n+1})-4w_j(t^n)+w_j(t^{n-1})}{2\Delta t}, l_h\right).
\end{aligned}\tag{3.11}
$$

Denote $e^n_{v,j} := v_j(t^n)-v^n_{j,h}$, $e^n_{w,j} := w_j(t^n)-w^n_{j,h}$. Subtracting (3.1) and (3.2) from equation (3.10) and (3.11) respectively, yields

$$
\begin{aligned}
&\left(\frac{3e^{n+1}_{j,v}-4e^n_{j,v}+e^{n-1}_{j,v}}{2\Delta t}, \chi_h\right)+\frac{\nu+\nu_m}{2}(\nabla e^{n+1}_{j,v},\nabla\chi_h)+\frac{\nu-\nu_m}{2}(\nabla(2e^n_{j,w}-e^{n-1}_{j,w}),\nabla\chi_h)\\
&\quad+((2e^n_{j,w}-e^{n-1}_{j,w})\cdot\nabla v_j(t^{n+1}),\chi_h)+((2w^n_{j,h}-w^{n-1}_{j,h})\cdot\nabla e^{n+1}_{j,v},\chi_h)\\
&\quad-(w'^n_{j,h}\cdot\nabla(e^{n+1}_{j,v}-2e^n_{j,v}+e^{n-1}_{j,v}),\chi_h)=-G_1(t,v_j,w_j,\chi_h),
\end{aligned}\tag{3.12}
$$

and

$$
\begin{aligned}
&\left(\frac{3e^{n+1}_{j,w}-4e^n_{j,w}+e^{n-1}_{j,w}}{2\Delta t}, l_h\right)+\frac{\nu+\nu_m}{2}(\nabla e^{n+1}_{j,w},\nabla l_h)+\frac{\nu-\nu_m}{2}(\nabla(2e^n_{j,v}-e^{n-1}_{j,v}),\nabla l_h)\\
&\quad+((2e^n_{j,v}-e^{n-1}_{j,v})\cdot\nabla w_j(t^{n+1}),l_h)+((2v^n_{j,h}-v^{n-1}_{j,h})\cdot\nabla e^{n+1}_{j,w},l_h)\\
&\quad-(v'^n_{j,h}\cdot\nabla(e^{n+1}_{j,w}-2e^n_{j,w}+e^{n-1}_{j,w}),l_h)=-G_2(t,v_j,w_j,l_h),
\end{aligned}\tag{3.13}
$$

where

$$
\begin{aligned}
G_1(t,v_j,w_j,\chi_h) := {}&((w_j(t^{n+1})-2w_j(t^n)+w_j(t^{n-1}))\cdot\nabla v_j(t^{n+1}),\chi_h)\\
&+(w'^n_{j,h}\cdot\nabla(v_j(t^{n+1})-2v_j(t^n)+v_j(t^{n-1})),\chi_h)\\
&+\frac{\nu-\nu_m}{2}(\nabla(w_j(t^{n+1})-2w_j(t^n)+w_j(t^{n-1})),\nabla\chi_h)\\
&+\left(v_{j,t}(t^{n+1})-\frac{3v_j(t^{n+1})-4v_j(t^n)+v_j(t^{n-1})}{2\Delta t},\chi_h\right)
\end{aligned}
$$

and

$$
\begin{aligned}
G_2(t,v_j,w_j,l_h) := {}&((v_j(t^{n+1})-2v_j(t^n)+v_j(t^{n-1}))\cdot\nabla w_j(t^{n+1}),l_h)\\
&+(v'^n_{j,h}\cdot\nabla(w_j(t^{n+1})-2w_j(t^n)+w_j(t^{n-1})),l_h)\\
&+\frac{\nu-\nu_m}{2}(\nabla(v_j(t^{n+1})-2v_j(t^n)+v_j(t^{n-1})),\nabla l_h)\\
&+\left(w_{j,t}(t^{n+1})-\frac{3w_j(t^{n+1})-4w_j(t^n)+w_j(t^{n-1})}{2\Delta t},l_h\right).
\end{aligned}
$$

Now we decompose the errors as

$$
e^n_{j,v} := v_j(t^n)-v^n_{j,h} = (v_j(t^n)-\tilde v^n_j)-(v^n_{j,h}-\tilde v^n_j) := \eta^n_{j,v}-\phi^n_{j,h},
$$

$$e^n_{j,w} := w_j(t^n) - w^n_{j,h} = (w_j(t^n) - \tilde{w}^n_j) - (w^n_{j,h} - \tilde{w}^n_j) := \eta^n_{j,w} - \psi^n_{j,h},$$

where $\tilde{v}^n_j := P^{L^2}_{V_h}(v_j(t^n)) \in V_h$ and $\tilde{w}^n_j := P^{L^2}_{V_h}(w_j(t^n)) \in V_h$ are the L^2 projections of $v_j(t^n)$ and $w_j(t^n)$ into V_h, respectively. Note that $(\eta^n_{j,v}, v_h) = (\eta^n_{j,w}, v_h) = 0\ \ \forall v_h \in V_h$. Rewriting, we have for $\chi_h, l_h \in V_h$

$$\begin{aligned}
\Big(\frac{3\phi^{n+1}_{j,h} - 4\phi^n_{j,h} + \phi^{n-1}_{j,h}}{2\Delta t}, \chi_h\Big) + \frac{\nu+\nu_m}{2}(\nabla\phi^{n+1}_{j,h}, \nabla\chi_h) + \frac{\nu-\nu_m}{2}(\nabla(2\psi^n_{j,h} - \psi^{n-1}_{j,h}), \nabla\chi_h)&\\
+((2\psi^n_{j,h} - \psi^{n-1}_{j,h})\cdot\nabla v_j(t^{n+1}), \chi_h) + ((2w^n_{j,h} - w^{n-1}_{j,h})\cdot\nabla\phi^{n+1}_{j,h}, \chi_h)&\\
-(w'^n_{j,h}\cdot\nabla(\phi^{n+1}_{j,h} - \phi^n_{j,h} + \phi^{n-1}_{j,h}), \chi_h) = \frac{\nu-\nu_m}{2}(\nabla(2\eta^n_{j,w} - \eta^{n-1}_{j,w}), \nabla\chi_h)&\\
+\frac{\nu+\nu_m}{2}(\nabla\eta^{n+1}_{j,v}, \nabla\chi_h) + ((2\eta^n_{j,w} - \eta^{n-1}_{j,w})\cdot\nabla v_j(t^{n+1}), \chi_h)&\\
+((2w^n_{j,h} - w^{n-1}_{j,h})\cdot\nabla\eta^{n+1}_{j,v}, \chi_h) - (w'^n_{j,h}\cdot\nabla(\eta^{n+1}_{j,v} - \eta^n_{j,v} + \eta^{n-1}_{j,v}), \chi_h) + G_1(t, v_j, w_j, \chi_h),&
\end{aligned} \tag{3.14}$$

and

$$\begin{aligned}
\Big(\frac{3\psi^{n+1}_{j,h} - 4\psi^n_{j,h} + \psi^{n-1}_{j,h}}{2\Delta t}, l_h\Big) + \frac{\nu+\nu_m}{2}(\nabla\psi^{n+1}_{j,h}, \nabla l_h) + \frac{\nu-\nu_m}{2}(\nabla(2\phi^n_{j,h} - \phi^{n-1}_{j,h}), \nabla l_h)&\\
+((2\phi^n_{j,h} - \phi^{n-1}_{j,h})\cdot\nabla w_j(t^{n+1}), l_h) + ((2v^n_{j,h} - v^{n-1}_{j,h})\cdot\nabla\psi^{n+1}_{j,h}, l_h)&\\
-(v'^n_{j,h}\cdot\nabla(\psi^{n+1}_{j,h} - \psi^n_{j,h} + \psi^{n-1}_{j,h}), l_h) = \frac{\nu-\nu_m}{2}(\nabla(2\eta^n_{j,v} - \eta^{n-1}_{j,v}), \nabla l_h)&\\
+\frac{\nu+\nu_m}{2}(\nabla\eta^{n+1}_{j,w}, \nabla l_h) + ((2\eta^n_{j,v} - \eta^{n-1}_{j,v})\cdot\nabla w_j(t^{n+1}), l_h)&\\
+((2v^n_{j,h} - v^{n-1}_{j,h})\cdot\nabla\eta^{n+1}_{j,w}, l_h) - (v'^n_{j,h}\cdot\nabla(\eta^{n+1}_{j,w} - \eta^n_{j,w} + \eta^{n-1}_{j,w}), l_h) + G_2(t, v_j, w_j, l_h),&
\end{aligned} \tag{3.15}$$

Choose $\chi_h = \phi^{n+1}_{j,h}, l_h = \psi^{n+1}_{j,h}$ and use the identity (3.4) in (3.14) and (3.15), to obtain

$$\begin{aligned}
\frac{1}{4\Delta t}(\|\phi^{n+1}_{j,h}\|^2 - \|\phi^n_{j,h}\|^2 + \|2\phi^{n+1}_{j,h} - \phi^n_{j,h}\|^2 - \|2\phi^n_{j,h} - \phi^{n-1}_{j,h}\|^2&\\
+\|\phi^{n+1}_{j,h} - 2\phi^n_{j,h} + \phi^{n-1}_{j,h}\|^2) + \frac{\nu+\nu_m}{2}\|\nabla\phi^{n+1}_{j,h}\|^2 + \frac{\nu-\nu_m}{2}(\nabla(2\psi^n_{j,h} - \psi^{n-1}_{j,h}), \nabla\phi^{n+1}_{j,h})&\\
+((2\psi^n_{j,h} - \psi^{n-1}_{j,h})\cdot\nabla v_j(t^{n+1}), \phi^{n+1}_{j,h}) - (w'^n_{j,h}\cdot\nabla(\phi^{n+1}_{j,h} - \phi^n_{j,h} + \phi^{n-1}_{j,h}), \phi^{n+1}_{j,h})&\\
= \frac{\nu+\nu_m}{2}(\nabla\eta^{n+1}_{j,v}, \nabla\phi^{n+1}_{j,h}) + \frac{\nu-\nu_m}{2}(\nabla(2\eta^n_{j,w} - \eta^{n-1}_{j,w}), \nabla\phi^{n+1}_{j,h})&\\
+((2\eta^n_{j,w} - \eta^{n-1}_{j,w})\cdot\nabla v_j(t^{n+1}), \phi^{n+1}_{j,h}) + ((2w^n_{j,h} - w^{n-1}_{j,h})\cdot\nabla\eta^{n+1}_{j,v}, \phi^{n+1}_{j,h})&\\
-(w'^n_{j,h}\cdot\nabla(\eta^{n+1}_{j,v} - \eta^n_{j,v} + \eta^{n-1}_{j,v}), \phi^{n+1}_{j,h}) + G_1(t, v_j, w_j, \phi^{n+1}_{j,h}),&
\end{aligned} \tag{3.16}$$

and

$$\frac{1}{4\Delta t}(\|\psi_{j,h}^{n+1}\|^2 - \|\psi_{j,h}^n\|^2 + \|2\psi_{j,h}^{n+1} - \psi_{j,h}^n\|^2 - \|2\psi_{j,h}^n - \psi_{j,h}^{n-1}\|^2$$
$$+\|\psi_{j,h}^{n+1} - 2\psi_{j,h}^n + \psi_{j,h}^{n-1}\|^2) + \frac{\nu+\nu_m}{2}\|\nabla\psi_{j,h}^{n+1}\|^2 + \frac{\nu-\nu_m}{2}\big(\nabla(2\phi_{j,h}^n - \phi_{j,h}^{n-1}), \nabla\psi_{j,h}^{n+1}\big)$$
$$+((2\phi_{j,h}^n - \phi_{j,h}^{n-1})\cdot\nabla w_j(t^{n+1}), \psi_{j,h}^{n+1}) - (v_{j,h}^{\prime n}\cdot\nabla(\psi_{j,h}^{n+1} - \psi_{j,h}^n + \psi_{j,h}^{n-1}), \psi_{j,h}^{n+1})$$
$$= \frac{\nu+\nu_m}{2}(\nabla\eta_{j,w}^{n+1}, \nabla\psi_{j,h}^{n+1}) + \frac{\nu-\nu_m}{2}\big(\nabla(2\eta_{j,v}^n - \eta_{j,v}^{n-1}), \nabla\psi_{j,h}^{n+1}\big)$$
$$+((2\eta_{j,v}^n - \eta_{j,v}^{n-1})\cdot\nabla w_j(t^{n+1}), \psi_{j,h}^{n+1}) + ((2v_{j,h}^n - v_{j,h}^{n-1})\cdot\nabla\eta_{j,w}^{n+1}, \psi_{j,h}^{n+1})$$
$$-(v_{j,h}^{\prime n}\cdot\nabla(\eta_{j,w}^{n+1} - \eta_{j,w}^n + \eta_{j,w}^{n-1}), \psi_{j,h}^{n+1}) + G_2(t, v_j, w_j, \psi_{j,h}^{n+1}), \tag{3.17}$$

We add equations (3.16) and (3.17), add and subtract the term $(\nu-\nu_m)(\nabla\phi_{j,h}^{n+1}, \nabla\psi_{j,h}^{n+1})$ and applying Cauchy-Schwarz results in,

$$\frac{1}{4\Delta t}(\|\phi_{j,h}^{n+1}\|^2 - \|\phi_{j,h}^n\|^2 + \|2\phi_{j,h}^{n+1} - \phi_{j,h}^n\|^2 - \|2\phi_{j,h}^n - \phi_{j,h}^{n-1}\|^2$$
$$+\|\psi_{j,h}^{n+1}\|^2 - \|\psi_{j,h}^n\|^2 + \|2\psi_{j,h}^{n+1} - \psi_{j,h}^n\|^2 - \|2\psi_{j,h}^n - \psi_{j,h}^{n-1}\|^2$$
$$+\|\phi_{j,h}^{n+1} - 2\phi_{j,h}^n + \phi_{j,h}^{n-1}\|^2 + \|\psi_{j,h}^{n+1} - 2\psi_{j,h}^n + \psi_{j,h}^{n-1}\|^2)$$
$$+\frac{\nu+\nu_m}{2}(\|\nabla\phi_{j,h}^{n+1}\|^2 + \|\nabla\psi_{j,h}^{n+1}\|^2) \le |\nu-\nu_m|\|\nabla\phi_{j,h}^{n+1}\|\|\nabla\psi_{j,h}^{n+1}\|$$
$$+\frac{|\nu-\nu_m|}{2}\|\nabla(\phi_{j,h}^{n+1} - 2\phi_{j,h}^n + \phi_{j,h}^{n-1})\|\|\nabla\psi_{j,h}^{n+1}\|$$
$$+\frac{|\nu-\nu_m|}{2}\|\nabla(\psi_{j,h}^{n+1} - 2\psi_{j,h}^n + \psi_{j,h}^{n-1})\|\|\nabla\phi_{j,h}^{n+1}\|$$
$$+\frac{\nu+\nu_m}{2}\big(\|\nabla\eta_{j,v}^{n+1}\|\|\nabla\phi_{j,h}^{n+1}\| + \|\nabla\eta_{j,w}^{n+1}\|\|\nabla\psi_{j,h}^{n+1}\|\big)$$
$$+\frac{|\nu-\nu_m|}{2}\|\nabla(2\eta_{j,w}^n - \eta_{j,w}^{n-1})\|\|\nabla\phi_{j,h}^{n+1}\| + \frac{|\nu-\nu_m|}{2}\|\nabla(2\eta_{j,v}^n - \eta_{j,v}^{n-1})\|\|\nabla\psi_{j,h}^{n+1}\|$$
$$+|(w_{j,h}^{\prime n}\cdot\nabla(\phi_{j,h}^{n+1} - \phi_{j,h}^n + \phi_{j,h}^{n-1}), \phi_{j,h}^{n+1})| + |(v_{j,h}^{\prime n}\cdot\nabla(\psi_{j,h}^{n+1} - \psi_{j,h}^n + \psi_{j,h}^{n-1}), \psi_{j,h}^{n+1})|$$
$$+|((2\eta_{j,w}^n - \eta_{j,w}^{n-1})\cdot\nabla v_j(t^{n+1}), \phi_{j,h}^{n+1})| + |((2\eta_{j,v}^n - \eta_{j,v}^{n-1})\cdot\nabla w_j(t^{n+1}), \psi_{j,h}^{n+1})|$$
$$+|((2w_{j,h}^n - w_{j,h}^{n-1})\cdot\nabla\eta_{j,v}^{n+1}, \phi_{j,h}^{n+1})| + |((2v_{j,h}^n - v_{j,h}^{n-1})\cdot\nabla\eta_{j,w}^{n+1}, \psi_{j,h}^{n+1})|$$
$$+|(w_{j,h}^{\prime n}\cdot\nabla(\eta_{j,v}^{n+1} - \eta_{j,v}^n + \eta_{j,v}^{n-1}), \phi_{j,h}^{n+1})| + |(v_{j,h}^{\prime n}\cdot\nabla(\eta_{j,w}^{n+1} - \eta_{j,w}^n + \eta_{j,w}^{n-1}), \psi_{j,h}^{n+1})|$$
$$+|((2\phi_{j,h}^n - \phi_{j,h}^{n-1})\cdot\nabla w_j(t^{n+1}), \psi_{j,h}^{n+1})| + |((2\psi_{j,h}^n - \psi_{j,h}^{n-1})\cdot\nabla v_j(t^{n+1}), \phi_{j,h}^{n+1})|$$
$$+|G_1(t, v_j, w_j, \phi_{j,h}^{n+1})| + |G_2(t, v_j, w_j, \psi_{j,h}^{n+1})|. \tag{3.18}$$

We now turn our attention to finding the bounds for the right-hand side terms in (3.18). Applying Cauchy-Schwarz and Young's inequalities on the first seven terms on left results in

$$|\nu-\nu_m|\|\nabla\phi_{j,h}^{n+1}\|\|\nabla\psi_{j,h}^{n+1}\| \le \frac{|\nu-\nu_m|}{2}\|\nabla\phi_{j,h}^{n+1}\|^2 + \frac{|\nu-\nu_m|}{2}\|\nabla\psi_{j,h}^{n+1}\|^2,$$
$$\frac{\nu+\nu_m}{2}\|\nabla\eta_{j,v}^{n+1}\|\|\nabla\phi_{j,h}^{n+1}\| \le \frac{\alpha}{36}\|\nabla\phi_{j,h}^{n+1}\|^2 + \frac{9(\nu+\nu_m)^2}{4\alpha}\|\nabla\eta_{j,v}^{n+1}\|^2,$$
$$\frac{\nu+\nu_m}{2}\|\nabla\eta_{j,w}^{n+1}\|\|\nabla\psi_{j,h}^{n+1}\| \le \frac{\alpha}{36}\|\nabla\psi_{j,h}^{n+1}\|^2 + \frac{9(\nu+\nu_m)^2}{4\alpha}\|\nabla\eta_{j,w}^{n+1}\|^2,$$

$$\frac{|\nu-\nu_m|}{2}\|\nabla(\phi_{j,h}^{n+1}-2\phi_{j,h}^{n}+\phi_{j,h}^{n-1})\|\|\nabla\psi_{j,h}^{n+1}\|$$
$$\leq \frac{\alpha}{36}\|\nabla\psi_{j,h}^{n+1}\|^2+\frac{9(\nu-\nu_m)^2}{4\alpha}\|\nabla(\phi_{j,h}^{n+1}-2\phi_{j,h}^{n}+\phi_{j,h}^{n-1})\|^2,$$

$$\frac{|\nu-\nu_m|}{2}\|\nabla(\psi_{j,h}^{n+1}-2\psi_{j,h}^{n}+\psi_{j,h}^{n-1})\|\|\nabla\phi_{j,h}^{n+1}\|$$
$$\leq \frac{\alpha}{36}\|\nabla\phi_{j,h}^{n+1}\|^2+\frac{9(\nu-\nu_m)^2}{4\alpha}\|\nabla(\psi_{j,h}^{n+1}-2\psi_{j,h}^{n}+\psi_{j,h}^{n-1})\|^2,$$

$$\frac{|\nu-\nu_m|}{2}\|\nabla(2\eta_{j,w}^{n}-\eta_{j,w}^{n-1})\|\|\nabla\phi_{j,h}^{n+1}\|\leq\frac{\alpha}{18}\|\nabla\phi_{j,h}^{n+1}\|^2+\frac{9(\nu-\nu_m)^2}{\alpha}\|\nabla\eta_{j,w}^{n}\|^2+\frac{9(\nu-\nu_m)^2}{4\alpha}\|\nabla\eta_{j,w}^{n-1}\|^2,$$
$$\frac{|\nu-\nu_m|}{2}\|\nabla(2\eta_{j,v}^{n}-\eta_{j,v}^{n-1})\|\|\nabla\psi_{j,h}^{n+1}\|\leq\frac{\alpha}{18}\|\nabla\psi_{j,h}^{n+1}\|^2+\frac{9(\nu-\nu_m)^2}{\alpha}\|\nabla\eta_{j,v}^{n}\|^2+\frac{9(\nu-\nu_m)^2}{4\alpha}\|\nabla\eta_{j,v}^{n-1}\|^2.$$

Apply Hölder and Young's inequalities with (2.1) on the following eight nonlinear terms yields

$$|(w_{j,h}^{\prime n}\cdot\nabla(\phi_{j,h}^{n+1}-2\phi_{j,h}^{n}+\phi_{j,h}^{n-1}),\phi_{j,h}^{n+1})|\leq\frac{\alpha}{36}\|\nabla\phi_{j,h}^{n+1}\|^2+\frac{9C^2}{4\alpha}\|\nabla w_{j,h}^{\prime n}\|^2\|\nabla(\phi_{j,h}^{n+1}-2\phi_{j,h}^{n}+\phi_{j,h}^{n-1})\|^2,$$
$$|(v_{j,h}^{\prime n}\cdot\nabla(\psi_{j,h}^{n+1}-2\psi_{j,h}^{n}+\psi_{j,h}^{n-1}),\psi_{j,h}^{n+1})|\leq\frac{\alpha}{36}\|\nabla\psi_{j,h}^{n+1}\|^2+\frac{9C^2}{4\alpha}\|\nabla v_{j,h}^{\prime n}\|^2\|\nabla(\psi_{j,h}^{n+1}-2\psi_{j,h}^{n}+\psi_{j,h}^{n-1})\|^2,$$
$$|((2\eta_{j,w}^{n}-\eta_{j,w}^{n-1})\cdot\nabla v_j(t^{n+1}),\phi_{j,h}^{n+1})|\leq\frac{\alpha}{36}\|\nabla\phi_{j,h}^{n+1}\|^2+\frac{9C^2}{4\alpha}\|\nabla(2\eta_{j,w}^{n}-\eta_{j,w}^{n-1})\|^2\|\nabla v_j(t^{n+1})\|^2,$$
$$|((2\eta_{j,v}^{n}-\eta_{j,v}^{n-1})\cdot\nabla w_j(t^{n+1}),\psi_{j,h}^{n+1})|\leq\frac{\alpha}{36}\|\nabla\psi_{j,h}^{n+1}\|^2+\frac{9C^2}{4\alpha}\|\nabla(2\eta_{j,v}^{n}-\eta_{j,v}^{n-1})\|^2\|\nabla w_j(t^{n+1})\|^2,$$
$$|((2w_{j,h}^{n}-w_{j,h}^{n-1})\cdot\nabla\eta_{j,v}^{n+1},\phi_{j,h}^{n+1})|\leq\frac{\alpha}{36}\|\nabla\phi_{j,h}^{n+1}\|^2+\frac{9C^2}{4\alpha}\|\nabla(2w_{j,h}^{n}-w_{j,h}^{n-1})\|^2\|\nabla\eta_{j,v}^{n+1}\|^2,$$
$$|((2v_{j,h}^{n}-v_{j,h}^{n-1})\cdot\nabla\eta_{j,w}^{n+1},\psi_{j,h}^{n+1})|\leq\frac{\alpha}{36}\|\nabla\psi_{j,h}^{n+1}\|^2+\frac{9C^2}{4\alpha}\|\nabla(2v_{j,h}^{n}-v_{j,h}^{n-1})\|^2\|\nabla\eta_{j,w}^{n+1}\|^2,$$
$$|(w_{j,h}^{\prime n}\cdot\nabla(\eta_{j,v}^{n+1}-\eta_{j,v}^{n}+\eta_{j,v}^{n-1}),\phi_{j,h}^{n+1})|\leq\frac{\alpha}{36}\|\nabla\phi_{j,h}^{n+1}\|^2+\frac{9C^2}{4\alpha}\|\nabla w_{j,h}^{\prime n}\|^2\|\nabla(\eta_{j,v}^{n+1}-\eta_{j,v}^{n}+\eta_{j,v}^{n-1})\|^2,$$
$$|(v_{j,h}^{\prime n}\cdot\nabla(\eta_{j,w}^{n+1}-\eta_{j,w}^{n}+\eta_{j,w}^{n-1}),\psi_{j,h}^{n+1})|\leq\frac{\alpha}{36}\|\nabla\psi_{j,h}^{n+1}\|^2+\frac{9C^2}{4\alpha}\|\nabla v_{j,h}^{\prime n}\|^2\|\nabla(\eta_{j,w}^{n+1}-\eta_{j,w}^{n}+\eta_{j,w}^{n-1})\|^2.$$

Apply Hölder's inequality, Sobolev embedding theorems, Poincare's and Young's inequalities with (2.1) on the following two nonlinear terms to reveal

$$|((2\psi_{j,h}^{n}-\psi_{j,h}^{n-1})\cdot\nabla v_j(t^{n+1}),\phi_{j,h}^{n+1})|\leq\frac{\alpha}{36}\|\nabla\phi_{j,h}^{n+1}\|^2+\frac{9C^2}{4\alpha}\|v_j(t^{n+1})\|_{H^2}^2\|2\psi_{j,h}^{n}-\psi_{j,h}^{n-1}\|^2,$$
$$|((2\phi_{j,h}^{n}-\phi_{j,h}^{n-1})\cdot\nabla w_j(t^{n+1}),\psi_{j,h}^{n+1})|\leq\frac{\alpha}{36}\|\nabla\psi_{j,h}^{n+1}\|^2+\frac{9C^2}{4\alpha}\|w_j(t^{n+1})\|_{H^2}^2\|2\phi_{j,h}^{n}-\phi_{j,h}^{n-1}\|^2.$$

Using Taylor's series, Cauchy-Schwarz, Poincare's and Young's inequalities the last two terms are evaluated as

$$|G_1(t,v_j,w_j,\phi_{j,h}^{n+1})|\leq\frac{\alpha}{36}\|\nabla\phi_{j,h}^{n+1}\|^2+\frac{36C^2}{\alpha}\Delta t^4\|\nabla w_{j,tt}(s^*)\|^2\|\nabla v_j(t^{n+1})\|^2$$
$$+\frac{36C^2}{\alpha}\Delta t^4\|\nabla w_{j,h}^{\prime n}\|^2\|\nabla v_{j,tt}(s^{**})\|^2+\frac{9(\nu-\nu_m)^2}{\alpha}\Delta t^4\|\nabla w_{j,tt}(s^*)\|^2+\frac{4C^2}{\alpha}\Delta t^4\|v_{j,ttt}(s^{***})\|^2,$$

$$|G_2(t,v_j,w_j,\psi_{j,h}^{n+1})| \le \frac{\alpha}{36}\|\nabla\psi_{j,h}^{n+1}\|^2 + \frac{36C^2}{\alpha}\Delta t^4\|\nabla v_{j,tt}(t^*)\|^2\|\nabla w_j(t^{n+1})\|^2$$
$$+\frac{36C^2}{\alpha}\Delta t^4\|\nabla v_{j,h}^{\prime n}\|^2\|\nabla w_{j,tt}(t^{**})\|^2 + \frac{9(\nu-\nu_m)^2}{\alpha}\Delta t^4\|\nabla v_{j,tt}(t^*)\|^2 + \frac{4C^2}{\alpha}\Delta t^4\|w_{j,ttt}(t^{***})\|^2$$

with $s^*, s^{**}, s^{***}, t^*, t^{**}, t^{***} \in [t^{n-1}, t^{n+1}]$. Using these estimates in (3.18) and reducing produces

$$\begin{aligned}
&\frac{1}{4\Delta t}(\|\phi_{j,h}^{n+1}\|^2 - \|\phi_{j,h}^n\|^2 + \|2\phi_{j,h}^{n+1}-\phi_{j,h}^n\|^2 - \|2\phi_{j,h}^n - \phi_{j,h}^{n-1}\|^2\\
&\quad+\|\psi_{j,h}^{n+1}\|^2 - \|\psi_{j,h}^n\|^2 + \|2\psi_{j,h}^{n+1}-\psi_{j,h}^n\|^2 - \|2\psi_{j,h}^n - \psi_{j,h}^{n-1}\|^2)\\
&+\Big[\frac{1}{4\Delta t} - \frac{9C_i(\nu-\nu_m)^2 + 9C_iC^2\|\nabla w_{j,h}^{\prime n}\|^2}{4\alpha h^2}\Big]\|\phi_{j,h}^{n+1} - 2\phi_{j,h}^n + \phi_{j,h}^{n-1}\|^2\\
&+\Big[\frac{1}{4\Delta t} - \frac{9C_i(\nu-\nu_m)^2 + 9C_iC^2\|\nabla v_{j,h}^{\prime n}\|^2}{4\alpha h^2}\Big]\|\psi_{j,h}^{n+1} - 2\psi_{j,h}^n + \psi_{j,h}^{n-1}\|^2\\
&+\frac{\alpha}{4}(\|\nabla\phi_{j,h}^{n+1}\|^2 + \|\nabla\psi_{j,h}^{n+1}\|^2) \le \frac{9(\nu+\nu_m)^2}{4\alpha}(\|\nabla\eta_{j,v}^{n+1}\|^2 + \|\nabla\eta_{j,w}^{n+1}\|^2)\\
&+\frac{9(\nu-\nu_m)^2}{\alpha}(\|\nabla\eta_{j,v}^{n}\|^2 + \|\nabla\eta_{j,w}^{n}\|^2) + \frac{9(\nu-\nu_m)^2}{4\alpha}(\|\nabla\eta_{j,v}^{n-1}\|^2 + \|\nabla\eta_{j,w}^{n-1}\|^2)\\
&+\frac{9C^2}{4\alpha}\|\nabla(2\eta_{j,w}^n - \eta_{j,w}^{n-1})\|^2\|\nabla v_j(t^{n+1})\|^2 + \frac{9C^2}{4\alpha}\|\nabla(2\eta_{j,v}^n - \eta_{j,v}^{n-1})\|^2\|\nabla w_j(t^{n+1})\|^2\\
&+\frac{9C^2}{4\alpha}\|\nabla(2w_{j,h}^n - w_{j,h}^{n-1})\|^2\|\nabla\eta_{j,v}^{n+1}\|^2 + \frac{9C^2}{4\alpha}\|\nabla(2v_{j,h}^n - v_{j,h}^{n-1})\|^2\|\nabla\eta_{j,w}^{n+1}\|^2\\
&+\frac{9C^2}{4\alpha}\|\nabla w_{j,h}^{\prime n}\|^2\|\nabla(\eta_{j,v}^{n+1} - \eta_{j,v}^n + \eta_{j,v}^{n-1})\|^2 + \frac{9C^2}{4\alpha}\|\nabla v_{j,h}^{\prime n}\|^2\|\nabla(\eta_{j,w}^{n+1} - \eta_{j,w}^n + \eta_{j,w}^{n-1})\|^2\\
&+\frac{9C^2}{4\alpha}\|v_j(t^{n+1})\|_{H^2}^2\|2\psi_{j,h}^n - \psi_{j,h}^{n-1}\|^2 + \frac{9C^2}{4\alpha}\|w_j(t^{n+1})\|_{H^2}^2\|2\phi_{j,h}^n - \phi_{j,h}^{n-1}\|^2\\
&+\frac{36C^2}{\alpha}\Delta t^4\|\nabla w_{j,tt}(s^*)\|^2\|\nabla v_j(t^{n+1})\|^2 + \frac{36C^2}{\alpha}\Delta t^4\|\nabla w_{j,h}^{\prime n}\|^2\|\nabla v_{j,tt}(s^{**})\|^2\\
&+\frac{9(\nu-\nu_m)^2}{\alpha}\Delta t^4\|\nabla w_{j,tt}(s^*)\|^2 + \frac{4C^2}{\alpha}\Delta t^4\|v_{j,ttt}(s^{***})\|^2\\
&+\frac{36C^2}{\alpha}\Delta t^4\|\nabla v_{j,tt}(t^*)\|^2\|\nabla w_j(t^{n+1})\|^2 + \frac{36C^2}{\alpha}\Delta t^4\|\nabla v_{j,h}^{\prime n}\|^2\|\nabla w_{j,tt}(t^{**})\|^2\\
&+\frac{9(\nu-\nu_m)^2}{\alpha}\Delta t^4\|\nabla v_{j,tt}(t^*)\|^2 + \frac{4C^2}{\alpha}\Delta t^4\|w_{j,ttt}(t^{***})\|^2
\end{aligned} \tag{3.19}$$

To make the third and fourth terms non-negative, we choose $\Delta t \le \frac{\alpha h^2}{9C_i(\nu-\nu_m)^2 + 9C_iC^2\max\left\{\|\nabla v_{j,h}^{\prime n}\|, \|\nabla w_{j,h}^{\prime n}\|\right\}}$. Drop the non-negative terms on the left-hand side, multiply both sides by $4\Delta t$, use the regularity assumption, $\|\phi^0_{j,h}\| = \|\psi^0_{j,h}\| = \|\phi^1_{j,h}\| = \|\psi^1_{j,h}\| = 0$, $\Delta t M = T$, and sum over the time steps to find

$$
\begin{aligned}
&\|\phi^M_{j,h}\|^2 + \|2\phi^M_{j,h} - \phi^{M-1}_{j,h}\|^2 + \|\psi^M_{j,h}\|^2 + \|2\psi^M_{j,h} - \psi^{M-1}_{j,h}\|^2 \\
&\quad + \alpha\Delta t \sum_{n=2}^{M} (\|\nabla\phi^n_{j,h}\|^2 + \|\nabla\psi^n_{j,h}\|^2) \leq C\frac{9(\nu+\nu_m)^2}{\alpha}\Delta t \sum_{n=2}^{M}(\|\nabla\eta^n_{j,v}\|^2 + \|\nabla\eta^n_{j,w}\|^2) \\
&\quad + \frac{36(\nu-\nu_m)^2}{\alpha}\Delta t \sum_{n=1}^{M-1}(\|\nabla\eta^n_{j,v}\|^2 + \|\nabla\eta^n_{j,w}\|^2) + \frac{9(\nu-\nu_m)^2}{\alpha}\Delta t \sum_{n=1}^{M-1}(\|\nabla\eta^{n-1}_{j,v}\|^2 + \|\nabla\eta^{n-1}_{j,w}\|^2) \\
&\quad + \frac{9C^2}{\alpha}\Delta t \sum_{n=1}^{M-1} \|\nabla(2\eta^n_{j,w} - \eta^{n-1}_{j,w})\|^2\|\nabla v_j(t^{n+1})\|^2 + \frac{9C^2}{\alpha}\Delta t \sum_{n=1}^{M-1} \|\nabla(2\eta^n_{j,v} - \eta^{n-1}_{j,v})\|^2\|\nabla w_j(t^{n+1})\|^2 \\
&\quad + \frac{9C^2}{\alpha}\Delta t \sum_{n=1}^{M-1} \|\nabla(2w^n_{j,h} - w^{n-1}_{j,h})\|^2\|\nabla\eta^{n+1}_{j,v}\|^2 + \frac{9C^2}{\alpha}\Delta t \sum_{n=1}^{M-1} \|\nabla(2v^n_{j,h} - v^{n-1}_{j,h})\|^2\|\nabla\eta^{n+1}_{j,w}\|^2 \\
&\quad + \frac{9C^2}{\alpha}\Delta t \sum_{n=1}^{M-1} \|\nabla w'^n_{j,h}\|^2\|\nabla(\eta^{n+1}_{j,v} - \eta^n_{j,v} + \eta^{n-1}_{j,v})\|^2 \\
&\quad + \frac{9C^2}{\alpha}\Delta t \sum_{n=1}^{M-1} \|\nabla v'^n_{j,h}\|^2\|\nabla(\eta^{n+1}_{j,w} - \eta^n_{j,w} + \eta^{n-1}_{j,w})\|^2 \\
&\quad + \frac{9C^2}{\alpha}\Delta t \sum_{n=1}^{M-1} \|v_j(t^{n+1})\|^2_{H^2}\|2\psi^n_{j,h} - \psi^{n-1}_{j,h}\|^2 + \frac{9C^2}{\alpha}\Delta t \sum_{n=1}^{M-1} \|w_j(t^{n+1})\|^2_{H^2}\|2\phi^n_{j,h} - \phi^{n-1}_{j,h}\|^2 \\
&\quad + \frac{144C^2}{\alpha}\Delta t^4\Delta t \sum_{n=1}^{M-1}(\|\nabla w'^n_{j,h}\|^2\|\nabla v_{j,tt}(t^{**})\|^2 + \|\nabla v'^n_{j,h}\|^2\|\nabla w_{j,tt}(t^{**})\|^2) \\
&\quad + \frac{144C^2}{\alpha}\Delta t^4\Delta t \sum_{n=1}^{M-1}(\|\nabla w_{j,tt}(s^*)\|^2\|\nabla v_j(t^{n+1})\|^2 + \|\nabla v_{j,tt}(t^*)\|^2\|\nabla w_j(t^{n+1})\|^2) \\
&\quad + \frac{9(\nu-\nu_m)^2}{\alpha}\Delta t^4\Delta t \sum_{n=1}^{M-1}(\|\nabla w_{j,tt}(s^*)\|^2 + \|\nabla v_{j,tt}(t^*)\|^2) \\
&\quad + \frac{16C^2}{\alpha}\Delta t^4\Delta t \sum_{n=1}^{M-1}(\|w_{j,ttt}(t^{***})\|^2 + \|v_{j,ttt}(s^{***})\|^2)
\end{aligned} \tag{3.20}
$$

Applying the regularity assumptions, stability bound, interpolation estimates for v_j, w_j

$$
\begin{aligned}
&\|\phi^M_{j,h}\|^2 + \|2\phi^M_{j,h} - \phi^{M-1}_{j,h}\|^2 + \|\psi^M_{j,h}\|^2 + \|2\psi^M_{j,h} - \psi^{M-1}_{j,h}\|^2 + \alpha\Delta t \sum_{n=2}^{M}(\|\nabla\phi^n_{j,h}\|^2 + \|\nabla\psi^n_{j,h}\|^2) \\
&\leq C\frac{9(\nu+\nu_m)^2}{\alpha}h^{2k} + C\frac{45(\nu-\nu_m)^2}{\alpha}h^{2k} + \frac{9C}{\alpha}\Delta t \sum_{n=1}^{M-1}(\|2\phi^n_{j,h} - \phi^{n-1}_{j,h}\|^2 + \|2\psi^n_{j,h} - \psi^{n-1}_{j,h}\|^2) \\
&+ \frac{9C}{\alpha}h^{2k} + \frac{304C}{\alpha}\Delta t^4 + \frac{9C(\nu-\nu_m)^2}{\alpha}\Delta t^4
\end{aligned} \tag{3.21}
$$

Applying the discrete Gronwall lemma, we have

$$\begin{aligned}\|\phi_{j,h}^M\|^2+&\|2\phi_{j,h}^M-\phi_{j,h}^{M-1}\|^2+\|\psi_{j,h}^M\|^2+\|2\psi_{j,h}^M-\psi_{j,h}^{M-1}\|^2+\alpha\Delta t\sum_{n=2}^{M}(\|\nabla\phi_{j,h}^n\|^2+\|\nabla\psi_{j,h}^n\|^2)\\ &\leq \frac{C}{\alpha}e^{\frac{9TC}{\alpha}}((\nu^2+\nu_m^2+1)h^{2k}+((\nu-\nu_m)^2+1)\Delta t^4) \qquad (3.22)\end{aligned}$$

Using the triangular inequality allows us to write

$$\begin{aligned}&\|e_{j,v}^M\|^2+\|e_{j,w}^M\|^2+\alpha\Delta t\sum_{n=2}^{M}(\|\nabla e_{j,v}^n\|^2+\|\nabla e_{j,w}^n\|^2)\leq 2\big(\|\phi_{j,h}^M\|^2+\|\psi_{j,h}^M\|^2\\ &+\alpha\Delta t\sum_{n=2}^{M}(\|\nabla\phi_{j,h}^n\|^2+\|\nabla\psi_{j,h}^n\|^2)+\|\eta_{j,v}^M\|^2+\|\eta_{j,w}^M\|^2+\alpha\Delta t\sum_{n=2}^{M}(\|\nabla\eta_{j,v}^n\|^2+\|\nabla\eta_{j,w}^n\|^2)\big)\\ &\leq \frac{C}{\alpha}e^{\frac{9TC}{\alpha}}((\nu^2+\nu_m^2+1)h^{2k}+((\nu-\nu_m)^2+1)\Delta t^4)+Ch^{2k+2} \qquad (3.23)\end{aligned}$$

Now summing over j and using the triangular inequality completes the proof. □

4 Numerical Experiments:

To test the proposed Algorithm 3.1 and theory, in this section we present results of numerical experiments. For the solution of the MHD systems it is considered important to enforce the solenoidal constraint $\nabla\cdot B=0$ in discrete level to the machine precision [23]. This is because the condition is induced by the induction equation, is a precise physical law. Moreover it has been shown that for MHD flow simulations $\nabla\cdot B\neq 0$ can produce large errors in the solution [8]. The Taylor Hood (TH) element is not pointwise divergence free, it is only weakly divergence free. Thus the use of TH-element directly is not appropriate. It has been proven [11] that TH approximations on a regular mesh converge to a pointwise divergence free solution as the grad-div stabilization parameter $\gamma\to\infty$. In all experiments, we used $((Q_2)^2, Q_1, (Q_2)^2, Q_1)$ Taylor Hood finite elements on regular quadrilateral meshes with appropriate large γ so that we can enforce point-wise divergence free constraints in our simulations. We use open source finite element library DealII[2] for all our experiments.

4.1 Convergence Rate Verification and Energy Plots:

To verify the predicted convergence rates of our analysis in Section 3.2, and to draw the energy plots, we begin these experiments with a manufactured analytical solution,

$$v=\begin{pmatrix}\cos y+(1+t)\sin y\\ \sin x+(1+t)\cos x\end{pmatrix},\; w=\begin{pmatrix}\cos y-(1+t)\sin y\\ \sin x-(1+t)\cos x\end{pmatrix},\; p=(x-y)(1+t),\; \lambda=0,$$

on the domain $\Omega=(0,1)^2$. Next, to create four different true solutions, we perturb the above solution introducing a parameter ϵ and defining as follows: $v_j:=\begin{cases}(1+(-1)^{j-1}\epsilon)v & 1\leq j<3\\ (1+(-1)^{j-1}2\epsilon)v & 3\leq j\leq 4\end{cases}$, and $w_j:=\begin{cases}(1+(-1)^{j-1}\epsilon)w & 1\leq j<3\\ (1+(-1)^{j-1}2\epsilon)w & 3\leq j\leq 4\end{cases}$ where $j\in\mathbb{N}$. Using these perturbed solutions, we compute right-hand side forcing terms. We consider the initial conditions $v_j(0)$ and $w_j(0)$. On the boundary of the unit square, Dirichlet conditions are used. The Algorithm 3.1 computes the discrete ensemble average $< v_h^n >$ and $< w_h^n >$, and these will be used to compare to the true ensemble average $< v(t^n) >$ and $< w(t^n) >$ respectively. We notate the ensemble average error as $< e_u >:=< u_h >^n - < u(t^n) >$. For our choice of elements, the theory predicts the $L^2(0,T;H^1(\Omega)^d)$ error to be $O(h^2 + \Delta t^2)$ provided $\Delta t < O(h^2)$. The grad-div stabilization parameter $\gamma = 10^5$ is chosen. We consider three different choices $\epsilon = 10^{-3}, 10^{-2}$ and 10^{-1} for the perturbation parameter herein and end time $T = 0.001$ for this test. For these choice of ϵ, Tables 1-2 exhibit errors and convergence rates, and we observe second order asymptotic convergence of our scheme as $\epsilon \to 0$, which is the expected convergence rate from our analysis.

		$\epsilon = 0.001$		$\epsilon = 0.01$		$\epsilon = 0.1$	
h	Δt	$\\|< e_v >\\|_{2,1}$	rate	$\\|< e_v >\\|_{2,1}$	rate	$\\|< e_v >\\|_{2,1}$	rate
$\frac{1}{2}$	$\frac{T}{4}$	3.650e-4		3.64973e-4		3.64973e-4	
$\frac{1}{4}$	$\frac{T}{8}$	1.008e-4	1.86	1.00764e-4	1.86	1.00764e-4	1.86
$\frac{1}{8}$	$\frac{T}{16}$	2.621e-5	1.94	2.62134e-5	1.94	2.62134e-5	1.94
$\frac{1}{16}$	$\frac{T}{32}$	6.670e-6	1.97	6.67033e-6	1.97	6.67034e-6	1.97
$\frac{1}{32}$	$\frac{T}{64}$	1.683e-6	1.99	1.69718e-6	1.97	1.72669e-6	1.95

Table 1: Errors and convergence rates for v with $\nu = 0.01$, $\nu_m = 0.001$.

		$\epsilon = 0.001$		$\epsilon = 0.01$		$\epsilon = 0.1$	
h	Δt	$\\|< e_w >\\|_{2,1}$	rate	$\\|< e_w >\\|_{2,1}$	rate	$\\|< e_w >\\|_{2,1}$	rate
$\frac{1}{2}$	$\frac{T}{4}$	7.168e-4		7.168e-4		7.168e-4	
$\frac{1}{4}$	$\frac{T}{8}$	1.930e-4	1.89	1.930e-4	1.89	1.930e-4	1.89
$\frac{1}{8}$	$\frac{T}{16}$	4.992e-5	1.95	4.992e-5	1.95	4.992e-5	1.95
$\frac{1}{16}$	$\frac{T}{32}$	1.268e-5	1.98	1.268e-5	1.98	1.268e-5	1.98
$\frac{1}{32}$	$\frac{T}{64}$	3.196e-6	1.99	3.197e-6	1.99	3.197e-6	1.99

Table 2: Errors and convergence rates for w with $\nu = 0.01$, $\nu_m = 0.001$.

We plot the energy curves corresponding to the Elsässer variable v for the ensemble average solution and the perturbed solutions in the Figure 1. Energy plots for w exhibit the similar behavior and are omitted. Clearly, the ensemble average solution is mean of the perturbed solutions.

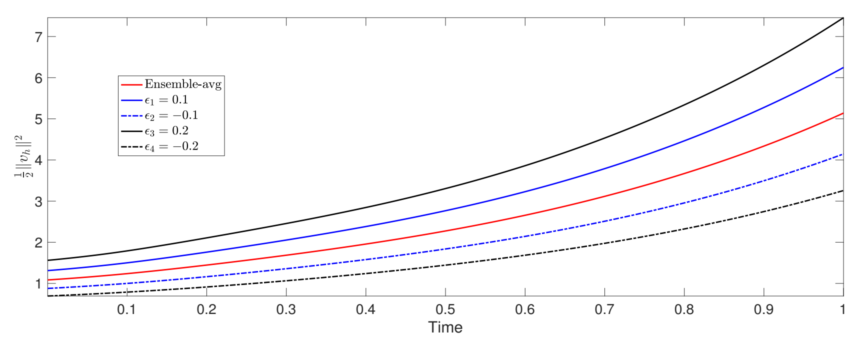

Figure 1: Energy plots of ensemble average and perturbed solutions with $\Delta t = 10^{-4}$, $h = \frac{1}{32}$, $\nu = 0.001$, and $\nu_m = 1$.

4.2 MHD Channel Flow over a Step:
Next, we consider a domain which is a 30×10 rectangular channel with a 1×1 step five units away from the inlet into the channel. No slip boundary condition

is prescribed for the velocity and $B =< 0, 1 >^T$ is enforced for the magnetic field on the walls and the step, $u =< y(10 - y)/25, 0 >^T$ and $B =< 0, 1 >^T$ at the inflow, and the outflow condition uses a channel of extension 10 units, and at the end of the extension, we set outflow velocity and magnetic field equal to the inflow.

An ensemble of four different solutions corresponding to the perturbed initial conditions $u_j(0) := \begin{cases}(1+(-1)^{j-1}\epsilon)u_0 & 1 \le j < 3\\ (1+(-1)^{j-1}2\epsilon)u_0 & 3 \le j \le 4\end{cases}$ and $B_j(0) := \begin{cases}(1+(-1)^{j-1}\epsilon)B_0 & 1 \le j < 3\\ (1+(-1)^{j-1}2\epsilon)B_0 & 3 \le j \le 4\end{cases}$ where $u_0 :=< y(10-y)/25, 0 >^T$ and $B_0 :=< 0, 1 >^T$, and a similar way perturbed inflow and outflow are considered. The grad-div stabilization parameter $\gamma = 10^5$ is used. As we used second order BDF-2 scheme to approximate time derivative, we used backward-Euler method at the first time step to get the second initial condition. A mesh of the domain with 37474 velocity degrees of freedom is shown in Figure 2, where some portions of the domain are structured mesh and other portions which are around the step are unstructured mesh. The simulations of the Algorithm 3.1 are done with the various values of ϵ until $T = 2$, with a time step of $\Delta t = 0.001$, which makes sure to satisfies the CFL like condition. Velocity and magnetic field ensemble average solutions for varying ϵ are plotted in Figure 3 and 4 respectively. To make a comparison, we also run the case with $\epsilon = 0$ (without perturbation) and compute the true ensemble average according to the formula (1.12). We observe that the ensemble average solutions appear to converge to the unperturbed solution as $\epsilon \to 0$ which is expected from our theory.

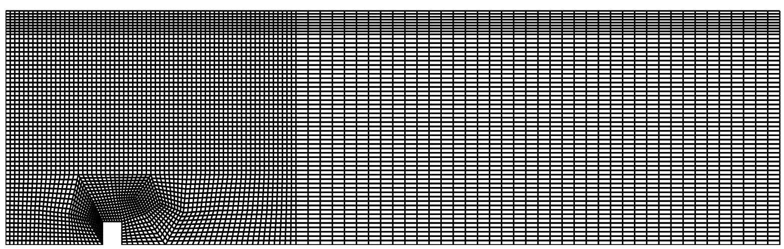

Figure 2: Mesh for the channel flow with a step example.

5 Conclusion:

In this paper, we proposed, analyzed and tested a second order efficient algorithm for MHD flow ensemble, which is a major improvement to the first order ensemble average algorithm proposed in [39]. The algorithm combines the breakthrough idea of Trenchea [46] to present a decoupled stable scheme in terms of Elsässer variables and the breakthrough idea for efficient computation of flow ensemble for Navier-Stokes [27] and extends it to MHD. The key features to the efficiency of the algorithm are (i) it is second order accurate stable decoupled method-split into two Oseen problems, which are much easier to solve and can be solved simultaneously (ii) at each time step, all J different linear systems share the same coefficient matrix, as a result storage requirement is reduced, a single assembly of the coefficient matrix is required instead of J times, preconditioners need to build once and can be reused. We proved the stability and second order convergence of the algorithm with respect to the time size. Numerical experiments were done on a unit square with a manufactured solution that verified the predicted convergence rates. Finally, we applied our scheme on a benchmark channel flow over a step problem and showed the method performed well.

In this paper, we consider flow ensemble subject to the different initial conditions, we plan to investigate flow ensemble behavior where the viscosities, the boundary conditions, and the body forces involve uncertainties. To reduced the computational cost, reduced order modeling (ROM) for the ensemble MHD flow computation will be the future research avenue. Recently, it has been shown the data-driven filtered ROM for flow problem [47] works well for the complex system. To reduced computation cost further to simulate ensemble MHD system as well as more accurate results, it is worth exploring in ROM with physically accurate data [40]. For our future investigation, we also plan to apply the recent advances [17] of nonintrusive filter-based stabilization of ROMs for uncertainty quantification of the MHD flow ensemble.

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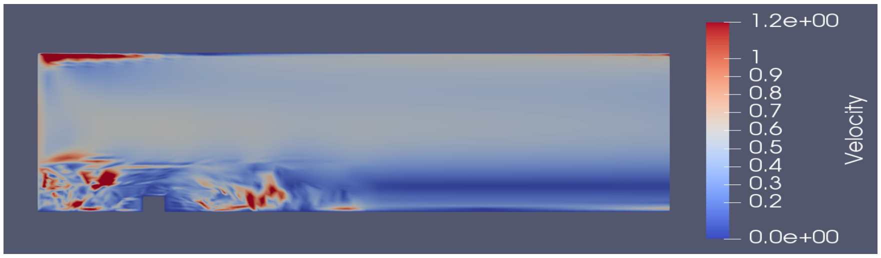

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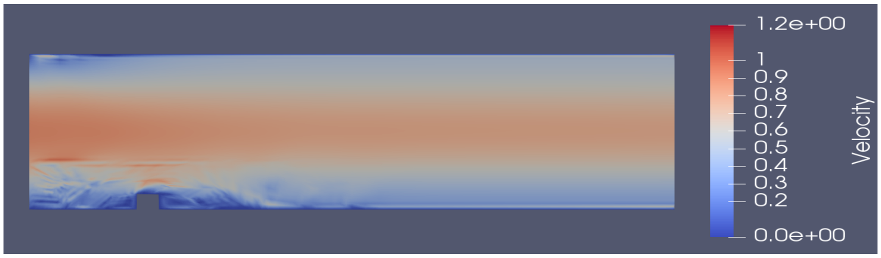

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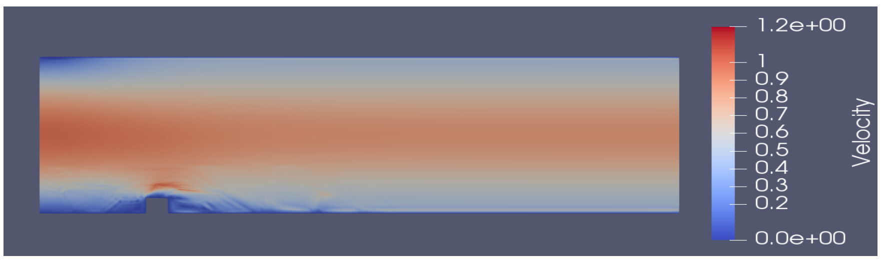

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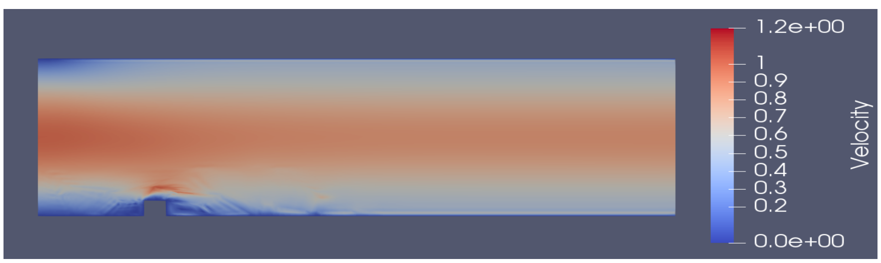

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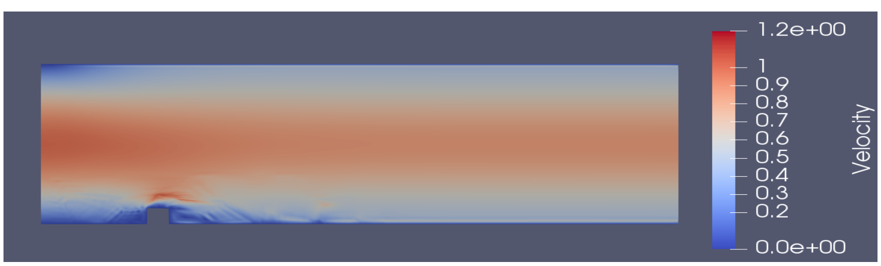
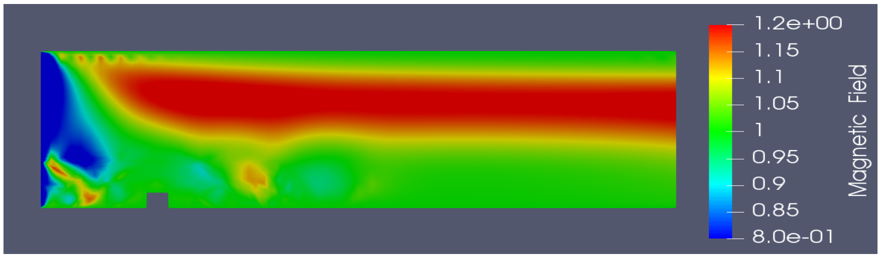


No perturbation

Figure 3: Shown above is $T = 2$, speed contours of the velocity ensemble solutions for MHD channel flow over a step with $\Delta t = 0.001$, $\nu = 0.001$, $\nu_m = 1$, and $J = 4$.



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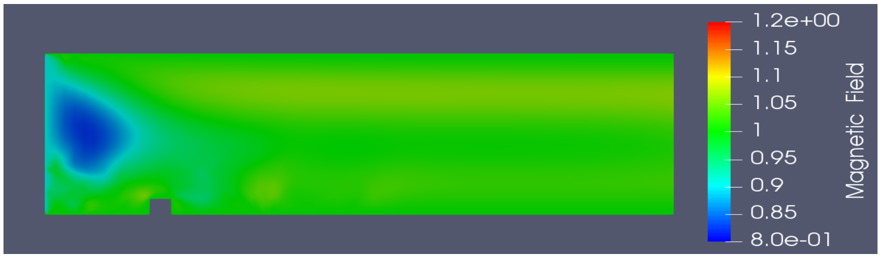
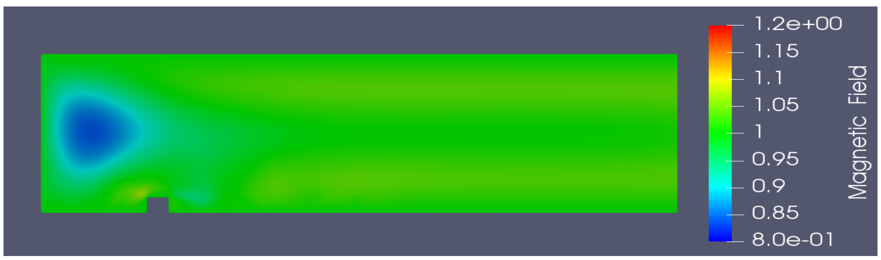
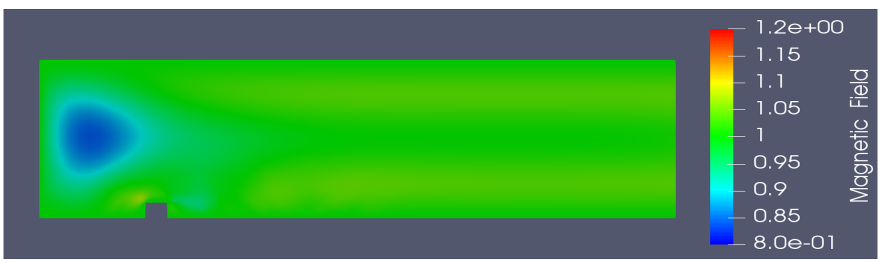


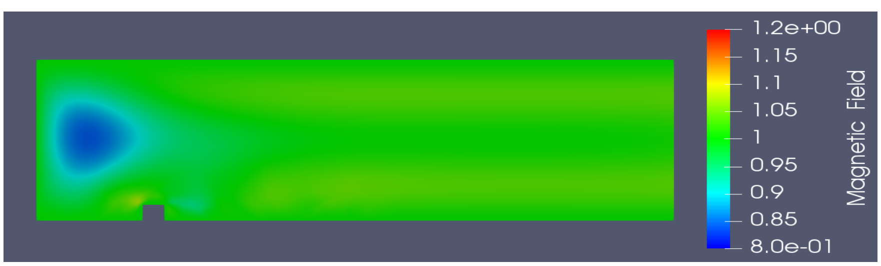
$\epsilon = 0.1$



$\epsilon = 0.01$



$\epsilon = 0.001$



No perturbation

Figure 4: Shown above is $T = 2$, magnitude of ensemble average magnetic field solutions for MHD channel flow over a step with $\Delta t = 0.001$, $\nu = 0.001$, $\nu_m = 1$, and $J = 4$.