A HIGH ORDER EFFICIENT GRAD-DIV STABILIZED ALGORITHM FOR PARAMETERIZED MAGNETOHYDRODYNAMIC FLOW ENSEMBLES SIMULATION

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Abstract.

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1. Introduction.

In this work, we consider the following set of J time-dependent, viscoresistive and incompressible dimensionless MHD equations [3, 5, 23, 31] for computing flow ensemble simulation of homogeneous Newtonian fluids:

$$\boldsymbol{u}_{j,t} + \boldsymbol{u}_j \cdot \nabla \boldsymbol{u}_j - s\boldsymbol{B}_j \cdot \nabla \boldsymbol{B}_j - \nu_j \Delta \boldsymbol{u}_j + \nabla p_j = \boldsymbol{f}_j(\boldsymbol{x}, t), \text{ in } \Omega \times (0, T], \tag{1.1}$$

$$\boldsymbol{B}_{j,t} + \boldsymbol{u}_j \cdot \nabla \boldsymbol{B}_j - \boldsymbol{B}_j \cdot \nabla \boldsymbol{u}_j - \nu_{m,j} \Delta \boldsymbol{B}_j + \nabla \lambda_j = \nabla \times \boldsymbol{g}_j(\boldsymbol{x}, t), \text{ in } \Omega \times (0, T], \tag{1.2}$$

$$\nabla \cdot \boldsymbol{u}_j = 0, \text{ in } \Omega \times (0, T], \tag{1.3}$$

$$\nabla \cdot \boldsymbol{B}_j = 0, \text{ in } \Omega \times (0, T], \tag{1.4}$$

$$\boldsymbol{u}_j(\boldsymbol{x}, 0) = \boldsymbol{u}_j^0(\boldsymbol{x}), \text{ in } \Omega, \tag{1.5}$$

$$\boldsymbol{B}_j(\boldsymbol{x}, 0) = \boldsymbol{B}_j^0(\boldsymbol{x}), \text{ in } \Omega, \tag{1.6}$$

where $\boldsymbol{u}_j$, $\boldsymbol{B}_j$, p_j, and λ_j denote the velocity, magnetic field, pressure, and artificial magnetic pressure solutions, respectively, for each $j = 1, 2, \cdots, J$, corresponding to distinct combination of kinematic viscosity ν_j, magnetic diffusivity $\nu_{m,j}$, body force $\boldsymbol{f}_j$, $\nabla \times \boldsymbol{g}_j$, and initial conditions $\boldsymbol{u}_j^0$, $\boldsymbol{B}_j^0$. The symbol Ω denotes the simulation domain (which we assume to be convex), t the time variable, $\boldsymbol{x}$ the spatial variable and T the simulation time. The coupling number s is the coefficient of the Lorentz force into the momentum equation (1.1). For simplicity of our analysis, we consider homogeneous Dirichlet boundary conditions.

Input data, e.g., initial and boundary conditions, viscosities, and body forces have a significant effect on simulations of complex dynamical systems, but the involvement of uncertainty in their measurements reduces the accuracy of final solutions. For a robust and high fidelity solution, computation of ensemble average solution is popular in many applications such as surface data assimilation [9], magnetohydrodynamics [21], porous media flow [20], weather forecasting [24, 27], spectral methods [28], sensitivity analyses [29], and hydrology [33]. Computing a quantity of interest by running a simulation subject to the ensemble average of a particular input data is not always the same as computing the ensemble average of the quantity of interest running the simulations for all different realizations of the input data first and then taking their average [11].

Computing long-time simulations of fully coupled MHD ensemble systems is computationally arduous and expensive. Therefore, decoupled algorithms which can reuse the global system matrix at each time-step for all J realizations are computationally attractive. First-order time-stepping partitioned algorithms with small time-step restrictions are studied at low magnetic Reynolds number in a reduced MHD system in [21]. Decoupled, and unconditionally stable algorithm for the evolutionary full MHD ensemble system in Elsässer variables are investigated in [31].

Viscosity parameters are the most important and sensitive input data, as they determine the flow characteristics. For example, as the Reynolds number $Re := UL/\nu$ grows, the laminar flow moves into a convective dominated regime and eventually becomes turbulent [37]. The situation is more complex in MHD flow with high magnetic Reynolds number $Re_m := UL/\nu_m$. Here, the contribution of the nonlinearity dominates the flow's development and evolution. Thus, for an accurate simulation, it is important to accurately account for their uncertainties. The above-mentioned MHD ensemble works

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[21, 31] were done assuming uncertainties only on the initial and boundary conditions, and forcing functions; no uncertainties are considered on the viscosity coefficients. In this paper, we propose an algorithm for the MHD flow ensemble in which not only the initial and boundary data, and forcing functions, but also the kinematic viscosity and magnetic diffusivity parameters are different from one ensemble member to another.

The proposed algorithm is based on Elsässer variables [6, 7, 25]. Recent studies show that instead of solving coupled MHD systems in primitive variables, using instead Elsässer variables can provide a decoupled stable MHD simulation algorithm, [1, 13, 30, 31, 34, 35]. Defining $\boldsymbol{v}_j := \boldsymbol{u}_j + \sqrt{s}\boldsymbol{B}_j$, $\boldsymbol{w}_j := \boldsymbol{u}_j - \sqrt{s}\boldsymbol{B}_j$, $\boldsymbol{f}_{1,j} := \boldsymbol{f}_j + \sqrt{s}\nabla\times\boldsymbol{g}_j$, $\boldsymbol{f}_{2,j} := \boldsymbol{f}_j - \sqrt{s}\nabla\times\boldsymbol{g}_j$, $q_j := p_j + \sqrt{s}\lambda_j$ and $r_j := p_j - \sqrt{s}\lambda_j$ produces the Elsässer variable formulation of the ensemble systems:

$$\boldsymbol{v}_{j,t} + \boldsymbol{w}_j\cdot\nabla\boldsymbol{v}_j - \frac{\nu_j+\nu_{m,j}}{2}\Delta\boldsymbol{v}_j - \frac{\nu_j-\nu_{m,j}}{2}\Delta\boldsymbol{w}_j + \nabla q_j = \boldsymbol{f}_{1,j}, \tag{1.7}$$
$$\boldsymbol{w}_{j,t} + \boldsymbol{v}_j\cdot\nabla\boldsymbol{w}_j - \frac{\nu_j+\nu_{m,j}}{2}\Delta\boldsymbol{w}_j - \frac{\nu_j-\nu_{m,j}}{2}\Delta\boldsymbol{v}_j + \nabla r_j = \boldsymbol{f}_{2,j}, \tag{1.8}$$
$$\nabla\cdot\boldsymbol{v}_j = \nabla\cdot\boldsymbol{w}_j = 0, \tag{1.9}$$

together with the initial and boundary conditions.

To reduce the immense computational cost for the above ensemble Elsässer variables system (1.7)-(1.9), we propose a decoupled scheme following the breakthrough idea from [18]. The decoupled feature enables the algorithm to solve two identical Oseen-type [38] sub-problems *simultaneously* at each time-step. Moreover, following the breakthrough idea [18], the two identical sub-problems are posed in such a way that, at each time-step the finite element assembly of the system provides the same coefficient matrix (which is independent of j) for each of the J realizations. Hence, to solve for the next time-step, one solves the following system of equations of the form $A[\boldsymbol{x}_1|\boldsymbol{x}_2|\cdots|\boldsymbol{x}_J] = [\boldsymbol{b}_1|\boldsymbol{b}_2|\cdots|\boldsymbol{b}_J]$. Therefore, a massive amount of computer memory is saved, and system matrix assembly and factorization/preconditioner are needed only once per time-step. Moreover, the algorithm can take advantage of block linear solvers [22]. This idea in [18] has been implemented for the solution of the heat equation with uncertain temperature-dependent conductivity [8], Navier-Stokes simulations [15, 16, 19, 32], magnetohydrodynamics [21, 31], parameterized flow problems [12, 26], and turbulence modeling [17].

The proposed efficient ensemble scheme is proven to be stable and convergent without any time-step restriction, and handles uncertainties in all input data. To the best of our knowledge, this scheme is novel for the uncertainty quantification of MHD flow ensembles.

The rest of the paper is organized as follows: To follow a smooth analysis, we provide necessary notations and mathematical preliminaries in Section 2. In Section 3, we present the proposed implicit-explicit efficient ensemble scheme, define the ensemble averages, and finally provide a corresponding fully discrete and decoupled algorithm. We analyze the discrete algorithm, state, and prove its stability and well-posedness in Section 4. To provide an error estimate, we state and prove the convergence theorem in Section **??**. To support the theoretical analysis, we compute the convergence rates, check the energy stability of the scheme, and test the scheme on benchmark problems in Section **??**. Finally, conclusions and future research avenues are given in Section **??**.

2. Notation and preliminaries.

Let $\Omega\subset\mathbb{R}^d$ $(d=2,3)$ be a convex polygonal or polyhedral domain with boundary $\partial\Omega$. The usual $L^2(\Omega)$ norm and inner product are denoted by $\|.\|$ and $(.,.)$, respectively. Similarly, the $L^p(\Omega)$ norms and the Sobolev $W_p^k(\Omega)$ norms are $\|.\|_{L^p}$ and $\|.\|_{W_p^k}$, respectively for $k\in\mathbb{N}$, $1\le p\le\infty$. The Sobolev space $W_2^k(\Omega)$ is represented by $H^k(\Omega)$ with norm $\|.\|_k$. The vector-valued spaces are

$$\boldsymbol{L}^p(\Omega) = (L^p(\Omega))^d, \text{ and } \boldsymbol{H}^k(\Omega) = (H^k(\Omega))^d.$$

For $\boldsymbol{X}$ being a normed function space in Ω, $L^p(0,T;\boldsymbol{X})$ is the space of all functions defined on $(0,T]\times\Omega$ for which the following norm

$$\|\boldsymbol{u}\|_{L^p(0,T;\boldsymbol{X})} = \left(\int_0^T \|\boldsymbol{u}\|_{\boldsymbol{X}}^p dt\right)^{\frac{1}{p}}, \quad p\in[1,\infty)$$

is finite. For $p = \infty$, the usual modification is used in the definition of this space. The natural function spaces for our problem are

$$\begin{aligned} \boldsymbol{X} &:= \boldsymbol{H}_0^1(\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega) : \nabla \boldsymbol{v} \in L^2(\Omega)^{d\times d}, \boldsymbol{v} = \boldsymbol{0} \text{ on } \partial\Omega\}, \\ Q &:= L_0^2(\Omega) = \{q \in L^2(\Omega) : \int_\Omega q \, d\boldsymbol{x} = 0\}. \end{aligned}$$

Recall the Poincaré inequality holds in $\boldsymbol{X}$: There exists C depending only on Ω satisfying for all $\boldsymbol{\varphi} \in \boldsymbol{X}$,

$$\|\boldsymbol{\varphi}\| \le C\|\nabla\boldsymbol{\varphi}\|.$$

The divergence-free velocity space is given by

$$\boldsymbol{V} := \{\boldsymbol{v} \in \boldsymbol{X} : (\nabla\cdot\boldsymbol{v}, q) = 0, \forall q \in Q\}.$$

We define the explicitly skew symmetrized trilinear form $b^* : \boldsymbol{X}\times\boldsymbol{X}\times\boldsymbol{X} \to \mathbb{R}$ by

$$b^*(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) := \frac{1}{2}\big((\boldsymbol{u}\cdot\nabla\boldsymbol{v},\boldsymbol{w}) - (\boldsymbol{u}\cdot\nabla\boldsymbol{w},\boldsymbol{v})\big).$$

Note that $b^*(\boldsymbol{u},\boldsymbol{v},\boldsymbol{v}) = 0$, and if $\|\nabla\cdot\boldsymbol{u}\| = 0$, then $(\boldsymbol{u}\cdot\nabla\boldsymbol{v},\boldsymbol{w}) = b^*(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})$. Moreover, $b^*(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})$ satisfies the following bound [10]:

$$|b^*(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})| \le C(\Omega)\|\nabla\boldsymbol{u}\|\|\nabla\boldsymbol{v}\|\|\nabla\boldsymbol{w}\|, \text{ for any } \boldsymbol{u},\boldsymbol{v},\boldsymbol{w} \in \boldsymbol{X}. \tag{2.1}$$

The conforming finite element spaces are denoted by $\boldsymbol{X}_h \subset \boldsymbol{X}$ and $Q_h \subset Q$, and we assume a regular triangulation $\tau_h(\Omega)$, where h is the maximum triangle diameter. We assume that $(\boldsymbol{X}_h, Q_h)$ satisfies the usual discrete *inf-sup* condition

$$\inf_{q_h\in Q_h}\sup_{\boldsymbol{v}_h\in\boldsymbol{X}_h}\frac{(q_h,\nabla\cdot\boldsymbol{v}_h)}{\|q_h\|\|\nabla\boldsymbol{v}_h\|} \ge \beta > 0, \tag{2.2}$$

where β is independent of h. The space of discretely divergence-free functions is defined as

$$\boldsymbol{V}_h := \{\boldsymbol{v}_h \in \boldsymbol{X}_h : (\nabla\cdot\boldsymbol{v}_h, q_h) = 0, \ \ \forall q_h \in Q_h\}.$$

For simplicity of our analysis, we will use the Scott-Vogelius (SV) finite element pair $(\boldsymbol{X}_h, Q_h) = \big((P_k)^d, P_{k-1}^{disc}\big)$, which satisfies the *inf-sup* condition under certain conditions, such as when the mesh is created as a barycenter refinement of a regular mesh and the polynomial degree $k \ge d$ [2, 36]. Our analysis can be extended without difficulty to any *inf-sup* stable element choice, although with minor additional technical detail. Since the SV element is point-wise divergence free, the discrete pressure terms vanish from the analysis. But if we use a finite element pair which is only weakly divergence free, for example, the Taylor-Hood element, then we must need to handle the pressure involving terms in the analysis. Since, in this case, the orthogonal property of the discrete pressure and velocity spaces may not hold true.

We have the following approximation properties in $(\boldsymbol{X}_h, Q_h)$: [4]

$$\inf_{\boldsymbol{v}_h\in\boldsymbol{X}_h}\|\boldsymbol{u}-\boldsymbol{v}_h\| \le Ch^{k+1}|\boldsymbol{u}|_{k+1}, \quad \boldsymbol{u}\in\boldsymbol{H}^{k+1}(\Omega), \tag{2.3}$$

$$\inf_{\boldsymbol{v}_h\in\boldsymbol{X}_h}\|\nabla(\boldsymbol{u}-\boldsymbol{v}_h)\| \le Ch^{k}|\boldsymbol{u}|_{k+1}, \quad \boldsymbol{u}\in\boldsymbol{H}^{k+1}(\Omega), \tag{2.4}$$

$$\inf_{q_h\in Q_h}\|p-q_h\| \le Ch^{k}|p|_{k}, \quad p\in H^{k}(\Omega), \tag{2.5}$$

where $|\cdot|_r$ denotes the H^r or $\boldsymbol{H}^r$ seminorm.

We will assume the mesh is sufficiently regular for the inverse inequality to hold, and with this and the *inf-sup* assumption, we have approximation properties

$$\|\nabla(\boldsymbol{u} - P_{\boldsymbol{V}_h}^{L^2}(\boldsymbol{u}))\| \le Ch^{k}|\boldsymbol{u}|_{k+1}, \quad \boldsymbol{u}\in\boldsymbol{H}^{k+1}(\Omega), \tag{2.6}$$

$$\inf_{\boldsymbol{v}_h\in\boldsymbol{V}_h}\|\nabla(\boldsymbol{u}-\boldsymbol{v}_h)\| \le Ch^{k}|\boldsymbol{u}|_{k+1}, \quad \boldsymbol{u}\in\boldsymbol{H}^{k+1}(\Omega), \tag{2.7}$$

where $P_{\boldsymbol{V}_h}^{L^2}(\boldsymbol{u})$ is the L^2 projection of $\boldsymbol{u}$ into $\boldsymbol{V}_h$.

The following lemma for the discrete Grönwall inequality was given in [14].

Lemma 2.1. *Let Δt, $\mathcal{D}$, a_n, b_n, c_n, d_n be non-negative numbers for $n = 1, \cdots, M$ such that*

$$a_M + \Delta t \sum_{n=1}^{M} b_n \leq \Delta t \sum_{n=1}^{M-1} d_n a_n + \Delta t \sum_{n=1}^{M} c_n + \mathcal{D} \quad \textit{for} \;\; M \in \mathbb{N},$$

then for all $\Delta t > 0$,

$$a_M + \Delta t \sum_{n=1}^{M} b_n \leq exp\left(\Delta t \sum_{n=1}^{M-1} d_n\right)\left(\Delta t \sum_{n=1}^{M} c_n + \mathcal{D}\right) \quad \textit{for} \;\; M \in \mathbb{N}.$$

3. The implicit-explicit ensemble scheme.

We suppress the spatial discretization momentarily to keep the focus on the main idea. We consider a uniform time-step size Δt and let $t^n = n\Delta t$ for $n = 0, 1, \cdots$, then computing the J solutions independently, takes the following form:

Step 1: For $j = 1, \cdots, J$,

$$\begin{aligned}
\frac{3\boldsymbol{v}_j^{n+1}}{2\Delta t} &+ <\boldsymbol{w}>^n \cdot \nabla \boldsymbol{v}_j^{n+1} - \frac{\bar{\nu}+\bar{\nu}_m}{2}\Delta \boldsymbol{v}_j^{n+1} + \gamma \nabla(\nabla\cdot \boldsymbol{v}_j^{n+1}) + \nabla q_j^{n+1} = \boldsymbol{f}_{1,j}(t^{n+1}) \\
&+ \frac{4\boldsymbol{v}_j^n - \boldsymbol{v}_j^{n-1}}{2\Delta t} - \boldsymbol{w}_j^{'n}\cdot\nabla(2\boldsymbol{v}_j^n - \boldsymbol{v}_j^{n-1}) + \frac{\nu_j^{'} + \nu_{m,j}^{'}}{2}\Delta(2\boldsymbol{v}_j^n - \boldsymbol{v}_j^{n-1}) \\
&+ (1-\theta)\frac{\nu_j - \nu_{m,j}}{2}\Delta \boldsymbol{w}_j^n + \theta \frac{\nu_j - \nu_{m,j}}{2}\Delta(2\boldsymbol{w}_j^n - \boldsymbol{w}_j^{n-1}),
\end{aligned} \tag{3.1}$$

$$\nabla \cdot \boldsymbol{v}_j^{n+1} = 0. \tag{3.2}$$

Step 2: For $j = 1, \cdots, J$,

$$\begin{aligned}
\frac{3\boldsymbol{w}_j^{n+1}}{2\Delta t} &+ <\boldsymbol{v}>^n \cdot \nabla \boldsymbol{w}_j^{n+1} - \frac{\bar{\nu}+\bar{\nu}_m}{2}\Delta \boldsymbol{w}_j^{n+1} + \gamma \nabla(\nabla\cdot \boldsymbol{w}_j^{n+1}) + \nabla r_j^{n+1} = \boldsymbol{f}_{2,j}(t^{n+1}) \\
&+ \frac{4\boldsymbol{w}_j^n - \boldsymbol{w}_j^{n-1}}{2\Delta t} - \boldsymbol{v}_j^{'n}\cdot\nabla(2\boldsymbol{w}_j^n - \boldsymbol{w}_j^{n-1}) + \frac{\nu_j^{'} + \nu_{m,j}^{'}}{2}\Delta(2\boldsymbol{w}_j^n - \boldsymbol{w}_j^{n-1}) \\
&+ (1-\theta)\frac{\nu_j - \nu_{m,j}}{2}\Delta \boldsymbol{v}_j^n + \theta \frac{\nu_j - \nu_{m,j}}{2}\Delta(2\boldsymbol{v}_j^n - \boldsymbol{v}_j^{n-1}),
\end{aligned} \tag{3.3}$$

$$\nabla \cdot \boldsymbol{w}_j^{n+1} = 0. \tag{3.4}$$

Here, $\boldsymbol{v}_j^n, \boldsymbol{w}_j^n, q_j^n$, and r_j^n denote approximations of $\boldsymbol{v}_j(\cdot, t^n), \boldsymbol{w}_j(\cdot, t^n), q_j(\cdot, t^n)$, and $r_j(\cdot, t^n)$, respectively. The ensemble average and fluctuation about the ensemble average are defined as follows:

$$\left.\begin{aligned}
<\boldsymbol{z}>^n :&= \frac{1}{J}\sum_{j=1}^{J}(2\boldsymbol{z}_j^n - \boldsymbol{z}_j^{n-1}), \quad \boldsymbol{z}_j^{'n} := 2\boldsymbol{z}_j^n - \boldsymbol{z}_j^{n-1} - <\boldsymbol{z}>^n, \\
\bar{\nu} :&= \frac{1}{J}\sum_{j=1}^{J}\nu_j, \quad \nu_j^{'} := \nu_j - \bar{\nu}, \\
\bar{\nu}_m :&= \frac{1}{J}\sum_{j=1}^{J}\nu_{m,j}, \quad \nu_{m,j}^{'} := \nu_{m,j} - \bar{\nu}_m.
\end{aligned}\right\} \tag{3.5}$$

We observe, at time $t = t^{n+1}$, the sub-problem (3.1)-(3.2) has unknowns $\boldsymbol{v}_j^{n+1}$, and q_j^{n+1} and on the other hand, the sub-problem (3.3)-(3.4) has unknowns $\boldsymbol{w}_j^{n+1}$, and r_j^{n+1}. Thus, the two sub-problems are decoupled and can be solved simultaneously. Moreover, the coefficient of each unknown in the two sub-problems does not depend on j, which allows having the same system matrix for all of the j realizations at each time-step after the finite element assembly. Consequently, we only need to solve a single linear system with J different right-hand-side vectors at each time-step. Therefore, in contrast to solving J different simulations independently, we only need a single LU decomposition or its variant if a direct solver is possible to use and a single preconditioner is needed to build if a block-iterative solver is used. This allows us to reduce a massive computational complexity for simulating convective-dominated MHD ensemble flow problems with low variability.

3.1. Fully discrete scheme. Using a finite element spatial discretization, we investigate the proposed decoupled ensemble scheme (3.1)-(3.4) in a fully discrete setting. Following the definitions in (3.5), we define the discrete ensemble average solutions as

$$<\boldsymbol{v}_h>^n:=\frac{1}{J}\sum_{j=1}^{J}(2\boldsymbol{v}_{j,h}^n-\boldsymbol{v}_{j,h}^{n-1}),\;\; <\boldsymbol{w}_h>^n:=\frac{1}{J}\sum_{j=1}^{J}(2\boldsymbol{w}_{j,h}^n-\boldsymbol{w}_{j,h}^{n-1}).$$

The fully discrete, and decoupled time-stepping scheme for computing MHD flow ensembles is defined below.

Algorithm 1: Fully discrete and decoupled ensemble scheme

Given time-step $\Delta t>0$, end time $T>0$, initial conditions $\boldsymbol{v}_j^0,\boldsymbol{w}_j^0,\boldsymbol{v}_j^1,\boldsymbol{w}_j^1\in\boldsymbol{H}^2\cup\boldsymbol{V}$ and $\boldsymbol{f}_{1,j},\boldsymbol{f}_{2,j}\in L^\infty\left(0,T;\boldsymbol{H}^{-1}(\Omega)\right)$ for $j=1,2,\cdots,J$. Set $M=T/\Delta t$ and for $n=1,\cdots,M-1$, compute: Find $\boldsymbol{v}_{j,h}^{n+1}\in\boldsymbol{V}_h$ satisfying, for all $\boldsymbol{\chi}_{j,h}\in\boldsymbol{V}_h$:

$$\begin{aligned}
&\left(\frac{3\boldsymbol{v}_{j,h}^{n+1}-4\boldsymbol{v}_{j,h}^n+\boldsymbol{v}_{j,h}^{n-1}}{2\Delta t},\boldsymbol{\chi}_{j,h}\right)+b^*\left(<\boldsymbol{w}_h>^n,\boldsymbol{v}_{j,h}^{n+1},\boldsymbol{\chi}_{j,h}\right)+b^*\left(\boldsymbol{w}_{j,h}^{'n},2\boldsymbol{v}_{j,h}^n-\boldsymbol{v}_{j,h}^{n-1},\boldsymbol{\chi}_{j,h}\right)\\
&+\frac{\bar\nu+\bar\nu_m}{2}\left(\nabla\boldsymbol{v}_{j,h}^{n+1},\nabla\boldsymbol{\chi}_{j,h}\right)+\frac{\nu_j^{'}+\nu_{m,j}^{'}}{2}\left(\nabla(2\boldsymbol{v}_{j,h}^n-\boldsymbol{v}_{j,h}^{n-1}),\nabla\boldsymbol{\chi}_{j,h}\right)\\
&+(1-\theta)\frac{\nu_j-\nu_{m,j}}{2}\left(\nabla\boldsymbol{w}_{j,h}^n,\nabla\boldsymbol{\chi}_{j,h}\right)+\theta\frac{\nu_j-\nu_{m,j}}{2}\left(\nabla(2\boldsymbol{w}_{j,h}^n-\boldsymbol{w}_{j,h}^{n-1}),\nabla\boldsymbol{\chi}_{j,h}\right)\\
&+\gamma\left(\nabla\cdot\boldsymbol{v}_{j,h}^{n+1},\nabla\cdot\boldsymbol{\chi}_{j,h}\right)=\left(\boldsymbol{f}_{1,j}(t^{n+1}),\boldsymbol{\chi}_{j,h}\right).
\end{aligned}\tag{3.6}$$

Find $\boldsymbol{w}_{j,h}^{n+1}\in\boldsymbol{V}_h$ satisfying, for all $\boldsymbol{l}_{j,h}\in\boldsymbol{V}_h$:

$$\begin{aligned}
&\left(\frac{3\boldsymbol{w}_{j,h}^{n+1}-4\boldsymbol{w}_{j,h}^n+\boldsymbol{w}_{j,h}^{n-1}}{2\Delta t},\boldsymbol{l}_{j,h}\right)+b^*\left(<\boldsymbol{v}_h>^n,\boldsymbol{w}_{j,h}^{n+1},\boldsymbol{l}_{j,h}\right)+b^*\left(\boldsymbol{v}_{j,h}^{'n},2\boldsymbol{w}_{j,h}^n-\boldsymbol{w}_{j,h}^{n-1},\boldsymbol{l}_{j,h}\right)\\
&+\frac{\bar\nu+\bar\nu_m}{2}\left(\nabla\boldsymbol{w}_{j,h}^{n+1},\nabla\boldsymbol{l}_{j,h}\right)+\frac{\nu_j^{'}+\nu_{m,j}^{'}}{2}\left(\nabla(2\boldsymbol{w}_{j,h}^n-\boldsymbol{w}_{j,h}^{n-1}),\nabla\boldsymbol{l}_{j,h}\right)\\
&+(1-\theta)\frac{\nu_j-\nu_{m,j}}{2}\left(\nabla\boldsymbol{v}_{j,h}^n,\nabla\boldsymbol{l}_{j,h}\right)+\theta\frac{\nu_j-\nu_{m,j}}{2}\left(\nabla(2\boldsymbol{v}_{j,h}^n-\boldsymbol{v}_{j,h}^{n-1}),\nabla\boldsymbol{l}_{j,h}\right)\\
&+\gamma\left(\nabla\cdot\boldsymbol{w}_{j,h}^{n+1},\nabla\cdot\boldsymbol{l}_{j,h}\right)=\left(\boldsymbol{f}_{2,j}(t^{n+1}),\boldsymbol{l}_{j,h}\right).
\end{aligned}\tag{3.7}$$

4. Stability analysis and well-posedness.

We now prove stability and well-posedness for the Algorithm 1. To simplify the notation, denote $\alpha_j:=\bar\nu+\bar\nu_m-|\nu_j-\nu_{m,j}|(1+2\theta)-3|\nu_j^{'}+\nu_{m,j}^{'}|>0$, for $j=1,2,\cdots,J$, which will allow us to choose the biggest possible θ on

$$\frac{\theta}{1+\theta}<\frac{\bar\nu}{\bar\nu_m}<\frac{1+\theta}{\theta}.$$

Theorem 4.1. *Suppose* $\boldsymbol{f}_{1,j}, \boldsymbol{f}_{2,j} \in L^{\infty}\left(0,T;\boldsymbol{H}^{-1}(\Omega)\right)$, *and* $\boldsymbol{v}_{j,h}^0$, $\boldsymbol{w}_{j,h}^0 \in \boldsymbol{H}^1(\Omega)$, *then the solutions to the Algorithm 1 are stable: For* $\Delta t \leq \frac{\alpha_{min} h^2}{C \max\limits_{1\leq j\leq J}\left\{\|\nabla \boldsymbol{v}'^{n}_{j,h}\|^2, \|\nabla \boldsymbol{w}'^{n}_{j,h}\|^2\right\}}$, *if* $\alpha_j > 0$

$$
\begin{aligned}
&\|\boldsymbol{v}_{j,h}^M\|^2 + \|2\boldsymbol{v}_{j,h}^M - \boldsymbol{v}_{j,h}^{M-1}\|^2 + \|\boldsymbol{w}_{j,h}^M\|^2 + \|2\boldsymbol{w}_{j,h}^M - \boldsymbol{w}_{j,h}^{M-1}\|^2 \\
&+ \alpha_j \Delta t \sum_{n=2}^{M}\Big(\|\nabla \boldsymbol{v}_{j,h}^n\|^2 + \|\nabla \boldsymbol{w}_{j,h}^n\|^2\Big) + 4\gamma\Delta t \sum_{n=2}^{M}\Big(\|\nabla\cdot \boldsymbol{v}_{j,h}^n\|^2 + \|\nabla\cdot \boldsymbol{w}_{j,h}^n\|^2\Big) \\
&\leq \|\boldsymbol{v}_{j,h}^0\|^2 + \|\boldsymbol{w}_{j,h}^0\|^2 + \|2\boldsymbol{v}_{j,h}^1 - \boldsymbol{v}_{j,h}^0\|^2 + \|2\boldsymbol{w}_{j,h}^1 - \boldsymbol{w}_{j,h}^0\|^2 \\
&+ (\bar{\nu} + \bar{\nu}_m)\Delta t\Big(\|\nabla \boldsymbol{v}_{j,h}^1\|^2 + \|\nabla \boldsymbol{w}_{j,h}^1\|^2 + \|\nabla \boldsymbol{v}_{j,h}^0\|^2 + \|\nabla \boldsymbol{w}_{j,h}^0\|^2\Big) \\
&+ \frac{8\Delta t}{\alpha_j}\sum_{n=1}^{M-1}\Big(\|\boldsymbol{f}_{1,j}(t^{n+1})\|_{-1}^2 + \|\boldsymbol{f}_{2,j}(t^{n+1})\|_{-1}^2\Big).
\end{aligned} \tag{4.1}
$$

Proof. The proof can be considered as 3 steps: In order to get the solution norms, we choose the test functions in the first step, in the second step we find the upper bounds of the terms on the right-hand-side, and then finally we combine the estimates, reduces terms and completes the proof as follows:

Step 1: Choose test functions to get solution norms. Choose $\boldsymbol{\chi}_{j,h} = \boldsymbol{v}_{j,h}^{n+1}$ in (3.6), and $\boldsymbol{l}_{j,h} = \boldsymbol{w}_{j,h}^{n+1}$ in (3.7), respectively to obtain

$$
\begin{aligned}
&\left(\frac{3\boldsymbol{v}_{j,h}^{n+1} - 4\boldsymbol{v}_{j,h}^n + \boldsymbol{v}_{j,h}^{n-1}}{2\Delta t}, \boldsymbol{v}_{j,h}^{n+1}\right) + b^*\left(\boldsymbol{w}'^{n}_{j,h}, 2\boldsymbol{v}_{j,h}^n - \boldsymbol{v}_{j,h}^{n-1}, \boldsymbol{v}_{j,h}^{n+1}\right) + \frac{\bar{\nu}+\bar{\nu}_m}{2}\|\nabla \boldsymbol{v}_{j,h}^{n+1}\|^2 \\
&+\gamma\|\nabla\cdot \boldsymbol{v}_{j,h}^{n+1}\|^2 = \left(\boldsymbol{f}_{1,j}(t^{n+1}), \boldsymbol{v}_{j,h}^{n+1}\right) - \frac{\nu_j' + \nu_{m,j}'}{2}\left(\nabla(2\boldsymbol{v}_{j,h}^n - \boldsymbol{v}_{j,h}^{n-1}), \nabla \boldsymbol{v}_{j,h}^{n+1}\right) \\
&-(1-\theta)\frac{\nu_j - \nu_{m,j}}{2}\left(\nabla \boldsymbol{w}_{j,h}^n, \nabla \boldsymbol{v}_{j,h}^{n+1}\right) - \theta\frac{\nu_j - \nu_{m,j}}{2}\left(\nabla(2\boldsymbol{w}_{j,h}^n - \boldsymbol{w}_{j,h}^{n-1}), \nabla \boldsymbol{v}_{j,h}^{n+1}\right),
\end{aligned} \tag{4.2}
$$

and

$$
\begin{aligned}
&\left(\frac{3\boldsymbol{w}_{j,h}^{n+1} - 4\boldsymbol{w}_{j,h}^n + \boldsymbol{w}_{j,h}^{n-1}}{2\Delta t}, \boldsymbol{w}_{j,h}^{n+1}\right) + b^*\left(\boldsymbol{v}'^{n}_{j,h}, 2\boldsymbol{w}_{j,h}^n - \boldsymbol{w}_{j,h}^{n-1}, \boldsymbol{w}_{j,h}^{n+1}\right) + \frac{\bar{\nu}+\bar{\nu}_m}{2}\|\nabla \boldsymbol{w}_{j,h}^{n+1}\|^2 \\
&+\gamma\|\nabla\cdot \boldsymbol{w}_{j,h}^{n+1}\|^2 = \left(\boldsymbol{f}_{2,j}(t^{n+1}), \boldsymbol{w}_{j,h}^{n+1}\right) - \frac{\nu_j' + \nu_{m,j}'}{2}\left(\nabla(2\boldsymbol{w}_{j,h}^n - \boldsymbol{w}_{j,h}^{n-1}), \nabla \boldsymbol{w}_{j,h}^{n+1}\right) \\
&-(1-\theta)\frac{\nu_j - \nu_{m,j}}{2}\left(\nabla \boldsymbol{v}_{j,h}^n, \nabla \boldsymbol{w}_{j,h}^{n+1}\right) - \theta\frac{\nu_j - \nu_{m,j}}{2}\left(\nabla(2\boldsymbol{v}_{j,h}^n - \boldsymbol{v}_{j,h}^{n-1}), \nabla \boldsymbol{w}_{j,h}^{n+1}\right).
\end{aligned} \tag{4.3}
$$

Using the following identity

$$
(3a - 4b + c, a) = \frac{a^2 + (2a-b)^2}{2} - \frac{b^2 + (2b-c)^2}{2} + \frac{(a-2b+c)^2}{2}, \tag{4.4}
$$

we write

$$
\begin{aligned}
&\frac{1}{4\Delta t}\Big(\|\boldsymbol{v}_{j,h}^{n+1}\|^2 - \|\boldsymbol{v}_{j,h}^n\|^2 + \|2\boldsymbol{v}_{j,h}^{n+1} - \boldsymbol{v}_{j,h}^n\|^2 - \|2\boldsymbol{v}_{j,h}^n - \boldsymbol{v}_{j,h}^{n-1}\|^2 + \|\boldsymbol{v}_{j,h}^{n+1} - 2\boldsymbol{v}_{j,h}^n + \boldsymbol{v}_{j,h}^{n-1}\|^2\Big) \\
&\qquad + b^*\left(\boldsymbol{w}'^{n}_{j,h}, 2\boldsymbol{v}_{j,h}^n - \boldsymbol{v}_{j,h}^{n-1}, \boldsymbol{v}_{j,h}^{n+1}\right) + \frac{\bar{\nu}+\bar{\nu}_m}{2}\|\nabla \boldsymbol{v}_{j,h}^{n+1}\|^2 + \gamma\|\nabla\cdot \boldsymbol{v}_{j,h}^{n+1}\|^2 \\
&\qquad = (\boldsymbol{f}_{1,j}(t^{n+1}), v_{j,h}^{n+1}) - \frac{\nu_j - \nu_{m,j}}{2}\left((1+\theta)\nabla \boldsymbol{w}_{j,h}^n - \theta\nabla \boldsymbol{w}_{j,h}^{n-1}, \nabla \boldsymbol{v}_{j,h}^{n+1}\right) \\
&\qquad - \frac{\nu_j' + \nu_{m,j}'}{2}\left(\nabla(2\boldsymbol{v}_{j,h}^n - \boldsymbol{v}_{j,h}^{n-1}), \nabla \boldsymbol{v}_{j,h}^{n+1}\right),
\end{aligned} \tag{4.5}
$$

and

$$\begin{aligned}
&\frac{1}{4\Delta t}\Big(\|\boldsymbol{w}_{j,h}^{n+1}\|^2-\|\boldsymbol{w}_{j,h}^{n}\|^2+\|2\boldsymbol{w}_{j,h}^{n+1}-\boldsymbol{w}_{j,h}^{n}\|^2-\|2\boldsymbol{w}_{j,h}^{n}-\boldsymbol{w}_{j,h}^{n-1}\|^2+\|\boldsymbol{w}_{j,h}^{n+1}-2\boldsymbol{w}_{j,h}^{n}+\boldsymbol{w}_{j,h}^{n-1}\|^2\Big)\\
&\qquad+b^*\left(\boldsymbol{v}_{j,h}^{'n},2\boldsymbol{w}_{j,h}^{n}-\boldsymbol{w}_{j,h}^{n-1},\boldsymbol{w}_{j,h}^{n+1}\right)+\frac{\bar{\nu}+\bar{\nu}_m}{2}\|\nabla\boldsymbol{w}_{j,h}^{n+1}\|^2+\gamma\|\nabla\cdot\boldsymbol{w}_{j,h}^{n+1}\|^2\\
&\qquad=(\boldsymbol{f}_{2,j}(t^{n+1}),w_{j,h}^{n+1})-\frac{\nu_j-\nu_{m,j}}{2}\big((1+\theta)\nabla\boldsymbol{v}_{j,h}^{n}-\theta\nabla\boldsymbol{v}_{j,h}^{n-1},\nabla\boldsymbol{w}_{j,h}^{n+1}\big)\\
&\qquad-\frac{\nu_j^{'}+\nu_{m,j}^{'}}{2}\left(\nabla(2\boldsymbol{w}_{j,h}^{n}-\boldsymbol{w}_{j,h}^{n-1}),\nabla\boldsymbol{w}_{j,h}^{n+1}\right). \qquad (4.6)
\end{aligned}$$

Next, following the property $b^*(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})=-b^*(\boldsymbol{u},\boldsymbol{w},\boldsymbol{v})$, using the bound in (2.1) and discrete inverse inequality, we have

$$\begin{aligned}
b^*\left(\boldsymbol{w}_{j,h}^{'n},2\boldsymbol{v}_{j,h}^{n}-\boldsymbol{v}_{j,h}^{n-1},\boldsymbol{v}_{j,h}^{n+1}\right)&=b^*\left(\boldsymbol{w}_{j,h}^{'n},\boldsymbol{v}_{j,h}^{n+1},\boldsymbol{v}_{j,h}^{n+1}-2\boldsymbol{v}_{j,h}^{n}+\boldsymbol{v}_{j,h}^{n-1}\right)\\
&\le C\|\nabla\boldsymbol{w}_{j,h}^{'n}\|\|\nabla\boldsymbol{v}_{j,h}^{n+1}\|\|\nabla\left(\boldsymbol{v}_{j,h}^{n+1}-2\boldsymbol{v}_{j,h}^{n}+\boldsymbol{v}_{j,h}^{n-1}\right)\|\\
&\le\frac{C}{h}\|\nabla\boldsymbol{w}_{j,h}^{'n}\|\|\nabla\boldsymbol{v}_{j,h}^{n+1}\|\|\boldsymbol{v}_{j,h}^{n+1}-2\boldsymbol{v}_{j,h}^{n}+\boldsymbol{v}_{j,h}^{n-1}\|. \qquad (4.7)
\end{aligned}$$

Using the above bound, adding equations (4.5) and (4.6), applying the Cauchy-Schwarz and Young's inequalities to the $\nu_j-\nu_{m,j}$, and $\nu_j^{'}+\nu_{m,j}^{'}$ terms, and rearranging, yields

$$\begin{aligned}
&\frac{1}{4\Delta t}\Big(\|\boldsymbol{v}_{j,h}^{n+1}\|^2-\|\boldsymbol{v}_{j,h}^{n}\|^2+\|2\boldsymbol{v}_{j,h}^{n+1}-\boldsymbol{v}_{j,h}^{n}\|^2-\|2\boldsymbol{v}_{j,h}^{n}-\boldsymbol{v}_{j,h}^{n-1}\|^2+\|\boldsymbol{v}_{j,h}^{n+1}-2\boldsymbol{v}_{j,h}^{n}+\boldsymbol{v}_{j,h}^{n-1}\|^2\\
&+\|\boldsymbol{w}_{j,h}^{n+1}\|^2-\|\boldsymbol{w}_{j,h}^{n}\|^2+\|2\boldsymbol{w}_{j,h}^{n+1}-\boldsymbol{w}_{j,h}^{n}\|^2-\|2\boldsymbol{w}_{j,h}^{n}-\boldsymbol{w}_{j,h}^{n-1}\|^2+\|\boldsymbol{w}_{j,h}^{n+1}-2\boldsymbol{w}_{j,h}^{n}+\boldsymbol{w}_{j,h}^{n-1}\|^2\Big)\\
&\qquad+\frac{\bar{\nu}+\bar{\nu}_m}{2}\left(\|\nabla\boldsymbol{v}_{j,h}^{n+1}\|^2+\|\nabla\boldsymbol{w}_{j,h}^{n+1}\|^2\right)+\gamma\left(\|\nabla\cdot\boldsymbol{v}_{j,h}^{n+1}\|^2+\|\nabla\cdot\boldsymbol{w}_{j,h}^{n+1}\|^2\right)\\
&\qquad\le\frac{|\nu_j-\nu_{m,j}|\,(1+2\theta)+3|\nu_j^{'}+\nu_{m,j}^{'}|}{4}\left(\|\nabla\boldsymbol{v}_{j,h}^{n+1}\|^2+\|\nabla\boldsymbol{w}_{j,h}^{n+1}\|^2\right)\\
&\qquad+\frac{|\nu_j-\nu_{m,j}|\,(1+\theta)+2|\nu_j^{'}+\nu_{m,j}^{'}|}{4}\left(\|\nabla\boldsymbol{v}_{j,h}^{n}\|^2+\|\nabla\boldsymbol{w}_{j,h}^{n}\|^2\right)\\
&\qquad+\frac{|\nu_j-\nu_{m,j}|\,\theta+|\nu_j^{'}+\nu_{m,j}^{'}|}{4}\left(\|\nabla\boldsymbol{v}_{j,h}^{n-1}\|^2+\|\nabla\boldsymbol{w}_{j,h}^{n-1}\|^2\right)\\
&\qquad+\frac{C}{h}\|\nabla\boldsymbol{w}_{j,h}^{'n}\|\|\nabla\boldsymbol{v}_{j,h}^{n+1}\|\|\boldsymbol{v}_{j,h}^{n+1}-2\boldsymbol{v}_{j,h}^{n}+\boldsymbol{v}_{j,h}^{n-1}\|\\
&\qquad+\frac{C}{h}\|\nabla\boldsymbol{v}_{j,h}^{'n}\|\|\nabla\boldsymbol{w}_{j,h}^{n+1}\|\|\boldsymbol{w}_{j,h}^{n+1}-2\boldsymbol{w}_{j,h}^{n}+\boldsymbol{w}_{j,h}^{n-1}\|\\
&\qquad+\|\boldsymbol{f}_{1,j}(t^{n+1})\|_{-1}\|\nabla\boldsymbol{v}_{j,h}^{n+1}\|+\|\boldsymbol{f}_{2,j}(t^{n+1})\|_{-1}\|\nabla\boldsymbol{w}_{j,h}^{n+1}\|. \qquad (4.8)
\end{aligned}$$

Next, we apply Young's inequality using $\alpha_j/8$ with the forcing terms, and non-linear terms, noting that $\alpha_j>0$ by the assumed choice of θ, and hiding terms on the left, and rearranging, we have

$$
\begin{aligned}
&\frac{1}{4\Delta t}\Big(\|\boldsymbol{v}_{j,h}^{n+1}\|^2-\|\boldsymbol{v}_{j,h}^{n}\|^2+\|2\boldsymbol{v}_{j,h}^{n+1}-\boldsymbol{v}_{j,h}^{n}\|^2-\|2\boldsymbol{v}_{j,h}^{n}-\boldsymbol{v}_{j,h}^{n-1}\|^2\\
&+\|\boldsymbol{w}_{j,h}^{n+1}\|^2-\|\boldsymbol{w}_{j,h}^{n}\|^2+\|2\boldsymbol{w}_{j,h}^{n+1}-\boldsymbol{w}_{j,h}^{n}\|^2-\|2\boldsymbol{w}_{j,h}^{n}-\boldsymbol{w}_{j,h}^{n-1}\|^2\Big)\\
&+\frac{\bar\nu+\bar\nu_m}{4}\left(\|\nabla\boldsymbol{v}_{j,h}^{n+1}\|^2+\|\nabla\boldsymbol{w}_{j,h}^{n+1}\|^2\right)+\gamma\left(\|\nabla\cdot\boldsymbol{v}_{j,h}^{n+1}\|^2+\|\nabla\cdot\boldsymbol{w}_{j,h}^{n+1}\|^2\right)\\
&+\left(\frac{1}{4\Delta t}-\frac{C}{\alpha_j h^2}\|\nabla\boldsymbol{w}_{j,h}'^{n}\|^2\right)\|\boldsymbol{v}_{j,h}^{n+1}-2\boldsymbol{v}_{j,h}^{n}+\boldsymbol{v}_{j,h}^{n-1}\|^2\\
&+\left(\frac{1}{4\Delta t}-\frac{C}{\alpha_j h^2}\|\nabla\boldsymbol{v}_{j,h}'^{n}\|^2\right)\|\boldsymbol{w}_{j,h}^{n+1}-2\boldsymbol{w}_{j,h}^{n}+\boldsymbol{w}_{j,h}^{n-1}\|^2\\
&\leq\frac{|\nu_j-\nu_{m,j}|(1+\theta)+2|\nu_j'+\nu_{m,j}'|}{4}\left(\|\nabla\boldsymbol{v}_{j,h}^{n}\|^2+\|\nabla\boldsymbol{w}_{j,h}^{n}\|^2\right)\\
&+\frac{|\nu_j-\nu_{m,j}|\theta+|\nu_j'+\nu_{m,j}'|}{4}\left(\|\nabla\boldsymbol{v}_{j,h}^{n-1}\|^2+\|\nabla\boldsymbol{w}_{j,h}^{n-1}\|^2\right)\\
&+\frac{2}{\alpha_j}\|\boldsymbol{f}_{1,j}(t^{n+1})\|_{-1}^2+\frac{2}{\alpha_j}\|\boldsymbol{f}_{2,j}(t^{n+1})\|_{-1}^2.
\end{aligned}\tag{4.9}
$$

Now we define, $\alpha_{\min}:=\min\limits_{1\leq j\leq J}\alpha_j$, and choose $\Delta t\leq\frac{\alpha_{\min}h^2}{C\max\limits_{1\leq j\leq J}\left\{\|\nabla\boldsymbol{v}_{j,h}'^{n}\|^2,\|\nabla\boldsymbol{w}_{j,h}'^{n}\|^2\right\}}$, drop non-negative terms from left, and rearrange, we have

$$
\begin{aligned}
&\frac{1}{4\Delta t}\Big(\|\boldsymbol{v}_{j,h}^{n+1}\|^2-\|\boldsymbol{v}_{j,h}^{n}\|^2+\|2\boldsymbol{v}_{j,h}^{n+1}-\boldsymbol{v}_{j,h}^{n}\|^2-\|2\boldsymbol{v}_{j,h}^{n}-\boldsymbol{v}_{j,h}^{n-1}\|^2\\
&+\|\boldsymbol{w}_{j,h}^{n+1}\|^2-\|\boldsymbol{w}_{j,h}^{n}\|^2+\|2\boldsymbol{w}_{j,h}^{n+1}-\boldsymbol{w}_{j,h}^{n}\|^2-\|2\boldsymbol{w}_{j,h}^{n}-\boldsymbol{w}_{j,h}^{n-1}\|^2\Big)\\
&+\frac{\bar\nu+\bar\nu_m}{4}\left(\|\nabla\boldsymbol{v}_{j,h}^{n+1}\|^2-\|\nabla\boldsymbol{v}_{j,h}^{n}\|^2+\|\nabla\boldsymbol{w}_{j,h}^{n+1}\|^2-\|\nabla\boldsymbol{w}_{j,h}^{n}\|^2\right)\\
&+\frac{\bar\nu+\bar\nu_m-|\nu_j-\nu_{m,j}|(1+\theta)-2|\nu_j'+\nu_{m,j}'|}{4}\left(\|\nabla\boldsymbol{v}_{j,h}^{n}\|^2-\|\nabla\boldsymbol{v}_{j,h}^{n-1}\|^2+\|\nabla\boldsymbol{w}_{j,h}^{n}\|^2-\|\nabla\boldsymbol{w}_{j,h}^{n-1}\|^2\right)\\
&+\frac{\bar\nu+\bar\nu_m-|\nu_j-\nu_{m,j}|(1+2\theta)-3|\nu_j'+\nu_{m,j}'|}{4}\left(\|\nabla\boldsymbol{v}_{j,h}^{n-1}\|^2+\|\nabla\boldsymbol{w}_{j,h}^{n-1}\|^2\right)\\
&+\gamma\left(\|\nabla\cdot\boldsymbol{v}_{j,h}^{n+1}\|^2+\|\nabla\cdot\boldsymbol{w}_{j,h}^{n+1}\|^2\right)\leq\frac{2}{\alpha_j}\|\boldsymbol{f}_{1,j}(t^{n+1})\|_{-1}^2+\frac{2}{\alpha_j}\|\boldsymbol{f}_{2,j}(t^{n+1})\|_{-1}^2.
\end{aligned}\tag{4.10}
$$

Multiplying both sides by $4\Delta t$, and summing over time-steps $n=1,\cdots,M-1$, dropping non-negative terms from the left hand sides completes the proof. □

5. Convergence.

In this section, we will prove the error estimate of the Algorithm 1.

Theorem 5.1. *Consider $m=\max\{2,k+1\}$, and $j=1,2,\cdots,J$, assume $(\boldsymbol{v}_j,\boldsymbol{w}_j,q_j)$ solves (1.7)-(1.9) and satisfies*

$$
\begin{gathered}
\boldsymbol{v}_j,\boldsymbol{w}_j\in L^\infty\big(0,T;\boldsymbol{H}^m(\Omega)^d\big),\quad q_j,r_j\in L^2\big(0,T,L^2(\Omega)^d\big),\\
\boldsymbol{v}_{j,t},\boldsymbol{v}_{j,t}\in L^\infty\big(0,T,\boldsymbol{H}^1(\Omega)^d\big),\boldsymbol{v}_{j,tt},\boldsymbol{v}_{j,tt}\in L^\infty\big(0,T,\boldsymbol{H}^1(\Omega)^d\big),\boldsymbol{v}_{j,ttt},\boldsymbol{v}_{j,ttt}\in L^\infty\big(0,T,\boldsymbol{L}^2(\Omega)^d\big).
\end{gathered}
$$

Then the solution $(\boldsymbol{v}_{j,h},\boldsymbol{w}_{j,h})$ to the Algorithm 1 converges to the true solution:

Proof. At first we build an error equation. Testing (1.7), and (1.8) with $\boldsymbol{\chi}_{j,h} \in \boldsymbol{V}_h$, and $\boldsymbol{l}_{j,h} \in \boldsymbol{V}_h$, respectively at the time level t^{n+1}, the continuous variational formulations can be written

$$\begin{aligned}
&\left(\frac{3\boldsymbol{v}_j(t^{n+1}) - 4\boldsymbol{v}_j(t^n) + \boldsymbol{v}_j(t^{n-1})}{2\Delta t}, \boldsymbol{\chi}_{j,h}\right) + b^*\big(\boldsymbol{w}_j(t^{n+1}), \boldsymbol{v}_j(t^{n+1}), \boldsymbol{\chi}_{j,h}\big) \\
&+\frac{\bar{\nu}+\bar{\nu}_m}{2}(\nabla \boldsymbol{v}_j(t^{n+1}), \nabla \boldsymbol{\chi}_{j,h}) + \frac{\nu_j^{'}+\nu_{m,j}^{'}}{2}(\nabla(2\boldsymbol{v}_j(t^n) - \boldsymbol{v}_j(t^{n-1})), \nabla \boldsymbol{\chi}_{j,h}) \\
&+\frac{\nu_j - \nu_{m,j}}{2}((1+\theta)\nabla \boldsymbol{w}_j(t^n) - \theta \nabla \boldsymbol{w}_j(t^{n-1}), \boldsymbol{\chi}_{j,h}) \\
&-\big(q_j(t^{n+1}), \nabla\cdot \boldsymbol{\chi}_{j,h}\big) + \gamma\big(\nabla\cdot \boldsymbol{v}_j(t^{n+1}), \nabla\cdot\boldsymbol{\chi}_{j,h}\big) = \big(\boldsymbol{f}_{1,j}(t^{n+1}), \boldsymbol{\chi}_{j,h}\big) \\
&-\frac{\nu_j^{'}+\nu_{m,j}^{'}}{2}\Big(\nabla\big(\boldsymbol{v}_j(t^{n+1}) - 2\boldsymbol{v}_j(t^n) + \boldsymbol{v}_j(t^{n-1})\big), \nabla \boldsymbol{\chi}_{j,h}\Big) \\
&-\frac{\nu_j - \nu_{m,j}}{2}(\nabla(\boldsymbol{w}_j(t^{n+1}) - (1+\theta)\boldsymbol{w}_j(t^n) + \theta \boldsymbol{w}_j(t^{n-1})), \boldsymbol{\chi}_{j,h}) \\
&-\left(\boldsymbol{v}_{j,t}(t^{n+1}) - \frac{3\boldsymbol{v}_j(t^{n+1}) - 4\boldsymbol{v}_j(t^n) + \boldsymbol{v}_j(t^{n-1})}{2\Delta t}, \boldsymbol{\chi}_{j,h}\right),
\end{aligned} \tag{5.1}$$

and

$$\begin{aligned}
&\left(\frac{3\boldsymbol{w}_j(t^{n+1}) - 4\boldsymbol{w}_j(t^n) + \boldsymbol{w}_j(t^{n-1})}{2\Delta t}, \boldsymbol{l}_{j,h}\right) + b^*\big(\boldsymbol{v}_j(t^{n+1}), \boldsymbol{w}_j(t^{n+1}), \boldsymbol{l}_{j,h}\big) \\
&+\frac{\bar{\nu}+\bar{\nu}_m}{2}(\nabla \boldsymbol{w}_j(t^{n+1}), \nabla \boldsymbol{l}_{j,h}) + \frac{\nu_j^{'}+\nu_{m,j}^{'}}{2}(\nabla(2\boldsymbol{w}_j(t^n) - \boldsymbol{w}_j(t^{n-1})), \nabla \boldsymbol{l}_{j,h}) \\
&+\frac{\nu_j - \nu_{m,j}}{2}((1+\theta)\nabla \boldsymbol{v}_j(t^n) - \theta \nabla \boldsymbol{v}_j(t^{n-1}), \boldsymbol{l}_{j,h}) \\
&-\big(r_j(t^{n+1}), \nabla\cdot \boldsymbol{l}_{j,h}\big) + \gamma\big(\nabla\cdot \boldsymbol{w}_j(t^{n+1}), \nabla\cdot\boldsymbol{l}_{j,h}\big) = \big(\boldsymbol{f}_{2,j}(t^{n+1}), \boldsymbol{l}_{j,h}\big) \\
&-\frac{\nu_j^{'}+\nu_{m,j}^{'}}{2}\Big(\nabla\big(\boldsymbol{w}_j(t^{n+1}) - 2\boldsymbol{w}_j(t^n) + \boldsymbol{w}_j(t^{n-1})\big), \nabla \boldsymbol{l}_{j,h}\Big) \\
&-\frac{\nu_j - \nu_{m,j}}{2}(\nabla(\boldsymbol{v}_j(t^{n+1}) - (1+\theta)\boldsymbol{v}_j(t^n) + \theta \boldsymbol{v}_j(t^{n-1})), \boldsymbol{l}_{j,h}) \\
&-\left(\boldsymbol{w}_{j,t}(t^{n+1}) - \frac{3\boldsymbol{w}_j(t^{n+1}) - 4\boldsymbol{w}_j(t^n) + \boldsymbol{w}_j(t^{n-1})}{2\Delta t}, \boldsymbol{l}_{j,h}\right).
\end{aligned} \tag{5.2}$$

Note $\big(\nabla\cdot\boldsymbol{v}_j(t^{n+1}), \nabla\cdot\boldsymbol{\chi}_{j,h}\big) = \big(\nabla\cdot\boldsymbol{w}_j(t^{n+1}), \nabla\cdot\boldsymbol{l}_{j,h}\big) = 0$. Denote $\boldsymbol{e}_{\boldsymbol{v},j}^n := \boldsymbol{v}_j(t^n) - \boldsymbol{v}_{j,h}^n$, and $\boldsymbol{e}_{\boldsymbol{w},j}^n := \boldsymbol{w}_j(t^n) - \boldsymbol{w}_{j,h}^n$. Subtracting (3.6) and (3.7) from equation (5.1) and (5.2), respectively yields

$$\begin{aligned}
&\left(\frac{3\boldsymbol{e}_{\boldsymbol{v},j}^{n+1} - 4\boldsymbol{e}_{\boldsymbol{v},j}^n + \boldsymbol{e}_{\boldsymbol{v},j}^{n-1}}{2\Delta t}, \boldsymbol{\chi}_{j,h}\right) + \frac{\bar{\nu}+\bar{\nu}_m}{2}(\nabla \boldsymbol{e}_{\boldsymbol{v},j}^{n+1}, \nabla\boldsymbol{\chi}_{j,h}) + \frac{\nu_j^{'}+\nu_{m,j}^{'}}{2}\Big(\nabla\big(2\boldsymbol{e}_{\boldsymbol{v},j}^n - \boldsymbol{e}_{\boldsymbol{v},j}^{n-1}\big), \nabla\boldsymbol{\chi}_{j,h}\Big) \\
&+\frac{\nu_j - \nu_{m,j}}{2}\big((1+\theta)\nabla \boldsymbol{e}_{\boldsymbol{w},j}^n - \theta\nabla \boldsymbol{e}_{\boldsymbol{w},j}^{n-1}, \nabla\boldsymbol{\chi}_{j,h}\big) + b^*\big(2\boldsymbol{e}_{\boldsymbol{w},j}^n - \boldsymbol{e}_{\boldsymbol{w},j}^{n-1}, \boldsymbol{v}_j(t^{n+1}), \boldsymbol{\chi}_{j,h}\big) \\
&+ b^*\big(2\boldsymbol{w}_{j,h}^n - \boldsymbol{w}_{j,h}^{n-1}, \boldsymbol{e}_{\boldsymbol{v},j}^{n+1}, \boldsymbol{\chi}_{j,h}\big) - b^*\big(\boldsymbol{w}_{j,h}^{'n}, \boldsymbol{e}_{\boldsymbol{v},j}^{n+1} - 2\boldsymbol{e}_{\boldsymbol{v},j}^n + \boldsymbol{e}_{\boldsymbol{v},j}^{n-1}, \boldsymbol{\chi}_{j,h}\big) \\
&-\big(q_j(t^{n+1}), \nabla\cdot\boldsymbol{\chi}_{j,h}\big) + \gamma\big(\nabla\cdot\boldsymbol{e}_{\boldsymbol{v},j}^{n+1}, \nabla\cdot\boldsymbol{\chi}_{j,h}\big) = -G_1(t, \boldsymbol{v}_j, \boldsymbol{w}_j, \boldsymbol{\chi}_{j,h}),
\end{aligned} \tag{5.3}$$

and

$$\begin{aligned}
&\left(\frac{3\boldsymbol{e}_{\boldsymbol{w},j}^{n+1} - 4\boldsymbol{e}_{\boldsymbol{w},j}^n + \boldsymbol{e}_{\boldsymbol{w},j}^{n-1}}{2\Delta t}, \boldsymbol{l}_{j,h}\right) + \frac{\bar{\nu}+\bar{\nu}_m}{2}(\nabla \boldsymbol{e}_{\boldsymbol{w},j}^{n+1}, \nabla\boldsymbol{l}_{j,h}) + \frac{\nu_j^{'}+\nu_{m,j}^{'}}{2}\Big(\nabla\big(2\boldsymbol{e}_{\boldsymbol{w},j}^n - \boldsymbol{e}_{\boldsymbol{w},j}^{n-1}\big), \nabla\boldsymbol{l}_{j,h}\Big) \\
&+\frac{\nu_j - \nu_{m,j}}{2}\big((1+\theta)\nabla \boldsymbol{e}_{\boldsymbol{v},j}^n - \theta\nabla \boldsymbol{e}_{\boldsymbol{v},j}^{n-1}, \nabla\boldsymbol{l}_{j,h}\big) + b^*\big(2\boldsymbol{e}_{\boldsymbol{w},j}^n - \boldsymbol{e}_{\boldsymbol{v},j}^{n-1}, \boldsymbol{w}_j(t^{n+1}), \boldsymbol{l}_{j,h}\big) \\
&+ b^*\big(2\boldsymbol{v}_{j,h}^n - \boldsymbol{v}_{j,h}^{n-1}, \boldsymbol{e}_{\boldsymbol{w},j}^{n+1}, \boldsymbol{l}_{j,h}\big) - b^*(\boldsymbol{v}_{j,h}^{'n}, \boldsymbol{e}_{\boldsymbol{w},j}^{n+1} - 2\boldsymbol{e}_{\boldsymbol{w},j}^n + \boldsymbol{e}_{\boldsymbol{w},j}^{n-1}, \boldsymbol{l}_{j,h}) \\
&-\big(r_j(t^{n+1}), \nabla\cdot\boldsymbol{l}_{j,h}\big) + \gamma\big(\nabla\cdot\boldsymbol{e}_{\boldsymbol{w},j}^{n+1}, \nabla\cdot\boldsymbol{l}_{j,h}\big) = -G_2(t, \boldsymbol{v}_j, \boldsymbol{w}_j, \boldsymbol{l}_{j,h}),
\end{aligned} \tag{5.4}$$

where

$$
\begin{aligned}
G_1(t,\boldsymbol{v}_j,\boldsymbol{w}_j,\boldsymbol{\chi}_{j,h}) &:= b^*\big(\boldsymbol{w}_j(t^{n+1}) - 2\boldsymbol{w}_j(t^n) + \boldsymbol{w}_j(t^{n-1}), \boldsymbol{v}_j(t^{n+1}), \boldsymbol{\chi}_{j,h}\big) \\
&+ b^*\big(\boldsymbol{w}_{j,h}^{'n}, \boldsymbol{v}_j(t^{n+1}) - 2\boldsymbol{v}_j(t^n) + \boldsymbol{v}_j(t^{n-1}), \boldsymbol{\chi}_{j,h}\big) \\
&+ \frac{\nu_j^{'} + \nu_{m,j}^{'}}{2}\Big(\nabla\big(\boldsymbol{v}_j(t^{n+1}) - 2\boldsymbol{v}_j(t^n) + \boldsymbol{v}_j(t^{n-1})\big), \nabla\boldsymbol{\chi}_{j,h}\Big) \\
&+ \frac{\nu_j - \nu_{m,j}}{2}\big(\nabla(\boldsymbol{w}_j(t^{n+1}) - (1+\theta)\boldsymbol{w}_j(t^n) + \theta\boldsymbol{w}_j(t^{n-1})), \nabla\boldsymbol{\chi}_{j,h}\big) \\
&+ \left(\boldsymbol{v}_{j,t}(t^{n+1}) - \frac{3\boldsymbol{v}_j(t^{n+1}) - 4\boldsymbol{v}_j(t^n) + \boldsymbol{v}_j(t^{n-1})}{2\Delta t}, \boldsymbol{\chi}_{j,h}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
G_2(t,\boldsymbol{v}_j,\boldsymbol{w}_j,\boldsymbol{l}_{j,h}) &:= b^*\big(\boldsymbol{v}_j(t^{n+1}) - 2\boldsymbol{v}_j(t^n) + \boldsymbol{v}_j(t^{n-1}), \boldsymbol{w}_j(t^{n+1}), \boldsymbol{l}_{j,h}\big) \\
&+ b^*\big(\boldsymbol{v}_{j,h}^{'n}, \boldsymbol{w}_j(t^{n+1}) - 2\boldsymbol{w}_j(t^n) + \boldsymbol{w}_j(t^{n-1}), \boldsymbol{l}_{j,h}\big) \\
&+ \frac{\nu_j^{'} + \nu_{m,j}^{'}}{2}\Big(\nabla\big(\boldsymbol{w}_j(t^{n+1}) - 2\boldsymbol{w}_j(t^n) + \boldsymbol{w}_j(t^{n-1})\big), \nabla\boldsymbol{l}_{j,h}\Big) \\
&+ \frac{\nu_j - \nu_{m,j}}{2}\big(\nabla(\boldsymbol{v}_j(t^{n+1}) - (1+\theta)\boldsymbol{v}_j(t^n) + \theta\boldsymbol{v}_j(t^{n-1})), \nabla\boldsymbol{l}_{j,h}\big) \\
&+ \left(\boldsymbol{w}_{j,t}(t^{n+1}) - \frac{3\boldsymbol{w}_j(t^{n+1}) - 4\boldsymbol{w}_j(t^n) + \boldsymbol{w}_j(t^{n-1})}{2\Delta t}, \boldsymbol{l}_{j,h}\right).
\end{aligned}
$$

Now we decompose the errors as

$$
\begin{aligned}
\boldsymbol{e}_{\boldsymbol{v},j}^n &:= \boldsymbol{v}_j(t^n) - \boldsymbol{v}_{j,h}^n = (\boldsymbol{v}_j(t^n) - \tilde{\boldsymbol{v}}_j^n) - (\boldsymbol{v}_{j,h}^n - \tilde{\boldsymbol{v}}_j^n) := \boldsymbol{\eta}_{\boldsymbol{v},j}^n - \boldsymbol{\varphi}_{j,h}^n, \\
\boldsymbol{e}_{\boldsymbol{w},j}^n &:= \boldsymbol{w}_j(t^n) - \boldsymbol{w}_{j,h}^n = (\boldsymbol{w}_j(t^n) - \tilde{\boldsymbol{w}}_j^n) - (\boldsymbol{w}_{j,h}^n - \tilde{\boldsymbol{w}}_j^n) := \boldsymbol{\eta}_{\boldsymbol{w},j}^n - \boldsymbol{\psi}_{j,h}^n,
\end{aligned}
$$

where $\tilde{\boldsymbol{v}}_j^n := P_{\boldsymbol{V}_h}^{L^2}(\boldsymbol{v}_j(t^n)) \in \boldsymbol{V}_h$ and $\tilde{\boldsymbol{w}}_j^n := P_{\boldsymbol{V}_h}^{L^2}(\boldsymbol{w}_j(t^n)) \in \boldsymbol{V}_h$ are the L^2 projections of $\boldsymbol{v}_j(t^n)$ and $\boldsymbol{w}_j(t^n)$ into $\boldsymbol{V}_h$, respectively. Note that $(\boldsymbol{\eta}_{\boldsymbol{v},j}^n, \boldsymbol{v}_{j,h}) = (\boldsymbol{\eta}_{\boldsymbol{w},j}^n, \boldsymbol{v}_{j,h}) = 0 \ \ \forall \boldsymbol{v}_{j,h} \in \boldsymbol{V}_h$. Rewriting, we have for $\boldsymbol{\chi}_{j,h}, \boldsymbol{l}_{j,h} \in \boldsymbol{V}_h$

$$
\begin{aligned}
&\left(\frac{3\boldsymbol{\varphi}_{j,h}^{n+1} - 4\boldsymbol{\varphi}_{j,h}^n + \boldsymbol{\varphi}_{j,h}^{n-1}}{2\Delta t}, \boldsymbol{\chi}_{j,h}\right) + \frac{\bar{\nu} + \bar{\nu}_m}{2}(\nabla\boldsymbol{\varphi}_{j,h}^{n+1}, \nabla\boldsymbol{\chi}_{j,h}) + \frac{\nu_j^{'} + \nu_{m,j}^{'}}{2}(\nabla(2\boldsymbol{\varphi}_{j,h}^n - \boldsymbol{\varphi}_{j,h}^{n-1}), \nabla\boldsymbol{\chi}_{j,h}) \\
&+ \frac{\nu_j - \nu_{m,j}}{2}\big((1+\theta)\nabla\boldsymbol{\psi}_{j,h}^n - \theta\nabla\boldsymbol{\psi}_{j,h}^{n-1}, \nabla\boldsymbol{\chi}_{j,h}\big) + b^*\big(2\boldsymbol{\psi}_{j,h}^n - \boldsymbol{\psi}_{j,h}^{n-1}, \boldsymbol{v}_j(t^{n+1}), \boldsymbol{\chi}_{j,h}\big) \\
&+ b^*(2\boldsymbol{w}_{j,h}^n - \boldsymbol{w}_{j,h}^{n-1}, \boldsymbol{\varphi}_{j,h}^{n+1}, \boldsymbol{\chi}_{j,h}) - b^*(\boldsymbol{w}_{j,h}^{'n}, \boldsymbol{\varphi}_{j,h}^{n+1} - 2\boldsymbol{\varphi}_{j,h}^n + \boldsymbol{\varphi}_{j,h}^{n-1}, \boldsymbol{\chi}_{j,h}) + \gamma(\nabla\cdot\boldsymbol{\varphi}_{j,h}^{n+1}, \nabla\cdot\boldsymbol{\chi}_{j,h}) \\
&= \frac{\nu_j - \nu_{m,j}}{2}\big((1+\theta)\nabla\boldsymbol{\eta}_{\boldsymbol{w},j}^n - \theta\nabla\boldsymbol{\eta}_{\boldsymbol{w},j}^{n-1}, \nabla\boldsymbol{\chi}_{j,h}\big) + \frac{\bar{\nu} + \bar{\nu}_m}{2}(\nabla\boldsymbol{\eta}_{\boldsymbol{v},j}^{n+1}, \nabla\boldsymbol{\chi}_{j,h}) \\
&+ \frac{\nu_j^{'} + \nu_{m,j}^{'}}{2}(\nabla(2\boldsymbol{\eta}_{\boldsymbol{v},j}^n - \boldsymbol{\eta}_{\boldsymbol{v},j}^{n-1}), \nabla\boldsymbol{\chi}_{j,h}) + b^*\big(2\boldsymbol{\eta}_{\boldsymbol{w},j}^n - \boldsymbol{\eta}_{\boldsymbol{w},j}^{n-1}, \boldsymbol{v}_j(t^{n+1}), \boldsymbol{\chi}_{j,h}\big) \\
&+ b^*(2\boldsymbol{w}_{j,h}^n - \boldsymbol{w}_{j,h}^{n-1}, \boldsymbol{\eta}_{\boldsymbol{v},j}^{n+1}, \boldsymbol{\chi}_{j,h}) - b^*(\boldsymbol{w}_{j,h}^{'n}, \boldsymbol{\eta}_{\boldsymbol{v},j}^{n+1} - 2\boldsymbol{\eta}_{\boldsymbol{v},j}^n + \boldsymbol{\eta}_{\boldsymbol{v},j}^{n-1}, \boldsymbol{\chi}_{j,h}) \\
&- \big(q_j(t^{n+1}), \nabla\cdot\boldsymbol{\chi}_{j,h}\big) + \gamma\big(\nabla\cdot\boldsymbol{\eta}_{\boldsymbol{v},j}^{n+1}, \nabla\cdot\boldsymbol{\chi}_{j,h}\big) + G_1(t,\boldsymbol{v}_j,\boldsymbol{w}_j,\boldsymbol{\chi}_{j,h}), \qquad (5.5)
\end{aligned}
$$

and

$$
\begin{aligned}
&\left(\frac{3\boldsymbol{\psi}_{j,h}^{n+1}-4\boldsymbol{\psi}_{j,h}^{n}+\boldsymbol{\psi}_{j,h}^{n-1}}{2\Delta t},\boldsymbol{l}_{j,h}\right)+\frac{\bar{\nu}+\bar{\nu}_m}{2}(\nabla\boldsymbol{\psi}_{j,h}^{n+1},\nabla\boldsymbol{l}_{j,h})+\frac{\nu_j^{'}+\nu_{m,j}^{'}}{2}(\nabla(2\boldsymbol{\psi}_{j,h}^{n}-\boldsymbol{\psi}_{j,h}^{n-1}),\nabla\boldsymbol{l}_{j,h})\\
&+\frac{\nu_j-\nu_{m,j}}{2}\big((1+\theta)\nabla\boldsymbol{\varphi}_{j,h}^{n}-\theta\nabla\boldsymbol{\varphi}_{j,h}^{n-1},\nabla\boldsymbol{l}_{j,h}\big)+b^*\big(2\boldsymbol{\varphi}_{j,h}^{n}-\boldsymbol{\varphi}_{j,h}^{n-1},\boldsymbol{w}_j(t^{n+1}),\boldsymbol{l}_{j,h}\big)\\
&+b^*(2\boldsymbol{v}_{j,h}^{n}-\boldsymbol{v}_{j,h}^{n-1},\boldsymbol{\psi}_{j,h}^{n+1},\boldsymbol{l}_{j,h})-b^*(\boldsymbol{v}_{j,h}^{'n},\boldsymbol{\psi}_{j,h}^{n+1}-2\boldsymbol{\psi}_{j,h}^{n}+\boldsymbol{\psi}_{j,h}^{n-1},\boldsymbol{l}_{j,h})+\gamma(\nabla\cdot\boldsymbol{\psi}_{j,h}^{n+1},\nabla\cdot\boldsymbol{l}_{j,h})\\
&=\frac{\nu_j-\nu_{m,j}}{2}\big((1+\theta)\nabla\boldsymbol{\eta}_{\boldsymbol{v},j}^{n}-\theta\nabla\boldsymbol{\eta}_{\boldsymbol{v},j}^{n-1},\nabla\boldsymbol{l}_{j,h}\big)+\frac{\bar{\nu}+\bar{\nu}_m}{2}(\nabla\boldsymbol{\eta}_{\boldsymbol{w},j}^{n+1},\nabla\boldsymbol{l}_{j,h})\\
&+\frac{\nu_j^{'}+\nu_{m,j}^{'}}{2}(\nabla(2\boldsymbol{\eta}_{\boldsymbol{w},j}^{n}-\boldsymbol{\eta}_{\boldsymbol{w},j}^{n-1}),\nabla\boldsymbol{l}_{j,h})+b^*\big(2\boldsymbol{\eta}_{\boldsymbol{v},j}^{n}-\boldsymbol{\eta}_{\boldsymbol{v},j}^{n-1},\boldsymbol{w}_j(t^{n+1}),\boldsymbol{l}_{j,h}\big)\\
&+b^*(2\boldsymbol{v}_{j,h}^{n}-\boldsymbol{v}_{j,h}^{n-1},\boldsymbol{\eta}_{\boldsymbol{w},j}^{n+1},\boldsymbol{l}_{j,h})-b^*(\boldsymbol{v}_{j,h}^{'n},\boldsymbol{\eta}_{\boldsymbol{w},j}^{n+1}-2\boldsymbol{\eta}_{\boldsymbol{w},j}^{n}+\boldsymbol{\eta}_{\boldsymbol{w},j}^{n-1},\boldsymbol{l}_{j,h})\\
&-\big(r_j(t^{n+1}),\nabla\cdot\boldsymbol{l}_{j,h}\big)+\gamma\big(\nabla\cdot\boldsymbol{\eta}_{\boldsymbol{w},j}^{n+1},\nabla\cdot\boldsymbol{l}_{j,h}\big)+G_2(t,\boldsymbol{v}_j,\boldsymbol{w}_j,\boldsymbol{l}_{j,h}). \qquad (5.6)
\end{aligned}
$$

Choose $\boldsymbol{\chi}_{j,h}=\boldsymbol{\varphi}_{j,h}^{n+1},\boldsymbol{l}_{j,h}=\boldsymbol{\psi}_{j,h}^{n+1}$, and use the polarization identity in (5.5) and (5.6), to obtain

$$
\begin{aligned}
&\frac{1}{4\Delta t}\left(\|\boldsymbol{\varphi}_{j,h}^{n+1}\|^2-\|\boldsymbol{\varphi}_{j,h}^{n}\|^2+\|2\boldsymbol{\varphi}_{j,h}^{n+1}-\boldsymbol{\varphi}_{j,h}^{n}\|^2-\|2\boldsymbol{\varphi}_{j,h}^{n}-\boldsymbol{\varphi}_{j,h}^{n-1}\|^2+\|\boldsymbol{\varphi}_{j,h}^{n+1}-2\boldsymbol{\varphi}_{j,h}^{n}+\boldsymbol{\varphi}_{j,h}^{n-1}\|^2\right)\\
&+\frac{\bar{\nu}+\bar{\nu}_m}{2}\|\nabla\boldsymbol{\varphi}_{j,h}^{n+1}\|^2+\gamma\|\nabla\cdot\boldsymbol{\varphi}_{j,h}^{n+1}\|^2\leq(1+\theta)\frac{|\nu_j-\nu_{m,j}|}{2}\Big\{|(\nabla\boldsymbol{\psi}_{j,h}^{n},\nabla\boldsymbol{\varphi}_{j,h}^{n+1})|+|(\nabla\boldsymbol{\eta}_{\boldsymbol{w},j}^{n},\nabla\boldsymbol{\varphi}_{j,h}^{n+1})|\Big\}\\
&+\theta\frac{|\nu_j-\nu_{m,j}|}{2}\Big\{|(\nabla\boldsymbol{\psi}_{j,h}^{n-1},\nabla\boldsymbol{\varphi}_{j,h}^{n+1})|+|(\nabla\boldsymbol{\eta}_{\boldsymbol{w},j}^{n-1},\nabla\boldsymbol{\varphi}_{j,h}^{n+1})|\Big\}+\frac{|\nu_j^{'}+\nu_{m,j}^{'}|}{2}|(\nabla(2\boldsymbol{\varphi}_{j,h}^{n}-\boldsymbol{\varphi}_{j,h}^{n-1}),\nabla\boldsymbol{\varphi}_{j,h}^{n+1})|\\
&+\frac{\bar{\nu}+\bar{\nu}_m}{2}|(\nabla\boldsymbol{\eta}_{\boldsymbol{v},j}^{n+1},\nabla\boldsymbol{\varphi}_{j,h}^{n+1})|+|b^*\big(2\boldsymbol{\eta}_{\boldsymbol{w},j}^{n}-\boldsymbol{\eta}_{\boldsymbol{w},j}^{n-1},\boldsymbol{v}_j(t^{n+1}),\boldsymbol{\varphi}_{j,h}^{n+1}\big)|+|b^*\big(2\boldsymbol{w}_{j,h}^{n}-\boldsymbol{w}_{j,h}^{n-1},\boldsymbol{\eta}_{\boldsymbol{v},j}^{n+1},\boldsymbol{\varphi}_{j,h}^{n+1}\big)|\\
&+|b^*\big(\boldsymbol{w}_{j,h}^{'n},\boldsymbol{\eta}_{\boldsymbol{v},j}^{n+1}-2\boldsymbol{\eta}_{\boldsymbol{v},j}^{n}+\boldsymbol{\eta}_{\boldsymbol{v},j}^{n-1},\boldsymbol{\varphi}_{j,h}^{n+1}\big)|+|b^*\big(2\boldsymbol{\psi}_{j,h}^{n}-\boldsymbol{\psi}_{j,h}^{n-1},\boldsymbol{v}_j(t^{n+1}),\boldsymbol{\varphi}_{j,h}^{n+1}\big)|\\
&+|b^*\big(\boldsymbol{w}_{j,h}^{'n},\boldsymbol{\varphi}_{j,h}^{n+1}-2\boldsymbol{\varphi}_{j,h}^{n}+\boldsymbol{\varphi}_{j,h}^{n-1},\boldsymbol{\varphi}_{j,h}^{n+1}\big)|+|\big(q_j(t^{n+1}),\nabla\cdot\boldsymbol{\varphi}_{j,h}^{n+1}\big)|+\gamma|\big(\nabla\cdot\boldsymbol{\eta}_{\boldsymbol{v},j}^{n+1},\nabla\cdot\boldsymbol{\varphi}_{j,h}^{n+1}\big)|\\
&+|G_1(t,\boldsymbol{v}_j,\boldsymbol{w}_j,\boldsymbol{\varphi}_{j,h}^{n+1})|, \qquad (5.7)
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{1}{4\Delta t}\left(\|\boldsymbol{\psi}_{j,h}^{n+1}\|^2-\|\boldsymbol{\psi}_{j,h}^{n}\|^2+\|2\boldsymbol{\psi}_{j,h}^{n+1}-\boldsymbol{\psi}_{j,h}^{n}\|^2-\|2\boldsymbol{\psi}_{j,h}^{n}-\boldsymbol{\psi}_{j,h}^{n-1}\|^2+\|\boldsymbol{\psi}_{j,h}^{n+1}-2\boldsymbol{\psi}_{j,h}^{n}+\boldsymbol{\psi}_{j,h}^{n-1}\|^2\right)\\
&+\frac{\bar{\nu}+\bar{\nu}_m}{2}\|\nabla\boldsymbol{\psi}_{j,h}^{n+1}\|^2+\gamma\|\nabla\cdot\boldsymbol{\psi}_{j,h}^{n+1}\|^2\leq(1+\theta)\frac{|\nu_j-\nu_{m,j}|}{2}\Big\{|(\nabla\boldsymbol{\varphi}_{j,h}^{n},\nabla\boldsymbol{\psi}_{j,h}^{n+1})|+|(\nabla\boldsymbol{\eta}_{\boldsymbol{v},j}^{n},\nabla\boldsymbol{\psi}_{j,h}^{n+1})|\Big\}\\
&+\theta\frac{|\nu_j-\nu_{m,j}|}{2}\Big\{|(\nabla\boldsymbol{\varphi}_{j,h}^{n-1},\nabla\boldsymbol{\psi}_{j,h}^{n+1})|+|(\nabla\boldsymbol{\eta}_{\boldsymbol{v},j}^{n-1},\nabla\boldsymbol{\psi}_{j,h}^{n+1})|\Big\}+\frac{|\nu_j^{'}+\nu_{m,j}^{'}|}{2}|(\nabla(2\boldsymbol{\psi}_{j,h}^{n}-\boldsymbol{\psi}_{j,h}^{n-1}),\nabla\boldsymbol{\psi}_{j,h}^{n+1})|\\
&+\frac{\bar{\nu}+\bar{\nu}_m}{2}|(\nabla\boldsymbol{\eta}_{\boldsymbol{w},j}^{n+1},\nabla\boldsymbol{\psi}_{j,h}^{n+1})|+|b^*\big(2\boldsymbol{\eta}_{\boldsymbol{v},j}^{n}-\boldsymbol{\eta}_{\boldsymbol{v},j}^{n-1},\boldsymbol{w}_j(t^{n+1}),\boldsymbol{\psi}_{j,h}^{n+1}\big)|+|b^*\big(2\boldsymbol{v}_{j,h}^{n}-\boldsymbol{v}_{j,h}^{n-1},\boldsymbol{\eta}_{\boldsymbol{w},j}^{n+1},\boldsymbol{\psi}_{j,h}^{n+1}\big)|\\
&+|b^*\big(\boldsymbol{v}_{j,h}^{'n},\boldsymbol{\eta}_{\boldsymbol{w},j}^{n+1}-2\boldsymbol{\eta}_{\boldsymbol{w},j}^{n}+\boldsymbol{\eta}_{\boldsymbol{w},j}^{n-1},\boldsymbol{\psi}_{j,h}^{n+1}\big)|+|b^*\big(2\boldsymbol{\varphi}_{j,h}^{n}-\boldsymbol{\varphi}_{j,h}^{n-1},\boldsymbol{w}_j(t^{n+1}),\boldsymbol{\psi}_{j,h}^{n+1}\big)|\\
&+|b^*\big(\boldsymbol{v}_{j,h}^{'n},\boldsymbol{\psi}_{j,h}^{n+1}-2\boldsymbol{\psi}_{j,h}^{n}+\boldsymbol{\psi}_{j,h}^{n-1},\boldsymbol{\psi}_{j,h}^{n+1}\big)|+|\big(r_j(t^{n+1}),\nabla\cdot\boldsymbol{\psi}_{j,h}^{n+1}\big)|+\gamma|\big(\nabla\cdot\boldsymbol{\eta}_{\boldsymbol{w},j}^{n+1},\nabla\cdot\boldsymbol{\psi}_{j,h}^{n+1}\big)|\\
&+|G_2(t,\boldsymbol{v}_j,\boldsymbol{w}_j,\boldsymbol{\psi}_{j,h}^{n+1})|. \qquad (5.8)
\end{aligned}
$$

We now turn our attention to finding bounds on the right-hand-side terms of (5.7). Applying Cauchy-Schwarz and Young's inequalities on the first six terms provides.

$$
\begin{aligned}
(1+\theta)\frac{|\nu_j-\nu_{m,j}|}{2}|(\nabla\boldsymbol{\psi}^n_{j,h},\nabla\boldsymbol{\varphi}^{n+1}_{j,h})| &\le (1+\theta)\frac{|\nu_j-\nu_{m,j}|}{4}\big(\|\nabla\boldsymbol{\psi}^n_{j,h}\|^2+\|\nabla\boldsymbol{\varphi}^{n+1}_{j,h}\|^2\big),\\
\theta\frac{|\nu_j-\nu_{m,j}|}{2}|(\nabla\boldsymbol{\psi}^{n-1}_{j,h},\nabla\boldsymbol{\varphi}^{n+1}_{j,h})| &\le \theta\frac{|\nu_j-\nu_{m,j}|}{4}\big(\|\nabla\boldsymbol{\psi}^{n-1}_{j,h}\|^2+\|\nabla\boldsymbol{\varphi}^{n+1}_{j,h}\|^2\big),\\
(1+\theta)\frac{|\nu_j-\nu_{m,j}|}{2}|(\nabla\boldsymbol{\eta}^n_{\boldsymbol{w},j},\nabla\boldsymbol{\varphi}^{n+1}_{j,h})| &\le \frac{\alpha_j}{36}\|\nabla\boldsymbol{\varphi}^{n+1}_{j,h}\|^2+\frac{9(1+\theta)^2(\nu_j-\nu_{m,j})^2}{4\alpha_j}\|\nabla\boldsymbol{\eta}^n_{\boldsymbol{w},j}\|^2,\\
\theta\frac{|\nu_j-\nu_{m,j}|}{2}|(\nabla\boldsymbol{\eta}^{n-1}_{\boldsymbol{w},j},\nabla\boldsymbol{\varphi}^{n+1}_{j,h})| &\le \frac{\alpha_j}{36}\|\nabla\boldsymbol{\varphi}^{n+1}_{j,h}\|^2+\frac{9\theta^2(\nu_j-\nu_{m,j})^2}{4\alpha_j}\|\nabla\boldsymbol{\eta}^{n-1}_{\boldsymbol{w},j}\|^2,\\
\frac{\nu_j^{'}+\nu_{m,j}^{'}}{2}|(\nabla(2\boldsymbol{\varphi}^n_{j,h}-\boldsymbol{\varphi}^{n-1}_{j,h}),\nabla\boldsymbol{\varphi}^{n+1}_{j,h})| &\le \frac{|\nu_j^{'}+\nu_{m,j}^{'}|}{4}(3\|\nabla\boldsymbol{\varphi}^{n+1}_{j,h}\|^2+2\|\nabla\boldsymbol{\varphi}^n_{j,h}\|^2+\|\nabla\boldsymbol{\varphi}^{n-1}_{j,h}\|^2),\\
\frac{\bar\nu+\bar\nu_m}{2}|(\nabla\boldsymbol{\eta}^{n+1}_{\boldsymbol{v},j},\nabla\boldsymbol{\varphi}^{n+1}_{j,h})| &\le \frac{\alpha_j}{36}\|\nabla\boldsymbol{\varphi}^{n+1}_{j,h}\|^2+\frac{9(\bar\nu+\bar\nu_m)^2}{4\alpha_j}\|\nabla\boldsymbol{\eta}^{n+1}_{\boldsymbol{v},j}\|^2.
\end{aligned}
$$

Applying Young's inequalities with (2.1) on the first three nonlinear terms yields

$$
\begin{aligned}
|b^*\big(2\boldsymbol{\eta}^n_{\boldsymbol{w},j}-\boldsymbol{\eta}^{n-1}_{\boldsymbol{w},j},\boldsymbol{v}_j(t^{n+1}),\boldsymbol{\varphi}^{n+1}_{j,h}\big)| &\le C\|\nabla(2\boldsymbol{\eta}^n_{\boldsymbol{w},j}-\boldsymbol{\eta}^{n-1}_{\boldsymbol{w},j})\|\|\nabla\boldsymbol{v}_j(t^{n+1})\|\|\nabla\boldsymbol{\varphi}^{n+1}_{j,h}\|\\
&\le \frac{\alpha_j}{36}\|\nabla\boldsymbol{\varphi}^{n+1}_{j,h}\|^2+\frac{C}{\alpha_j}\|\nabla(2\boldsymbol{\eta}^n_{\boldsymbol{w},j}-\boldsymbol{\eta}^{n-1}_{\boldsymbol{w},j})\|^2\|\nabla\boldsymbol{v}_j(t^{n+1})\|^2,\\
|b^*\big(2\boldsymbol{w}^n_{j,h}-\boldsymbol{w}^{n-1}_{j,h},\boldsymbol{\eta}^{n+1}_{\boldsymbol{v},j},\boldsymbol{\varphi}^{n+1}_{j,h}\big)| &\le C\|\nabla(2\boldsymbol{w}^n_{j,h}-\boldsymbol{w}^{n-1}_{j,h})\|\|\nabla\boldsymbol{\eta}^{n+1}_{\boldsymbol{v},j}\|\|\nabla\boldsymbol{\varphi}^{n+1}_{j,h}\|\\
&\le \frac{\alpha_j}{36}\|\nabla\boldsymbol{\varphi}^{n+1}_{j,h}\|^2+\frac{C}{\alpha_j}\|\nabla(2\boldsymbol{w}^n_{j,h}-\boldsymbol{w}^{n-1}_{j,h})\|^2\|\nabla\boldsymbol{\eta}^{n+1}_{\boldsymbol{v},j}\|^2,\\
|b^*\big(\boldsymbol{w}^{'n}_{j,h},\boldsymbol{\eta}^{n+1}_{\boldsymbol{v},j}-\boldsymbol{\eta}^n_{\boldsymbol{v},j}+\boldsymbol{\eta}^{n-1}_{\boldsymbol{v},j},\boldsymbol{\varphi}^{n+1}_{j,h}\big)| &\le C\|\nabla\boldsymbol{w}^{'n}_{j,h}\|\|\nabla(\boldsymbol{\eta}^{n+1}_{\boldsymbol{v},j}-\boldsymbol{\eta}^n_{\boldsymbol{v},j}+\boldsymbol{\eta}^{n-1}_{\boldsymbol{v},j})\|\|\nabla\boldsymbol{\varphi}^{n+1}_{j,h}\|\\
&\le \frac{\alpha_j}{36}\|\nabla\boldsymbol{\varphi}^{n+1}_{j,h}\|^2+\frac{C}{\alpha_j}\|\nabla\boldsymbol{w}^{'n}_{j,h}\|^2\|\nabla(\boldsymbol{\eta}^{n+1}_{\boldsymbol{v},j}-\boldsymbol{\eta}^n_{\boldsymbol{v},j}+\boldsymbol{\eta}^{n-1}_{\boldsymbol{v},j})\|^2.
\end{aligned}
$$

For the fourth nonlinear term, we use Hölder's inequality, Sobolev embedding theorems, Agmon's, Poincare's and Young's inequalities to reveal

$$
\begin{aligned}
&|b^*\big(2\boldsymbol{\psi}^n_{j,h}-\boldsymbol{\psi}^{n-1}_{j,h},\boldsymbol{v}_j(t^{n+1}),\boldsymbol{\varphi}^{n+1}_{j,h}\big)|\\
&\le C\|2\boldsymbol{\psi}^n_{j,h}-\boldsymbol{\psi}^{n-1}_{j,h}\|\|\nabla\boldsymbol{v}_j(t^{n+1})\|_{\boldsymbol{L}^6}\|\boldsymbol{\varphi}^{n+1}_{j,h}\|_{\boldsymbol{L}^3}+C\|2\boldsymbol{\psi}^n_{j,h}-\boldsymbol{\psi}^{n-1}_{j,h}\|\|\nabla\boldsymbol{\varphi}^{n+1}_{j,h}\|\|\boldsymbol{v}_j(t^{n+1})\|_\infty\\
&\le C\|2\boldsymbol{\psi}^n_{j,h}-\boldsymbol{\psi}^{n-1}_{j,h}\|\|\boldsymbol{v}_j(t^{n+1})\|_{\boldsymbol{H}^2}\|\boldsymbol{\varphi}^{n+1}_{j,h}\|^{1/2}\|\nabla\boldsymbol{\varphi}^{n+1}_{j,h}\|^{1/2}+C\|2\boldsymbol{\psi}^n_{j,h}-\boldsymbol{\psi}^{n-1}_{j,h}\|\|\nabla\boldsymbol{\varphi}^{n+1}_{j,h}\|\|\boldsymbol{v}_j(t^{n+1})\|_{\boldsymbol{H}^2}\\
&\le C\|2\boldsymbol{\psi}^n_{j,h}-\boldsymbol{\psi}^{n-1}_{j,h}\|\|\boldsymbol{v}_j(t^{n+1})\|_{\boldsymbol{H}^2}\|\nabla\boldsymbol{\varphi}^{n+1}_{j,h}\|\\
&\le \frac{\alpha_j}{36}\|\nabla\boldsymbol{\varphi}^{n+1}_{j,h}\|^2+\frac{C}{\alpha_j}\|\boldsymbol{v}_j(t^{n+1})\|^2_{\boldsymbol{H}^2}\|2\boldsymbol{\psi}^n_{j,h}-\boldsymbol{\psi}^{n-1}_{j,h}\|^2. \qquad (5.9)
\end{aligned}
$$

Applying inverse and Young's inequalities with (2.1) on the fifth nonlinear terms yields

$$
\begin{aligned}
|b^*\big(\boldsymbol{w}^{'n}_{j,h},\boldsymbol{\varphi}^{n+1}_{j,h}-2\boldsymbol{\varphi}^n_{j,h}+\boldsymbol{\varphi}^{n-1}_{j,h},\boldsymbol{\varphi}^{n+1}_{j,h}\big)| &\le C\|\nabla\boldsymbol{w}^{'n}_{j,h}\|\|\nabla(\boldsymbol{\varphi}^{n+1}_{j,h}-2\boldsymbol{\varphi}^n_{j,h}+\boldsymbol{\varphi}^{n-1}_{j,h})\|\|\nabla\boldsymbol{\varphi}^{n+1}_{j,h}\|\\
&\le \frac{\alpha_j}{36}\|\nabla\boldsymbol{\varphi}^{n+1}_{j,h}\|^2+\frac{C}{\alpha_j h^2}\|\nabla\boldsymbol{w}^{'n}_{j,h}\|^2\|\boldsymbol{\varphi}^{n+1}_{j,h}-2\boldsymbol{\varphi}^n_{j,h}+\boldsymbol{\varphi}^{n-1}_{j,h}\|^2. \qquad (5.10)
\end{aligned}
$$

Since $\boldsymbol{\varphi}^{n+1}_{j,h}\in\boldsymbol{V}_h$, we rewrite the pressure term, apply Cauchy-Schwarz and Young's inequalities to get

$$
\begin{aligned}
|\big(q_j(t^{n+1}),\nabla\cdot\boldsymbol{\varphi}^{n+1}_{j,h}\big)| &= |\big(q_j(t^{n+1})-q^{n+1}_{j,h},\nabla\cdot\boldsymbol{\varphi}^{n+1}_{j,h}\big)|\\
&\le \frac{\gamma}{4}\|\nabla\cdot\boldsymbol{\varphi}^{n+1}_{j,h}\|^2+\gamma^{-1}\|q_j(t^{n+1})-q^{n+1}_{j,h}\|^2, \quad \forall q^{n+1}_{j,h}\in Q_h.
\end{aligned}
$$

Next, we find the upper bound of the term with coefficient γ using Cauchy-Schwarz and Young's inequalities as

$$
\begin{aligned}
\gamma|\big(\nabla\cdot\boldsymbol{\eta}^{n+1}_{\boldsymbol{v},j},\nabla\cdot\boldsymbol{\varphi}^{n+1}_{j,h}\big)| &\le \gamma\|\nabla\cdot\boldsymbol{\eta}^{n+1}_{\boldsymbol{v},j}\|\|\nabla\cdot\boldsymbol{\varphi}^{n+1}_{j,h}\|\\
&\le \frac{\gamma}{2}\|\nabla\cdot\boldsymbol{\varphi}^{n+1}_{j,h}\|^2+\frac{\gamma}{2}\|\nabla\cdot\boldsymbol{\eta}^{n+1}_{\boldsymbol{v},j}\|^2.
\end{aligned}
$$

Using Taylor's series expansion about time-step size, Cauchy-Schwarz, Poincare's and Young's inequalities, the upper bound of the last term can be found as

$$\begin{aligned}|G_1(t,\boldsymbol{v}_j,\boldsymbol{w}_j,\boldsymbol{\varphi}_{j,h}^{n+1})| &\leq \frac{\alpha_j}{36}\|\nabla\boldsymbol{\varphi}_{j,h}^{n+1}\|^2+(\Delta t)^4\frac{45(\nu_j^{'}+\nu_{m,j}^{'})^2}{4\alpha_j}\|\nabla\boldsymbol{v}_{j,tt}(s_1^*)\|^2\\&+(\Delta t)^2\frac{45(\nu_j-\nu_{m,j})^2(1-\theta)^2}{4\alpha_j}\|\nabla\boldsymbol{w}_{j,t}(s_2^*)\|^2\\&+(\Delta t)^4\frac{C}{\alpha_j}\Big(\|\nabla\boldsymbol{w}_{j,tt}(s_3^*)\|^2\|\nabla\boldsymbol{v}_j(t^{n+1})\|^2+\|\nabla\boldsymbol{w}_{j,h}^{'n}\|^2\|\nabla\boldsymbol{v}_{j,tt}(s_4^*)\|^2+\|\boldsymbol{v}_{j,ttt}(s_5^*)\|^2\Big)\end{aligned}$$

with $s_1^*,s_2^*,s_3^*,s_4^*,s_5^*\in[t^{n-1},t^{n+1}]$. Using these estimates in (5.7) and reducing produces

$$\begin{aligned}&\frac{1}{4\Delta t}\Big(\|\boldsymbol{\varphi}_{j,h}^{n+1}\|^2-\|\boldsymbol{\varphi}_{j,h}^{n}\|^2+\|2\boldsymbol{\varphi}_{j,h}^{n+1}-\boldsymbol{\varphi}_{j,h}^{n}\|^2-\|2\boldsymbol{\varphi}_{j,h}^{n}-\boldsymbol{\varphi}_{j,h}^{n-1}\|^2\Big)\\&+\Big(\frac{1}{4\Delta t}-\frac{C}{\alpha_j h^2}\|\nabla\boldsymbol{w}_{j,h}^{'n}\|^2\Big)\|\boldsymbol{\varphi}_{j,h}^{n+1}-2\boldsymbol{\varphi}_{j,h}^{n}+\boldsymbol{\varphi}_{j,h}^{n-1}\|^2+\frac{\bar\nu+\bar\nu_m}{4}\|\nabla\boldsymbol{\varphi}_{j,h}^{n+1}\|^2+\frac{\gamma}{4}\|\nabla\cdot\boldsymbol{\varphi}_{j,h}^{n+1}\|^2\\&\leq(1+\theta)\frac{|\nu_j-\nu_{m,j}|}{4}\|\nabla\boldsymbol{\psi}_{j,h}^{n}\|^2+\theta\frac{|\nu_j-\nu_{m,j}|}{4}\|\nabla\boldsymbol{\psi}_{j,h}^{n-1}\|^2+\frac{9(1+\theta)^2(\nu_j-\nu_{m,j})^2}{4\alpha_j}\|\nabla\boldsymbol{\eta}_{\boldsymbol{w},j}^{n}\|^2\\&+\frac{9\theta^2(\nu_j-\nu_{m,j})^2}{4\alpha_j}\|\nabla\boldsymbol{\eta}_{\boldsymbol{w},j}^{n-1}\|^2+\frac{|\nu_j^{'}+\nu_{m,j}^{'}|}{4}(2\|\nabla\boldsymbol{\varphi}_{j,h}^{n}\|^2+\|\nabla\boldsymbol{\varphi}_{j,h}^{n-1}\|^2)+\frac{9(\bar\nu+\bar\nu_m)^2}{4\alpha_j}\|\nabla\boldsymbol{\eta}_{\boldsymbol{v},j}^{n+1}\|^2\\&+\frac{C}{\alpha_j}\|\nabla(2\boldsymbol{\eta}_{\boldsymbol{w},j}^{n}-\boldsymbol{\eta}_{\boldsymbol{w},j}^{n-1})\|^2\|\nabla\boldsymbol{v}_j(t^{n+1})\|^2+\frac{C}{\alpha_j}\|\nabla(2\boldsymbol{w}_{j,h}^{n}-\boldsymbol{w}_{j,h}^{n-1})\|^2\|\nabla\boldsymbol{\eta}_{\boldsymbol{v},j}^{n+1}\|^2\\&+\frac{C}{\alpha_j}\|\nabla\boldsymbol{w}_{j,h}^{'n}\|^2\|\nabla(\boldsymbol{\eta}_{\boldsymbol{v},j}^{n+1}-\boldsymbol{\eta}_{\boldsymbol{v},j}^{n}+\boldsymbol{\eta}_{\boldsymbol{v},j}^{n-1})\|^2+\frac{C}{\alpha_j}\|\boldsymbol{v}_j(t^{n+1})\|_{H^2}^2\|2\boldsymbol{\psi}_{j,h}^{n}-\boldsymbol{\psi}_{j,h}^{n-1}\|^2\\&+(\Delta t)^4\frac{45(\nu_j^{'}+\nu_{m,j}^{'})^2}{4\alpha_j}\|\nabla\boldsymbol{v}_{j,tt}(s_1^*)\|^2+(\Delta t)^2\frac{45(\nu_j-\nu_{m,j})^2(1-\theta)^2}{4\alpha_j}\|\nabla\boldsymbol{w}_{j,t}(s_2^*)\|^2\\&+\frac{\gamma}{2}\|\nabla\cdot\boldsymbol{\eta}_{\boldsymbol{v},j}^{n+1}\|^2+\gamma^{-1}\|q_j(t^{n+1})-q_{j,h}^{n+1}\|^2\\&+(\Delta t)^4\frac{C}{\alpha_j}\Big(\|\nabla\boldsymbol{w}_{j,tt}(s_3^*)\|^2\|\nabla\boldsymbol{v}_j(t^{n+1})\|^2+\|\nabla\boldsymbol{w}_{j,h}^{'n}\|^2\|\nabla\boldsymbol{v}_{j,tt}(s_4^*)\|^2+\|\boldsymbol{v}_{j,ttt}(s_5^*)\|^2\Big),\end{aligned}\tag{5.11}$$

Applying similar techniques to (5.8), we get

$$\begin{aligned}&\frac{1}{4\Delta t}\Big(\|\boldsymbol{\psi}_{j,h}^{n+1}\|^2-\|\boldsymbol{\psi}_{j,h}^{n}\|^2+\|2\boldsymbol{\psi}_{j,h}^{n+1}-\boldsymbol{\psi}_{j,h}^{n}\|^2-\|2\boldsymbol{\psi}_{j,h}^{n}-\boldsymbol{\psi}_{j,h}^{n-1}\|^2\Big)\\&+\Big(\frac{1}{4\Delta t}-\frac{C}{\alpha_j h^2}\|\nabla\boldsymbol{v}_{j,h}^{'n}\|^2\Big)\|\boldsymbol{\psi}_{j,h}^{n+1}-\boldsymbol{\psi}_{j,h}^{n}+\boldsymbol{\psi}_{j,h}^{n-1}\|^2+\frac{\bar\nu+\bar\nu_m}{4}\|\nabla\boldsymbol{\psi}_{j,h}^{n+1}\|^2+\frac{\gamma}{4}\|\nabla\cdot\boldsymbol{\psi}_{j,h}^{n+1}\|^2\\&\leq(1+\theta)\frac{|\nu_j-\nu_{m,j}|}{4}\|\nabla\boldsymbol{\varphi}_{j,h}^{n}\|^2+\theta\frac{|\nu_j-\nu_{m,j}|}{4}\|\nabla\boldsymbol{\varphi}_{j,h}^{n-1}\|^2+\frac{9(1+\theta)^2(\nu_j-\nu_{m,j})^2}{4\alpha_j}\|\nabla\boldsymbol{\eta}_{\boldsymbol{v},j}^{n}\|^2\\&+\frac{9\theta^2(\nu_j-\nu_{m,j})^2}{4\alpha_j}\|\nabla\boldsymbol{\eta}_{\boldsymbol{v},j}^{n-1}\|^2+\frac{|\nu_j^{'}+\nu_{m,j}^{'}|}{4}(2\|\nabla\boldsymbol{\psi}_{j,h}^{n}\|^2+\|\nabla\boldsymbol{\psi}_{j,h}^{n-1}\|^2)+\frac{9(\bar\nu+\bar\nu_m)^2}{4\alpha_j}\|\nabla\boldsymbol{\eta}_{\boldsymbol{w},j}^{n+1}\|^2\\&+\frac{C}{\alpha_j}\|\nabla(2\boldsymbol{\eta}_{\boldsymbol{v},j}^{n}-\boldsymbol{\eta}_{\boldsymbol{v},j}^{n-1})\|^2\|\nabla\boldsymbol{w}_j(t^{n+1})\|^2+\frac{C}{\alpha_j}\|\nabla(2\boldsymbol{v}_{j,h}^{n}-\boldsymbol{v}_{j,h}^{n-1})\|^2\|\nabla\boldsymbol{\eta}_{\boldsymbol{w},j}^{n+1}\|^2\\&+\frac{C}{\alpha_j}\|\nabla\boldsymbol{v}_{j,h}^{'n}\|^2\|\nabla(\boldsymbol{\eta}_{\boldsymbol{w},j}^{n+1}-\boldsymbol{\eta}_{\boldsymbol{w},j}^{n}+\boldsymbol{\eta}_{\boldsymbol{w},j}^{n-1})\|^2+\frac{C}{\alpha_j}\|\boldsymbol{w}_j(t^{n+1})\|_{H^2}^2\|2\boldsymbol{\varphi}_{j,h}^{n}-\boldsymbol{\varphi}_{j,h}^{n-1}\|^2\\&+(\Delta t)^4\frac{45(\nu_j^{'}+\nu_{m,j}^{'})^2}{4\alpha_j}\|\nabla\boldsymbol{w}_{j,tt}(t_1^*)\|^2+(\Delta t)^2\frac{45(\nu_j-\nu_{m,j})^2(1-\theta)^2}{4\alpha_j}\|\nabla\boldsymbol{v}_{j,t}(t_2^*)\|^2\\&+\frac{\gamma}{2}\|\nabla\cdot\boldsymbol{\eta}_{\boldsymbol{w},j}^{n+1}\|^2+\gamma^{-1}\|r_j(t^{n+1})-r_{j,h}^{n+1}\|^2\\&+(\Delta t)^4\frac{C}{\alpha_j}\Big(\|\nabla\boldsymbol{v}_{j,tt}(t_3^*)\|^2\|\nabla\boldsymbol{w}_j(t^{n+1})\|^2+\|\nabla\boldsymbol{v}_{j,h}^{'n}\|^2\|\nabla\boldsymbol{w}_{j,tt}(t_4^*)\|^2+\|\boldsymbol{w}_{j,ttt}(t_5^*)\|^2\Big),\end{aligned}\tag{5.12}$$

with $t_1^*, t_2^*, t_3^*, t_4^*, t_5^* \in [t^{n-1}, t^{n+1}]$. Now add (5.11) and (5.12), assume

$$\Delta t \leq \frac{\alpha_{\min} h^2}{C \max\limits_{1\leq j\leq J} \left\{ \|\nabla \boldsymbol{v}'^{n}_{j,h}\|^2, \|\nabla \boldsymbol{w}'^{n}_{j,h}\|^2 \right\}}, \tag{5.13}$$

drops non-negative terms from left-hand-side, and rearrange, we have

$$\begin{aligned}
\frac{1}{4\Delta t}&\left(\|\boldsymbol{\varphi}^{n+1}_{j,h}\|^2 - \|\boldsymbol{\varphi}^{n}_{j,h}\|^2 + \|2\boldsymbol{\varphi}^{n+1}_{j,h} - \boldsymbol{\varphi}^{n}_{j,h}\|^2 - \|2\boldsymbol{\varphi}^{n}_{j,h} - \boldsymbol{\varphi}^{n-1}_{j,h}\|^2\right)\\
+\frac{1}{4\Delta t}&\left(\|\boldsymbol{\psi}^{n+1}_{j,h}\|^2 - \|\boldsymbol{\psi}^{n}_{j,h}\|^2 + \|2\boldsymbol{\psi}^{n+1}_{j,h} - \boldsymbol{\psi}^{n}_{j,h}\|^2 - \|2\boldsymbol{\psi}^{n}_{j,h} - \boldsymbol{\psi}^{n-1}_{j,h}\|^2\right)\\
&+\frac{\bar{\nu}+\bar{\nu}_m}{4}\left(\|\nabla\boldsymbol{\varphi}^{n+1}_{j,h}\|^2 - \|\nabla\boldsymbol{\varphi}^{n}_{j,h}\|^2 + \|\nabla\boldsymbol{\psi}^{n+1}_{j,h}\|^2 - \|\nabla\boldsymbol{\psi}^{n}_{j,h}\|^2\right)\\
+\frac{\alpha_j + \theta|\nu_j - \nu_{m,j}| + |\nu_j' + \nu_{m,j}'|}{4}&\left(\|\nabla\boldsymbol{\varphi}^{n}_{j,h}\|^2 - \|\nabla\boldsymbol{\varphi}^{n-1}_{j,h}\|^2 + \|\nabla\boldsymbol{\psi}^{n}_{j,h}\|^2 - \|\nabla\boldsymbol{\psi}^{n-1}_{j,h}\|^2\right)\\
&+\frac{\alpha_j}{4}\left(\|\nabla\boldsymbol{\varphi}^{n-1}_{j,h}\|^2 + \|\nabla\boldsymbol{\psi}^{n-1}_{j,h}\|^2\right) + \frac{\gamma}{4}\left(\|\nabla\cdot\boldsymbol{\varphi}^{n+1}_{j,h}\|^2 + \|\nabla\cdot\boldsymbol{\psi}^{n+1}_{j,h}\|^2\right)\\
&\leq \frac{9(1+\theta)^2(\nu_j - \nu_{m,j})^2}{4\alpha_j}\left(\|\nabla\boldsymbol{\eta}^{n}_{\boldsymbol{v},j}\|^2 + \|\nabla\boldsymbol{\eta}^{n}_{\boldsymbol{w},j}\|^2\right)\\
+\frac{9\theta^2(\nu_j-\nu_{m,j})^2}{4\alpha_j}&\left(\|\nabla\boldsymbol{\eta}^{n-1}_{\boldsymbol{v},j}\|^2 + \|\nabla\boldsymbol{\eta}^{n-1}_{\boldsymbol{w},j}\|^2\right) + \frac{9(\bar{\nu}+\bar{\nu}_m)^2}{4\alpha_j}\left(\|\nabla\boldsymbol{\eta}^{n+1}_{\boldsymbol{v},j}\|^2 + \|\nabla\boldsymbol{\eta}^{n+1}_{\boldsymbol{w},j}\|^2\right)\\
&+\frac{C}{\alpha_j}\|\nabla(2\boldsymbol{\eta}^{n}_{\boldsymbol{w},j} - \boldsymbol{\eta}^{n-1}_{\boldsymbol{w},j})\|^2\|\nabla\boldsymbol{v}_j(t^{n+1})\|^2 + \frac{C}{\alpha_j}\|\nabla(2\boldsymbol{\eta}^{n}_{\boldsymbol{v},j} - \boldsymbol{\eta}^{n-1}_{\boldsymbol{v},j})\|^2\|\nabla\boldsymbol{w}_j(t^{n+1})\|^2\\
&+\frac{C}{\alpha_j}\|\nabla(2\boldsymbol{w}^{n}_{j,h} - \boldsymbol{w}^{n-1}_{j,h})\|^2\|\nabla\boldsymbol{\eta}^{n+1}_{\boldsymbol{v},j}\|^2 + \frac{C}{\alpha_j}\|\nabla(2\boldsymbol{v}^{n}_{j,h} - \boldsymbol{v}^{n-1}_{j,h})\|^2\|\nabla\boldsymbol{\eta}^{n+1}_{\boldsymbol{w},j}\|^2\\
+\frac{C}{\alpha_j}&\|\nabla\boldsymbol{w}'^{n}_{j,h}\|^2\|\nabla(\boldsymbol{\eta}^{n+1}_{\boldsymbol{v},j} - \boldsymbol{\eta}^{n}_{\boldsymbol{v},j} + \boldsymbol{\eta}^{n-1}_{\boldsymbol{v},j})\|^2 + \frac{C}{\alpha_j}\|\nabla\boldsymbol{v}'^{n}_{j,h}\|^2\|\nabla(\boldsymbol{\eta}^{n+1}_{\boldsymbol{w},j} - \boldsymbol{\eta}^{n}_{\boldsymbol{w},j} + \boldsymbol{\eta}^{n-1}_{\boldsymbol{w},j})\|^2\\
&+\frac{C}{\alpha_j}\|\boldsymbol{v}_j(t^{n+1})\|^2_{H^2}\|2\boldsymbol{\psi}^{n}_{j,h} - \boldsymbol{\psi}^{n-1}_{j,h}\|^2 + \frac{C}{\alpha_j}\|\boldsymbol{w}_j(t^{n+1})\|^2_{H^2}\|2\boldsymbol{\varphi}^{n}_{j,h} - \boldsymbol{\varphi}^{n-1}_{j,h}\|^2\\
&+(\Delta t)^4\frac{45(\nu_j' + \nu_{m,j}')^2}{4\alpha_j}\left(\|\nabla\boldsymbol{v}_{j,tt}(s_1^*)\|^2 + \|\nabla\boldsymbol{w}_{j,tt}(t_1^*)\|^2\right)\\
&+(\Delta t)^2\frac{45(\nu_j - \nu_{m,j})^2(1-\theta)^2}{4\alpha_j}\left(\|\nabla\boldsymbol{v}_{j,t}(t_2^*)\|^2 + \|\nabla\boldsymbol{w}_{j,t}(s_2^*)\|^2\right)\\
+\frac{\gamma}{2}&\left(\|\nabla\cdot\boldsymbol{\eta}^{n+1}_{\boldsymbol{v},j}\|^2 + \|\nabla\cdot\boldsymbol{\eta}^{n+1}_{\boldsymbol{w},j}\|^2\right) + \gamma^{-1}\left(\|q_j(t^{n+1}) - q^{n+1}_{j,h}\|^2 + \|r_j(t^{n+1}) - r^{n+1}_{j,h}\|^2\right)\\
+(\Delta t)^4\frac{C}{\alpha_j}&\Big(\|\nabla\boldsymbol{w}_{j,tt}(s_3^*)\|^2\|\nabla\boldsymbol{v}_j(t^{n+1})\|^2 + \|\nabla\boldsymbol{w}'^{n}_{j,h}\|^2\|\nabla\boldsymbol{v}_{j,tt}(s_4^*)\|^2 + \|\boldsymbol{v}_{j,ttt}(s_5^*)\|^2\Big)\\
+(\Delta t)^4\frac{C}{\alpha_j}&\Big(\|\nabla\boldsymbol{v}_{j,tt}(t_3^*)\|^2\|\nabla\boldsymbol{w}_j(t^{n+1})\|^2 + \|\nabla\boldsymbol{v}'^{n}_{j,h}\|^2\|\nabla\boldsymbol{w}_{j,tt}(t_4^*)\|^2 + \|\boldsymbol{w}_{j,ttt}(t_5^*)\|^2\Big).
\end{aligned}\tag{5.14}$$

Multiply both sides by $4\Delta t$, use stability and regularity assumptions, $\|\boldsymbol{\varphi}^0_{j,h}\| = \|\boldsymbol{\psi}^0_{j,h}\| = \|\boldsymbol{\varphi}^1_{j,h}\| =$

$\|\boldsymbol{\psi}^1_{j,h}\| = 0$, $\Delta t M = T$, and sum over the time steps to find

$$\begin{aligned}
&\|\boldsymbol{\varphi}^M_{j,h}\|^2 + \|2\boldsymbol{\varphi}^M_{j,h} - \boldsymbol{\varphi}^{M-1}_{j,h}\|^2 + \|\boldsymbol{\psi}^M_{j,h}\|^2 + \|2\boldsymbol{\psi}^M_{j,h} - \boldsymbol{\psi}^{M-1}_{j,h}\|^2 \\
&+ \alpha_j \Delta t \sum_{n=2}^{M} (\|\nabla \boldsymbol{\varphi}^n_{j,h}\|^2 + \|\nabla \boldsymbol{\psi}^n_{j,h}\|^2) + \gamma \Delta t \sum_{n=2}^{M} \left(\|\nabla\cdot \boldsymbol{\varphi}^n_{j,h}\|^2 + \|\nabla\cdot \boldsymbol{\psi}^n_{j,h}\|^2\right) \\
&\le \frac{9(1+\theta)^2(\nu_j - \nu_{m,j})^2}{\alpha_j} \Delta t \sum_{n=1}^{M-1} (\|\nabla \boldsymbol{\eta}^n_{\boldsymbol{v},j}\|^2 + \|\nabla \boldsymbol{\eta}^n_{\boldsymbol{w},j}\|^2) \\
&+ \frac{9\theta^2(\nu_j - \nu_{m,j})^2}{\alpha_j} \Delta t \sum_{n=1}^{M-1} (\|\nabla \boldsymbol{\eta}^{n-1}_{\boldsymbol{v},j}\|^2 + \|\nabla \boldsymbol{\eta}^{n-1}_{\boldsymbol{w},j}\|^2) \\
&+ \frac{9(\bar\nu + \bar\nu_m)^2}{\alpha_j} \Delta t \sum_{n=1}^{M-1} (\|\nabla \boldsymbol{\eta}^{n+1}_{\boldsymbol{v},j}\|^2 + \|\nabla \boldsymbol{\eta}^{n+1}_{\boldsymbol{w},j}\|^2) \\
&+ \frac{C}{\alpha_j} \Delta t \sum_{n=1}^{M-1} \left\{ \|\nabla(2\boldsymbol{\eta}^n_{\boldsymbol{w},j} - \boldsymbol{\eta}^{n-1}_{\boldsymbol{w},j})\|^2 \|\nabla \boldsymbol{v}_j(t^{n+1})\|^2 + \|\nabla(2\boldsymbol{\eta}^n_{\boldsymbol{v},j} - \boldsymbol{\eta}^{n-1}_{\boldsymbol{v},j})\|^2 \|\nabla \boldsymbol{w}_j(t^{n+1})\|^2 \right\} \\
&+ \frac{C}{\alpha_j} \Delta t \sum_{n=1}^{M-1} \left\{ \|\nabla(2\boldsymbol{w}^n_{j,h} - \boldsymbol{w}^{n-1}_{j,h})\|^2 \|\nabla \boldsymbol{\eta}^{n+1}_{j,\boldsymbol{v}}\|^2 + \|\nabla(2\boldsymbol{v}^n_{j,h} - \boldsymbol{v}^{n-1}_{j,h})\|^2 \|\nabla \boldsymbol{\eta}^{n+1}_{\boldsymbol{w},j}\|^2 \right\} \\
&+ \frac{C}{\alpha_j} \Delta t \sum_{n=1}^{M-1} \left\{ \|\nabla \boldsymbol{w}'^n_{j,h}\|^2 \|\nabla(\boldsymbol{\eta}^{n+1}_{\boldsymbol{v},j} - \boldsymbol{\eta}^n_{\boldsymbol{v},j} + \boldsymbol{\eta}^{n-1}_{\boldsymbol{v},j})\|^2 + \|\nabla \boldsymbol{v}'^n_{j,h}\|^2 \|\nabla(\boldsymbol{\eta}^{n+1}_{\boldsymbol{w},j} - \boldsymbol{\eta}^n_{\boldsymbol{w},j} + \boldsymbol{\eta}^{n-1}_{\boldsymbol{w},j})\|^2 \right\} \\
&+ \frac{C}{\alpha_j} \Delta t \sum_{n=1}^{M-1} \left\{ \|\boldsymbol{v}_j(t^{n+1})\|^2_{H^2} \|2\boldsymbol{\psi}^n_{j,h} - \boldsymbol{\psi}^{n-1}_{j,h}\|^2 + \|\boldsymbol{w}_j(t^{n+1})\|^2_{H^2} \|2\boldsymbol{\varphi}^n_{j,h} - \boldsymbol{\varphi}^{n-1}_{j,h}\|^2 \right\} \\
&+ C\big((\Delta t)^4 + (\nu_j - \nu_{m,j})(1-\theta)^2(\Delta t)^2\big) \\
&+ 2\gamma \Delta t \sum_{n=1}^{M-1} \left(\|\nabla\cdot \boldsymbol{\eta}^{n+1}_{\boldsymbol{v},j}\|^2 + \|\nabla\cdot \boldsymbol{\eta}^{n+1}_{\boldsymbol{w},j}\|^2\right) \\
&+ \frac{4\Delta t}{\gamma} \sum_{n=1}^{M-1} \left(\|q_j(t^{n+1}) - q^{n+1}_{j,h}\|^2 + \|r_j(t^{n+1}) - r^{n+1}_{j,h}\|^2 \right). \qquad (5.15)
\end{aligned}$$

$$\begin{aligned}
&\|\boldsymbol{\varphi}^M_{j,h}\|^2 + \|2\boldsymbol{\varphi}^M_{j,h} - \boldsymbol{\varphi}^{M-1}_{j,h}\|^2 + \|\boldsymbol{\psi}^M_{j,h}\|^2 + \|2\boldsymbol{\psi}^M_{j,h} - \boldsymbol{\psi}^{M-1}_{j,h}\|^2 \\
&+ \alpha_j \Delta t \sum_{n=2}^{M} (\|\nabla \boldsymbol{\varphi}^n_{j,h}\|^2 + \|\nabla \boldsymbol{\psi}^n_{j,h}\|^2) + \gamma \Delta t \sum_{n=2}^{M} \left(\|\nabla\cdot \boldsymbol{\varphi}^n_{j,h}\|^2 + \|\nabla\cdot \boldsymbol{\psi}^n_{j,h}\|^2\right) \\
&\le \frac{C(1+\theta)^2(\nu_j - \nu_{m,j})^2}{\alpha_j} h^{2k} + \frac{C\theta^2(\nu_j - \nu_{m,j})^2}{\alpha_j} h^{2k} + \frac{C(\bar\nu + \bar\nu_m)^2}{\alpha_j} h^{2k} \\
&+ \frac{CT}{\alpha_j} h^{2k} \left(\|\boldsymbol{v}_j(t)\|^2_{L^\infty\left(0,T,H^1(\Omega)\right)} + \|\boldsymbol{w}_j(t)\|^2_{L^\infty\left(0,T,H^1(\Omega)\right)} \right) + \frac{C}{\alpha_j} h^{2k} \\
&+ \frac{C}{\alpha_j} \Delta t \sum_{n=1}^{M-1} \left\{ \|\boldsymbol{v}_j(t)\|^2_{L^\infty(0,T,H^2(\Omega))} \|2\boldsymbol{\psi}^n_{j,h} - \boldsymbol{\psi}^{n-1}_{j,h}\|^2 + \|\boldsymbol{w}_j(t)\|^2_{L^\infty(0,T,H^2(\Omega))} \|2\boldsymbol{\varphi}^n_{j,h} - \boldsymbol{\varphi}^{n-1}_{j,h}\|^2 \right\} \\
&+ C\big((\Delta t)^4 + (\nu_j - \nu_{m,j})(1-\theta)^2(\Delta t)^2\big) + C\gamma h^{2k} \\
&+ \frac{CT}{\gamma} h^{2k} \sum_{n=1}^{M-1} \left(\|q_j(t^{n+1}) - q^{n+1}_{j,h}\|^2 + \|r_j(t^{n+1}) - r^{n+1}_{j,h}\|^2 \right). \qquad (5.16)
\end{aligned}$$

Using triangular and Young's inequalities, dropping non-negative terms from left-hand-side, and simplifying yield

$$
\begin{aligned}
\|\boldsymbol{\varphi}_{j,h}^M\|^2+&\|\boldsymbol{\psi}_{j,h}^M\|^2+\alpha_j\Delta t\sum_{n=2}^{M}(\|\nabla\boldsymbol{\varphi}_{j,h}^n\|^2+\|\nabla\boldsymbol{\psi}_{j,h}^n\|^2)+\gamma\Delta t\sum_{n=2}^{M}\big(\|\nabla\cdot\boldsymbol{\varphi}_{j,h}^n\|^2+\|\nabla\cdot\boldsymbol{\psi}_{j,h}^n\|^2\big)\\
&\le\Delta t\sum_{n=2}^{M-1}\frac{C}{\alpha_j}\big(\|\boldsymbol{\varphi}_{j,h}^n\|^2+\|\boldsymbol{\psi}_{j,h}^n\|^2\big)+C\big(h^{2k}+(\Delta t)^4+(\nu_j-\nu_{m,j})(1-\theta)^2(\Delta t)^2\big).
\end{aligned}
\tag{5.17}
$$

Applying the discrete Gronwall lemma 2.1, we have

$$
\begin{aligned}
\|\boldsymbol{\varphi}_{j,h}^M\|^2+\|\boldsymbol{\psi}_{j,h}^M\|^2+&\alpha_j\Delta t\sum_{n=2}^{M}(\|\nabla\boldsymbol{\varphi}_{j,h}^n\|^2+\|\nabla\boldsymbol{\psi}_{j,h}^n\|^2)+\gamma\Delta t\sum_{n=2}^{M}\big(\|\nabla\cdot\boldsymbol{\varphi}_{j,h}^n\|^2+\|\nabla\cdot\boldsymbol{\psi}_{j,h}^n\|^2\big)\\
&\le C\exp\left(\frac{CT}{\alpha_j}\right)\big(h^{2k}+(\Delta t)^4+(\nu_j-\nu_{m,j})(1-\theta)^2(\Delta t)^2\big).
\end{aligned}
\tag{5.18}
$$

Sum over $j=1,2,\cdots,J$ to obtain

$$
\begin{aligned}
&\sum_{j=1}^{J}\|\boldsymbol{\varphi}_{j,h}^M\|^2+\sum_{j=1}^{J}\|\boldsymbol{\psi}_{j,h}^M\|^2+\alpha_{\min}\Delta t\sum_{j=1}^{J}\sum_{n=2}^{M}(\|\nabla\boldsymbol{\varphi}_{j,h}^n\|^2+\|\nabla\boldsymbol{\psi}_{j,h}^n\|^2)\\
&+\gamma\Delta t\sum_{j=1}^{J}\sum_{n=2}^{M}\big(\|\nabla\cdot\boldsymbol{\varphi}_{j,h}^n\|^2+\|\nabla\cdot\boldsymbol{\psi}_{j,h}^n\|^2\big)\\
&\le CJ\exp\left(\frac{CT}{\alpha_{\min}}\right)\big(h^{2k}+(\Delta t)^4+(\bar{\nu}-\bar{\nu}_m)(1-\theta)^2(\Delta t)^2\big).
\end{aligned}
\tag{5.19}
$$

Define the average error as $\bar{\boldsymbol{e}}_{\boldsymbol{v}}^n:=\frac{1}{J}\sum\limits_{j=1}^{J}\boldsymbol{e}_{\boldsymbol{v},j}^n$, and use the triangle and Young's inequality, we can write

$$
\begin{aligned}
\|\bar{\boldsymbol{e}}_{\boldsymbol{v}}^M\|^2+\|\bar{\boldsymbol{e}}_{\boldsymbol{w}}^M\|^2&+\alpha_{\min}\Delta t\sum_{n=2}^{M}\big(\|\nabla\bar{\boldsymbol{e}}_{\boldsymbol{v}}^n\|^2+\|\nabla\bar{\boldsymbol{e}}_{\boldsymbol{w}}^n\|^2\big)+\gamma\Delta t\sum_{n=2}^{M}\big(\|\nabla\cdot\bar{\boldsymbol{e}}_{\boldsymbol{v}}^n\|^2+\|\nabla\cdot\bar{\boldsymbol{e}}_{\boldsymbol{w}}^n\|^2\big)\\
\le 2\Bigg(&\sum_{j=1}^{J}\|\boldsymbol{e}_{\boldsymbol{v},j}^M\|^2+\sum_{j=1}^{J}\|\boldsymbol{e}_{\boldsymbol{w},j}^M\|^2+\alpha_{\min}\Delta t\sum_{j=1}^{J}\sum_{n=2}^{M}(\|\nabla\boldsymbol{e}_{\boldsymbol{v},j}^n\|^2+\|\nabla\boldsymbol{e}_{\boldsymbol{w},j}^n\|^2)\\
&+\gamma\Delta t\sum_{j=1}^{J}\sum_{n=2}^{M}\big(\|\nabla\cdot\boldsymbol{e}_{\boldsymbol{v},j}^n\|^2+\|\nabla\cdot\boldsymbol{e}_{\boldsymbol{w},j}^n\|^2\big)\Bigg)\\
\le 4\Bigg(&\sum_{j=1}^{J}\|\boldsymbol{\varphi}_{j,h}^M\|^2+\sum_{j=1}^{J}\|\boldsymbol{\psi}_{j,h}^M\|^2+\alpha_{\min}\Delta t\sum_{j=1}^{J}\sum_{n=2}^{M}(\|\nabla\boldsymbol{\varphi}_{j,h}^n\|^2+\|\nabla\boldsymbol{\psi}_{j,h}^n\|^2)\\
&+\gamma\Delta t\sum_{j=1}^{J}\sum_{n=2}^{M}\big(\|\nabla\cdot\boldsymbol{\varphi}_{j,h}^n\|^2+\|\nabla\cdot\boldsymbol{\psi}_{j,h}^n\|^2\big)+\sum_{j=1}^{J}\|\boldsymbol{\eta}_{\boldsymbol{v},j}^M\|^2+\sum_{j=1}^{J}\|\boldsymbol{\eta}_{\boldsymbol{w},j}^M\|^2\\
&+\alpha_{\min}\Delta t\sum_{j=1}^{J}\sum_{n=2}^{M}(\|\nabla\boldsymbol{\eta}_{\boldsymbol{v},j}^n\|^2+\|\nabla\boldsymbol{\eta}_{\boldsymbol{w},j}^n\|^2)+\gamma\Delta t\sum_{j=1}^{J}\sum_{n=2}^{M}\big(\|\nabla\cdot\boldsymbol{\eta}_{\boldsymbol{v},j}^n\|^2+\|\nabla\cdot\boldsymbol{\eta}_{\boldsymbol{w},j}^n\|^2\big)\Bigg)\\
&\le CJ\exp\left(\frac{CT}{\alpha_{\min}}\right)\big(h^{2k}+(\Delta t)^4+(\bar{\nu}-\bar{\nu}_m)(1-\theta)^2(\Delta t)^2\big).
\end{aligned}
\tag{5.20}
$$

□

REFERENCES

[1] M. Akbas, S. Kaya, M. Mohebujjaman, and L. Rebholz. Numerical analysis and testing of a fully discrete, decoupled penalty-projection algorithm for MHD in elsässer variable. International Journal of Numerical Analysis & Modeling, 13(1):90–113, 2016.

[2] D. Arnold and J. Qin. Quadratic velocity/linear pressure Stokes elements. Advances in Computer Methods for Partial Differential Equations, 7:28–34, 1992.
[3] D. Biskamp. Magnetohydrodynamic Turbulence. Cambridge University Press, Cambridge, 2003.
[4] S. C. Brenner and L. R. Scott. The Mathematical Theory of Finite Element Methods. Texts in Applied Mathematics, 15. Springer Science+Business Media, LLC, 2008.
[5] P. A. Davidson. An introduction to magnetohydrodynamics. Cambridge Texts in Applied Mathematics, Cambridge University Press, Cambridge, 2001.
[6] W. M. Elsässer. The hydromagnetic equations. Phys. Rev., 79:183, 1950.
[7] D. Erkmen, S. Kaya, and A. Çıbık. A second order decoupled penalty projection method based on deferred correction for MHD in Elsässer variable. Journal of Computational and Applied Mathematics, 371:112694, 2020.
[8] J. Fiordilino and M. Winger. Unconditionally energy stable and first-order accurate numerical schemes for the heat equation with uncertain temperature-dependent conductivity. arXiv preprint arXiv:2106.02754, 2021.
[9] T. Fujita, D. J. Stensrud, and D. C. Dowell. Surface data assimilation using an ensemble Kalman filter approach with initial condition and model physics uncertainties. Monthly Weather Review, 135(5):1846–1868, 2007.
[10] V. Girault and P.-A.Raviart. Finite Element Methods for Navier-Stokes Equations: Theory and Algorithms. Springer-Verlag, 1986.
[11] M. Gunzburger, N. Jiang, and Z. Wang. An efficient algorithm for simulating ensembles of parameterized flow problems. IMA Journal of Numerical Analysis, 39(3):1180–1205, 2019.
[12] M. Gunzburger, N. Jiang, and Z. Wang. A second-order time-stepping scheme for simulating ensembles of parameterized flow problems. Computational Methods in Applied Mathematics, 19(3):681–701, 2019.
[13] T. Heister, M. Mohebujjaman, and L. Rebholz. Decoupled, unconditionally stable, higher order discretizations for MHD flow simulation. Journal of Scientific Computing, 71:21–43, 2017.
[14] J. G. Heywood and R. Rannacher. Finite-Element approximation of the nonstationary Navier-Stokes problem part IV: error analysis for second-order time discretization. SIAM Journal on Numerical Analysis, 27:353–384, 1990.
[15] N. Jiang. A higher order ensemble simulation algorithm for fluid flows. Journal of Scientific Computing, 64:264–288, 2015.
[16] N. Jiang. A second-order ensemble method based on a blended backward differentiation formula timestepping scheme for time-dependent Navier–Stokes equations. Numerical Methods for Partial Differential Equations, 33(1):34–61, 2017.
[17] N. Jiang, S. Kaya, and W. Layton. Analysis of model variance for ensemble based turbulence modeling. Computational Methods in Applied Mathematics, 15:173–188, 2015.
[18] N. Jiang and W. Layton. An algorithm for fast calculation of flow ensembles. International Journal for Uncertainty Quantification, 4:273–301, 2014.
[19] N. Jiang and W. Layton. Numerical analysis of two ensemble eddy viscosity numerical regularizations of fluid motion. Numerical Methods for Partial Differential Equations, 31:630–651, 2015.
[20] N. Jiang, Y. Li, and H. Yang. An artificial compressibility Crank–Nicolson leap-frog method for the Stokes–Darcy model and application in ensemble simulations. SIAM Journal on Numerical Analysis, 59(1):401–428, 2021.
[21] N. Jiang and M. Schneier. An efficient, partitioned ensemble algorithm for simulating ensembles of evolutionary MHD flows at low magnetic Reynolds number. Numerical Methods for Partial Differential Equations, 34(6):2129–2152, 2018.
[22] L. Ju, W. Leng, Z. Wang, and S. Yuan. Numerical investigation of ensemble methods with block iterative solvers for evolution problems. Discrete & Continuous Dynamical Systems-B, 25(12):4905, 2020.
[23] L. D. Landau and E. M. Lifshitz. Electrodynamics of Continuous Media. Pergamon Press, Oxford, 1960.
[24] J. M. Lewis. Roots of ensemble forecasting. Monthly Weather Review, 133:1865 – 1885, 2005.
[25] Y. Li and C. Trenchea. Partitioned second order method for magnetohydrodynamics in elsässer variables. Discrete & Continuous Dynamical Systems-B, 23(7):2803, 2018.
[26] N. Jiang M. Gunzburger and Z. Wang. A second-order time-stepping scheme for simulating ensembles of parameterized flow problems. Computational Methods in Applied Mathematics, 1(4):349–364, 1988.
[27] T. N. Palmer M. Leutbecher. Ensemble forecasting. Journal of Computational Physics, 227:3515–3539, 2008.
[28] O. P. L. Maître and O. M. Knio. Spectral methods for uncertainty quantification. Springer, 2010.
[29] W. J. Martin and M. Xue. Sensitivity analysis of convection of the 24 May 2002 IHOP case using very large ensembles. Monthly Weather Review, 134(1):192–207, 2006.
[30] M. Mohebujjaman. High order efficient algorithm for computation of MHD flow ensembles. Advances in Applied Mathematics and Mechanics, in press, 2021.
[31] M. Mohebujjaman and L. G. Rebholz. An efficient algorithm for computation of MHD flow ensembles. Computational Methods in Applied Mathematics, 17:121–137, 2017.
[32] M. Neda, A. Takhirov, and J. Waters. Ensemble calculations for time relaxation fluid flow models. Numerical Methods for Partial Differential Equations, 32(3):757–777, 2016.
[33] J. D. Giraldo Osorio and S. G. Garcia Galiano. Building hazard maps of extreme daily rainy events from PDF ensemble, via REA method, on Senegal river basin. Hydrology and Earth System Sciences, 15:3605 – 3615, 2011.
[34] C. Trenchea. Unconditional stability of a partitioned IMEX method for magnetohydrodynamic flows. Applied Mathematics Letters, 27:97–100, 2014.
[35] N. Wilson, A. Labovsky, and C. Trenchea. High accuracy method for magnetohydrodynamics system in Elsässer variables. Computational Methods in Applied Mathematics, 15(1):97–110, 2015.
[36] S. Zhang. A new family of stable mixed finite elements for the 3D Stokes equations. Mathematics of Computation, 74:543–554, 2005.
[37] Y. Zhang. Critical transition Reynolds number for plane channel flow. Applied Mathematics and Mechanics, 38(10):1415–1424, 2017.
[38] B. Zheng and Y. Shang. Two-level defect-correction stabilized algorithms for the simulation of 2D/3D steady Navier-

Stokes equations with damping. Applied Numerical Mathematics, 163:182–203, 2021.