Research Article

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An Efficient Algorithm for Computation of MHD Flow Ensembles

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Abstract: An efficient algorithm is proposed and studied for computing flow ensembles of incompressible magnetohydrodynamic (MHD) flows under uncertainties in initial or boundary data. The ensemble average of J realizations is approximated through a clever algorithm (adapted from a breakthrough idea of Jiang and Layton [23]) that, at each time step, uses the same matrix for each of the J systems solves. Hence, preconditioners need to be built only once per time step, and the algorithm can take advantage of block linear solvers. Additionally, an Elsässer variable formulation is used, which allows for a stable decoupling of each MHD system at each time step. We prove stability and convergence of the algorithm, and test it with two numerical experiments.

Keywords: Magnetohydrodynamics, Uncertainty Quantification, Fast Ensemble Calculation, Finite Element Method, Elsässer Variables

MSC 2010: 65M12, 65M60, 76W05

1 Introduction

The flow of electrically conducting fluids in the presence of magnetic field is called magnetohydrodynamics (MHD) flow, and arises in a wide range of applications in, e.g., astrophysics and geophysics [4, 5, 9, 10, 20, 32], liquid metal cooling of nuclear reactors [3, 16, 33], and process metallurgy [8]. The physical principles governing these flows are that when an electrically conducting fluid moves in a magnetic field, the magnetic field induces current in the fluid, which in turn creates forces on the fluid and also alters the magnetic field. The governing equations of the model consist of a nonlinear system of partial differential equations (PDEs) that couple the Navier–Stokes equations for fluid flow to the Maxwell's equations for electromagnetics, and are given by

$$\begin{aligned} &u_t + u \cdot \nabla u - sB \cdot \nabla B - \nu \Delta u + \nabla p = f, \\ &B_t + u \cdot \nabla B - B \cdot \nabla u - \nu_m \Delta B + \nabla \lambda = \nabla \times g, \\ &\nabla \cdot u = \nabla \cdot B = 0, \end{aligned}$$

in $\Omega \times (0, T)$, where $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) is the domain of the fluid (which we assume to be convex), s is the coupling number, ν is the kinematic viscosity, ν_m is the magnetic resistivity, f is body forces, $\nabla \times g$ is the forcing on the magnetic field, and λ can be considered as a Lagrange multiplier corresponding to the solenoidal constraint on the magnetic field. We note that the curl formulation of the Maxwell's equation is avoided by assuming smooth domains, which is a common assumption in, e.g., applications in geophysics and astrophysics.

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In the numerical simulation of MHD flows, uncertainties arise due to both the lack of data, and the inherent irregularity of the physical process involved [11, 12]. Uncertainty quantification (UQ) is the process of characterizing the impacts of uncertainty on the final quantities of interest, and in MHD flow simulations with incomplete data, UQ plays an important role in the validation of simulation methodologies and aims at developing rigorous methods to characterize the effect of the variability. A typical approach involves computing flow ensembles [7, 13, 27–29, 31], where J separate realizations of the problem are solved, and are then used to calculate means and sensitivities. This leads to J separate MHD systems needing solved. If we denote the solutions of realization j by u_j, B_j and p_j, the systems take the following form, for $j = 1, 2, \dots, J$:

$$\begin{aligned}
&u_{j,t} + u_j \cdot \nabla u_j - sB_j \cdot \nabla B_j - \nu\Delta u_j + \nabla p_j = f_j(x, t) \quad \text{in } \Omega \times (0, T),\\
&B_{j,t} + u_j \cdot \nabla B_j - B_j \cdot \nabla u_j - \nu_m \Delta B_j + \nabla \lambda_j = \nabla \times g_j(x, t) \quad \text{in } \Omega \times (0, T),\\
&\nabla \cdot u_j = 0 \quad \text{in } \Omega \times (0, T),\\
&\nabla \cdot B_j = 0 \quad \text{in } \Omega \times (0, T),\\
&u_j(x, 0) = u^0(x) \quad \text{in } \Omega,\\
&B_j(x, 0) = B_j^0(x) \quad \text{in } \Omega.
\end{aligned}$$

For simplicity we equip both velocity and magnetic fields with homogeneous Dirichlet boundary conditions (our analysis and results will still hold, although with minor modifications, in the case of periodic boundary conditions or inhomogeneous Dirichlet boundary conditions).

In the recent works [1, 34], it was shown that algorithms based on the Elsässer variable formulation of MHD lead to more efficient algorithms, as they can be decoupled in a stable way so that two Oseen-type problems need to be solved at each time step, instead of a fully coupled linear system. Defining

$$\begin{aligned}
v_j &:= u_j + \sqrt{s}B_j, & w_j &:= u_j - \sqrt{s}B_j,\\
f_{1,j} &:= f_j + \sqrt{s}\nabla \times g, & f_{2,j} &:= f_j - \sqrt{s}\nabla \times g,\\
q_j &:= p_j + \sqrt{s}\lambda_j, & r_j &:= p_j - \sqrt{s}\lambda_j
\end{aligned}$$

produces the following Elsässer variable formulation of the J realizations:

$$v_{j,t} + w_j \cdot \nabla v_j + \nabla q_j - \frac{\nu + \nu_m}{2}\Delta v_j - \frac{\nu - \nu_m}{2}\Delta w_j = f_{1,j}, \tag{1.1}$$

$$w_{j,t} + v_j \cdot \nabla w_j + \nabla r_j - \frac{\nu + \nu_m}{2}\Delta w_j - \frac{\nu - \nu_m}{2}\Delta v_j = f_{2,j}, \tag{1.2}$$

$$\nabla \cdot v_j = \nabla \cdot w_j = 0, \tag{1.3}$$

together with initial and boundary conditions.

It is the purpose of this paper to develop and study efficient algorithms for computing (1.1)–(1.3), in particular for the purpose of efficiently computing ensemble averages of the J solutions. The key ideas we use follow in the same spirit as those used for Navier–Stokes simulations in [21, 22, 24, 30], in particular we will create an algorithm which solves for all J solutions together, where the matrices that arise at each time step are the same for all J simulations. Thus, preconditioners need to be developed only once, and one may also take advantage of block solvers. This leads to a simulation far more efficient than computing J solutions independently, and takes the following form (suppressing the spatial discretization):

Step 1: for $j = 1, \dots, J$,

$$\begin{aligned}
\frac{v_j^{n+1} - v_j^n}{\Delta t} &+ \nabla q_j^{n+1} - \frac{\nu + \nu_m}{2}\Delta v_j^{n+1} - \frac{\nu - \nu_m}{2}\Delta w_j^n + \langle w\rangle^n \cdot \nabla v_j^{n+1}\\
&+ (w_j^n - \langle w\rangle^n) \cdot \nabla v_j^n - \nabla \cdot (2\nu_T(w' t)\nabla v_j^{n+1}) = f_{1,j}(t^{n+1}).
\end{aligned} \tag{1.4}$$

Step 2: for $j = 1, \dots, J$,

$$\begin{aligned}
\frac{w_j^{n+1} - w_j^n}{\Delta t} &+ \nabla r_j^{n+1} - \frac{\nu + \nu_m}{2}\Delta w_j^{n+1} - \frac{\nu - \nu_m}{2}\Delta v_j^n + \langle v\rangle^n \cdot \nabla w_j^{n+1}\\
&+ (v_j^n - \langle v\rangle^n) \cdot \nabla w_j^n - \nabla \cdot (2\nu_T(v' t)\nabla w_j^{n+1}) = f_{2,j}(t^{n+1}).
\end{aligned} \tag{1.5}$$

Here the ensemble mean, fluctuation and its magnitude are denoted by $\langle v\rangle$, v_j' and $|v'|$, respectively, and defined as follows:

$$\text{mean } \langle v\rangle := \frac{1}{J}\sum_{j=1}^{J} v_j, \quad \text{fluctuation } v_j' := v_j - \langle v\rangle, \quad \text{magnitude } |v'| := \sqrt{\sum_{j=1}^{J}|v_j'|^2}.$$

The ν_T terms represent $O(\Delta t)$ eddy viscosity terms based on mixing lengths, and are included to provide stability for flows that are not resolvable on particular meshes, following ideas in [24]. The precise definitions for these terms are given in Section 3. With these stabilization terms, we are able to prove unconditional (with respect to the time step size) stability of the algorithm, but without them, stability requires a time step restriction.

A key feature of the method above is that the MHD systems decouple into two Oseen problems, and further the matrices that arise at each time step are identical; only the right-hand sides change at each time step. We will prove that the ensemble of the J computed solutions converges (as the spatial and temporal mesh width tend to zero) to the ensemble solution of the J MHD solutions. Numerical tests will be given that verify our theoretical results.

This paper is arranged as follows. In Section 2, we give notation and mathematical preliminaries to allow for a smooth presentation and analysis to follow. Section 3 presents and analyzes a fully discrete algorithm corresponding to (1.4)–(1.5), and proves it is stable and convergent. Numerical tests are presented in Section 4, and finally conclusions are drawn in Section 5.

2 Notation and Preliminaries

Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be a convex polygonal or polyhedral domain in $\mathbb{R}^d$ $(d = 2, 3)$ with boundary $\partial\Omega$. The usual $L^2(\Omega)$ norm and inner product are denoted by $\|\cdot\|$ and $(\cdot,\cdot)$, respectively. Similarly, the $L^p(\Omega)$ norms and the Sobolev $W_p^k(\Omega)$ norms are $\|\cdot\|_{L^p}$ and $\|\cdot\|_{W_p^k}$, respectively, for $k \in \mathbb{N}$, $1 \le p \le \infty$. The Sobolev space $W_2^k(\Omega)$ is represented by $H^k(\Omega)$ with norm $\|\cdot\|_k$. The natural function spaces for our problem are

$$X := H_0^1(\Omega) = \{v \in (L^p(\Omega))^d : \nabla v \in L^2(\Omega)^{d\times d},\ v = 0 \text{ on } \partial\Omega\},$$
$$Q := L_0^2(\Omega) = \Big\{q \in L^2(\Omega) : \int_\Omega q\,dx = 0\Big\}.$$

Recall that the Poincaré inequality holds in X: there exists C depending only on the size of Ω satisfying, for all $\phi \in X$,

$$\|\phi\| \le C\|\nabla\phi\|.$$

The divergence free velocity space is given by

$$V := \{v \in X : (\nabla\cdot v, q) = 0 \text{ for all } q \in Q\}.$$

We define the trilinear form $b : X \times X \times X \to \mathbb{R}$ by

$$b(u, v, w) := (u\cdot\nabla v, w),$$

and recall from [14] that $b(u, v, v) = 0$ if $u \in V$, and

$$|b(u, v, w)| \le C(\Omega)\|\nabla u\|\|\nabla v\|\|\nabla w\| \quad \text{for any } u, v, w \in X. \tag{2.1}$$

The conforming finite element spaces are denoted by $X_h \subset X$ and $Q_h \subset Q$, and we assume a regular triangulation $\tau_h(\Omega)$, where the maximum triangle diameter is h. We assume that (X_h, Q_h) satisfies the usual discrete inf-sup condition

$$\inf_{q_h\in Q_h}\sup_{v_h\in X_h}\frac{(q_h, \nabla\cdot v_h)}{\|q_h\|\|\nabla v_h\|} \ge \beta > 0,$$

where β is independent of h.

The space of discretely divergence free functions is defined as

$$V_h := \{v_h \in X_h : (\nabla \cdot v_h, q_h) = 0 \text{ for all } q_h \in Q_h\}.$$

For MHD flows, the enforcement of the solenoidal constraints is believed critical, for (at least) two reasons. Enforcing $\nabla \cdot u = 0$ discretely is now widely known to be important for general incompressible flows [25]. The $\nabla \cdot B = 0$ constraint is a *mathematical* constraint in the sense that the magnetic field is derived as the curl of the electric field, and therefore the solenoidal constraint arises because 'div-curl' is zero. Although at the continuous level, a divergence-free initial condition B_0 will guarantee a divergence-free magnetic field at all times, this is not typically the case for Galerkin discretizations. Also, in many MHD formulations, the pressure term includes contributions of the magnetic field, which can potentially create a large and complex pressure; recent work in mixed finite element theory has shown that weak mass conservation may not be enough to sufficiently eliminate the effect of such modified pressures on velocity and magnetic field errors [25]. Due to these issues, it is natural to choose strongly divergence-free elements in this setting, and we choose the $(X_h, Q_h) = (P_k, P_{k-1}^{\mathrm{disc}})$ Scott–Vogelius (SV) finite element pair for both our analysis and computations. This element satisfies the inf-sup condition when the mesh is created as a barycenter refinement of a regular mesh, and the polynomial degree $k \geq d$; see [2, 35]. Our analysis can be extended without difficulty to any inf-sup stable element choice, however, there will be additional terms that appear in the convergence analysis if non-divergence-free elements are chosen. In particular, pressure robustness of the convergence estimates will be lost, as the error will be dependent on the size of true solution pressure derivatives.

The following lemma for the discrete Gronwall inequality was given in [19].

Lemma 2.1. *Let Δt, H, a_n, b_n, c_n, d_n be non-negative numbers for $n = 1, \dots, M$ with $M \in \mathbb{N}$ such that*

$$a_M + \Delta t \sum_{n=1}^{M} b_n \leq \Delta t \sum_{n=1}^{M-1} d_n a_n + \Delta t \sum_{n=1}^{M} c_n + H.$$

Then, for all $\Delta t > 0$,

$$a_M + \Delta t \sum_{n=1}^{M} b_n \leq \exp\Big(\Delta t \sum_{n=1}^{M-1} d_n\Big)\Big(\Delta t \sum_{n=1}^{M} c_n + H\Big).$$

3 Fully Discrete Scheme and Analysis of Ensemble Eddy Viscosity

We are now ready to present the fully discrete scheme for efficient MHD ensemble calculations. It equips (1.1)–(1.3) with a finite element spatial discretization. The eddy viscosity term is defined using mixing length phenomenology, following [24], and is given by

$$\nu_T(u_h', t^n) := \mu \Delta t \max_j |(u_{j,h}^n)'|^2.$$

There are different ways to define the mixing length, and multiple definitions are studied in [24]. We chose this one due to its simplicity, and the fact that it leads to a stable and optimally convergent algorithm. The scheme is defined as follows.

Algorithm 3.1. Given time step $\Delta t > 0$, end time $T > 0$, initial conditions $v_j^0, w_j^0 \in V_h$ and forcings $f_{1,j}, f_{2,j} \in L^\infty(0, T; H^{-1}(\Omega)^d)$ for $j = 1, 2, \dots, J$. Set $M = T/\Delta t$ and for $n = 1, \dots, M-1$, compute:

Find $v_{j,h}^{n+1} \in V_h$ satisfying, for all $\chi_h \in V_h$,

$$\begin{aligned}\Big(\frac{v_{j,h}^{n+1} - v_{j,h}^n}{\Delta t}, \chi_h\Big) + \frac{\nu + \nu_m}{2}(\nabla v_{j,h}^{n+1}, \nabla \chi_h) + \frac{\nu - \nu_m}{2}(\nabla w_{j,h}^n, \nabla \chi_h) + (\langle w_h \rangle^n \cdot \nabla v_{j,h}^{n+1}, \chi_h)\\ + ((w_{j,h}^n - \langle w_h \rangle^n) \cdot \nabla v_{j,h}^n, \chi_h) + (2\nu_T(w_h', t)\nabla v_{j,h}^{n+1}, \nabla \chi_h) = (f_{1,j}(t^{n+1}), \chi_h).\end{aligned} \tag{3.1}$$

Find $w_{j,h}^{n+1} \in V_h$ satisfying, for all $l_h \in V_h$,

$$\Big(\frac{w_{j,h}^{n+1} - w_{j,h}^{n}}{\Delta t}, l_h\Big) + \frac{\nu + \nu_m}{2}(\nabla w_{j,h}^{n+1}, \nabla l_h) + \frac{\nu - \nu_m}{2}(\nabla v_{j,h}^{n}, \nabla l_h) + (\langle v_h \rangle^n \cdot \nabla w_{j,h}^{n+1}, l_h)$$
$$+ ((v_{j,h}^n - \langle v_h \rangle^n) \cdot \nabla w_{j,h}^n, l_h) + (2\nu_T(v_h', t)\nabla w_{j,h}^{n+1}, \nabla l_h) = (f_{2,j}(t^{n+1}), l_h). \tag{3.2}$$

Remark 3.2. At each time step, all J realizations for step 1 will have the same matrix for the linear systems that arise, and similarly for step 2. Thus, block solvers can be taken advantage of, and matrices and preconditioners need to be built just once instead of J times. The key idea is a particular explicit treatment of part of the nonlinear term for each realization, and the stabilization term is used to stabilize this explicit treatment.

Remark 3.3. For simplicity of notation, the algorithm is presented in a V_h formulation. While this is equivalent to an (X_h, Q_h) formulation and is more convenient for analysis (see e.g. [26]), implementation should be performed using the (X_h, Q_h) formulation since it is unknown how to efficiently construct a basis for V_h.

We now prove that Algorithm 3.1 is unconditionally stable with respect to the time step size, provided $\mu \ge \frac{1}{2}$.

Theorem 3.4 (Unconditional Stability). *Suppose $f_{1,j}, f_{2,j} \in L^\infty(0, T; H^{-1}(\Omega)^d)$ and $v_{j,h}^0, w_{j,h}^0 \in V_h$. Then for any $\Delta t > 0$ and $\mu \ge \frac{1}{2}$, solutions to (1.4)–(1.5) satisfy*

$$\|v_{j,h}^M\|^2 + \|w_{j,h}^M\|^2 + \frac{(\nu - \nu_m)^2}{2(\nu + \nu_m)}\Delta t(\|\nabla v_{j,h}^M\|^2 + \|\nabla w_{j,h}^M\|^2)$$
$$\le \|v_{j,h}^0\|^2 + \|w_{j,h}^0\|^2 + \frac{(\nu - \nu_m)^2}{2(\nu + \nu_m)}\Delta t(\|\nabla v_{j,h}^0\|^2 + \|\nabla w_{j,h}^0\|^2) + \frac{\nu + \nu_m}{2\nu\nu_m}\Delta t \sum_{n=0}^{M-1}(\|f_{1,j}(t^{n+1})\|_{-1}^2 + \|f_{2,j}(t^{n+1})\|_{-1}^2).$$

Proof. Choose $\chi_h = v_{j,h}^{n+1}$ in (3.1). The first nonlinear term vanishes and we obtain

$$\frac{1}{\Delta t}(v_{j,h}^{n+1} - v_{j,h}^n, v_{j,h}^{n+1}) + \frac{\nu + \nu_m}{2}(\nabla v_{j,h}^{n+1}, \nabla v_{j,h}^{n+1}) + \frac{\nu - \nu_m}{2}(\nabla w_{j,h}^n, \nabla v_{j,h}^{n+1})$$
$$+ ((w_{j,h}^n - \langle w_h \rangle^n) \cdot \nabla v_{j,h}^n, v_{j,h}^{n+1}) + (2\nu_T(w_h', t)\nabla v_{j,h}^{n+1}, \nabla v_{j,h}^{n+1}) = (f_{1,j}(t^{n+1}), v_{j,h}^{n+1}).$$

Using the polarization identity and that

$$(2\nu_T(w_{j,h}', t)\nabla v_{j,h}^{n+1}, \nabla v_{j,h}^{n+1}) = 2\mu\Delta t\| |w_{j,h}'| |\nabla v_{j,h}^{n+1}| \|^2,$$

we get

$$\frac{1}{2\Delta t}(\|v_{j,h}^{n+1} - v_{j,h}^n\|^2 + \|v_{j,h}^{n+1}\|^2 - \|v_{j,h}^n\|^2) + \frac{\nu + \nu_m}{2}\|\nabla v_{j,h}^{n+1}\|^2 + \frac{\nu - \nu_m}{2}(\nabla w_{j,h}^n, \nabla v_{j,h}^{n+1})$$
$$+ ((w_{j,h}^n - \langle w_h \rangle^n) \cdot \nabla v_{j,h}^n, v_{j,h}^{n+1}) + 2\mu\Delta t\| |w_{j,h}'^n| |\nabla v_{j,h}^{n+1}| \|^2 \le (f_{1,j}(t^{n+1}), v_{j,h}^{n+1}). \tag{3.3}$$

Similarly, choosing $l_h = w_{j,h}^{n+1}$ in (3.2), we have

$$\frac{1}{2\Delta t}(\|w_{j,h}^{n+1} - w_{j,h}^n\|^2 + \|w_{j,h}^{n+1}\|^2 - \|w_{j,h}^n\|^2) + \frac{\nu + \nu_m}{2}\|\nabla w_{j,h}^{n+1}\|^2 + \frac{\nu - \nu_m}{2}(\nabla v_{j,h}^n, \nabla w_{j,h}^{n+1})$$
$$+ ((v_{j,h}^n - \langle v_h \rangle^n) \cdot \nabla w_{j,h}^n, w_{j,h}^{n+1}) + 2\mu\Delta t\| |v_{j,h}'^n| |\nabla w_{j,h}^{n+1}| \|^2 \le (f_{2,j}(t^{n+1}), w_{j,h}^{n+1}). \tag{3.4}$$

Adding equations (3.3) and (3.4) yields

$$\frac{1}{2\Delta t}(\|v_{j,h}^{n+1} - v_{j,h}^n\|^2 + \|v_{j,h}^{n+1}\|^2 - \|v_{j,h}^n\|^2 + \|w_{j,h}^{n+1} - w_{j,h}^n\|^2 + \|w_{j,h}^{n+1}\|^2 - \|w_{j,h}^n\|^2)$$
$$+ \frac{\nu + \nu_m}{2}(\|\nabla v_{j,h}^{n+1}\|^2 + \|\nabla w_{j,h}^{n+1}\|^2) + \frac{\nu - \nu_m}{2}\{(\nabla w_{j,h}^n, \nabla v_{j,h}^{n+1}) + (\nabla v_{j,h}^n, \nabla w_{j,h}^{n+1})\}$$
$$+ ((w_{j,h}^n - \langle w_h \rangle^n) \cdot \nabla v_{j,h}^n, v_{j,h}^{n+1}) + ((v_{j,h}^n - \langle v_h \rangle^n) \cdot \nabla w_{j,h}^n, w_{j,h}^{n+1})$$
$$+ 2\mu\Delta t\| |w_{j,h}'^n| |\nabla v_{j,h}^{n+1}| \|^2 + 2\mu\Delta t\| |v_{j,h}'^n| |\nabla w_{j,h}^{n+1}| \|^2 \le (f_{1,j}(t^{n+1}), v_{j,h}^{n+1}) + (f_{2,j}(t^{n+1}), w_{j,h}^{n+1}). \tag{3.5}$$

Next, using

$$
\begin{aligned}
((w^n_{j,h} - \langle w_h\rangle^n)\cdot\nabla v^n_{j,h}, v^{n+1}_{j,h}) &= -((w^n_{j,h} - \langle w_h\rangle^n)\cdot\nabla v^{n+1}_{j,h}, v^n_{j,h})\\
&= ((w^n_{j,h} - \langle w_h\rangle^n)\cdot\nabla v^{n+1}_{j,h}, v^{n+1}_{j,h} - v^n_{j,h})\\
&\le \|(w^n_{j,h} - \langle w_h\rangle^n)\cdot\nabla v^{n+1}_{j,h}\|\,\|v^{n+1}_{j,h} - v^n_{j,h}\|,
\end{aligned}
$$

we reduce (3.5) to

$$
\begin{aligned}
&\frac{1}{2\Delta t}(\|v^{n+1}_{j,h} - v^n_{j,h}\|^2 + \|v^{n+1}_{j,h}\|^2 - \|v^n_{j,h}\|^2 + \|w^{n+1}_{j,h} - w^n_{j,h}\|^2 + \|w^{n+1}_{j,h}\|^2 - \|w^n_{j,h}\|^2)\\
&\qquad + \frac{\nu+\nu_m}{2}(\|\nabla v^{n+1}_{j,h}\|^2 + \|\nabla w^{n+1}_{j,h}\|^2) + 2\mu\Delta t\||w'^n_{j,h}|\,|\nabla v^{n+1}_{j,h}|\|^2 + 2\mu\Delta t\||v'^n_{j,h}|\,|\nabla w^{n+1}_{j,h}|\|^2\\
&\le \frac{\nu+\nu_m}{4}(\|\nabla w^{n+1}_{j,h}\|^2 + \|\nabla v^{n+1}_{j,h}\|^2) + \frac{(\nu-\nu_m)^2}{4(\nu+\nu_m)}(\|\nabla v^n_{j,h}\|^2 + \|\nabla w^n_{j,h}\|^2) + \|f_{1,j}(t^{n+1})\|_{-1}\|\nabla v^{n+1}_{j,h}\|\\
&\qquad + \|f_{2,j}(t^{n+1})\|_{-1}\|\nabla w^{n+1}_{j,h}\| + \||w'^n_{j,h}|\,|\nabla v^{n+1}_{j,h}|\|\,\|v^{n+1}_{j,h} - v^n_{j,h}\| + \||v'^n_{j,h}|\,|\nabla w^{n+1}_{j,h}|\|\,\|w^{n+1}_{j,h} - w^n_{j,h}\|.
\end{aligned}
$$

After using Young's inequality again on the last two terms, we are able to hide these terms that arise from the explicit treatment of part of the nonlinearity in the positive stabilization terms on the left-hand side:

$$
\begin{aligned}
&\frac{1}{4\Delta t}(\|v^{n+1}_{j,h} - v^n_{j,h}\|^2 + \|w^{n+1}_{j,h} - w^n_{j,h}\|^2) + \frac{1}{2\Delta t}(\|v^{n+1}_{j,h}\|^2 - \|v^n_{j,h}\|^2 + \|w^{n+1}_{j,h}\|^2 - \|w^n_{j,h}\|^2)\\
&\qquad + \frac{\nu+\nu_m}{4}(\|\nabla v^{n+1}_{j,h}\|^2 + \|\nabla w^{n+1}_{j,h}\|^2) + (2\mu-1)\Delta t(\||w'^n_{j,h}|\,|\nabla v^{n+1}_{j,h}|\|^2 + \||v'^n_{j,h}|\,|\nabla w^{n+1}_{j,h}|\|^2)\\
&\le \frac{(\nu-\nu_m)^2}{4(\nu+\nu_m)}(\|\nabla v^n_{j,h}\|^2 + \|\nabla w^n_{j,h}\|^2) + \|f_{1,j}(t^{n+1})\|_{-1}\|\nabla v^{n+1}_{j,h}\| + \|f_{2,j}(t^{n+1})\|_{-1}\|\nabla w^{n+1}_{j,h}\|.
\end{aligned}
$$

Dropping the non-negative terms $\|v^{n+1}_{j,h} - v^n_{j,h}\|^2$ and $\|w^{n+1}_{j,h} - w^n_{j,h}\|^2$, and using Young's inequality, we get

$$
\begin{aligned}
&\frac{1}{2\Delta t}(\|v^{n+1}_{j,h}\|^2 - \|v^n_{j,h}\|^2 + \|w^{n+1}_{j,h}\|^2 - \|w^n_{j,h}\|^2) + (2\mu-1)\Delta t(\||w'^n_{j,h}|\,|\nabla v^{n+1}_{j,h}|\|^2 + \||v'^n_{j,h}|\,|\nabla w^{n+1}_{j,h}|\|^2)\\
&\qquad + \frac{(\nu-\nu_m)^2}{4(\nu+\nu_m)}(\|\nabla v^{n+1}_{j,h}\|^2 - \|\nabla v^n_{j,h}\|^2 + \|\nabla w^{n+1}_{j,h}\|^2 - \|\nabla w^n_{j,h}\|^2)\\
&\le \frac{\nu+\nu_m}{4\nu\nu_m}(\|f_{1,j}(t^{n+1})\|^2_{-1} + \|f_{2,j}(t^{n+1})\|^2_{-1}).
\end{aligned}
$$

The term $(2\mu-1)\Delta t(\||w'^n_j|\,|\nabla v^{n+1}_j|\|^2 + \||v'^n_j|\,|\nabla w^{n+1}_j|\|^2)$ can have two signs. To make it non-negative, we choose $\mu \ge \frac{1}{2}$, and then drop it. Next, multiply both sides by $2\Delta t$ and sum over time steps:

$$
\begin{aligned}
&\|v^M_{j,h}\|^2 + \|w^M_{j,h}\|^2 + \frac{(\nu-\nu_m)^2}{2(\nu+\nu_m)}\Delta t(\|\nabla v^M_{j,h}\|^2 + \|\nabla w^M_{j,h}\|^2)\\
&\quad \le \|v^0_{j,h}\|^2 + \|w^0_{j,h}\|^2 + \frac{(\nu-\nu_m)^2}{2(\nu+\nu_m)}\Delta t(\|\nabla v^0_{j,h}\|^2 + \|\nabla w^0_{j,h}\|^2) + \frac{\nu+\nu_m}{2\nu\nu_m}\Delta t\sum_{n=0}^{M-1}(\|f_{1,j}(t^{n+1})\|^2_{-1} + \|f_{2,j}(t^{n+1})\|^2_{-1}).
\end{aligned}
$$

This finishes the proof. □

We will now give a full error analysis of the proposed algorithm which converges in space and in time, provided sufficient smoothness of the true solutions.

Theorem 3.5. *For (v_j, w_j, q_j, r_j) satisfying (1.1)–(1.3) and regularity assumptions for $m = \max\{3, k+1\}$, $v_j, w_j \in L^\infty(0, T; H^m(\Omega)^d)$, $v_{j,t}, w_{j,t} \in L^\infty(0, T; H^1(\Omega)^d)$, and $v_{j,tt}, w_{j,tt} \in L^\infty(0, T; L^2(\Omega)^d)$, the ensemble solution $(\langle v_h\rangle, \langle w_h\rangle)$ to Algorithm 3.1 unconditionally converges to the true ensemble solution: for any $\Delta t > 0$ and $\mu > \frac{1}{2}$, one has*

$$
\begin{aligned}
&\|\langle v\rangle(T) - \langle v_h\rangle^M\|^2 + \|\langle w\rangle(T) - \langle w_h\rangle^M\|^2 + \frac{3\alpha}{4}\Delta t\sum_{j=1}^{J}\sum_{n=1}^{M}(\|\nabla(\langle v\rangle(t^n) - \langle v_h\rangle^n)\|^2 + \|\nabla(\langle w\rangle(t^n) - \langle w_h\rangle^n)\|^2)\\
&\qquad \le Ce^{\frac{CT(1+\Delta t^2)}{\alpha}}(h^{2k} + h^{2k-1}\Delta t + \Delta t^2(1 + h^{2-d}) + h^{2k}\Delta t^2),
\end{aligned}
$$

where $\alpha := \nu + \nu_m - |\nu - \nu_m|$.

Remark 3.6. In 3D, this result predicts the temporal convergence rate could be reduced to $O(\Delta t(1 + h^{-1/2}))$, which is less than the optimal rate of $O(\Delta t)$. This reduction in error comes from the use of the inverse inequality in the analysis of the stabilization term. This can be improved to $O(\Delta t)$ by removing the stabilization term, but that will in turn cause a time step restriction for stability and convergence results.

Proof. We begin by obtaining error equations. Testing (1.1) and (1.2) with $\chi_h, l_h \in V_h$ at $t = t^{n+1}$ yields

$$\begin{aligned}\frac{1}{\Delta t}(v_j(t^{n+1}) - v_j(t^n), \chi_h) &+ (w_j(t^{n+1}) \cdot \nabla v_j(t^{n+1}), \chi_h) + \frac{\nu + \nu_m}{2}(\nabla v_j(t^{n+1}), \nabla \chi_h) + \frac{\nu - \nu_m}{2}(\nabla w_j(t^n), \nabla \chi_h)\\ &= -\Big(v_{j,t}(t^{n+1}) - \frac{v_j(t^{n+1}) - v_j(t^n)}{\Delta t}, \chi_h\Big) - \frac{\nu - \nu_m}{2}(\nabla w_j(t^{n+1}) - \nabla w_j(t^n), \nabla \chi_h) + (f_{1,j}(t^{n+1}), \chi_h) \end{aligned} \tag{3.6}$$

and

$$\begin{aligned}\frac{1}{\Delta t}(w_j(t^{n+1}) - w_j(t^n), l_h) &+ (v_j(t^{n+1}) \cdot \nabla w_j(t^{n+1}), l_h) + \frac{\nu + \nu_m}{2}(\nabla w_j(t^{n+1}), \nabla l_h) + \frac{\nu - \nu_m}{2}(\nabla v_j(t^n), \nabla l_h)\\ &= -\Big(w_{j,t}(t^{n+1}) - \frac{w_j(t^{n+1}) - w_j(t^n)}{\Delta t}, l_h\Big) - \frac{\nu - \nu_m}{2}(\nabla v_j(t^{n+1}) - \nabla v_j(t^n), \nabla l_h) + (f_{2,j}(t^{n+1}), l_h). \end{aligned} \tag{3.7}$$

Denote

$$e^n_{j,v} := v_j(t^n) - v^n_{h,j}, \quad e^n_{j,w} := w_j(t^n) - w^n_{h,j}, \quad \langle e_v\rangle^n := \frac{1}{J}\sum_{j=1}^J e^n_{j,v}, \quad \langle e_w\rangle^n := \frac{1}{J}\sum_{j=1}^J e^n_{j,w}.$$

Subtracting (1.4) and (1.5) from (3.6) and (3.7), respectively, produces

$$\begin{aligned}\frac{1}{\Delta t}(e^{n+1}_{v,j} - e^n_{v,j}, \chi_h) &+ (\langle e_w\rangle^n \cdot \nabla(v_j(t^{n+1}) - v_j(t^n)), \chi_h) + (\langle w_h\rangle^n \cdot \nabla e^{n+1}_{v,j}, \chi_h)\\ &+ \frac{\nu + \nu_m}{2}(\nabla e^{n+1}_{v,j}, \nabla \chi_h) + \frac{\nu - \nu_m}{2}(\nabla e^n_{w,j}, \nabla \chi_h) + (w^{n'}_{j,h} \cdot \nabla e^n_{v,j}, \chi_h) + 2\mu\Delta t(|w^{n'}_{j,h}|^2 \nabla e^{n+1}_{v,j}, \nabla \chi_h)\\ &+ (e^n_{w,j} \cdot \nabla v_j(t^n), \chi_h) - 2\mu\Delta t(|w^{n'}_{j,h}|^2 \nabla v_j(t^{n+1}), \nabla \chi_h) = -G_1(t, v_j, w_j, \chi_h)\end{aligned}$$

and

$$\begin{aligned}\frac{1}{\Delta t}(e^{n+1}_{w,j} - e^n_{w,j}, l_h) &+ (\langle e_v\rangle^n \cdot \nabla(w_j(t^{n+1}) - w_j(t^n)), l_h) + (\langle v_h\rangle^n \cdot \nabla e^{n+1}_{w,j}, l_h)\\ &+ \frac{\nu + \nu_m}{2}(\nabla e^{n+1}_{w,j}, \nabla l_h) + \frac{\nu - \nu_m}{2}(\nabla e^n_{v,j}, \nabla l_h) + (v^{n'}_{j,h} \cdot \nabla e^n_{w,j}, l_h) + 2\mu\Delta t(|v^{n'}_{j,h}|^2 \nabla e^{n+1}_{w,j}, \nabla l_h)\\ &+ (e^n_{v,j} \cdot \nabla w_j(t^n), l_h) - 2\mu\Delta t(|v^{n'}_{j,h}|^2 \nabla w_j(t^{n+1}), \nabla l_h) = -G_2(t, v_j, w_j, l_h),\end{aligned}$$

where

$$\begin{aligned}G_1(t, v_j, w_j, \chi_h) &= \Big(v_{j,t}(t^{n+1}) - \frac{v_j(t^{n+1}) - v_j(t^n)}{\Delta t}, \chi_h\Big) + \frac{\nu - \nu_m}{2}(\nabla w_j(t^{n+1}) - \nabla w_j(t^n), \nabla \chi_h)\\ &+ ((w_j(t^{n+1}) - w_j(t^n)) \cdot \nabla v_j(t^{n+1}), \chi_h) + ((w_j(t^n) - \langle w(t^n)\rangle) \cdot \nabla(v_j(t^{n+1}) - v_j(t^n)), \chi_h)\end{aligned}$$

and

$$\begin{aligned}G_2(t, v_j, w_j, l_h) &= \Big(w_{j,t}(t^{n+1}) - \frac{w_j(t^{n+1}) - w_j(t^n)}{\Delta t}, l_h\Big) + \frac{\nu - \nu_m}{2}(\nabla v_j(t^{n+1}) - \nabla v_j(t^n), \nabla l_h)\\ &+ ((v_j(t^{n+1}) - v_j(t^n)) \cdot \nabla w_j(t^{n+1}), l_h) + ((v_j(t^n) - \langle v(t^n)\rangle) \cdot \nabla(w_j(t^{n+1}) - w_j(t^n)), l_h).\end{aligned}$$

Now we decompose the errors as

$$\begin{aligned}e^n_{v,j} &:= v_j(t^n) - v^n_{j,h} = (v_j(t^n) - \tilde{v_j}^n) - (v^n_{j,h} - \tilde{v_j}^n) := \eta^n_{v,j} - \phi^n_{j,h},\\ e^n_{w,j} &:= w_j(t^n) - w^n_{j,h} = (w_j(t^n) - \tilde{w_j}^n) - (w^n_{j,h} - \tilde{w_j}^n) := \eta^n_{w,j} - \psi^n_{j,h},\end{aligned}$$

where $\tilde{v_j}^n = P^{L^2}_{V_h}(v_j(t^n)) \in V_h$ and $\tilde{w_j}^n = P^{L^2}_{V_h}(w_j(t^n)) \in V_h$ are the L^2 projections of $v_j(t^n)$ and $w_j(t^n)$ into V_h, respectively. Note that $(\eta^n_{v,j}, v_h) = 0$ for all $v_h \in V_h$ and $(\eta^n_{w_j}, v_h) = 0$ for all $v_h \in V_h$. Rewriting, for $\chi_h, l_h \in V_h$, we have

$$\begin{aligned}
&\frac{1}{\Delta t}(\phi_{j,h}^{n+1}-\phi_{j,h}^{n},\chi_h)+(\langle\psi_h\rangle^n\cdot\nabla(v_j(t^{n+1})-v_j(t^n)),\chi_h)+(\langle w_h\rangle^n\cdot\nabla\phi_{j,h}^{n+1},\chi_h)\\
&\qquad+\frac{\nu+\nu_m}{2}(\nabla\phi_{j,h}^{n+1},\nabla\chi_h)+\frac{\nu-\nu_m}{2}(\nabla\psi_{j,h}^{n},\nabla\chi_h)+(w_{j,h}^{n'}\cdot\nabla\phi_{j,h}^{n},\chi_h)+(\psi_{j,h}^{n}\cdot\nabla v_j(t^n),\chi_h)\\
&\qquad+2\mu\Delta t(|w_{j,h}^{n'}|^2\nabla\phi_{j,h}^{n+1},\nabla\chi_h)\\
&=(\langle\eta_w\rangle^n\cdot\nabla(v_j(t^{n+1})-v_j(t^n)),\chi_h)+(\langle w_h\rangle^n\cdot\nabla\eta_{v,j}^{n+1},\chi_h)\\
&\qquad+\frac{\nu+\nu_m}{2}(\nabla\eta_{v,j}^{n+1},\nabla\chi_h)+\frac{\nu-\nu_m}{2}(\nabla\eta_{w,j}^{n},\nabla\chi_h)+(w_{j,h}^{n'}\cdot\nabla\eta_{v,j}^{n},\chi_h)+(\eta_{w,j}^{n}\cdot\nabla v_j(t^n),\chi_h)\\
&\qquad+2\mu\Delta t(|w_{j,h}^{n'}|^2\nabla\eta_{v,j}^{n+1},\nabla\chi_h)+2\mu\Delta t\big|(|w_{j,h}^{n'}|^2\nabla v_j(t^{n+1}),\nabla\chi_h)\big|+|G_1(t,v_j,w_j,\chi_h)| \qquad (3.8)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{\Delta t}(\psi_{j,h}^{n+1}-\psi_{j,h}^{n},l_h)+(\langle\phi_h\rangle^n\cdot\nabla(w_j(t^{n+1})-w_j(t^n)),l_h)+(\langle v_h\rangle^n\cdot\nabla\psi_{j,h}^{n+1},l_h)\\
&\qquad+\frac{\nu+\nu_m}{2}(\nabla\psi_{j,h}^{n+1},\nabla l_h)+\frac{\nu-\nu_m}{2}(\nabla\phi_{j,h}^{n},\nabla l_h)+(v_{j,h}^{n'}\cdot\nabla\psi_{j,h}^{n},l_h)+((\phi_{j,h}^{n}-\langle\phi_h\rangle^n)\cdot\nabla w_j(t^n),l_h)\\
&\qquad+2\mu\Delta t(|v_{j,h}'|^2\nabla\psi_{j,h}^{n+1},\nabla l_h)\\
&=(\langle\eta_v\rangle^n\cdot\nabla(w_j(t^{n+1})-w_j(t^n)),l_h)+(\langle v_h\rangle^n\cdot\nabla\eta_{w,j}^{n+1},l_h)\\
&\qquad+\frac{\nu+\nu_m}{2}(\nabla\eta_{w,j}^{n+1},\nabla l_h)+\frac{\nu-\nu_m}{2}(\nabla\eta_{v,j}^{n},\nabla l_h)+(v_{j,h}^{n'}\cdot\nabla\eta_{w,j}^{n},l_h)+(\eta_{v,j}^{n}\cdot\nabla w_j(t^n),l_h)\\
&\qquad+2\mu\Delta t(|v_{j,h}^{n'}|^2\nabla\eta_{w,j}^{n+1},\nabla l_h)+2\mu\Delta t\big|(|v_{j,h}^{n'}|^2\nabla w_j(t^{n+1}),\nabla l_h)\big|+|G_2(t,v_j,w_j,l_h)|. \qquad (3.9)
\end{aligned}$$

Choose $\chi_h=\phi_{j,h}^{n+1}$, $l_h=\psi_{j,h}^{n+1}$ and use the polarization identity in (3.8) and (3.9), to obtain

$$\begin{aligned}
&\frac{1}{2\Delta t}(\|\phi_{j,h}^{n+1}\|^2-\|\phi_{j,h}^{n}\|^2+\|\phi_{j,h}^{n+1}-\phi_{j,h}^{n}\|^2)+\frac{\nu+\nu_m}{2}\|\nabla\phi_{j,h}^{n+1}\|^2+2\mu\Delta t\||w_{j,h}^{n'}|\,|\nabla\phi_{j,h}^{n+1}|\|^2\\
&\quad\le\frac{|\nu-\nu_m|}{2}\big|(\nabla\psi_{j,h}^{n},\nabla\phi_{j,h}^{n+1})\big|+\frac{\nu+\nu_m}{2}\big|(\nabla\eta_{v,j}^{n+1},\nabla\phi_{j,h}^{n+1})\big|+\frac{|\nu-\nu_m|}{2}\big|(\nabla\eta_{w,j}^{n},\nabla\phi_{j,h}^{n+1})\big|\\
&\qquad+2\mu\Delta t\big|(|w_{j,h}^{n'}|^2\nabla\eta_{v,j}^{n+1},\nabla\phi_{j,h}^{n+1})\big|+2\mu\Delta t\big|(|w_{j,h}^{n'}|^2\nabla v_j(t^{n+1}),\nabla\phi_{j,h}^{n+1})\big|\\
&\qquad+\big|(\langle\psi_h\rangle^n\cdot\nabla(v_j(t^{n+1})-v_j(t^n)),\phi_{j,h}^{n+1})\big|+\big|(\psi_{j,h}^{n}\cdot\nabla v_j(t^n),\phi_{j,h}^{n+1})\big|+\big|(w_{j,h}^{n'}\cdot\nabla\phi_{j,h}^{n},\phi_{j,h}^{n+1})\big|\\
&\qquad+\big|(\langle\eta_w\rangle^n\cdot\nabla(v_j(t^{n+1})-v_j(t^n)),\phi_{j,h}^{n+1})\big|+\big|(\langle w_h\rangle^n\cdot\nabla\eta_{v,j}^{n+1},\phi_{j,h}^{n+1})\big|+\big|(w_{j,h}^{n'}\cdot\nabla\eta_{v,j}^{n},\phi_{j,h}^{n+1})\big|\\
&\qquad+\big|(\eta_{w,j}^{n}\cdot\nabla v_j(t^n),\phi_{j,h}^{n+1})\big|+|G_1(t,v_j,w_j,\phi_{j,h}^{n+1})| \qquad (3.10)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{2\Delta t}(\|\psi_{j,h}^{n+1}\|^2-\|\psi_{j,h}^{n}\|^2+\|\psi_{j,h}^{n+1}-\psi_{j,h}^{n}\|^2)+\frac{\nu+\nu_m}{2}\|\nabla\psi_{j,h}^{n+1}\|^2+2\mu\Delta t\||v_{j,h}^{n'}|\,|\nabla\psi_{j,h}^{n+1}|\|^2\\
&\quad\le\frac{|\nu-\nu_m|}{2}\big|(\nabla\phi_{j,h}^{n},\nabla\psi_{j,h}^{n+1})\big|+\frac{\nu+\nu_m}{2}\big|(\nabla\eta_{w,j}^{n+1},\nabla\psi_{j,h}^{n+1})\big|+\frac{|\nu-\nu_m|}{2}\big|(\nabla\eta_{v,j}^{n},\nabla\psi_{j,h}^{n+1})\big|\\
&\qquad+2\mu\Delta t\big|(|v_{j,h}^{n'}|^2\nabla\eta_{w,j}^{n+1},\nabla\psi_{j,h}^{n+1})\big|+2\mu\Delta t\big|(|v_{j,h}^{n'}|^2\nabla w_j(t^{n+1}),\nabla\psi_{j,h}^{n+1})\big|\\
&\qquad+\big|(\langle\phi_h\rangle^n\cdot\nabla(w_j(t^{n+1})-w_j(t^n)),\psi_{j,h}^{n+1})\big|+\big|(\phi_{j,h}^{n}\cdot\nabla w_j(t^n),\psi_{j,h}^{n+1})\big|+\big|(v_{j,h}^{n'}\cdot\nabla\psi_{j,h}^{n},\psi_{j,h}^{n+1})\big|\\
&\qquad+\big|(\langle\eta_v\rangle^n\cdot\nabla(w_j(t^{n+1})-w_j(t^n)),\psi_{j,h}^{n+1})\big|+\big|(\langle v_h\rangle^n\cdot\nabla\eta_{w,j}^{n+1},\psi_{j,h}^{n+1})\big|+\big|(v_{j,h}^{n'}\cdot\nabla\eta_{w,j}^{n},\psi_{j,h}^{n+1})\big|\\
&\qquad+\big|(\eta_{v,j}^{n}\cdot\nabla w_j(t^n),\psi_{j,h}^{n+1})\big|+|G_2(t,v_j,w_j,\psi_{j,h}^{n+1})|. \qquad (3.11)
\end{aligned}$$

Let us define $\alpha:=\nu+\nu_m-|\nu-\nu_m|>0$ and assume $\mu>\frac{1}{2}$. We turn our attention to finding bounds for the terms on the right-hand side of (3.10) (the estimates for (3.11) are similar). Applying Cauchy–Schwarz and Young's inequalities on the first three terms results in

$$\begin{aligned}
\frac{|\nu-\nu_m|}{2}\big|(\nabla\psi_{j,h}^{n},\nabla\phi_{j,h}^{n+1})\big|&\le\frac{|\nu-\nu_m|}{4}\|\nabla\phi_{j,h}^{n+1}\|^2+\frac{|\nu-\nu_m|}{4}\|\nabla\psi_{j,h}^{n}\|^2,\\
\frac{\nu+\nu_m}{2}\big|(\nabla\eta_{v,j}^{n+1},\nabla\phi_{j,h}^{n+1})\big|&\le\frac{\alpha}{32}\|\nabla\phi_{j,h}^{n+1}\|^2+\frac{6(\nu+\nu_m)^2}{\alpha}\|\nabla\eta_{v,j}^{n+1}\|^2,\\
\frac{|\nu-\nu_m|}{2}\big|(\nabla\eta_{w,j}^{n},\nabla\phi_{j,h}^{n+1})\big|&\le\frac{\alpha}{32}\|\nabla\phi_{j,h}^{n+1}\|^2+\frac{6(\nu-\nu_m)^2}{\alpha}\|\nabla\eta_{w,j}^{n}\|^2.
\end{aligned}$$

The fourth and fifth right-hand side terms of (3.10) are less standard. For the fourth term, we apply Cauchy–Schwarz and Young's inequalities to obtain

$$
\begin{aligned}
2\mu\Delta t\big|(|w_{j,h}^{n'}|^2\nabla\eta_{v,j}^{n+1},\nabla\phi_{j,h}^{n+1})\big| &= 2\mu\Delta t(|w_{j,h}^{n'}|\nabla\eta_{v,j}^{n+1},|w_{j,h}^{n'}|\nabla\phi_{j,h}^{n+1})\\
&\le \frac{2\mu-1}{4}\Delta t\||w_{j,h}^{n'}|\nabla\phi_{j,h}^{n+1}\|^2+\frac{4\mu^2\Delta t}{2\mu-1}\||w_{j,h}^{n'}|\nabla\eta_{v,j}^{n+1}\|^2.
\end{aligned}
$$

For the fifth term, we use Hölder's inequality and the regularity assumptions of the true solution to get

$$
\begin{aligned}
2\mu\Delta t\big|(|w_{j,h}^{n'}|^2\nabla v_j(t^{n+1}),\nabla\phi_{j,h}^{n+1})\big| &\le C\mu\Delta t\|\nabla v_j(t^{n+1})\|_{L^\infty}\|w_{j,h}^{n'}\|_{L^4}^2\|\nabla\phi_{j,h}^{n+1}\|\\
&\le \frac{\alpha}{32}\|\nabla\phi_{j,h}^{n+1}\|^2+C\frac{\mu^2\Delta t^2}{\alpha}\|w_{j,h}^{n'}\|_{L^4}^4.
\end{aligned}
$$

For the first and second nonlinear terms, we use Hölder's inequality, Sobolev embedding theorems, Poincaré's and Young's inequalities to reveal

$$
\begin{aligned}
\big|(\langle\psi_h\rangle^n\cdot\nabla(v_j(t^{n+1})-v_j(t^n)),\phi_{j,h}^{n+1})\big| &\le \|\langle\psi_h\rangle^n\|\,\|\nabla(v_j(t^{n+1})-v_j(t^n))\|_{L^6}\|\phi_{j,h}^{n+1}\|_{L^3}\\
&\le C\|\langle\psi_h\rangle^n\|\,\|v_j(t^{n+1})-v_j(t^n)\|_2\|\phi_{j,h}^{n+1}\|^{1/2}\|\nabla\phi_{j,h}^{n+1}\|^{1/2}\\
&\le C\|\langle\psi_h\rangle^n\|\,\|v_j(t^{n+1})-v_j(t^n)\|_2\|\nabla\phi_{j,h}^{n+1}\|\\
&\le \frac{\alpha}{32}\|\nabla\phi_{j,h}^{n+1}\|^2+\frac{C\Delta t^2}{\alpha}\|\langle\psi_h\rangle^n\|^2\|v_{j,t}(t^*)\|_2^2,\\
\big|(\psi_{j,h}^n\cdot\nabla v_j(t^n),\phi_{j,h}^{n+1})\big| &\le \frac{\alpha}{32}\|\nabla\phi_{j,h}^{n+1}\|^2+\frac{C}{\alpha}\|\psi_{j,h}^n\|^2\|v_j(t^n)\|_2^2.
\end{aligned}
$$

For the third nonlinear term, rearranging and applying Cauchy–Schwarz and Young's inequalities yields

$$
\begin{aligned}
\big|(w_{j,h}^{n'}\cdot\nabla\phi_{j,h}^n,\phi_{j,h}^{n+1})\big| = \big|-(w_{j,h}^{n'}\cdot\nabla\phi_{j,h}^{n+1},\phi_{j,h}^n)\big| &= \big|(w_{j,h}^{n'}\cdot\nabla\phi_{j,h}^{n+1},\phi_{j,h}^{n+1}-\phi_{j,h}^n)\big|\\
&\le \||w_{j,h}^{n'}|\,|\nabla\phi_{j,h}^{n+1}|\|\,\|\phi_{j,h}^{n+1}-\phi_{j,h}^n\|\\
&\le \frac{1}{4\Delta t}\|\phi_{j,h}^{n+1}-\phi_{j,h}^n\|^2+\Delta t\||w_{j,h}^{n'}|\,|\nabla\phi_{j,h}^{n+1}|\|^2.
\end{aligned}
$$

For the fourth, fifth, sixth and seventh nonlinear terms, we apply Hölder and Young's inequalities with (2.1) to obtain

$$
\begin{aligned}
\big|(\langle\eta_w\rangle^n\cdot\nabla(v_j(t^{n+1})-v_j(t^n)),\phi_{j,h}^{n+1})\big| &\le C\|\nabla\langle\eta_{w,j}\rangle^n\|\|\nabla(v_j(t^{n+1})-v_j(t^n))\|\|\nabla\phi_{j,h}^{n+1}\|\\
&\le \frac{\alpha}{32}\|\nabla\phi_{j,h}^{n+1}\|^2+\frac{C\Delta t^2}{\alpha}\|\nabla\langle\eta_w\rangle^n\|^2\|\nabla v_{j,t}(t^{**})\|^2,\\
\big|(\langle w_h\rangle^n\cdot\nabla\eta_{v,j}^{n+1},\phi_{j,h}^{n+1})\big| &\le C\|\nabla\langle w_h\rangle^n\|\|\nabla\eta_{v,j}^{n+1}\|\|\nabla\phi_{j,h}^{n+1}\|\\
&\le \frac{\alpha}{32}\|\nabla\phi_{j,h}^{n+1}\|^2+\frac{C}{\alpha}\|\nabla\langle w_h\rangle^n\|^2\|\nabla\eta_{v,j}^{n+1}\|^2,\\
\big|(w_{j,h}^{n'}\cdot\nabla\eta_{v,j}^n,\phi_{j,h}^{n+1})\big| &\le C\|\nabla w_{j,h}^{n'}\|\|\nabla\eta_{v,j}^n\|\|\nabla\phi_{j,h}^{n+1}\|\\
&\le \frac{\alpha}{32}\|\nabla\phi_{j,h}^{n+1}\|^2+\frac{C}{\alpha}\|\nabla w_{j,h}^{n'}\|^2\|\nabla\eta_{v,j}^n\|^2,\\
\big|(\eta_{w,j}^n\cdot\nabla v_j(t^n),\phi_{j,h}^{n+1})\big| &\le C\|\nabla\eta_{w,j}^n\|\|\nabla v_j(t^n)\|\|\nabla\phi_{j,h}^{n+1}\|\\
&\le \frac{\alpha}{32}\|\nabla\phi_{j,h}^{n+1}\|^2+\frac{C}{\alpha}\|\nabla\eta_{w,j}^n\|^2\|\nabla v_j(t^n)\|^2.
\end{aligned}
$$

Using Taylor's series, Cauchy–Schwarz and Young's inequalities, we can evaluate the last term as

$$
\begin{aligned}
|G_1(t,v_j,w_j,\phi_{j,h}^{n+1})| \le \frac{\alpha}{24}\|\nabla\phi_{j,h}^{n+1}\|^2+\frac{\Delta t^2C}{\alpha}\big(&\|v_{j,tt}(t^*)\|^2+\|\nabla w_{j,t}(s^*)\|^2\\
&+\|\nabla w_{j,t}(s^*)\|^2\|\nabla v_j(t^{n+1})\|^2+\|\nabla(w_j(t^n)-\langle w(t^n)\rangle)\|^2\|\nabla v_{j,t}(t^*)\|^2\big),
\end{aligned}
$$

with $s^*, t^* \in [t^{n-1}, t^{n+1}]$. Using these estimates in (3.10) and reducing produces

$$\begin{aligned}
&\frac{1}{2\Delta t}(\|\phi_{j,h}^{n+1}\|^2 - \|\phi_{j,h}^n\|^2) + \frac{1}{4\Delta t}\|\phi_{j,h}^{n+1} - \phi_{j,h}^n\|^2 + \frac{\alpha + 2(\nu+\nu_m)}{8}\|\nabla\phi_{j,h}^{n+1}\|^2 + \frac{(2\mu-1)\Delta t}{2}\||w_{j,h}^{n'}||\nabla\phi_{j,h}^{n+1}|\|^2 \\
&\quad\le \frac{|\nu-\nu_m|}{4}\|\nabla\psi_{j,h}^n\|^2 + \frac{6(\nu-\nu_m)^2}{\alpha}\|\nabla\eta_{w,j}^n\|^2 + \frac{6(\nu+\nu_m)^2}{\alpha}\|\nabla\eta_{v,j}^{n+1}\|^2 \\
&\qquad + \frac{C}{\alpha}(\|\psi_{j,h}^n\|^2\|v_j(t^n)\|_2^2 + \|\nabla\langle w_h\rangle^n\|^2\|\nabla\eta_{v,j}^{n+1}\|^2 + \|\nabla w_{j,h}^{n'}\|^2\|\nabla\eta_{v,j}^n\|^2 + \|\nabla\eta_{w,j}^n\|^2\|\nabla v_j(t^n)\|^2) \\
&\qquad + \frac{4\mu^2\Delta t}{2\mu-1}\||w_{j,h}^{n'}|\nabla\eta_{v,j}^{n+1}\|^2 + C\frac{\mu^2\Delta t^2}{\alpha}\|w_{j,h}^{n'}\|_{L^4}^4 \\
&\qquad + \frac{C\Delta t^2}{\alpha}(\|\langle\psi_h\rangle^n\|^2\|v_{j,t}(t^*)\|_2^2 + \|\nabla\langle\eta_w\rangle^n\|^2\|\nabla v_{j,t}(t^*)\|^2 + \|v_{j,tt}(t^*)\|^2 + \|\nabla w_{j,t}(s^*)\|^2 \\
&\qquad + \|\nabla w_{j,t}(s^*)\|^2\|\nabla v_j(t^{n+1})\|^2 + \|\nabla(w_j(t^n) - \langle w(t^n)\rangle)\|^2\|\nabla v_{j,t}(t^*)\|^2).
\end{aligned} \tag{3.12}$$

Applying similar techniques to (3.11), we get

$$\begin{aligned}
&\frac{1}{2\Delta t}(\|\psi_{j,h}^{n+1}\|^2 - \|\psi_{j,h}^n\|^2) + \frac{1}{4\Delta t}\|\psi_{j,h}^{n+1} - \psi_{j,h}^n\|^2 + \frac{\alpha + 2(\nu+\nu_m)}{8}\|\nabla\psi_{j,h}^{n+1}\|^2 + \frac{(2\mu-1)\Delta t}{2}\||v_{j,h}^{n'}||\nabla\psi_{j,h}^{n+1}|\|^2 \\
&\quad\le \frac{|\nu-\nu_m|}{4}\|\nabla\phi_{j,h}^n\|^2 + \frac{6(\nu-\nu_m)^2}{\alpha}\|\nabla\eta_{v,j}^n\|^2 + \frac{6(\nu+\nu_m)^2}{\alpha}\|\nabla\eta_{w,j}^{n+1}\|^2 \\
&\qquad + \frac{C}{\alpha}(\|\phi_{j,h}^n\|^2\|w_j(t^n)\|_2^2 + \|\nabla\langle v_h\rangle^n\|^2\|\nabla\eta_{w,j}^{n+1}\|^2 + \|\nabla v_{j,h}^{n'}\|^2\|\nabla\eta_{w,j}^n\|^2 + \|\nabla\eta_{v,j}^n\|^2\|\nabla w_j(t^n)\|^2) \\
&\qquad + \frac{4\mu^2\Delta t}{2\mu-1}\||v_{j,h}^{n'}|\nabla\eta_{w,j}^{n+1}\|^2 + C\frac{\mu^2\Delta t^2}{\alpha}\|v_{j,h}^{n'}\|_{L^4}^4 \\
&\qquad + \frac{C\Delta t^2}{\alpha}(\|\langle\phi_h\rangle^n\|^2\|w_{j,t}(s^{**})\|_2^2 + \|\nabla\langle\eta_v\rangle^n\|^2\|\nabla w_{j,t}(s^{**})\|^2 + \|w_{j,tt}(s^{**})\|^2 + \|\nabla v_{j,t}(t^{**})\|^2 \\
&\qquad + \|\nabla v_{j,t}(t^{**})\|^2\|\nabla w_j(t^{n+1})\|^2 + \|\nabla(v_j(t^n) - \langle v(t^n)\rangle)\|^2\|\nabla w_{j,t}(s^{**})\|^2)
\end{aligned} \tag{3.13}$$

with $s^{**}, t^{**} \in [t^{n-1}, t^{n+1}]$. Dropping non-negative terms on the left-hand side and adding inequalities (3.12) and (3.13), multiplying by $2\Delta t$, using regularity assumptions, $\|\phi_{j,h}^0\| = \|\psi_{j,h}^0\| = \|\nabla\phi_{j,h}^0\| = \|\nabla\psi_{j,h}^0\| = 0$, $\Delta tM = T$, and summing over the time steps yields

$$\begin{aligned}
&\|\phi_{j,h}^M\|^2 + \|\psi_{j,h}^M\|^2 + \frac{3\alpha}{4}\Delta t\sum_{n=1}^{M}(\|\nabla\phi_{j,h}^n\|^2 + \|\nabla\psi_{j,h}^n\|^2) \\
&\quad\le \frac{12\Delta t(\nu-\nu_m)^2}{\alpha}\sum_{n=0}^{M-1}(\|\nabla\eta_{v,j}^n\|^2 + \|\nabla\eta_{w,j}^n\|^2) + \frac{8\mu^2\Delta t^2}{2\mu-1}\sum_{n=0}^{M-1}(\||v_{j,h}^{n'}|\nabla\eta_{w,j}^{n+1}\|^2 + \||w_{j,h}^{n'}|\nabla\eta_{v,j}^{n+1}\|^2) \\
&\qquad + C\Delta t\frac{\mu^2\Delta t^2}{\alpha}\sum_{n=0}^{M-1}(\|w_{j,h}^{n'}\|_{L^4}^4 + \|v_{j,h}^{n'}\|_{L^4}^4) + \frac{12\Delta t(\nu+\nu_m)^2}{\alpha}\sum_{n=0}^{M-1}(\|\nabla\eta_{v,j}^{n+1}\|^2 + \|\nabla\eta_{w,j}^{n+1}\|^2) \\
&\qquad + \frac{C\Delta t}{\alpha}\sum_{n=0}^{M-1}(\|\phi_{j,h}^n\|^2\|w_j(t)\|_{L^\infty(0,T,H^2)} + \|\psi_{j,h}^n\|^2\|v_j(t)\|_{L^\infty(0,T,H^2)} + \|\nabla\langle v_h\rangle^n\|^2\|\nabla\eta_{w,j}^{n+1}\|^2 \\
&\qquad + \|\nabla\langle w_h\rangle^n\|^2\|\nabla\eta_{v,j}^{n+1}\|^2 + \|\nabla v_{j,h}^{n'}\|^2\|\nabla\eta_{w,j}^n\|^2 + \|\nabla w_{j,h}^{n'}\|^2\|\nabla\eta_{v,j}^n\|^2) \\
&\qquad + \frac{C\Delta t^2}{\alpha}\sum_{n=0}^{M-1}\Delta t(\|\langle\phi_h\rangle^n\|^2\|w_{j,t}(t)\|_{L^\infty(0,T,H^2)}^2 + \|\langle\psi_h\rangle^n\|^2\|v_{j,t}(t)\|_{L^\infty(0,T,H^2)}^2 \\
&\qquad + \|\nabla\langle\eta_v\rangle^n\|^2 + \|\nabla\langle\eta_w\rangle^n\|^2) + C(h^{2k} + \Delta t^2).
\end{aligned}$$

The second sum on the right-hand side is nonstandard. For the first part of it (the second follows analogously), we begin with Hölder's inequality and the generalized inverse inequality [6]:

$$\begin{aligned}
\Delta t^2\sum_{n=0}^{M-1}\||v_{j,h}^{n'}|\nabla\eta_{w,j}^{n+1}\|^2 &\le \Delta t^2\sum_{n=0}^{M-1}\|v_{j,h}^{n'}\|_\infty^2\|\nabla\eta_{w,j}^{n+1}\|^2 \\
&\le Ch^{-1}\Delta t^2\sum_{n=0}^{M-1}\|\nabla v_{j,h}^{n'}\|^2\|\nabla\eta_{w,j}^{n+1}\|^2
\end{aligned}$$

$$
\begin{aligned}
&\leq Ch^{2k-1}\Delta t^2 \sum_{n=0}^{M-1} \|\nabla v_{j,h}^{n'}\|^2 |w_j^{n+1}|_{k+1}^2 \\
&\leq Ch^{2k-1}\Delta t,
\end{aligned}
$$

with the last two steps following from standard estimates of the L^2 projection error in the H^1 norm for finite element functions, and the stability estimate. For the third sum on the right-hand side, we get different bounds for 2D and 3D due to different Sobolev embeddings:

$$
\begin{aligned}
\text{2D:} &\quad \|w_{j,h}^{n'}\|_{L^4}^4 \leq C\|w_{j,h}^{n'}\|^2 \|\nabla w_{j,h}^{n'}\|^2 \leq C\|\nabla w_{j,h}^{n'}\|^2, \\
\text{3D:} &\quad \|w_{j,h}^{n'}\|_{L^4}^4 \leq C\|w_{j,h}^{n'}\| \|\nabla w_{j,h}^{n'}\|^3 \leq C\|\nabla w_{j,h}^{n'}\|^3,
\end{aligned}
$$

and similarly for $v_{j,h}^{n'}$, with the second upper bound in each inequality coming from the stability lemma. With the inverse inequality and the stability bound (used on the L^2 norm), we can bound

$$
\|\nabla w_{j,h}^{n'}\| \leq Ch^{-1}\|w_{j,h}^{n'}\| \leq Ch^{-1},
$$

and thus we obtain the bounds for both 2D or 3D:

$$
\begin{aligned}
\|w_{j,h}^{n'}\|_{L^4}^4 &\leq Ch^{2-d}\|\nabla w_{j,h}^{n'}\|^2, \\
\|v_{j,h}^{n'}\|_{L^4}^4 &\leq Ch^{2-d}\|\nabla v_{j,h}^{n'}\|^2.
\end{aligned}
$$

Using these bounds and the stability bound, the third sum is bounded as

$$
C\Delta t \frac{\mu^2 \Delta t^2}{\alpha} \sum_{n=0}^{M-1} (\|w_{j,h}^{n'}\|_{L^4}^4 + \|v_{j,h}^{n'}\|_{L^4}^4) \leq Ch^{2-d}\Delta t \frac{\mu^2\Delta t^2}{\alpha} \sum_{n=0}^{M-1} (\|\nabla w_{j,h}^{n'}\|^2 + \|\nabla v_{j,h}^{n'}\|^2) \leq Ch^{2-d}\frac{\mu^2\Delta t^2}{\alpha}.
$$

Now, summing over j and using the standard bounds for $\|\nabla \eta_v\|$ and $\|\nabla \eta_w\|$, we obtain

$$
\begin{aligned}
&\sum_{j=1}^{J}\|\phi_{j,h}^M\|^2 + \sum_{j=1}^{J}\|\psi_{j,h}^M\|^2 + \frac{3\alpha}{4}\Delta t \sum_{j=1}^{J}\sum_{n=1}^{M}(\|\nabla\phi_{j,h}^n\|^2 + \|\nabla\psi_{j,h}^n\|^2) \\
&\qquad \leq C(h^{2k} + h^{2k-1}\Delta t + \Delta t^2(1 + h^{2-d}) + h^{2k}\Delta t^2) + \sum_{n=0}^{M-1}\frac{C}{\alpha}(\Delta t + \Delta t^3)\Big(\sum_{j=1}^{J}\|\phi_{j,h}^n\|^2 + \sum_{j=1}^{J}\|\psi_{j,h}^n\|^2\Big).
\end{aligned}
$$

Applying the discrete Gronwall lemma, we have

$$
\begin{aligned}
&\sum_{j=1}^{J}\|\phi_{j,h}^M\|^2 + \sum_{j=1}^{J}\|\psi_{j,h}^M\|^2 + \frac{3\alpha}{4}\Delta t \sum_{j=1}^{J}\sum_{n=1}^{M}(\|\nabla\phi_{j,h}^n\|^2 + \|\nabla\psi_{j,h}^n\|^2) \\
&\qquad \leq Ce^{\frac{CT(1+\Delta t^2)}{\alpha}}(h^{2k} + h^{2k-1}\Delta t + \Delta t^2(1 + h^{2-d}) + h^{2k}\Delta t^2).
\end{aligned}
$$

Using the triangle inequality allows us to write

$$
\begin{aligned}
&\sum_{j=1}^{J}\|e_{v,j}^M\|^2 + \sum_{j=1}^{J}\|e_{w,j}^M\|^2 + \frac{3\alpha}{4}\Delta t \sum_{j=1}^{J}\sum_{n=1}^{M}(\|\nabla e_{v,j}^n\|^2 + \|\nabla e_{w,j}^n\|^2) \\
&\quad \leq 2\Big(\sum_{j=1}^{J}\|\eta_{v,j}^M\|^2 + \sum_{j=1}^{J}\|\phi_{j,h}^M\|^2 + \sum_{j=1}^{J}\|\psi_{j,h}^M\|^2 + \sum_{j=1}^{J}\|\eta_{w,j}^M\|^2 \\
&\qquad + \frac{3\alpha}{4}\Delta t \sum_{j=1}^{J}\sum_{n=1}^{M}(\|\nabla\eta_{v,j}^n\|^2 + \|\nabla\phi_{j,h}^n\|^2 + \|\nabla\psi_{j,h}^n\|^2 + \|\nabla\eta_{w,j}^n\|^2)\Big) \\
&\quad \leq 2C^*(h^{2k+2} + h^{2k}) + 2Ce^{\frac{CT(1+\Delta t^2)}{\alpha}}\Big(h^{2k} + h^{2k}\frac{\Delta t}{h} + \Delta t^2(1 + h^{2-d}) + h^{2k}\Delta t^2\Big),
\end{aligned}
$$

which implies

$$\sum_{j=1}^{J}\|e_{v,j}^M\|^2+\sum_{j=1}^{J}\|e_{w,j}^M\|^2+\frac{3\alpha}{4}\Delta t\sum_{j=1}^{J}\sum_{n=1}^{M}(\|\nabla e_{v,j}^n\|^2+\|\nabla e_{w,j}^n\|^2)$$
$$\leq Ce^{\frac{CT(1+\Delta t^2)}{\alpha}}(h^{2k}+h^{2k-1}\Delta t+\Delta t^2(1+h^{2-d})+h^{2k}\Delta t^2).$$

Using again the triangle inequality yields

$$\|\langle e_v\rangle^T\|^2+\|\langle e_w\rangle^T\|^2+\frac{3\alpha}{4}\Delta t\sum_{j=1}^{J}\sum_{n=1}^{M}(\|\nabla\langle e_v\rangle^n\|^2+\|\nabla\langle e_w\rangle^n\|^2)$$
$$\leq Ce^{\frac{CT(1+\Delta t^2)}{\alpha}}(h^{2k}+h^{2k-1}\Delta t+\Delta t^2(1+h^{2-d})+h^{2k}\Delta t^2),$$

which completes the proof. □

4 Numerical Experiments

This section presents results of numerical experiments used to test the proposed scheme and theory. In all tests, we used (P_2, P_1^{disc}) finite elements on barycenter refined, regular triangular meshes. The tuning parameter is $\mu = 1$ in all tests. The choice of $J = 4$ is made in all of our simulations; although in practice we expect much larger J, our intent here is for a first proof of concept. The codes were written in FreeFem++ [17], and since the experiments are essentially proof of concept tests in 2D, we used the UMFPACK direct solver built into FreeFem++ for the individual systems. In practice, for larger problems and especially in 3D, it is critical to make the solvers more efficient, by using block solver algorithms to simultaneously solve $Ax = b$ with multiple right-hand sides, or to reuse efficient preconditioners; see [15, 22, 23] for more discussion of this important step.

4.1 Convergence Rate Verification

Our first experiment tests the convergence rates predicted by the theory in Section 3, which proved the $L^2(0,T;H^1(\Omega))$ error to be $O(\Delta t+h^2+h^{3/2}\Delta t^{1/2})$ in two dimensions, due to our choice of elements. Provided $\Delta t < O(h)$ or $h < \Delta t^{1/3}$, the predicted error becomes $O(\Delta t+h^2)$; in our tests, we ensure these criteria are met.

We begin this test by selecting an analytical function,

$$v=\begin{pmatrix}\cos y+(1+t)\sin y\\ \sin x+(1+t)\cos x\end{pmatrix},\quad w=\begin{pmatrix}\cos y-(1+t)\sin y\\ \sin x-(1+t)\cos x\end{pmatrix},\quad p=(x-y)(1+t),\quad \lambda=0.$$

Next, we choose four perturbed solutions, which are defined by $v_j=(1\pm\epsilon)v$, $w_j=(1\pm\epsilon)w$, for $j=1,2$, and $v_j=(1\pm2\epsilon)v$, $w_j=(1\pm2\epsilon)w$, for $j=3,4$. From these perturbed solutions and the choices $\nu=0.01$, $\nu_m=0.1$, initial conditions, Dirichlet boundary conditions and right-hand side forcing terms are calculated. The ensemble scheme will be used to calculate $\langle w_h^n\rangle$ and $\langle v_h^n\rangle$, and we will compare that to the true average $\langle w(t^n)\rangle$ and $\langle v(t^n)\rangle$. Note that errors and rates for w are very similar to those of v, and are omitted.

We first test the temporal convergence. To do so, we fix $h=1/64$ and end time $T=1$, and compute solutions with varying Δt. For several choices of ϵ, we show errors and convergence rates in Table 1, and observe first order temporal convergence.

To test spatial convergence, we fix $T=0.001$ and $\Delta t=\frac{T}{8}$, and compute on varying mesh widths. Errors and rates are shown in Table 2 for several choices of ϵ, and in all cases we observe second order convergence.

	$\epsilon = 0.1$		$\epsilon = 0.01$		$\epsilon = 0.001$	
Δt	$\\|\langle v\rangle - \langle v_h\rangle\\|_{2,1}$	rate	$\\|\langle v\rangle - \langle v_h\rangle\\|_{2,1}$	rate	$\\|\langle v\rangle - \langle v_h\rangle\\|_{2,1}$	rate
$T/4$	3.241e-2		2.201e-2		2.292e-2	
$T/8$	2.340e-2	0.47	1.774e-2	0.31	1.781e-2	0.36
$T/16$	1.582e-2	0.56	9.447e-3	0.91	9.488e-3	0.91
$T/32$	9.968e-3	0.67	4.923e-3	0.94	4.944e-3	0.94
$T/64$	5.852e-3	0.77	2.517e-3	0.97	2.527e-3	0.97
$T/128$	3.238e-3	0.85	1.273e-3	0.98	1.277e-3	0.98
$T/256$	1.718e-3	0.91	6.403e-4	0.99	6.426e-4	0.99

Table 1. Temporal convergence rates for $\nu = 0.01$, $\nu_m = 0.1$, and fixed $h = 1/64$.

	$\epsilon = 0.1$		$\epsilon = 0.01$		$\epsilon = 0.001$	
h	$\\|\langle v\rangle - \langle v_h\rangle\\|_{2,1}$	rate	$\\|\langle v\rangle - \langle v_h\rangle\\|_{2,1}$	rate	$\\|\langle v\rangle - \langle v_h\rangle\\|_{2,1}$	rate
1/4	1.363e-4		1.363e-4		1.363e-4	
1/8	3.405e-5	2.00	3.408e-5	2.00	3.408e-5	2.00
1/16	8.512e-6	2.00	8.531e-6	2.00	8.531e-6	2.00
1/32	2.128e-6	2.00	2.136e-6	2.00	2.135e-6	2.00
1/64	5.320e-7	2.00	5.345e-7	2.00	5.334e-7	2.00

Table 2. Spatial convergence rates for $\nu = 0.01$, $\nu_m = 0.1$, and fixed $T = 0.001$, $\Delta t = T/8$.

4.2 MHD Channel Flow over a Step

For our second test, we consider channel flow in a 30×10 rectangular domain with a 1×1 step five units into the channel, in the presence of a magnetic field. No slip boundary conditions are enforced for velocity components and $B = \langle 0, 1\rangle^T$ is used on the walls and step, $u = \langle y(10-y)/25, 0\rangle^T$ and $B = \langle 0, 1\rangle^T$ at the inflow, and the outflow condition uses a channel extension of 10 units, and at the end of the extension we set the outflow velocity and magnetic field equal to the inflow. We set $s = 0.01$, $\nu = 1/1000$ and $\nu_m = 0.1$. The initial conditions are $u_0 = \langle y(10-y)/25, 0\rangle^T$ and $B_0 = \langle 0, 1\rangle^T$.

We consider an ensemble of four different solutions with the initial and boundary conditions perturbed by multiplicative factors of $(1 \pm \epsilon)$ and $(1 \pm 2\epsilon)$. The simulations are carried out for various choices of ϵ using Algorithm 3.1 until $T = 40$, with a time step of $\Delta t = 0.05$, and a mesh that provided 75 222 degrees of freedom. Plots of ensemble velocity solutions for varying ϵ are shown in Figure 1, and magnetic field solutions in Figure 2. For comparison, we also give results of a single run with $\epsilon = 0$ (single solution with no perturbation). As expected, we observe that as $\epsilon \to 0$, ensemble solutions appear to converge to the unperturbed solution. We also observe that the ensemble solution for all choices of ϵ matches the unperturbed solution well.

5 Conclusion

This paper represents an extension of the breakthrough idea for efficient flow ensemble calculation of Jiang and Layton [23], originally performed for Navier–Stokes, to MHD. We have developed herein an efficient algorithm for calculating ensemble averages of MHD flows. The keys to the efficiency are (i) at each time step, each of the J realizations solves linear systems with the same matrices – this means assembly needs done once instead of J times, block linear solvers can potentially be used, and preconditioners can be reused; and (ii) due to use of the Elsässer variable formulation, the linear systems at each time step are not fully coupled, but instead split into two Oseen problems, which are much easier to solve.

The algorithm is proven to be unconditionally stable with respect to the time step size, which is somewhat remarkable since the systems are split into two Oseen problems at each time step, and the schemes are such that some of the nonlinearity is treated explicitly at each time step. We also prove the method is

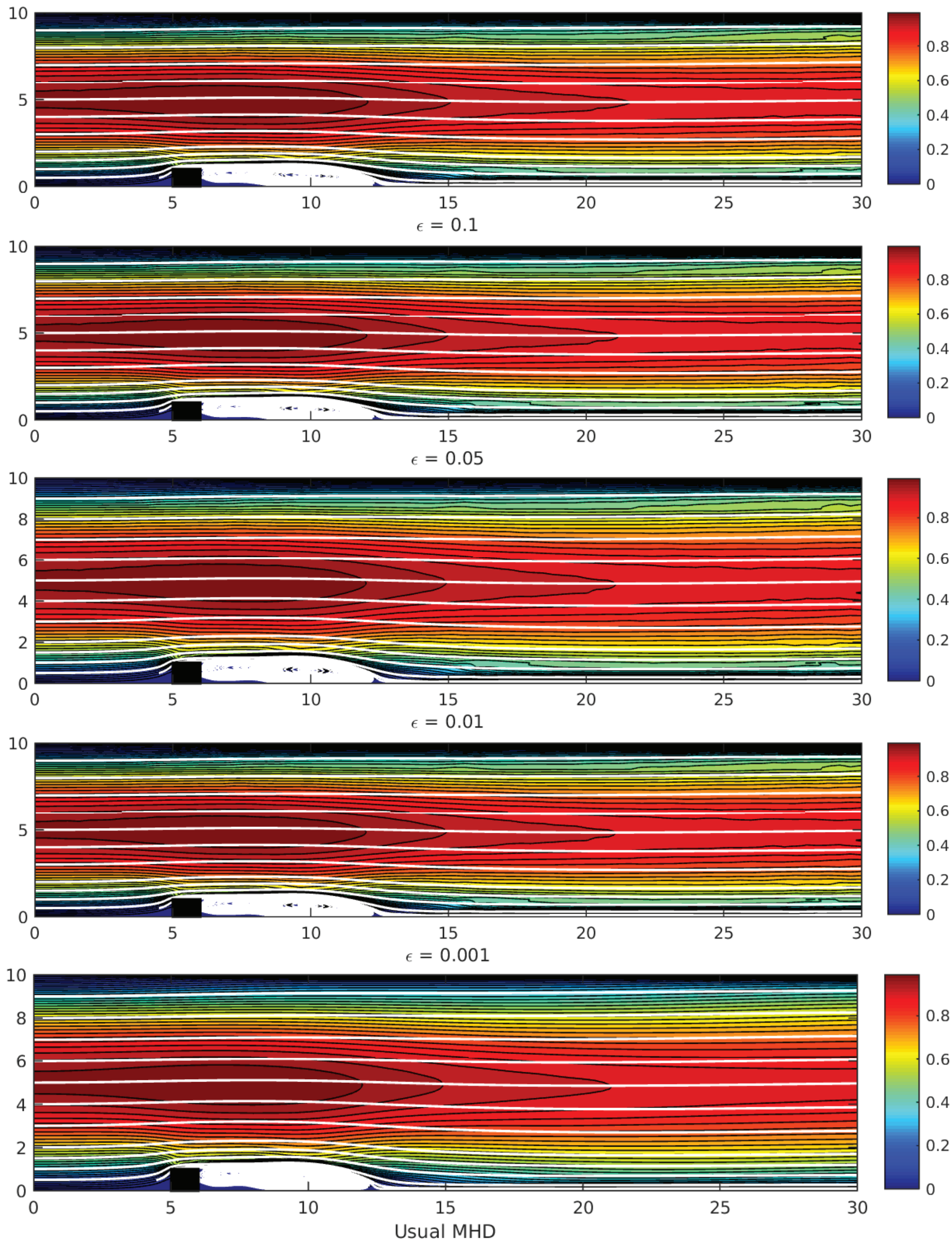

Figure 1. $T = 40$, velocity ensemble solutions (shown as streamlines over speed contours) for MHD channel flow over a step with $dt = 0.05$, $s = 0.01$ and dof $= 75222$.

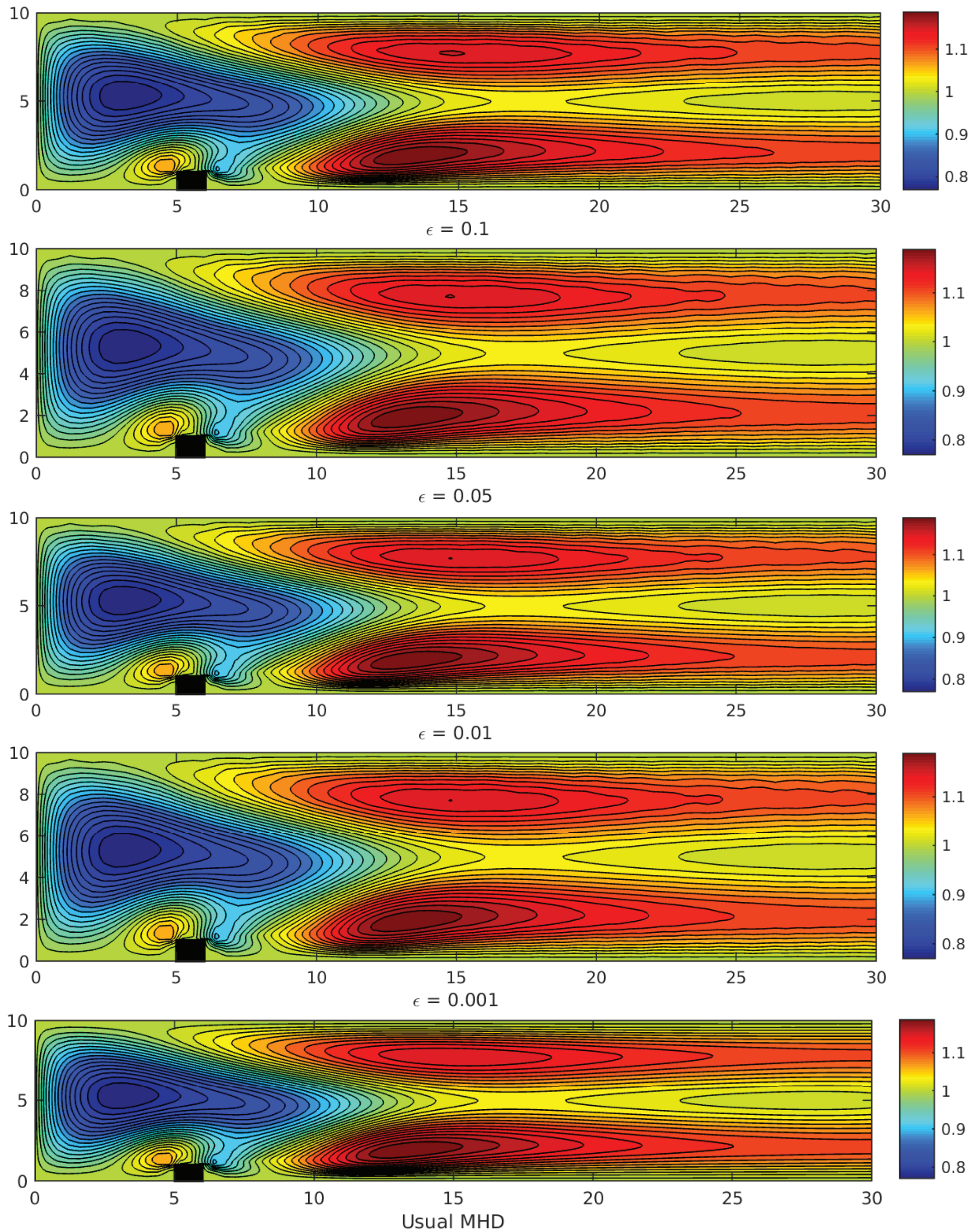

Figure 2. $T = 40$, magnitudes of ensemble magnetic field solutions (magnitude) for MHD channel flow over a step with $dt = 0.05$, $s = 0.01$ and dof = 75222.

convergent; in 2D, the convergence is optimal under very mild criteria, but in 3D the convergence is proven to be possibly suboptimal due to a $\Delta t/h^{1/2}$ term that arises (instead of Δt, which would be optimal) due to a worse 3D Sobolev embedding used in the analysis. Numerical tests were performed that verified the predicted convergence rates, and also showed the method performed well on a more physically relevant test problem.

An important next step would be to apply the ideas herein to the higher order decoupled MHD scheme proposed in [18], which would present significantly more challenges in the analysis.

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