Math Notes

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CONTENTS

Notational Notes

I don't use the regular notation for vector derivatives of their components. I find them confusing because sometimes I want to refer to the component by the subscript and not be mistaken for referring to the derivative.

1. r'_x refers to the vector component of x after a derivative was taken.
2. r_x refers to the vector component of x.

This shouldn't be confused with taking partial derivatives of these vectors. Note, you could still form a vector function with two variables.

$$\vec{\phi}(x,y) = \phi_x(x,y)\hat{i} + \phi_y(x,y)\hat{j} + \phi_z(x,y)\hat{k}$$

The partial $\partial_x \phi$ refers to taking the partial derivative with respect to x to all vector components.

CHAPTER **1**

SYSTEM OF LINEAR EQUATIONS

System of linear equations are equations such that each unknown variable are in a linear form. Let's start with some definitions,

Definition (1): Linear Combinations and Equations

A linear combination of the variables $x_1...x_n$, has the form of

$$a_1x_1 + a_2x_2 + ... + a_nx_n$$

where the numbers $a_1...a_n$ are considered to be constant coefficients of $\mathbb{R}^n$. A linear equation takes a linear combination an assign some value to it, for now noted as D.

$$a_1x_1 + a_2x_2 + ... + a_nx_n = D$$

We note that a linear system takes such a form of,

$$a_1x_1 + a_2x_2 + ... + a_nx_n = D_n$$
$$a_{2,1}x_{2,1} + a_{2,2}x_{2,2} + ... + a_{2,n}x_{2,n} = D_{2,n}$$

such at a set of solutions of $s_1, ..., s_2$ in a order form for the corresponding variables $x_1, ..., x_n$, is considered to be an $n-$tuple.

These are basic terms we are going to be using in the linear algebra section.

1.1 Row Operations

Let's note that we have a linear system of equations,

$$\begin{array}{ccccc} a_1x_1 & a_2x_2 & .... & a_nx_n & = D \\ a_{2,1}x_{2,1} & a_{2,2}x_{2,2} & .... & a_{2,n}x_{2,n} & = D_2 \\ a_{m,1}x_{m,1} & ... & .... & a_{m,n}x_{m,n} & = D_n \end{array}$$

To find the solution said system, we have methods called row operations. These operations can convert these linear equations down into simple forms that allow for easy backwards substitutions.

1. Multiply a row by a non-zero constant.
2. Interchange a row with another row.
3. Add a constant multiple of one row to another row.

1.1.1 Augmented Matrix

Let **A** note a matrix that represents linear system.

$$\left(\begin{array}{ccc|c} a1 & a2 & a3 & D_1 \\ a1 & a2 & a3 & D_1 \\ a1 & a2 & a3 & D_1 \end{array}\right)$$

1.2 Gaussian Elimination

Gaussian Elimination is the method used to solve systems of linear equations. The goal is put a matrix in a form such that the solutions are easily seen.

1.2.1 Row Echelon Form

A matrix in row echelon form requires 3 things,

1. If a row doesn't consist of all zeros, the first non-zero number in the row has to be a one, noted as the **leading 1**.
2. If there is any row that is all zeros, they are moved towards the bottom of the matrix.
3. In any two successive rows that do not consist of all zeros, the leading 1 in the leading row is to the left of the rows that come right after it.

A **reduced row echelon form** is one such that every **column** that contains a leading 1 will have zeros everywhere else in that column. We note that this is an example of a matrix in echelon form,

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & D_1 \\ 0 & 1 & 4 & D_2 \\ 0 & 0 & 0 & D_3 \end{array}\right)$$

while this is an example of a matrix in reduced row echelon form,

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & D_1 \\ 0 & 1 & 0 & D_2 \\ 0 & 0 & 1 & D_3 \end{array}\right)$$

1.2.2 Solutions and Augmented Matrix

Let's note that there are two different cases of solutions to any system (ones that actually exist). The unique solution and the solution that generates multiple other solutions. Note a matrix in reduced row echelon form, $\mathbf{A}'$.

$$\left(\begin{array}{ccc|c} 1 & 0 & 3 & -1 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The equation that you can gather from this system will show that you would get a system of equations with one free variable. For example,

$$x = -1 - 3z$$
$$y = 2 + 4z$$

Notice how the leading 1 gives way to dependent variables, while the you have z being a free variable.

1.2.3 Gauss's Method

Gaussian Elimination will take any matrix and produce a reduced row echelon matrix.

1. Locate the leftmost column that doesn't consist of entirely of zeros.
2. Interchange the top row with another row, if necessary, to bring a non-zero entry to the top of the column.
3. If that entry is "a", multiple that row by the constant $\frac{1}{a}$ to introduce a leading one.
4. Multiply constant multiple of the top row to other rows so the column underneath the leading one is just full of zeros.
5. Move to the right and down a row. Repeat the same steps for the other rows as above.
6. If you have a leading one, where you need to introduce zeros above it's position in the column, multiply constant multiples of that row to above rows to get those zeros.

Steps 1-5 are considered to be the forward phase of the method, and those steps will reduce the matrix to row echelon form. While step 6 will reduce the matrix to reduce row echelon form.

1.3 Homogeneous Linear Systems

A system is considered homogeneous if the constants terms of that system are all zeros.

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = 0$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = 0$$

Each homogeneous system has two possible configurations for it's solutions. It can either be a trivial set of n-tuple of zeros. Or it can be a set of trivial solutions plus a non-trivial solution such that there is a possibility of infinitely many solutions.

1.3.1 Free Variable Theorem

Consider a reduced row echelon form of a homogeneous system.

$$\left(\begin{array}{cccc|c} x_{k_1} & 0 & 0 & \Sigma() & 0 \\ 0 & x_{k_2} & 0 & \Sigma() & 0 \\ 0 & 0 & x_{k_r} & \Sigma() & 0 \\ .. & .. & .. & .. & .. \end{array}\right)$$

In this case, the Σ denotes any leftover free variables from the row operations. We note that the row operations don't change the columns of zeros that make this system homogeneous. It actually seems like the row operations don't really move columns so that make sense. This system will have n-unknowns with r-non-zero rows. That general form of a reduced row echelon matrix shows that any homogeneous system will have $n - r$ free variables. This is because the n represents the number of columns while r represents the number of rows and equations we are provided. Since there are going to be cases were $n > r$, that means we are going to have left-over variables based on the amount of r equations we have. If you have less equations then unknowns, that means there are going to be less conditions imposed on some variables.

Now consider m equations with n-unknowns, $m < n$ so that also means $r < n$. This is because all $m = r$ realistically. This means that since $r < n$, all homogeneous equations with less equations that unknowns will have infinitely many solutions.

1.3.2 Pivot Columns and Rows

When a matrix is in row reduced echelon form, the positions of the leading 1's stay the same for any row and reduced row echelon matrix. We call the pivot column to be the column containing the leading 1 while the pivot row is the row that contains the leading one. The pivot position are the same positions as the leading 1's.

CHAPTER **2**

MATRIX OPERATIONS

This section will go in depth over the concept of the matrix. In linear algebra, we mostly deal with matrices. Here we will define different types of matrix operations. A matrix is a rectangular array of numbers. The size of a matrix is defined by the number of rows vs the number of columns. The first number of a matrix's size will always be the row number.

2.1 Simple Matrix Operations

The simplest operation we can considered is the matrix addition and subtraction. As along as the matrices share the same size, you can add them together by adding the individual elements together. Say we have matrix **A** and **B**, both size $m \times n$.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \pm \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a \pm e & b \pm f \\ c \pm g & d \pm h \end{pmatrix}$$

We say that two matrices are equal to each other if they share the same size and the components are equal to each other. Matrices can be multiplied by a constant. This just scales up the individual components.

2.2 Matrix Multiplication

Matrix multiplication can get pretty complicated. First I'm going to provided by own interpretation of the operation. Then I will go over the more in depth methods. My method is basically the linear combination perspective of matrix multiplication but a bit expanded.

2.2.1 Linear Combination Viewpoint

Geometrically we can view matrix multiplication as a collection of transforms. More specifically we can look it as a collection of linear transforms, as these transformations will keep parallel lines parallel. Let's note something first, a vector is just a single column matrix.

$$\mathbf{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

If $\mathbf{v}$ notes a position matrix in $\mathbb{R}^3$, then we can apply a linear transformation to this vector in form of multiplying it by a matrix.

$$\mathbf{v}' = \mathbf{A}\mathbf{v} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = a \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} + b \begin{pmatrix} \beta \\ \delta \end{pmatrix} = \begin{pmatrix} a\alpha + b\beta \\ a\gamma + b\delta \end{pmatrix}$$

It look likes the column matrix's row elements are scaling the individual columns of the square matrix.

There is something important to note from this example, one is the size of the matrices. Even though it's a column matrix, it still has a size, in this case 2×1. The square matrix is 2×2. Note that **column** of the square matrix is the same size of **row** of the vector. This is an important rule in any matrix multiplication, given an $m \times r$ multiplied by a $r \times n$, the product will be a $m \times n$ matrix. From the example, the rows from the column matrix can be seen as scaling the columns of the square matrix. This is why we have the rule, we must have enough rows for columns. Otherwise we would have an undefined matrix.

Now we can look at two 2×2 matrices to see this effect. By our previous rule we know that it's going to produce another 2×2 matrix. But since we considered a matrix a linear transformation that can be applied to a vector, we can consider two matrices being multiplied together as two succeeding transformations. Let's note a 2×2 matrix as $\mathbf{A}$ and the other one as $\mathbf{B}$.

$$\mathbf{AB} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

First note that the transformations are combined from right to left, so it's $\mathbf{B}$ applied to $\mathbf{A}$. We first look at the $\mathbf{B}$ as two different column matrices. We first apply the first column to $\mathbf{A}$ then the second column. We don't add them together but instead the columns of the product will be those single column transforms. I'm using commas to denote this point

$$\mathbf{AB} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e \\ g \end{pmatrix}, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} f \\ h \end{pmatrix} = \begin{pmatrix} ae + gb \\ ce + gd \end{pmatrix}, \begin{pmatrix} af + bh \\ cf + dh \end{pmatrix} = \begin{pmatrix} ae + gb & af + bh \\ ce + gd & cf + dh \end{pmatrix}$$

Of course we can go further and simplify it down but the point of this view is that we have a very mental image of what's going with matrix multiplication. This is a very generic and quick way to multiply matrices by hand and gives a geometric insight to what actually is going on. Let's try it with a 3×2 by 2×3, which will produced a 3×3.

$$\mathbf{AB} = \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} \begin{pmatrix} g & h & i \\ j & k & l \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} \begin{pmatrix} g \\ j \end{pmatrix}, \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} \begin{pmatrix} h \\ k \end{pmatrix}, \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} \begin{pmatrix} i \\ l \end{pmatrix}$$

From above a single column in the $\mathbf{AB}$ product will be like this,

$$\begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} \begin{pmatrix} g \\ j \end{pmatrix} = \begin{pmatrix} ag + bj \\ cg + dj \\ eg + fj \end{pmatrix}$$

2.2.2 Multiplication by Columns and by Rows

First let's defined what a Partitioned Matrices are. Since matrices are arrays of numbers, we can divide them into sub-divisions.

$$A = \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} \end{array}\right] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

$$A = \left[\begin{array}{cccc} a_{11} & a_{12} & a_{13} & a_{14} \\ \hline a_{21} & a_{22} & a_{23} & a_{24} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} \end{array}\right] = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \end{bmatrix}$$

$$A = \left[\begin{array}{c|c|c|c} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{array}\right] = [\mathbf{c}_1 \quad \mathbf{c}_2 \quad \mathbf{c}_3 \quad \mathbf{c}_4]$$

Simply just noting the sub-divisions as their own sub-matrices or row/column matrices. Now for the multiplication by column. Let's note with **AB** as two matrices with column partitions. We can use this to find specific columns or rows with computing the whole matrix product,

$$\mathbf{AB} = \mathbf{A} \begin{pmatrix} b_1 & b_2 & \dots & b_n \end{pmatrix}$$

Where j is the column you want to compute so it would be,

$$\mathbf{AB}_j = \mathbf{A} \begin{pmatrix} b_j \end{pmatrix}$$

Where b_j is the column sub-matrix of the j-th column of $\mathbf{B}$. Where a_i is the row sub-matrix of the the i-th row of $\mathbf{A}$.

$$\mathbf{AB}_i = \begin{pmatrix} a_i \end{pmatrix} \mathbf{B}$$

2.3 Transpose of Matrix

The definition of a transpose is given a matrix that is $m \times n$, the transpose is defined to be a $n \times m$ matrix such matrix operation turns columns into rows and rows into columns. Given a $\mathbf{A}$, the notation for the transpose is $\mathbf{A}^T$. For example,

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}^T = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}$$

Now for square matrices, you can use the diagonal to swap the positions. This is because the diagonal will remain the same during a transpose of a square matrix.

2.4 Trace of Matrix

The trace of a matrix is notated as **tr** (**A**) such that the trace is defined to be the sum of the main diagonal of a square matrix. The main diagonal is defined to be the diagonal that is left to right, not right to left.

vector calculus

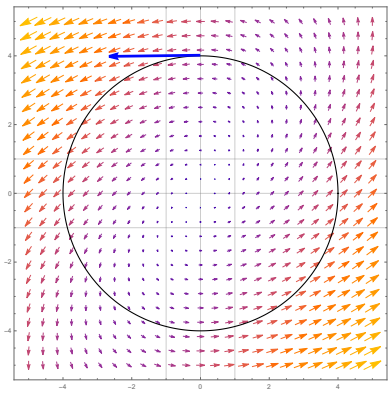
CHAPTER **3**

LINE INTEGRALS

Before we define what a line integral is, we need to consider the derivative operators over a vector field. First we must talk about **flow**, and how it relates to a vector field. When you drop a leaf into a river, the leaf flows according to the current (motion) of the river. If you define a vector field to represent the motion of said river, you have points in that river that rotate around some points, and other points where the leaf will just follow the flow of the river. I'm only considering the integrals over vector fields. Scaler fields are not as important to me right now and they are honestly pretty simple.

3.1 Flow and Circulation, Integral

Consider the unit tangent vector, $\hat{T}$. If we project $\hat{T}$ onto the vector field across some particle's path curve C on the vector field, we will get the total flow of that particle across that path.



Circular Flow

The blue tangent vector will be projected onto the field itself, giving an example of flow. Now if we take a line integral over this curve C, and apply $\vec{\mathrm{F}} \cdot \hat{\mathrm{T}}$, then we will get whats known as a **flow integral**.

$$\int_C \vec{\mathrm{F}} \cdot \hat{\mathrm{T}} \, \mathrm{d}s$$

Now $\mathrm{d}s$ is a infinitesimal measure of an arc length, so $\mathrm{d}s = |\vec{\mathrm{r}'}(t)|\mathrm{d}t$. That means we can write the flow integral as:

$$\int_C \vec{\mathrm{F}} \cdot \vec{\mathrm{r}'}\,\mathrm{d}t = \int_C \vec{\mathrm{F}} \cdot\ \mathrm{dr}$$

Definition (2): Flow Integral

Given a vector field in $\mathbb{R}^n$, where $\vec{\mathrm{F}}$ is such a field, the total flow across some curve C is equal to,

$$\int_C \vec{\mathrm{F}} \cdot \hat{\mathrm{T}}\,\mathrm{d}s = \int_C \vec{\mathrm{F}} \cdot\ \mathrm{d}\vec{\mathrm{r}}$$

If said curve is **closed**, where $\vec{\mathrm{r}}(b) = \vec{\mathrm{r}}(a)$, then the integral is called a circulation integral,

$$\oint_C \vec{\mathrm{F}} \cdot\ \mathrm{d}\vec{\mathrm{r}}$$

3.1.1 Green's Theorem for Flow Integrals

Definition (3): Green's Theorem (P1)

Given a positively oriented, smooth, simple, closed curve that is a boundary to a region, we can relate the circulation integral to the total internal circulation of that closed region.

$$\oint_C \vec{\mathrm{F}} \cdot\ \mathrm{d}\vec{\mathrm{r}} = \iint_D (\vec{\nabla} \times \vec{\mathrm{F}}) \cdot \hat{k}\ \mathrm{d}a$$

First, recognize that $\vec{F} \cdot\ \mathrm{d}\vec{\mathrm{r}}$ represents an infinitesimal circulation of the vector field. For example, how much work a vector field does on a particle. Now Green's theorem basically says that is equivalent to total infinitesimal circulation in that region. Now that makes a bit of sense, considering that we know, from the Fundamental Theorem of Calculus, which states

$$\frac{d}{dx}\int_c^x f\mathrm{d}t = \frac{d}{dx}(F(x) - F(c)) = f(x) - f(c)$$

The FTC basically relates that the change in area with respect to it's input on an interval is equal to the change in the function itself. Now Green's theorem relates can be seen as a higher version of this FTC. It's right side relates a derivative operation (like FTC's right left side) to an integral, consider the $\vec{\nabla}$ as a derivative operator. You can see it more when the derivative operator of FTC is inside the integral.

$$\int_c^x \frac{d}{dx}\mathbf{f}\,\mathrm{d}t \to \iint_D (\vec{\nabla} \times \vec{\mathrm{F}}) \cdot \hat{k}\ \mathrm{d}a$$

3.2 Flux (2D)

The second version of a line integral is the **Flux Integral**. The flux integral measures the total "flux" out of a curve in a vector field. First look at $\vec{\mathrm{F}} \cdot \hat{n}$, where $\hat{n}$ is the orientation of the surface.

Definition (4): Orientation

If we considered a normal vector to a surface where we can construct a field of $\hat{n}$ that points in the same "general" direction all over the surface, we say that surface is orientable. Now let's considered we have a curve $C = \partial S$. If we walk along C with our head in the direction of $\hat{n}$ counterclockwise then the surface will be to our right. That's considered positively orientated.

The key is thinking about the direction of $\hat{n}$ and if it's counterclockwise or not. If the direction of the normal is positive (outwards), and we are going counterclockwise then the surface will always be on our left hand side. While if we went counterclockwise but the normal was negative (inwards), then the surface will be to our right hand side, meaning that it's negatively orientated.

Now we can consider the flux:

$$\vec{\mathrm{F}} \cdot \hat{n}$$

We are projecting the field at that point to the positive normal vector. This will give us a scaler value representing total outward flow from that specific point in that region. Now we can define a flux integral.

Definition (5): Flux Integral (2D)

$$\int_C \vec{\mathrm{F}} \cdot \hat{n} \, \mathrm{d}s$$

Where $\hat{n}$ is the normal vector to that surface or region. The $\mathrm{d}s$ is an infinitesimal measure of arc length so $|r'(\vec{t})|dt$, were $r(\vec{t})$ is the vector function that defines the curve C. We will now have to construct $\hat{n}$. For other surfaces, $\hat{n}$ is typical constructed by the $\frac{\partial \vec{r}}{\partial \mathrm{x}} \times \frac{\partial \vec{r}}{\partial \mathrm{y}}$, since you can parameterize with two parameters. But in the planer case, we construct $\hat{n}$ from the cross product of $\hat{T}$ with $\hat{k}$.

$$\int_C F_x \, \mathrm{d}y - F_y \mathrm{d}x$$

Where $\mathrm{d}y$ is basically shorthand for $r'_y(t)$.

3.2.1 Flux Integral Derivation

Now let's try deriving the flux integral equation from above. For planer equations $\hat{n}$ is the cross product between $\hat{T}$ and $\hat{k}$.

$$\begin{bmatrix} i & j & k \\ \frac{r'_x}{|\vec{r}|} & \frac{r'_y}{|\vec{r}|} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \langle \frac{r'_y}{|\vec{r}|}, -\frac{r'_x}{|\vec{r}|}, 0 \rangle$$

Now let $\vec{\mathrm{F}} = \langle F_x, F_y \rangle$,

$$\int_C \langle F_x, F_y \rangle \cdot \langle \frac{r'_y}{|\vec{r}|}, -\frac{r'_x}{|\vec{r}|}, 0 \rangle \, |r'(\vec{t})| \, \mathrm{d}t$$

$$\int_C F_x(r'_y) - F_y(r'_x) \, \mathrm{d}t$$

3.2.2 Flux Integral: Green's Theorem

The thing for simple line integrals is the Green's Theorem part 2. This is for Flux Integrals and it relates the derivative operator of divergence to flux.

Definition (6): Green's Theorem Part 2

Given a simple, positively oriented, closed boundary curve $C = \partial D$, represented by the vector equation $r(\vec{t})$, the flux integral inside a vector field $\vec{\mathrm{F}}$ is given by this relationship

$$\int_C \vec{\mathrm{F}} \cdot \hat{n} \, \mathrm{d}s = \iint_{\partial D} \vec{\nabla} \cdot \vec{\mathrm{F}} \, \mathrm{d}A$$

Note how this makes sense, the divergence measures how much a field flows out of a point. Sort of like how the flux measures the outward flow from a surface. This is another version (or generalization) of the FTC like before. This does show how there are two different types of derivatives in vector fields. The tangential ones (dot products), and the normal ones (cross products).

CHAPTER 4

SURFACE INTEGRALS

4.1 Parameterization of Surfaces

We can parameterize a surface or any surfaces using a vector equation.

$$\vec{\phi}(u,v) = \phi_x(u,v)\hat{i} + \phi_y(u,v)\hat{j} + \phi_z(u,v)\hat{k}$$

It makes sense to have two parameters as we can have two "directions" to travel along in any given surface. This is were things get messy, considering the notation of any given surface integral. First step would be to parameterize the surface, using the symbol ϕ. Make sure to use this notation because you don't want to confuse the surface parameterization with the field vector.

4.2 Surface Integral Over Scaler Field

I choose to write about this integral's scaler field version because it has some insights into why there isn't really a circulation version of this for surfaces. There is a flux integral, the 3D version of the previous flux integral we have seen, but there isn't a surface integral version of a circulation line integral.

Definition (7): Surface Integral (Scaler)

Given a surface Σ over a scaler function $f(x,y,z)$.

$$\iint_{\Sigma} f(\vec{\phi}(u,v))\,\mathrm{d}\Sigma$$

Where Σ represents the surface that is being parameterized by ϕ. We note the infinitesimal $\mathrm{d}\Sigma$, here as a measure of surface area (like how line integrals have infinitesimals of arc length). Change in surface area of a surface is found by taking the cross product of the partials of ϕ. This is because the abs value of the cross product will be the area of a parallelogram. The patch of surface can be represented as the parallelogram, I.E so the area can be represented as the cross product.

$$\mathrm{d}\Sigma = |\partial_u\phi \times \partial_v\phi|\mathrm{d}A$$

The **MISTAKE** I make is confusing the parametrization and the vector or scaler field equations. As when $f(\phi(u,v))$ that means, $x = \phi_x(u,v)$.

4.2.1 Surface Integrals Over Graphs

Given a $f(x,y,z)$ where $z = \phi(x,y)$, we can write $f(x,y,\phi(x,y))$. The function is already parameterized by ϕ, so now we can try finding the magnitude of the cross product.

$$\vec{\phi}(x,y) = x\hat{i} + y\hat{j} + \phi(x,y)\hat{k}$$
$$\partial_x\vec{\phi} = \hat{i} + 0\hat{j} + \frac{\partial\phi}{\partial \mathrm{x}}\hat{k}$$
$$\partial_y\vec{\phi} = 0\hat{i} + 1\hat{j} + \frac{\partial\phi}{\partial \mathrm{y}}\hat{k}$$

The magnitude of the cross will be

$$\sqrt{(\frac{\partial\phi}{\partial \mathrm{x}})^2 + (\frac{\partial\phi}{\partial \mathrm{y}})^2 + 1}$$

Now we get the equation for a surface integral over a graph

$$\iint_S f(x,y,\phi(x,y))\sqrt{(\frac{\partial\phi}{\partial \mathrm{x}})^2 + (\frac{\partial\phi}{\partial \mathrm{y}})^2 + 1}\,\mathrm{d}A$$

4.3 Surface Integral Over Vector Fields

This is considered the **Flux Integral** 3D version. We first have to consider, like the last flux integral, the orientation of a surface. We define the $\hat{n}$ as the normal vector for any point on that surface. If we think of the surface area patch as a parallelogram, the unit normal vector would be $\frac{\partial\vec{\phi}}{\partial \mathrm{x}} \times \frac{\partial\vec{\phi}}{\partial \mathrm{y}} / |\frac{\partial\vec{\phi}}{\partial \mathrm{x}} \times \frac{\partial\vec{\phi}}{\partial \mathrm{y}}|$. Where the $\vec{\phi}$ is the parameterization of the surface.

Definition (8): Flux Integral (3D)

Given a surface that is closed, positively oriented, and smooth, the flux integral over that said surface is defined to be,

$$\iint_\Sigma \vec{\mathrm{F}} \cdot \,\mathrm{d}\vec{S}$$

Now the $\mathrm{d}\vec{S}$ is defined to be $\hat{n}\mathrm{d}S$, where $\mathrm{d}S$ is the infinitesimal change in surface area. We already know how surface area is defined for a general parameterized surface, so we can define the $\mathrm{d}\vec{S}$

$$\mathrm{d}\vec{S} = (\partial_u\vec{\phi} \times \partial_v\vec{\phi})\,\mathrm{d}A$$

So the surface integral becomes

$$\iint_\Sigma \vec{\mathrm{F}} \cdot \,\mathrm{d}\vec{S} = \iint_\Sigma \vec{\mathrm{F}}(\phi(\vec{u},v) \cdot (\partial_u\vec{\phi} \times \partial_v\vec{\phi})\,\mathrm{d}A$$

CHAPTER 5

FTC: GENERALIZED

With all these types of derivatives and integrals, it would be better to organize the notes for all the theorems of calculus in one place. Note, I don't think it will be example heavy, as these theorems are pretty simple to understand.

5.1 Fundamental Theorem of Line Integrals: Gradient Theorem

Before we define what the FTC is for line integrals, we have to consider path independence. Path independence is kind of funny to think about without a physical analogy. Work is the best case for this. If there is no friction in a force field, all paths that start at the same place (A) and end at the another same place (B), then the work done those paths by the field will be the same for all paths from A to B.

5.1.1 Path Independence

Let's say we have a vector field $\vec{F}$, if that field is a gradient field then it will be a conservative vector field. That's the strange thing, because this implies that there won't be any curl within the field for the most part in $\mathbb{R}^3$. If there is a hole though (places were something is undefined), you could have a curl free vector field. Really the best test would be to quickly do a closed loop line integral around the vector field. If it yields zero, because using the work example, there is no displacement from where you started and ended.

Definition (9): Potential Function

A vector field has a potential function if that said function's gradient field is the vector field itself.

$$\vec{\mathrm{F}} = \vec{\nabla}\mathbf{f}$$

This means that $\vec{\mathrm{F}}$ is always parallel to $\vec{\nabla} f$ at all points, so the curl of any point in $\vec{\mathrm{F}}$ will always be zero.

$$\vec{\nabla} \times \vec{\mathrm{F}} = \vec{0}.$$

5.1.2 Finding the Potential Function

Finding a potential function isn't a clear algebraic process. Consider $\vec{\mathrm{F}}$, where it's a gradient field of some function.

$$\vec{\mathrm{F}} = \vec{\nabla} f$$

We know that F_{x_i} will equal the gradient functions' component. We then have a relationship with something like this.

$$\frac{\partial f}{\partial \mathrm{x}} = \mathrm{F}_x$$

We solve it like a simple differential equation,

$$\partial f = \int \mathrm{F}_x \, \mathrm{d}x$$
$$\partial f = f_x + H(y, z)$$

We are left with the function $H(y, z)$ because we have three cases that it could be due to it being a partial derivative. It can be a constant, a variable of y, or a variable of z. So we aren't done yet.

$$\frac{\partial}{\partial \mathrm{y}}(f_x + H(y, z)) = F_y$$

Because it's a potential function, the y component of the vector field will be equal to the partial derivative with respect to y. Solve for the $H(y, z)$ and integrate with respect to y, and you will be left with a function of $H(z)$ that you will have to find again by using the F_z component.

Or, you could be like me and consider..

$$\frac{\partial f}{\partial \mathrm{x}} = F_x \quad \frac{\partial f}{\partial \mathrm{y}} = F_y \quad \frac{\partial f}{\partial \mathrm{z}} = F_z$$

If you solved both like a differential equation, you will notice some common terms and uncommon terms. Those uncommon terms are the $H(y, z)$, $H(x, z)$, and $H(y, x)$. So you could easily form the potential function from that.

5.1.3 Fundamental Theorem of Line Integrals

Definition (10): FTC: Line Integral

Given a conservative field $\vec{F}$, where $\vec{F} = \vec{\nabla}\mathbf{f}$, the line integral between the two points, A and B of a curve C are equal to the potential function at those points.

$$\int_C \vec{F} \cdot \, \mathrm{d}\vec{r} = f(\vec{r}(B)) - f(\vec{r}(A))$$

5.2 Stokes' Theorem

EXAMPLES

These are some interesting problems I found, and the notes and mistakes I had with them. There is no real order to how the notes are placed. I could be making mistakes on interpreting these problems so these notes are subjected to change.

6.1 E&M: $\vec{\nabla}$ Operator

Consider that you had $\vec{\ell}$ which represent the separation vector between two points. Note how the $\vec{\nabla}$ operator functions on the different versions of $|\ell|^n$.

$$|\ell|^n = \sqrt{(x-x')^2+(y-y')^2+(z-z')^2}^n$$

When we apply the ∇, we consider first the components

$$\begin{aligned}
\frac{\partial \ell}{\partial \mathrm{x}} &= (\frac{n}{2})((x-x')^2+(y-y')^2+(z-z')^2)^{\frac{n}{2}-1}2(x-x') \\
\frac{\partial \ell}{\partial \mathrm{x}} &= (n)((x-x')^2+(y-y')^2+(z-z')^2)^{\frac{n}{2}-1}(x-x') \\
\frac{\partial \ell}{\partial \mathrm{x}} &= (n)\frac{((x-x')^2+(y-y')^2+(z-z')^2)^{\frac{n}{2}}(x-x')}{(x-x')^2+(y-y')^2+(z-z')^2} \\
\frac{\partial \ell}{\partial \mathrm{x}} &= (n)\frac{\ell^n}{\ell^2}(x-x') \\
\frac{\partial \ell}{\partial \mathrm{x}} &= (n)\ell^{n-1}\frac{(x-x')}{\ell}
\end{aligned}$$

Since the first part is just a scaler, the second part can be rewritten as $\hat{\ell_x}$ We end up with this statement for ∇ operators:

$$\vec{\nabla}|\ell|^n = n|\ell|^{n-1}\hat{\ell}$$

It looks sort of like how a regular derivative operates. But there is a quicker way to get to the same conclusion.

$$\vec{\nabla}|\ell|^n = n|\ell|^{n-1}\frac{\partial \ell}{\partial \mathrm{x_i}}$$

Note the implicit differentiation with the ∇, the only thing left to is evaluate the $\frac{\partial \ell}{\partial \mathrm{x_i}}$.

$$\begin{aligned}
\frac{\partial \ell}{\partial \mathrm{x}} &= \frac{\partial}{\partial \mathrm{x}}((x-x')^2 + ... + (x_i + x_i')^2)^{\frac{1}{2}} \\
\frac{\partial \ell}{\partial \mathrm{x}} &= ((x-x')^2 + ... + (x_i + x_i')^2)^{\frac{1}{2}-1}(x-x') \\
\frac{\partial \ell}{\partial \mathrm{x}} &= \hat{\ell_x}
\end{aligned}$$

Leading you to the final conclusion of:

$$\vec{\nabla}|\ell|^n = n|\ell|^{n-1}\hat{\ell}$$

6.2 Surface Integrals

Evaluate $\iint_\Sigma \vec{F} \cdot \, \mathrm{d}\vec{\Sigma}$, where $\vec{F} = xy\hat{i} - 2y\hat{j} + 3x\hat{k}$, and Σ is the sphere with an outward orientation. Since it's a sphere, we have to parameterize the sphere using spherical coordinates.

$$\vec{\Phi}(\theta, \phi) = \langle \sin\theta\cos\phi, \sin\theta\sin\phi, \sin\phi \rangle$$

When we take the cross,

$$\vec{\Phi}_{cross}(\theta, \phi) = -4\sin^2\phi\cos\theta\hat{i} - 4\sin^2\phi\sin\theta\hat{j} - 4\sin\phi\sin\theta\hat{k}$$

When we consider the normal, we note that since all components are negative, meaning the normal has an inward orientation. We wanted to be outwards, so we take $-\vec{\Phi}_{cross}(\theta, \phi)$. The bounds for the surface is

$$\begin{aligned}
0 \leq \theta \leq 2\pi \\
0 \leq \phi \leq 2\pi
\end{aligned}$$

Now we set up the surface integral by first parameterizing the vector field,

$$\vec{F}(\vec{\Phi}) = \sin^2\theta\cos\phi\sin\phi\hat{i} - 2\sin\theta\sin\phi\hat{j} + 3\sin\theta\cos\phi\hat{k}$$

Now we can setup the surface integral,

$$\int_0^\pi \int_0^{2\pi} (\sin^2\theta\cos\phi\sin\phi\hat{i} - 2\sin\theta\sin\phi\hat{j} + 3\sin\theta\cos\phi\hat{k}) \cdot (-4\sin^2\phi\cos\theta\hat{i} - 4\sin^2\phi\sin\theta\hat{j} - 4\sin\phi\sin\theta\hat{k}) \, \mathrm{d}\theta\mathrm{d}\phi$$

CHAPTER 7

DIFFERENTIAL EQUATIONS

A differential equation just relates the change in some variable with respect to another to another quantity. They are useful tools to model real life phenomena. A first order differential equation is one of which it's the first derivative that is involved in the equation.

$$\frac{\mathrm{dy}}{\mathrm{dx}} = \mathrm{f}(x, y)$$

In this case, these functions aren't really multi-variable equations, y and x share a dependence relationship. Basically most of the time these equations are implicit derivatives of some solution equation.

7.1 Classification of Types

Differential equations can be classified into different categories, each having different methods of solving them.Ordinary differential equations are ones that deal with regular one-dimensional derivatives.

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 5y = 7\phi(x)$$

Partial differential equations are ones that deal with partials derivatives.

$$\frac{\partial y}{\partial \mathrm{x}} + 5y = 7\frac{\partial \phi(x)}{\partial \mathrm{x}}$$

You can classify differential equations by order, usually the highest order of the a derivative in the equation. Like how you classify polynomials with highest exponent. Other classifications will come up and their solutions would be given their own sections.

7.2 Differential Form

Recall that a differential is considered to be $\mathrm{d}y = y'\mathrm{d}x$. When you divide a differential by another differential of a different variable, you can arrive back at the differential equation. The differential form is considered to be very important. When solving equations, typically you want the equation to be first in differential form. This standard example of a differential form is:

$$\phi_1(x, y)\mathrm{d}x + \phi_1(x, y)\mathrm{d}y = 0$$

The differential equation, expressed as a function is $F(x, y, y', ..., y^{(n)}) = 0$. This is called the general form. It's a bit different from what a differential form is, for example,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} + 3x = 0$$

would be considered a general form of the differential equation. The normal form is when we solve for one of the derivatives,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{\mathrm{d}y}{\mathrm{d}x} - 3x$$

7.3 General Solutions and Unique Solutions

When we talk about solutions to differential equations, we mostly talk about functions that when plugged into the equation, will result in a true expression. For example, when given $y = f(x)$, and the differential equation $y' = \phi(x)$, if we took the derivative of y, then we would see that $y' = \phi(x)$. The function $y = f(x)$ is what we consider to be a solution to the differential equation.

7.3.1 Interval of Solution

The interval of solution, I, is the interval of the solution's domain that satisfy the differential equation. That unique means that even if we get a function that is defined on some higher interval I, the actual domain of the solution might be a sub-set of I. For example,

$$y'x + y = 0$$

has a solution of $y = 1/x$, but y's domain is $(-\infty, 0) \cup (0, \infty)$. But the solution must be differentiable on the solution interval, so we can't include 0 in any of our domain. Any I that contains a solution must not contain 0, so any solution *curve* must be defined on that interval. We would have to break the interval into smaller pieces, and find the largest of pieces. For this differential equation, two I's exist, $(-\infty, 0)$ or $(0, \infty)$ each with their own solution curve.

7.3.2 Family of Solutions

When you solve a differential equation, you notice that there are some constants left over. These constants can take on any value. So what you have is a common list of functions that all differ by a constant or some value of that constant. First order equations usually only contain a single constant parameter, so we call solutions that contain these parameters **one-parameter family of solutions**. When you are given an initial condition, and you solve for that constant, what you did was find a **particular solution**.

7.3.3 Existence and Uniqueness Theorem

Given $y' = f(x, y)$ and you impose an initial condition $y(x_0) = y_0$, if $f(x, y)$ and it's partial derivative with respect to y, $\frac{\partial f}{\partial y}$, are both continuous on the same input space $\alpha < x < \beta$, $\gamma < y < \sigma$, and as along as the input space contains (x_0, y_0), then there is some unique solution to the equation in some interval $x_0 - h < x < x_0 + h$ that is contained in a higher interval of $\alpha < x < \beta$.

When we say a unique solution, that means from a graphical point of view, that solution will never touch another curve. There is a concept called interval of validity is like the interval of solution but must contain the (x_0, y_0) and no discontinuous inputs. It would be hard to find the exact interval of validity without solving the equations. But we can guess what interval it would be using this theorem. **NOTE**, to use this theorem just find the shared domain of both $f(x, y)$ and $D_y f$.

7.3.4 General Solution

A general solution is any solution that doesn't take into account of an initial condition. Really the hard part of find a general solution is that we have to show this solution contains all solutions in some interval I. If we can find that all solutions can be obtained from an n-order differential equation in some n-parameter family, that solution family is consider a general solution. In linear equations, imposing conditions on it makes sure all those conditions are passed along to the solutions. This is why a general solution to the differential equation can be found strictly. It's not always the case that you can get all the solutions from a n-parameter family.

7.4 Initial Value Problems

The most common types of problems that you will deal with in differential equations is the initial value problem. Consider you have an differential equation $y' = f(x, y)$, and you are given that a solution to the differential equation will have to also fulfill a initial condition, such as $y(x_0) = y_0$. It's usually as simple as find the value for the constant of integration, but sometimes it's a bit more complicated.

7.4.1 Explicit vs Implicit Equations

An explicit relationship in math is exactly what the name implies. A direct relationship that relates two quantities together. For example,

$$y = x$$

is an example of a explicit relationship. But the issue is this equation,

$$y - x = 0$$

which is an example of implicit relationship because it relates x, y to a different quantity, 0. More complicated implicit equations show up as solutions to differential equations,

$$\ln|y| = f(x)$$

Here we have a relationship that has $y \in \mathbb{R}$. But it's implicit because $\ln|y|$ is relating to $f(x)$, not y itself. If we solve for y, we have a relationship the violates the original range. So we have to be careful about finding the constant of integration with implicit relationships and domain restrictions.

7.4.2 Example

Given $x^2y' = y - xy$, and the initial condition $y(-1) = 4$, find a solution that satisfy the differential equation.

$$\begin{aligned}
\frac{1}{y}\mathrm{dy} &= \frac{1-x}{x^2}\mathrm{dx} \\
\ln|y| &= \int \frac{1}{x^2} - \frac{1}{x}\,\mathrm{dx} \\
\ln|y| &= -x^{-1} - \ln|x| + C
\end{aligned}$$

At this point, we can find an explicit relationship by solving for y, but we won't be able to use our initial condition, because -1 isn't in the domain of the solution. So we will use the implicit relationship,

$$\ln|-4| = 1 - 0 + C$$

giving us the solution of

$$y = e^{-x^{-1} - \ln|x| + \ln|4| - 1}$$

CHAPTER 8

FIRST ORDER DIFFERENTIAL EQUATIONS

8.1 Separable Equations

Separable equations are the simplest first order equations to solve. A separable equation y' is ensure to have terms that can be separated into factors. Since these are meant to be separate, we can just leave it in normal form and solve it by integration.

$$\begin{aligned} y' &= g(x)h(y) \\ h(y)^{-1}\frac{\mathrm{d}y}{\mathrm{d}x} &= g(x) \\ \int \frac{1}{h(y)}\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x &= \int g(x)\,\mathrm{d}x \\ H(y) &= G(x) + C \end{aligned}$$

Now it step 3, the reason why the derivative and the differential dx cancels out is because if $y = f(x)$ is some solution, then $h(f(x))^{-1}f'(x) = g(x)$. This condition is imposed from u-sub method of solving integration. And so looking at step 3 again, noting that $f'(x)\mathrm{d}x = \mathrm{d}y$

$$\int \frac{1}{h(f(x))}f'(x)\,\mathrm{d}x = \int h_1(y)\,\mathrm{d}y$$

8.1.1 Losing A Solution

Like in algebra, regular solutions can be lost by dividing out a function. For example, take this differential equation,

$$\frac{\mathrm{dy}}{\mathrm{dx}} = y^2 - 4$$

The solutions $y = 2, -2$ are constant solutions to the equation. But if you solve the equation, you get something like this,

$$y = 2\frac{1 + ce^{4x}}{1 - ce^{4x}}$$

Notice how you can get 2 back from the equation but not negative 2. Since, they are constant solutions, you can get 2 from any x value, but the -2 is impossible to get from the solution. So you lost a solution by dividing it out.

8.2 Autonomous Differential Equations

Autonomous equations are differential equations that just dependent on the dependent variable. For example,

$$\frac{\mathrm{dy}}{\mathrm{dx}} = y\ln(y+2)$$

The issue with autonomous differential equations is that they can be pretty difficult to solve. We can construct graphical solutions to the differential equation. So instead, we can try developing what sketch of the graphical solution. What we first do is consider the critical points of the differential equation, I.E, where the differential equation is equal to zero. You can use this to create a phase line of the equation.

8.3 First Order Linear Differential Equations

Classification of differential equations in linearly are when the y, dependent variable, is a linear variable. Meaning that it isn't a function of some exponential, log, or trig, function. First order linear equations are in standard form of,

$$\frac{\mathrm{dy}}{\mathrm{dx}} + p_1(x)\mathrm{y} = p_2(x)$$

We assume that for a non-trivial case, p_2 is a non-zero function of x. If it was, then the equation would just become a separable equation.

8.3.1 General Solution to First Order Linear Differential Equations

Consider the above equation, it's not separable in a non-trivial case. We are going to have to transform the left-hand side into a form that would be considered separable. Let's call this transforming function, $v(x)$, the integration factor.

$$v(x)(\frac{\mathrm{dy}}{\mathrm{dx}} + p_1(x)y) = v(x)p_2(x)$$

If the integration factor actually separates the function as we wanted it to, we impose this condition on the left-hand side,

$$\frac{\mathrm{d}}{\mathrm{dx}}(v(x) \times y) = v(x)(\frac{\mathrm{dy}}{\mathrm{dx}} + p_1(x)y) = v(x)p_2(x)$$
$$\frac{\mathrm{d}}{\mathrm{dx}}(v(x) \times y) = v(x)p_2(x)$$

The last part gives us a separable differential equation, and doing our integration on both sides with respect to x, gives us the final general solution.

$$y = \frac{1}{v(x)} \int v(x)p_2(x)\,\mathrm{d}x$$

8.3.2 Integration Factor

The integration factor, $v(x)$, can be found by taking the left-hand condition that we imposed and solving for v.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{dx}}(v(x) \times y) &= v(x)(\frac{\mathrm{dy}}{\mathrm{dx}} + p_1(x)y) \\
v'y + y'v &= vy' + p_1vy \\
v' &= p_1v \\
v &= e^{\int p_1\,\mathrm{d}x}
\end{aligned}$$

The constant in the integration factor cancels out, so we only worry about the constants in the other integration in the general solution.

8.3.3 Singular Points and Transient Terms

The reason why we can consider the general solution to the first order linear equation to be a general solution is because we have conditions that make it so. Consider the standard form, solving it means that we have to make sure $p_1(x)$ is continuous. Discontinuous points will carry throughout the all the solutions if p_1 is discontinuous. That's also because of the integration factor, since p_1 is what is being used to transform the

equation into a separable equation. Points where p_1 are discontinuous are considered to be **singular points**, and should be used to determine the interval of solution. Consider this solution to a differential equation,

$$y = ax + Ce^{-x}$$

As $\lim_{x\to\infty}$, the function takes on the value of ax, the other term doesn't matter. Ce^{-x} is what's called a **transient term**.

8.4 Exact Differential Equations

Recall what a differential is,

$$\mathrm{dz} = \frac{\partial f}{\partial \mathrm{x}}\mathrm{dx} + \frac{\partial f}{\partial \mathrm{y}}\mathrm{dy}$$

Now let's say we have a function $f(x, y) = C$, where C is some constant over $\mathbb{R}$, then the differential of f, would be

$$0 = \frac{\partial f}{\partial \mathrm{x}}\mathrm{dx} + \frac{\partial f}{\partial \mathrm{y}}\mathrm{dy}$$

Now this makes it interesting, as since we gone from having a function $f(x, y)$ to a differential, we could take that differential and go backwards to find $f(x, y)$, I.E a solution to a differential equation. But before we can go backwards, we have to make sure equation is **exact**. Exact equation refers to a differential equation that has a corresponding differential of a function. Meaning if $f(x, y)$ was able to construct a differential df, that df would be considered an exact differential, so it's differential equation will be able to yield $f(x, y)$ going backwards.

8.4.1 Proof of Exact Equation Condition

To show that a differential equation and a function's differential is exact, consider,

$$M(x, y)\mathrm{dx} + N(x, y)\mathrm{dy} = \frac{\partial f}{\partial \mathrm{x}}\mathrm{dx} + \frac{\partial f}{\partial \mathrm{y}}\mathrm{dy}$$

We consider the $M(x, y)$ to equal $\frac{\partial f}{\partial \mathrm{x}}$ and same thing for $N(x, y)$ to $\frac{\partial f}{\partial \mathrm{y}}$. We impose the condition that $\frac{\partial M}{\partial \mathrm{y}}$ will equal $\frac{\partial N}{\partial \mathrm{x}}$. Now this is the condition to determine if an equation is exact or not because if this statement wasn't true, it would impossible for anyone get $f(x, y)$. Note, this requires, the partials to be continuous, otherwise, the mixed partials wouldn't always equal each other.

8.4.2 Connection to Conservative Functions

To actually solve Exact equations, you use the same method to find the potential function from conservative functions. This makes sense if you think about it. The df and $\vec{\mathrm{F}} = \nabla f$ are both differentials. If the condition is that conservative functions all have potential functions, that means that a differential ∇f has a $f(x, y)$ that you can get. Meaning exactness and conservative functions have the same condition, $\nabla \times \vec{\mathrm{F}} = 0$.

8.4.3 Transforming an Differential into An Exact Differential

Let's consider a non-exact differential. To transform it, we do the same process as the integration factor.

$$\frac{\partial M}{\partial \mathrm{y}} \neq \frac{\partial N}{\partial \mathrm{x}}$$

Assume $\mu(x, y)$ is a function that transforms a component of that differential into a exact component. That means $\frac{\partial}{\partial \mathrm{y}}(\mu(x, y) \times M) = \frac{\partial}{\partial \mathrm{x}}(\mu(x, y) \times N)$. This statement makes sense, as if it was an exact transformer, applying to both functions should yield the same thing.

$$\begin{aligned}
\frac{\partial}{\partial \mathrm{y}}(\mu(x, y) \times M) &= \frac{\partial}{\partial \mathrm{x}}(\mu(x, y) \times N) \\
\frac{\partial \mu}{\partial \mathrm{y}} M + \frac{\partial M}{\partial \mathrm{y}} \mu &= \frac{\partial \mu}{\partial \mathrm{x}} N + \frac{\partial N}{\partial \mathrm{x}} \mu \\
\frac{\partial \mu}{\partial \mathrm{x}} N - \frac{\partial \mu}{\partial \mathrm{y}} N &= \mu(\frac{\partial M}{\partial \mathrm{y}} - \frac{\partial N}{\partial \mathrm{x}})
\end{aligned}$$

Notice how this becomes a partial differential equation, which we can't solve with any techniques we know of right now. We make the assumption to simplify our case by making μ a single variable function of either x or y. When we assume x as our independent variable for μ, then some of the partials cancel out.

$$\begin{aligned}
\frac{\mathrm{d}\mu}{\mathrm{dx}} N &= \mu(\frac{\partial M}{\partial \mathrm{y}} - \frac{\partial N}{\partial \mathrm{x}}) \\
\frac{1}{\mu} \frac{\mathrm{d}\mu}{\mathrm{dx}} &= \frac{1}{N}(\frac{\partial M}{\partial \mathrm{y}} - \frac{\partial N}{\partial \mathrm{x}}) \\
\mu &= e^{\int \frac{1}{N}(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x})\,\mathrm{dx}}
\end{aligned}$$

The process is the same for the assumption when μ is only in terms of y.

8.5 Solving Differential Equations Using U-Subs

This section goes over types of equations that we can solve using subs.

8.5.1 Homogeneous Coefficients

Given a function $f(x, y)$, where we can rewrite it as $f(tx, ty) = t^\alpha f(x, y)$, that function is considered to have homogeneous coefficients of degree α. Now consider a first order differential equation in differential form,

$$M(x, y)\mathrm{dx} + N(x, y)\mathrm{dy} = 0$$

where M and N have homogeneous coefficients of the same degree α. We can preform a sub of $y = xu$ or $x = vy$ on the equation to turn it into,

$$x^\alpha M(1, u)\mathrm{dx} + x^\alpha N(1, u)\mathrm{dy} = 0$$

or

$$y^\alpha M(v, 1)\mathrm{dx} + y^\alpha N(v, 1)\mathrm{dy} = 0$$

The homogeneous coefficients cancel out. Leaving us with the differential in terms of u or v. Then by subing in the differential $\mathrm{dy} = u\mathrm{dx} + x\mathrm{du}$ or $\mathrm{dx} = v\mathrm{dy} + y\mathrm{dv}$, you get something like this,

$$M(1, u)\mathrm{dx} + N(1, u)\mathrm{dy}[u\mathrm{dx} + x\mathrm{du}] = 0$$

. At this point, this function is solvable with the other methods we have learned already.

8.5.2 Bernoulli's Equations

Consider this equation,

$$\frac{\mathrm{dy}}{\mathrm{dx}} + P(x)y = f(x)y^n$$

where $n \neq 0, 1$. Preforming a sub of $u = y^{1-n}$ will transform the equation into a first order linear equation. Which you can solve in terms of u and then sub back into y.

8.5.3 Polynomial in a non-linear function

$$\frac{\mathrm{dy}}{\mathrm{dx}} = f(Ax + By + C)$$

can be solved using the sub of $u = Ax + By + C$.

CHAPTER 9

HOMOGENEOUS EQUATIONS

A linear n-th order differential equation is **homogeneous** if it takes a form of

$$H(x,y) = a_n(x)\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + a_{n-1}(x)\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + ... + a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y = 0$$

A **non-homogeneous** equation is

$$H(x,y) = g(x)$$

on a interval where $g(x) \neq 0$. It will be shown that an associated homogeneous equation will need to be solved first before we can solve the non-homogeneous equation. We will make two simple assumptions, one the coefficient functions and $g(x)$ are continuous. The second that $a_n(x) \neq 0$ for all x in a common interval I.

9.1 Notational Differential Operators

A differential operator is a operator that turns a differentiable function into a function. That sounds exactly like what we do with the regular differential operator, and the point is that this is the regular differential operator. We let the differential operator D be exactly that.

$$D^n\{f(x)\} = f'(x) = \frac{\mathrm{df(x)}}{\mathrm{dx}}$$

The properties of differentiation imposes two useful rules. The first is the ability to factor out a constant coefficient outside of the operator.

$$cD^n\{f(x)\} = D^n\{c \times f(x)\}$$

The second is the sum and difference rule,

$$D^n\{f(x) \pm g(x)\} = D^n\{f(x)\} \pm D^n\{g(x)\}$$

These two rules allows us to write the homogeneous expression in terms of a polynomial differential operator.

$$\mathrm{L} = a_n(x)D^n + a_{n-1}(x)D^{n-1} + ... + a_1 D + a_0(x)$$

9.2 Superposition Principle

The sum or superposition of two or more solutions to a homogeneous linear differential equation is also consider to be a solution to a differential equation. This is an interesting concept because it actual limits to the amount of useful solutions we want. Consider that we could super impose solutions that were just multiple factors of each other. That doesn't do much for us, so we could instead super impose just two solutions who are linearly independent of each other.

But let's just consider the simple case, where $\gamma_1, \gamma_2, \gamma_3, ..., \gamma_n$ are considered to be solutions to a differential equation on some common interval I. The linear combination

$$y = c_1\gamma_1 + c_2\gamma_2 + c_3\gamma_3 + ... + c_n\gamma_n$$

is considered to be a solution with $c_1...c_n$ are considered to be arbitrary constants. We can compactly write this as

$$y = \sum_{i=0}^{k} c_i\gamma_i(x)$$

9.2.1 Proof of Principle

A good way to test the usage of our L is using it to prove the superposition principle. Consider the case $k = 2$, we let L be a polynomial differential operator and let γ_1,γ_2 be solutions to the differential equation, expressed in L,

$$\mathrm{L}\{y\} = 0$$

We define $c_1\gamma_1 + c_2\gamma_2 = y$ and by linearity,

$$\begin{aligned} \mathrm{L}\{c_1\gamma_1 + c_2\gamma_2\} &= \mathrm{L}\{y\} = 0 \\ c_1\mathrm{L}\{\gamma_1\} + c_2\mathrm{L}\{\gamma_2\} &= 0 \\ c_1 \times 0 + c_2 \times 0 &= 0 \end{aligned}$$

Since γ_1 and γ_2 are known solutions to $\mathrm{L}\{y\} = 0$, we know that they evaluate to zero. There by proving our superposition principle.

9.3 Solutions of Differential Equations

We are interested in linearly independent solutions. Note the superposition principle allows us to super impose any solution we have with independent solutions. To determine if a set of solutions are linearly independent, we take the Worskin Detriment,

$$\mathbf{W} = \begin{bmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{bmatrix}$$

If we have a set of functions $f_1, f_2, f_3...$, that are a solution to a homogeneous function, that set is independent linearly if and only if $\mathbf{W} \neq 0$ for all $x \in I$.

9.4 General Solution of a Homogeneous Equation

Let $y(x)$ be a solution and let y_1, y_2 be known solutions to a homogeneous equation L(y) = 0. Suppose on the common interval I, $x = t$ where $\mathbf{W}(y_1, y_2) \neq 0$ for all $x = t$. That means y_1 and y_2 are solutions that can be used to form a superposition,

$$y(x) = c_1 y_1 + c_2 y_2$$

9.5 General Solution of a Non-Homogeneous Equation

Any function free of arbitrary parameters, that satisfies a non-homogeneous equation is said to be a particular solution of the equation. Now if you had a set of solutions, the linear combination of that set and the y_p is also a solution. This means that if you took the solution set from a homogeneous equation, the fundamental solution set for the associated homogeneous equation.

$$y(x) = c_1 y_1 + c_2 y_2 + c_3 y_p$$

9.5.1 Reduction of Order

CHAPTER **10**

NUMERICAL METHODS OF SOLVING DIFFERENTIAL EQUATION

There are points were you can't solve a differential equation by normal means. You can use methods listed here to solve it numerical, meaning trying to find an approximate solution. These methods will also include methods of finding the error in the measurements.

10.1 Euler's Method

10.2 Taylor Series Expansion

LAPLACE TRANSFORM

We note that a transform takes one function and converts it to another. The Laplace transform is an integral transform. Given any $f(x, y)$, by taking a derivative with respect to one variable will always lead to an equation (assuming the integral is definite and convergent) within terms of the other variable.

Given a interval of integration bounded below, $[0, \infty)$, the integral transform is usually defined like this,

$$\int_a^b K(s,t)f(t)\,\mathrm{dt}$$

Where $f(t)$ is defined on the interval of integration. When the integral converges we will get a function that is defined in terms of s. We will look at a special case of the integral transform where $K = e^{-st}$.

11.1 Laplace Transform Integral

Given $f(t)$, where it's defined on $[0, \infty)$, the Laplace Transform for this function is defined

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}f(t)\,\mathrm{dt} = F(s)$$

Note that computing the Laplace transform will always take some integration by parts, so to review a quick method of computing the integration by parts, check the integration methods section. Other than that, here is a table of general common Laplace Transforms.

$$\begin{aligned}
\mathcal{L}\{1\} &= \frac{1}{s} \\
\mathcal{L}\{t^n\} &= \frac{n!}{s^{n+1}} \quad n = 1, 2, 3, ... \\
\mathcal{L}\{e^{at}\} &= \frac{1}{s-a} \\
\mathcal{L}\{\sin(kt)\} &= \frac{k}{s^2+k^2} \\
\mathcal{L}\{\cos(kt)\} &= \frac{s}{s^2+k^2}
\end{aligned}$$

11.1.1 Conditions For Existence

These conditions are not absolutely needed for the existence of a Laplace Transform but still can be used to see if one will exist. First $f(t)$ must be piecewise continuous. Basically continuous for piecewise functions. A general idea is that all the pieces are continuous, and the finite amount of discontinuities that exist are really just ones that are jumping.

The formal definition is, on the interval I, f is piecewise continuous if $0 \leq a \leq t \leq b$, where t_k, $k = 1, 2, 3, ..., n$, and $t_{k-1} < t_k$ where at $f(t_k)$ exist a finite amount of discontinuities but are continuous on the open intervals of (t_{k-1}, t_k).

$f(t)$ also must be of exponential order, meaning that it doesn't grow faster then a exponential function.

$$|f(t)| \leq Me^{ct}$$

where $M > 0$, $T > 0$, and C is a positive constant, on the interval of $t > T$.

11.1.2 Laplace Inverse Transform

Inverse transforms are simply reverses the transform,

$$\begin{aligned}
\mathcal{L}^{-1}\{\frac{1}{s}\} &= 1 \\
\mathcal{L}^{-1}\{\frac{n!}{s^{n+1}}\} &= t^n \;\; n = 1, 2, 3, ... \\
\mathcal{L}^{-1}\{\frac{1}{s-a}\} &= e^{at} \\
\mathcal{L}^{-1}\{\frac{k}{s^2+k^2}\} &= \sin(kt) \\
\mathcal{L}^{-1}\{\frac{s}{s^2+k^2}\} &= \cos(kt)
\end{aligned}$$

The key is to algebraically simplify the expression or dividing or multiplying by a constant to make it look like one of the table functions.

$$\mathcal{L}^{-1}\{\frac{1}{s^5}\}$$

We know that it looks like one of our table functions but it's missing the $n!$, so lets consider this

$$\frac{1}{n!}\mathcal{L}^{-1}\frac{n!}{s^5} = \mathcal{L}^{-1}\{\frac{1}{s^5}\}$$

Because $\mathcal{L}$ is a integral transform it has linearity with constants. We know the inverse of the right hand side, so we find the solution of

$$\frac{1}{n!}\mathcal{L}^{-1}\frac{n!}{s^5} = \frac{1}{4!}t^4$$

11.2 Volterra Integral Equation

Solve for $f(t)$ using this Volterra Integral equation by way of a Laplace Transform.

$$f(t) = 4t - 16\int_0^t \sin(t) f(t-\tau)\,\mathrm{d}\tau$$

First we apply the $\mathcal{L}$,

$$\begin{aligned}
F &= \frac{4}{s^2} - 16\mathcal{L}\{\int_0^t \sin(t) f(t-\tau)\,\mathrm{d}\tau\} \\
F &= \frac{4}{s^2} - 16\mathcal{L}\{\sin(t)\}F \\
F + 16\frac{1}{s^2+1}F &= \frac{4}{s^2} \\
F(1+\frac{16}{s^2+1}) = F\frac{s^2+17}{s^2+1} &= \frac{4}{s^2} \\
F &= \frac{4}{s^2}\frac{s^2+1}{s^2+17} \\
F &= 4(\frac{1}{s^2(s^2+17)} + \frac{s^2}{s^2(s^2+17)}) \\
F &= 4(\frac{1}{s^2(s^2+17)} + \frac{1}{(s^2+17)}) \\
F &= 4(\frac{\frac{1}{s(s^2+17)}}{s} + \frac{1}{s^2+17})
\end{aligned}$$

Now we apply the $\mathcal{L}^{-1}$,

$$\begin{aligned}
f(t) &= 4\mathcal{L}^{-1}\{\frac{\frac{1}{s(s^2+17)}}{s}\} + 4\mathcal{L}^{-1}\{\frac{1}{s^2+17}\} \\
f(t) &= 4\frac{1}{\sqrt{17}}\sin(\sqrt{17}t) + 4\mathcal{L}^{-1}\{\frac{\frac{1}{s(s^2+17)}}{s}\}
\end{aligned}$$

For the second inverse, we must first apply the second inverse convulsion theorem,

$$\mathcal{L}^{-1}\{\frac{\frac{1}{s(s^2+17)}}{s}\} = \mathcal{L}^{-1}\{\frac{F_1(s)}{s}\} = \int_0^t f_1(\tau)\,\mathrm{d}\tau$$

Where $f_1(t) = \mathcal{L}^{-1}\{F(s)\}$, by the first inverse convulsion theorem,

$$f_1(t) = \mathcal{L}^{-1}\{\frac{1}{s}\frac{1}{s^2+17}\} = 1 * (\frac{1}{\sqrt{17}}\sin(\sqrt{17}t)) = \int_0^t \sqrt{17}\sin(\sqrt{17}\tau)\,\mathrm{d}\tau$$

$$f_1(t) = -\frac{1}{17}\cos(\sqrt{17}t) + \frac{1}{17}$$

$$\begin{aligned}
\mathcal{L}^{-1}\{\frac{\frac{1}{s(s^2+17)}}{s}\} &= \int_0^t -\frac{1}{17}\cos(\sqrt{17}\tau) + \frac{1}{17}\,\mathrm{d}\tau \\
&= \frac{-1}{17\sqrt{17}}\sin(\sqrt{17}t) + \frac{1}{17}t \\
f(t) = 4\frac{1}{\sqrt{17}}\sin(\sqrt{17}t) + 4(\frac{-1}{17\sqrt{17}}\sin(\sqrt{17}t) + \frac{1}{17}) &= \frac{64\sqrt{17}}{289}\sin\left(\sqrt{17}t\right) + \frac{4t}{17}
\end{aligned}$$

CHAPTER **12**

DIFFERENTIAL EQUATION APPLICATIONS

This chapter will go over all the application problems you might have in the class. Before we over that, important ass note here. Conservations factors in this class aren't as simple as metric systems. For example mass in imperial is equal F/g, because force is lb-force and g=$32 ft/s^2$.

12.1 Growth and Decay

All these problems deal with simple growth and decay. This includes logistic models, Newton's law of cooling, and simple growth and decay.

12.1.1 Simple Growth and Decay

Given $y' = kx$, you have a simple exponential growth problem. Given $y(x_0) = y_0$, the general solution would be,

$$y(x) = C_0 e^{kx}$$

Solving, for the constant of integration, you would get

$$\begin{aligned} y_0 &= C_0 e^{kx_0} \\ \frac{y_0}{e^{kx_0}} &= C_0 \end{aligned}$$

12.1.2 Newton's Law of Cooling

Newton's law of cooling is defined in this general differential equation,

$$\frac{\mathrm{dT}}{\mathrm{dt}} = k(T - T_M)$$

We assume T_M is a given constant, the solution to this differential equation is

$$\begin{aligned} \frac{1}{T - T_m}\frac{\mathrm{dT}}{\mathrm{dt}} &= k \\ \ln|T - T_m| &= kt + C \\ T - T_m &= e^{kt+C} \\ T &= Ce^{kt} + T_m \end{aligned}$$

As you can see the constant of integration isn't exactly T_m, solving for the constant of integration would,

$$\frac{T_0 - T_m}{e^{kt_0}} = C$$

12.1.3 Logistic Growth Models

A logistic growth model is defined to be this,

$$\frac{\mathrm{dP}}{\mathrm{dt}} = P(r - \frac{r}{k}P)$$

Where we assume r to be the rate of proportionality and k to be the max carrying capacity. If we simplify it down,

$$\frac{\mathrm{dP}}{\mathrm{dt}} = aP - bP^2$$

where $a = r$, and $b = r/k$. We can now work on solving this differential equation.

$$\frac{\mathrm{dP}}{aP - bP^2} = \mathrm{dt}$$
$$(\frac{1/a}{P} + \frac{b/a}{a - bP})\mathrm{dP} = \mathrm{dt}$$

After this point, it's just algebra to solve the equation, so the given solution is

$$P(t) = \frac{aP_0}{bP_0 + (a - bP_0)e^{-at}}$$

You can also have the growth rate written in terms of things that are also harvested or restocked at a different rate function of $h(t)$.

$$P' = P(r - \frac{r}{k}P) \pm h(t)$$

where we can assume $h(t)$ will be a constant or a continuous function.

12.2 Mixtures and Physics Applications

All these problems are dealing with physical applications of differential equations with respect to chemistry and physics.

12.2.1 Mixtures

When you are mixing two things together, the simple expression can be written down to determine how much of something you have when a mixture is leaving a container.

$$\frac{\mathrm{dA}}{\mathrm{dt}} = R_{in} - R_{out}$$

The hard part is finding out what goes in and what goes out. Let's deal with question from an exam,

A large tank is partially filled with 100 gallons of fluid in which 10 pounds of salt are dissolved. Brine (a salt solution in water) containing (1/2) lb. of salt per gallon is pumped into the tank at a rate of 6 gal/minute. The well mixed solution is them pumped out at a slower rate of 4 gal/min. Find the amount of salt (in pounds) in the tank after 30-minutes. Round your answer to three digits after the decimal sign.

We know that half a pound of salt goes in to the tank at at rate of 6 gallons per minute. So that means R_{in} is $\frac{1}{2}\frac{lb}{gal} * 6\frac{gal}{min}$. R_{out} is a bit more tricky. Considered that we are trying to find how much salt is leaving the tank. Since there is a 100 gallon of fluid, we know that A, our variable representing the amount of salt in the container is proportional to the total amount of fluid. So $A/100$ or the amount of salt per gallon of fluid. We know that mixture is leaving at at given rate, so $R_{out} = A/100 * 4gal/min$, the ending differential equation, with the given initial solution is

$$\frac{\mathrm{dA}}{\mathrm{dt}} = R_{in} - R_{out} = 3 - \frac{4A}{100} \quad , A(0) = 10$$

When the rate going out is higher than the rate going in, you have to account for the decrease in fluid. Let $r_{outflow}$ be the rate at which fluid leaves the tank (difference between R_{in} and R_{pump}), R_{pump} is how much the solution is leaving.

$$R_{out} = \frac{A}{T_w - r_{outflow}} \times R_{pump}$$

CHAPTER **13**

ELECTROSTATICS

In electrostatics, the main force we deal with is the electrostatic force (Coulomb's force) that is gained from charged matters. Unlike gravity, when mass is presented, only attracts, there are two different types of charges. Positive and negatively charged matters. The charge is a quality of the proton and electron. Where net charge is the total charge in a body. Like most things, charge is conserved and obey the conservation principles.

13.1 Charge By Induction

A object is only charged when is there is a net unbalance of charges. Meaning that if a object is uncharged it still has charge but the net sum of all the charges cancel out and evaluate to zero. Usually you cause this by moving some amount of electrons from one object to another. In physics, we consider protons to be the one that actually moves (all fields are defined by a positive moving charge), though in real life it's only the electron that moves. **Induction** is a method of charging a object without even coming into contact with one another. Say you have two uncharged spheres that are grounded by a insulating rod. That means they are completely isolated from the world. The spheres are in contact with each other. Then you bring a negatively charge rod close to, but not touching, one of the spheres. Then you move the sphere furthest away from the rod, away from the other sphere. The sphere closest to the rod gains a positive charge, the sphere furthest from the rod gains a negative charge. That's because, even though they weren't touching the sphere, the attraction between the negatively charged rod and the sphere's positively charged "protons" moved all the positive charges from both spheres into one sphere. Leaving the other sphere negatively charged.

13.2 Coulomb's Law

There is a force between charges that there attracts or repeals. The force is defined by Coulomb's Law,

$$\vec{\mathbf{F}} = \frac{1}{4\pi\epsilon_0}\frac{|q_1 q_2|}{r^2}\hat{r}$$

The abs matters as while $\vec{\mathbf{F}} \propto q_1 q_2$, the direction is dependent on the signs of the charges.

13.2.1 Principle of Superposition

The idea behind superposition is that we could evaluate the total force, field, or something at a certain point by just taking the superposition (or the sum) of all the total forces or fields at that point. For example, given $\mathbf{F_1}, \mathbf{F_2}, \mathbf{F_3}$ forces placed at some random points in space $\mathbb{R}^3$, the force on a charge q is equal to $\sum \mathbf{F_q i}$. That seems natural and doesn't really need to be stated, but this principle allows us to ignore things when they don't suit our calculations at that time and evaluate them later in different parts.

13.2.2 Newton's Third Law

In a system of charges, the simplest being two charges, we can see that as one charge feels some attraction or repulsive force from the other, that other charge will feel the same magnitude of force but in the opposite direction. This is true from a Newton's law's point of view and also the same from a Coulomb's law point of view. In Newton's corner, when one charge applies a force on another charge, that charge applies the same force on the other charge. Using Coulomb's law with the subscripts,

$$\mathbf{F}_{21} = \frac{1}{4\pi\epsilon_0}\frac{|q_1 q_2|}{r^2}$$

$$\mathbf{F}_{12} = \frac{1}{4\pi\epsilon_0}\frac{|q_1 q_2|}{r^2}$$

Where it reads, the force on 2 by 1. The magnitude of the forces doesn't change depending on the perspective of who's doing what.

CHAPTER 14

ELECTRIC FIELD

14.1 Introduction

Let's construct a vector field all over some random space $\mathbb{R}^3$. Every point on this vector field will give a vector that represents "the amount of force a single charge (Coulomb) would feel". Given the Coulomb force,

$$\vec{\mathbf{F}} = \frac{1}{4\pi\epsilon_0}\frac{|q_1 q_2|}{r^2}\hat{r}$$

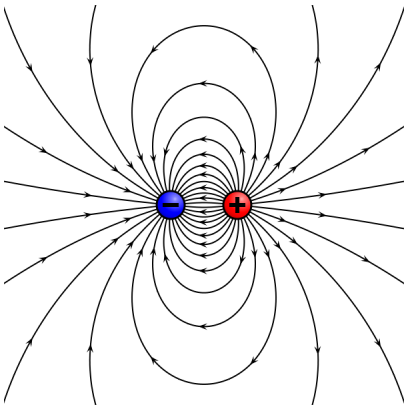
The quantity we want would be $\frac{\vec{\mathbf{F}}}{\mathbf{Q}}$, where $\mathbf{Q}$ is some random positive test charge. That means a charge could exist at that random point, but really we don't factor that into our calculations. The quantity should only be dependent on the distance between from the source charge to the test charge, which means that the quantity would only be a property of the space around the source charge. This is called an **electric field**, and the source charge produces the electric field. **NOTE:** that the $\hat{r}$ is the displacement vector from the source charge to the test charge!

$$\vec{\mathrm{E}}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{|q_1|}{r^2}\hat{r}$$

If we gather the electric field in a certain region or area, the density of "electric field lines" or the lines representing the "flow", would be equal to the magnitude of the electric field. The direction would be the same as the direction of the electric field. **NOTE:** Since we defined the electric field based on a positive test charge, the field flows in the direction of the positive charges. Meaning, if we had a negative charge, it would just be the opposite direction.

14.2 Electric Field Lines

First let's some rules for the electric field from how we defined it. The field is a vector value at any given point in space that gives a force vector for a single test charge. The force is due to the source charge that generates a electric field. Since we define the electric field to be based on a positive test charge, that means that electric flow for anything in electrostatics would be based on positive charge movement. We can draw what's called electric field lines. These lines are based on the flow of the electric field lines.

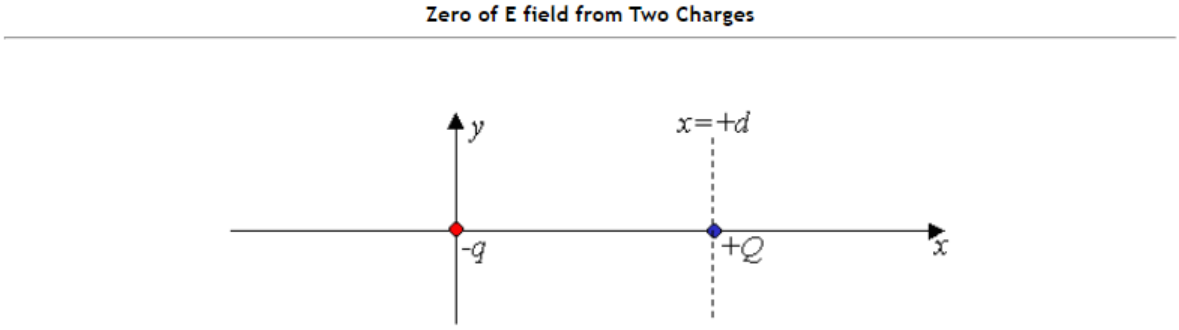


Electric Dipole Field

The **density** of field lines at any given point for a select area is given by the magnitude of the electric field at that point. The **direction** is from the direction of the electric field.

14.3 Example: Finding the Zero of Electric Field

Given this set up, find the point or region on the x-axis where the electric field would be zero.



Electrostatics Setup

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First let's establish a bunch of possible regions that electric field could be 0. The first region would be in interval $x \in \{0, d\}$. We can calculate the total electric field in this region by way of superposition,

$$\vec{\mathrm{E}} = \frac{1}{4\pi\epsilon_0}\left(\frac{-q}{x^2} + \frac{Q}{(d-x)^2}\right)\hat{\mathbf{x}}$$

Before we do any math to simplify, let's consider what's happening. The negative charge is pulling our E-field towards the left, and the positive charge is pushing our E-field towards the left. The vector sum wouldn't cancel out to 0, because both charges aren't opposing each other. This gives us a hint. Since Q is

larger than q, we need to find a region where the distance from Q to the zero point x_0 is farther then the distance between the q and x_0. This gives us the interval of $-\infty, 0$ because any point on that side would be closer to q than Q. Giving us a chance for the field to be 0.

14.4 Superposition of Charges

We are going to be going over differential versions of principle of superposition in order to develop equations for continuous structures of charges. This should be used as a pre-text for Gauss's Law, which simplifies this process even further.

If we consider a discrete structure, with infinite amount of points,

$$\sum_{i=0}^{\infty} \vec{\mathrm{E}}_i \to \sum \frac{1}{4\pi\epsilon_0} \frac{|q_i|}{r^2} \hat{r}$$

We can easily transform into integral expression.

$$\int \mathrm{d}\vec{\mathrm{E}}$$

where the differential notes the electric field due to a differential of charge dx. We take the sum of all individual fields due to some small section of charge.

$$\int \mathrm{d}\vec{\mathrm{E}} = \frac{1}{4\pi\epsilon_0} \int \frac{1}{r^2} \hat{r} \, \mathrm{dq}$$

14.4.1 Superposition of Line Charge

We consider a charge that is uniformly placed and distributed over the x-axis, with a length of spanning of L, with a charge per unit length of λ. We consider a test charge placed on the $+y$-axis, centered at $x = 0$. We choose the origin to be the coordinate points $(0, 0)$.

We note the distance from any point on the x-axis to a positive $y = +R$, would be $\sqrt{x^2 + R^2}$. The $\hat{r}$, from the source charges of dq would be $\hat{r} = \frac{1}{\sqrt{x^2+R^2}} \langle -x, R \rangle$. At this point, we can setup most of the equation for the super position. We note that the differential of charge would be λdx, because if we took a small section of the line, the charge contained would equal to our charge per unit length times the length of the section.

$$\vec{\mathrm{E}} = \frac{\lambda}{4\pi\epsilon_0} \int \frac{-x}{(x^2 + R^2)^{3/2}} \hat{\mathbf{x}} + \frac{R}{(x^2 + R^2)^{3/2}} \hat{\mathbf{y}} \ \mathrm{dx}$$

$$\vec{\mathrm{E}} = \frac{\lambda L}{2\pi\epsilon_0 R\sqrt{L^2 + R^2}} \hat{\mathbf{y}}$$

As you can see, due to symmetry a perpendicular test charge above a line charge will face no force in the direction parallel to the line. We can find some general results from the this equation by taking two different limits.

14.5 Gauss' Law

CHAPTER **15**

ELECTROSTATICS AND KINEMATICS

CHAPTER **16**

ELECTRICAL ENERGY

16.1 Capacitors

CHAPTER **17**

ELECTRICAL CURRENT

17.1 Resistors

17.2 Ohm's Law

CHAPTER **18**

MAGNETISM

Certain metals you find will exhibit what's known as magnetism. Magnetic objects, like electrical charges, will show levels of attractiveness towards "opposite charges". Magnetism, and it's forces and fields are directly related to electricity. First let's note the difference between magnetism and electricity. The first thing to note, all magnetic objects that have currently been observed are **dipoles** compare to singularly charged parties. They have a north and south ends. The north and south ends attract each other oppositely. If you break a magnet in half, it will just split into different north and south poles. Oddly enough our discussion of magnetism will start with the magnetic field.

18.1 Magnetic Field

A **magnetic field is created when a moving charged particle or current is present**. Remember that fields are a property of the surrounding space. There is going to be a magnetic field and electric field at the same time because if there is a charge present then of course you will have an electric field present. Now the difference is that a magnetic field is only present if the charge is moving or you have a current (moving charges). The electric field is always present and not dependent on if the charge is moving or not. The second thing to note is that the magnetic field exerts a force $\vec{\mathrm{F}}$ on other moving charges or currents in the space of the field. I don't know if at this point if we know that the magnetic field is self-interacting with the moving particle that creates it.

18.1.1 Properties of the Magnetic Force

We note two things done by experiments to determine the properties of the magnetic force. First, that there is a direct relationship between strength of the magnetic force and the amount of charge present. Second, the force is solely perpendicular to the velocity of the charged particle and the magnetic field present. The cross-product shares the same properties, and we can write the Lorenz's Force for magnetism,

$$\vec{\mathrm{F}_{\mathrm{b}}} = q\vec{v} \times \vec{\mathrm{B}}$$

18.1.2 Magnetic Field Lines and Flux

First, note that the magnetic field lines are not lines of force. The magnetic force is perpendicular to those field lines. Second, we note the Gauss's Law of Magnetic Flux, because there isn't a single monopole of magnetism discovered.

$$\phi_B = \int \vec{\mathrm{B}} \cdot \mathrm{d}\tilde{\mathrm{S}} = 0$$

18.2 Motion in a Uniform Magnetic Field

We consider a uniform magnetic field. Let's say that it points out in the direction of $-\hat{i}$ at all points in the region described by $x = a$ and $y = a$. There is a particle that enters the field with a certain velocity with a direction of positive $\hat{k}$. First with uniform magnetic field, the force is going to be centripetal. It's always going to be perpendicular to the field and the velocity. So we know we are going to have uniform rotational motion. By Lorenz's Force Law,

$$\vec{\mathrm{F}_{\mathrm{b}}} = q\vec{v} \times \vec{\mathrm{B}}$$

We also know that for uniform rotational,

$$\mathrm{F}_b = \frac{mv^2}{R} = qv\mathrm{B}$$

We can get the radius of curvature from this equation,

$$R = \frac{mv}{q\mathrm{B}}$$

Now we can form our equations of motion, based on the fact that we have R. We know the equation of a circle,

$$(x-h)^2 + (y-k)^2 = R^2$$

depending on how the circle is centered.

CHAPTER **19**

MAGNETIC FORCE: CURRENT

We are going to develop several laws and applications for magnetism applied to currents. First we developed a formula that applies to current loops for magnetic forces. We first note what the Lorentz's Force Law states,

$$\begin{aligned}\vec{\mathrm{F}_{\mathrm{b}}} &= q\vec{v} \times \vec{\mathrm{B}} \\ \vec{\mathrm{F}_{\mathrm{b}}} &= q\sum_i \vec{v_i} \times \vec{\mathrm{B}} \\ \vec{\mathrm{F}_{\mathrm{b}}} &= I\vec{l} \times \vec{\mathrm{B}}\end{aligned}$$

The derivation notes that $N = nAL$ and $I = qnAv_{avg}$. We take the fact that currents are going be always describes by moving charge particles.

19.1 Magnetic Forces on Current Loops

Start developing some important ideas. First current loops, we can gather the force from any shaped current loop by this equation,

$$\vec{\mathrm{F}} = \int_C I\mathrm{d}\vec{\ell} \times \vec{\mathrm{B}}$$

This is a line integral result, so it be useful to break it up the loop path into different segments.

$$|\vec{\mathrm{F}}| = \int_C I\mathrm{B}\sin(\theta)\ \mathrm{d}\ell$$

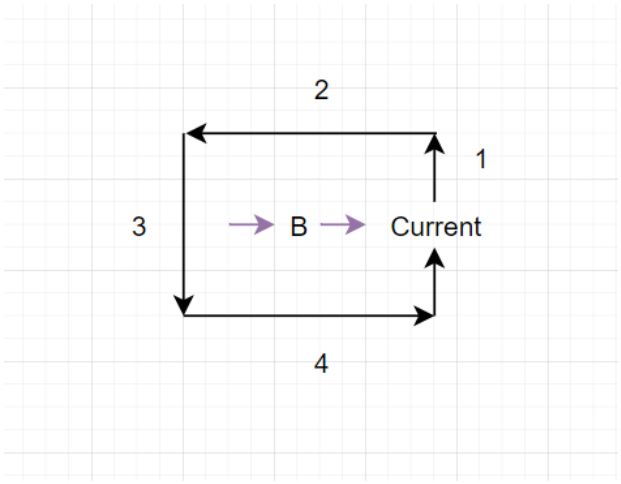
This is a general result for any path that describes a current loop that is closed. Any **closed** current loop that exist within a uniform magnetic field, will have a net force equal to zero.

$$\oint \mathrm{d}\vec{\mathrm{F}_{\mathrm{b}}} = \oint I\mathrm{d}\vec{\ell} \times \vec{\mathrm{B}} = 0$$

19.2 Magnetic Torque

Even though a closed current loop will have a zero net force, the will still be a non-zero torque. Consider the square loop, centered at the origin, current going counterclockwise, in a magnetic field that is pointing to the right. We consider the moment axis to be centered around the origin. Consider the torque from all sides.

$$\vec{\tau} = \vec{r} \times \vec{\mathrm{F}}$$



Example Current Loop

First the forces on the current loop. Side 1 will be pointing inwards, while side 3 will be pointing outwards. Sides 2 and 4 will have no force because the currents they are charing will be parallel to the magnetic field. That means the torque we are going to have will be dependent on the vertical sides. The $\hat{r}$ is going to be the position from our moment axis. We choose the moment axis to be centered at the origin. The torque contributed from side 1 will be upwards, because the $\hat{r}$ is point to the left and the $\vec{\mathrm{F}}$ from side 1 is pointing inwards. The torque contributed from side 3 will be upwards as well because $\hat{r}$ will be pointing to the right and the force vector is outwards. The total torque is upwards.

19.2.1 Magnetic Dipole Moment

To simplify our work, we can express the torque from a magnetic field on a current loop by using it's dipole moment. Consider the total magnitude of the torque from the square loop.

$$\tau = 2(r\mathrm{F})$$

We know that $\mathrm{F} = I\mathrm{B}\ell$, where r is $\frac{1}{2}\ell$, the expression becomes

$$\tau = 2(\frac{1}{2}\ell)I\mathrm{B}\ell$$
$$\tau = I(\ell)^2\mathrm{B}$$
$$\tau = I(A)\mathrm{B}$$

We can express the IA to be consider the magnetic dipole moment, and we can express this with the vector form of the dipole moment.

$$\tau = \vec{\mu} \times \vec{\mathrm{B}}$$

We also consider the amount of loops there is, so we redefined $\mu = IN\vec{\mathrm{A}}$, and the direction of the $\vec{\mu}$ is going to be considered like a normal surface vector.

19.3 Potential Energy for Magnetic Dipoles

When you have a current loop in a magnetic field, while the net force by the field is zero, the change in orientation caused by the torque suggest that the field is doing work on the loop. We can derive the equation for potential energy of a dipole by consider the total amount of work done by the torque. We know the work done by a torque is going to be,

$$\begin{aligned} W &= \int \tau \ \mathrm{d}\theta \\ W &= \int |\ \vec{\mu} \times \vec{\mathrm{B}}\ |\ \mathrm{d}\theta \\ W &= \int \mu \mathrm{B} \sin(\theta)\ \mathrm{d}\theta \\ W &= \mu \mathrm{B} \cos(\theta) \\ W &= \vec{\mu} \cdot \vec{\mathrm{B}} \end{aligned}$$

Work stays positive because we are taking the magnitude of the cross, so that's why the integral stays positive. But since we know the change in potential energy is equal to negative work,

$$\Delta U = -\vec{\mu} \cdot \vec{\mathrm{B}}$$

Where we define, U_0 to be when μ is perpendicular to $\vec{\mathrm{B}}$. Please **NOTE** that potential energy is "how much stuff might happen" versus work's "how much stuff is happening". When you think about when potential energy is the highest is when μ is anti-parallel with $\vec{\mathrm{B}}$ because at that point, there is a potential for the field to do work to change it's orientation.

19.4 Biot-Savart Law

We can find the magnitude of a magnetic field caused by a moving charge at a given instant by a source point and a test point in space. This is like how we developed the electric field concept. Experiments and previous observations note that magnetic field from a moving source charge at a given instant is equal to,

$$\vec{\mathrm{B}} = \frac{\mu_0}{4\pi} \frac{|q| \vec{v} \times \hat{r}}{r^2}$$

Where $\hat{r}$ is the vector from the source point to the test point. The constant μ_0 is called the magnetic constant, and it has the value of $4\pi \times 10^{-7} T \cdot \mathrm{m/A}$. We can move this further with general current shapes. Consider a current carrying rod, if we took a segment of length $\mathrm{d}\ell$ times it's cross-sectional area A, then times a charged particle per unit volume, n, we get the differential for our change in charge, $dQ = nqA\mathrm{d}\ell$,

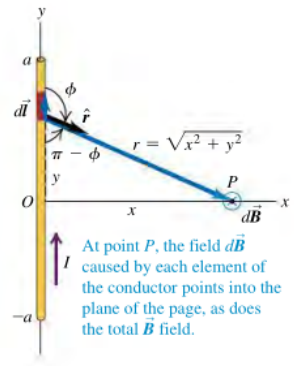
$$\begin{aligned} \mathrm{d}\tilde{\mathrm{B}} &= \frac{\mu_0}{4\pi} \frac{|dQ| \vec{v} \times \hat{r}}{r^2} \\ \mathrm{d}\tilde{\mathrm{B}} &= \frac{\mu_0}{4\pi} \frac{n|q|A\mathrm{d}\ell \vec{v} \times \hat{r}}{r^2} \\ \mathrm{d}\tilde{\mathrm{B}} &= \frac{\mu_0}{4\pi} \frac{I\mathrm{d}\vec{\ell} \times \hat{r}}{r^2} \end{aligned}$$

The final law for the total field is,

$$\vec{\mathrm{B}} = \frac{\mu_0}{4\pi} \int_C \frac{I\mathrm{d}\vec{\ell} \times \hat{r}}{r^2}$$

19.4.1 Magnetic Field of a Straight Current-Carrying Conductor

We use the Biot-Savart Law on a straight current-carrying conductor.



Current Carrying Conductor

We know that $\mathrm{d}\ell \times \hat{r}$ is forms a plane along inwards in the straight path of the conductor. That means all the directions of the magnetic filed is going to be the same, so we can just add their magnitudes together. We know that $r = \sqrt{x^2 + y^2}$, $\mathrm{d}\ell = I\mathrm{dy}$, and the $\sin(\pi - \theta) = x\sqrt{x^2 + y^2}^{-1}$, forming the integral expression,

$$\mathrm{B} = \frac{\mu_0}{4\pi} \int_{-a}^{a} \frac{Ix}{(x^2 + y^2)^{3/2}} \ \mathrm{d}\ell$$

Which simplifies down to,

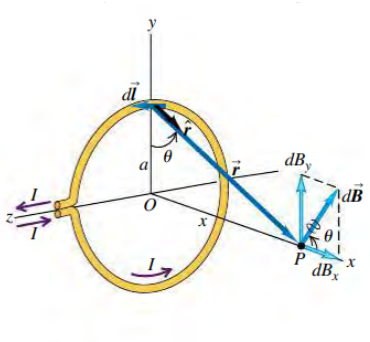
$$\mathrm{B} = \frac{\mu_0 I}{4\pi} \frac{2a}{x\sqrt{x^2 + a^2}}$$

When we are very far away from the conductor, or when $a \to \infty$, the expression becomes

$$\mathrm{B} = \frac{\mu_0 I}{2\pi r}$$

19.4.2 Magnetic Field of A Coil Loop

We are going to use BS's law to develop a formulation for conducting coils. Coils are going to play an important part in later parts. First part, is figuring out the curl of $Id\ell \times \hat{r}$. Since we know that the curl will always point in the constant direction, we can just sum of the total magnitude of the $\hat{r}$.



Single Loop

We easily can find the magnitude of the infinitesimal, and will just apply the components related to it. Since we know that the y-direction will have no magnetic field by way of symmetry, we just can use the cosine.

$$\mathrm{dB}\cos\theta = \frac{\mu_0 I}{4\pi}\frac{\mathrm{d}\ell}{(x^2+a^2)}\frac{a}{\sqrt{x^2+a^2}}$$

The integral will be defined as this,

$$\int \frac{\mu_0 I}{4\pi}\frac{\mathrm{d}\ell}{(x^2+a^2)}\frac{a}{\sqrt{x^2+a^2}} = \frac{\mu_0 I a^2}{2(x^2+a^2)^{3/2}}$$

When we are close to the coil (at the center of it), we find the expression simplifies down to

$$\mathrm{B} = \frac{\mu_0 I N}{2a}$$

where N is the amount of turns the coil has, since we could just stack bunch of coils of infinitesimal loop.

19.4.3 Parallel Conductors

When we consider two rods, r_1 and r_2, the force felt by the magnetic field produced by both is dependent on their currents. They both feel the force for the other's respect magnetic field. Let's look at the force of r_1. The lower rod would produce a magnetic field who's magnitude would be,

$$\mathrm{B} = \frac{\mu_0 I}{2\pi r}$$

Let I' be the current of the top rod. We can figure out that the force on the top rod by the second rod is equal to,

$$F_b = I' L\mathrm{B} = I' L\frac{\mu_0 I}{2\pi r}$$

Where the force per unit length is just equal to

$$F_b = I'\frac{\mu_0 I}{2\pi r}$$

Now we have to consider the vector form,

$$\vec{\mathrm{F}} = I'\vec{\ell} \times \vec{\mathrm{B}}$$

We see if that I' and I share the same direction, the two rods, by the right hand rule, will attract each other. If they have different current directions, the rods will repel each other.

19.5 Ampere's Law

Ampere's Law can be used to gather the magnetic field from a single path integral. Like how Gauss's used imagery surfaces, Ampere's Law states that for any closed loop path, the circulation integral will equal the enclosed current in surface created by that path.

$$\oint \vec{\mathrm{B}} \cdot \ \mathrm{d}\vec{\ell} = \mu_o I_{enclosed}$$

But of course, like Gaussian surfaces, the loops must exhibit proper symmetries so we can use it. These loops are called Amperian loops. The direction of $\mathrm{d}\vec{\mathrm{B}}$ and $\mathrm{d}\vec{s}$ should be the same for all segments of the loop. That way you can pull the magnetic field out of the integral and solve for it.

19.5.1 Ampere's Law for Solenoids

Solenoids are another important shape for magnetic fields. If you have an infinite solenoid, you can find that inside the solenoid, the magnetic field would be uniform. The outside of the solenoid would have no magnetic field. The line integral for Ampere's Law would just yield a BL because the $\vec{\mathrm{B}}$ inside would just be uniform.

$$\oint_C \vec{\mathrm{B}} \cdot \mathrm{d}\vec{\ell} = \mathrm{B}\ell$$

For the right-hand side of Ampere's Law, we note that the integration path is a rectangular path that goes inside and outside of the solenoid. The solenoid, we assume, makes a constant amount of turns per length, n. That means, if the solenoid carried a uniform charge of I, the enclosed amount of charge in that area would be $n\ell I$.

$$\mathrm{B} = \mu_0 n I$$

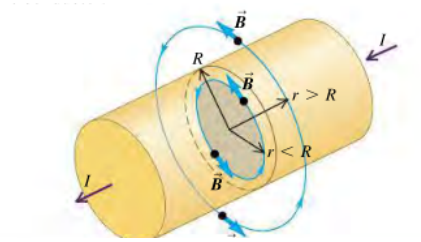
19.5.2 Ampere's Law for Toroidal Solenoid

The toroidal solenoid is another common shape that we can consider. First we note that the surface (and outside) will not have a magnetic field. Because of the winding of solenoid, the current passes through the surface twice, equal in opposite directions. Meaning the net enclosed charge will be zero. The only way we would have a non-zero magnetic field is if we consider the field inside of the toroidal volume. We know that the enclosed amount would be NI, where N is the amount of turns (not to be confused with amount of turns per length). Solving Ampere's Law will yield,

$$\mathrm{B} = \frac{\mu_0 N I}{2\pi r}$$

19.5.3 Example of Ampere's Law

A cylindrical conductor with radius R carries a current I. The current is uniformly distributed over the cross-sectional area of the conductor. Find the magnetic field as a function of the distance r from the conductor axis for points both inside ($r < R$) and out side ($r > R$) the conductor.



Conductor

Let's first consider the inside of the conductor. Consider the picture first, we need to know how much current is in the desired radius. We know that the total current is I, the current density $J = I/A$. Where $A = \pi R^2$, so $J = \frac{I}{\pi R^2}$.

$$I_{enclosed} = \pi r^2 \frac{I}{\pi R^2} = \frac{r^2}{R^2} I$$

Let's take the path integral,

$$\oint \vec{\mathrm{B}} \cdot \mathrm{d}\vec{\ell} = \mu_0 I_{enclosed}$$

Since we know that the $\vec{\mathrm{B}}$ is constant and pointing a constant direction, we can pull it out, and the path integral will just yield the total perimeter (or circumstance in this case).

$$\mathrm{B} \oint \mathrm{d}\ell = \mathrm{B}(2\pi r) = \mu_0 \frac{r^2}{R^2} I$$

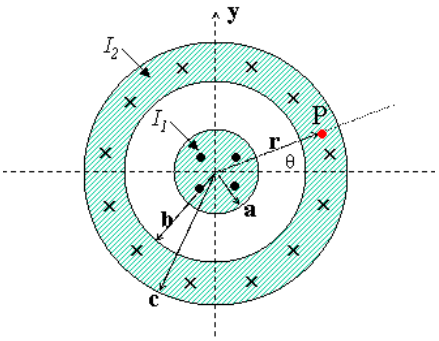
$$\mathrm{B} = \frac{\mu_0}{2\pi} \frac{r}{R^2} I \quad r < R$$

The process is the same with $r > R$, but since we know the enclosed charge is just going to be I, the expression because much simpler

$$\mathrm{B} = \frac{\mu_0}{2\pi} \frac{I}{r}$$

The same expression as we got with Biot-Savart Law but with a much faster approach since we had the symmetry.

19.5.4 Example 2 of Ampere's Law



Conductor

Two very long coaxial cylindrical conductors are shown in cross-section above. The inner cylinder has radius $a = 2$ cm and caries a total current of $I_1 = 1.2$ A in the positive z-direction (pointing out of the screen). The outer cylinder has an inner radius $b = 4$ cm, outer radius $c = 6$ cm and carries a current of $I_2 = 2.4$ A in the negative z-direction (pointing into the screen). You may assume that the current is uniformly distributed over the cross-sectional area of the conductors. What is B_x, the x-component of the magnetic field at point P which is located at a distance $r = 5$ cm from the origin and makes an angle of 30 degrees with the x-axis?

Let's consider the Ampere's Law,

$$\oint \vec{\mathrm{B}} \cdot \mathrm{d}\vec{\mathrm{B}} = \mu_0 I_{enclosed}$$

We choose a path, that is a circle with radius P. There is no need to break the integral down if we consider the signs of how currents. We note that the current direction is always given by the cross of $\mathrm{d}\ell$ and $\hat{r}$. We find that current I_1 is going have a positive counterclockwise direction. We find that current I_2 is going to have a negative clockwise direction. To properly find the enclosed current, we need to first find the current density of I_2.

$$J = \frac{I_2}{\pi(c^2 - b^2)}$$
$$I_{2,enc} = \frac{I_2}{\pi(c^2 - b^2)}\pi(r^2 - b^2) = \frac{I_2}{(c^2 - b^2)}(r^2 - b^2)$$

We know that the total enclosed current will equal,

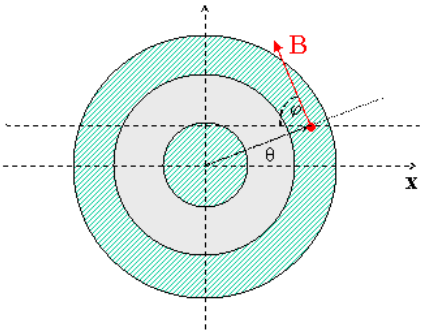
$$I_1 - I_{2,enc}$$

You would note that we get a positive value, suggesting that $\vec{\mathrm{B}}$ will point towards a positive counterclockwise direction, which is important to note when we are finding the $\vec{\mathrm{B}}_x$.

We finish Ampere's Law,

$$\mathrm{B}(2\pi)r = \mu_0(I_1 - \frac{I_2}{(c^2 - b^2)}(r^2 - b^2))$$
$$\mathrm{B} = \frac{1}{2\pi r}\mu_0(I_1 - \frac{I_2}{(c^2 - b^2)}(r^2 - b^2))$$

We note that the final assumed drawing,



Final Step

That the angle $\gamma = 90 - \theta$ by geometry. Which gives us our final expression of the B_x component.

$$B_x = -\frac{1}{2\pi r}\mu_0(I_1 - \frac{I_2}{(c^2 - b^2)}(r^2 - b^2))cos(\gamma)$$

CHAPTER **20**

ELECTROMAGNETIC INDUCTION

To recap, we note that a magnetic field is created by moving charges or currents. We didn't talk about how a magnetic field can create a **induced** current or an $\mathcal{E}$. So we are going in reverse, we are trying to figure out how a magnetic field creates a current. Through experiments, we find that the induced $\mathcal{E}$ is proportional to the rate of change of magnetic flux, Φ_{B}.

20.1 Faraday's Law

As we note, for any general flux integral,

$$\Phi = \iint_S \vec{\mathrm{F}} \cdot \mathrm{d}\vec{\mathrm{S}}$$

This hold true for magnetic flux but don't confuse it with a closed surface integral. This integral applies to non-closed surfaces for non-zero magnetic flux. We note from the introduction that changes to the magnetic flux will yield a $\mathcal{E}$. Since Φ_{B} is dependent on $\vec{\mathrm{S}}$, changes to the surface will also yield a $\mathcal{E}$. I think we assume a constant $\vec{\mathrm{B}}$ for now but both $\vec{\mathrm{B}}$ and the area can be dependent on time.

Faraday's Law states,

$$\mathcal{E} = -\frac{\mathrm{d}\Phi_{\mathrm{B}}}{\mathrm{dt}}$$

That the induced $\mathcal{E}$ is equal to a changing magnetic flux over time. The direction of the $\mathcal{E}$ and current is determined by this procedure. First consider the positive direction for the normal surface vector. Second determine the sign of the $\mathcal{E}$, where if the $\frac{\mathrm{d}\Phi_{\mathrm{B}}}{\mathrm{dt}}$ is positive, then $\mathcal{E}$ is negative, or visa versa. Then determine the cross of normal surface vector and the magnetic field vector. If the $\mathcal{E}$ is positive, the direction (or rotation) of the cross will be the same direction for $\mathcal{E}$, if it's negative, the direction (or rotation) is the opposite of the cross. If the current loop is a coil that has n identical turns, and Φ_{B} remains the same through each turn,

$$\mathcal{E} = -N\frac{\mathrm{d}\Phi_{\mathrm{B}}}{\mathrm{dt}}$$

20.1.1 Lenz's Law

Consider the previous section's method on determine the direction of the induced current. Lenz's Law is a quick way to summarize those results. The law states that the direction of any magnetic induction effect is such as to oppose the cause of the effect. This means if either $\vec{\mathrm{B}}$ or $\vec{\mathrm{A}}$ changes in any direction, the magnetic induction effect will be in the opposite direction (that's the $\mathcal{E}$ or current). For example, let's say

the magnetic field is pointing upwards, coaxial with a circular loop. By Lenz's Law, the opposing direction for the increasing magnetic field is downwards (we don't know the magnitude yet). Using the right hand rule, we find that the opposing magnetic field causes a clockwise rotation for the induced current and $\mathcal{E}$. Remember that Lenz's Law relates this concept to changing magnetic flux, not the flux itself. This law comes naturally from relating the energy conservation laws. If the law was reversed, that means the induced current would be in the direction of the induction. Meaning the induced current will create a magnetic field that will add to the magnetic field that caused the induction, and in turn create a larger induction current, and so on. You just created a stronger magnetic field from nothing which is a clear violation of energy conservation.

20.2 Motional Electromotive Force

Let's consider that we attached a conducting rod to some rail. If we had the conducting rod move in a magnetic field, $\vec{\mathrm{B}}$, then we would create an $\mathcal{E}$. The reason why this emf is created is because the magnetic field creates an electrical field inside the length of the rod. The magnetic field would separate the positive and negative charges. This means that there is going to be a potential difference between the top and the bottom of the rod as well an electrical field. As we defined electrical potential,

$$V_{ab} = -\int \vec{\mathrm{E}} \cdot \mathrm{d}\ell$$

the magnitude of the potential would be $V = \mathrm{E}\ell$. Let's assume that we are able to maintain a constant velocity. The only that would be possible the electrical field cancels out the magnetic field.

$$\vec{\mathrm{F}}_b = q\vec{\mathrm{v}} \times \vec{\mathrm{B}} = \vec{\mathrm{F}}_e = q\vec{\mathrm{E}}$$

We can use this to rewrite the potential in terms of the magnetic field and velocity.

$$\mathcal{E} = \mathrm{B}v\ell$$

Where ℓ is the length of the rod.

20.2.1 General Form of EMF

We can use the previous statement to generalize the emf produced for any shape loop.

$$\mathcal{E} = \oint (\vec{\mathrm{v}} \times \vec{\mathrm{B}}) \cdot \mathrm{d}\vec{\ell}$$

It's easy enough to see how we got this result from looking at the previous expressions.

20.3 Induced Electric Field

The electric field that is induced isn't exactly the same as an electric field caused by a point or charge distribution. Consider a solenoid that carries a current I that changes with respect to time ($\frac{\mathrm{dI}}{\mathrm{dt}}$). The magnetic flux in the solenoid is increasing, causing an induced $\mathcal{E}$. We know that inside a solenoid, the magnetic field is uniform and is equal to,

$$\mathrm{B} = \mu_0 n I$$

We can figure out the magnetic flux,

$$\Phi = \vec{\mathrm{B}} \cdot \vec{\mathrm{A}}$$

where we take $\vec{\mathrm{A}}$ to be the normal surface vector of thin slices of the solenoid. The magnetic flux would be equal to

$$\Phi = BA = \mu_0 n I A$$

because the magnetic field and the normal surface vector are parallel with each other. Faraday's Law states the induced $\mathcal{E}$,

$$\mathcal{E} = -\frac{\mathrm{d}\Phi_\mathrm{b}}{\mathrm{dt}} = -\mu_0 n A \frac{\mathrm{dI}}{\mathrm{dt}}$$

We also cannot forget that there is an induced current as well that differs from the I. The solenoid would create an induced current on any other conducting loop near it. We note that I' (induced current) is equal to $\frac{\mathcal{E}}{R}$, were R is the resistance of the loop. But what's not as clear is why there is a induced current. Remember that outside of a solenoid the magnetic field isn't present. We conclude that there must be an electric field that is induced by the emf created in the solenoid. The electric field does work on the charges, causing it to go around the loop, meaning if we work backwards, the induced electric field must equal the emf. Which makes senses, the force that causes the charges to go around is equal to the emf that created the said force. This electric field isn't conservative because if we generalize the loop integral,

$$\oint \vec{\mathrm{E}} \cdot \mathrm{d}\vec{\ell} = \mathcal{E} = -\frac{\mathrm{d}\Phi}{\mathrm{dt}}$$

We can see that it isn't equal to zero, which violates the rule for closed line integrals in conservative fields. We call this type of field a **non-electrostatic field**. This is because you will never be able to create this type of non-conservative field with a static charge. Also this means, if you have a magnetic field that changes over time (and note that a magnetic field is only caused by moving charges) you will create a electric field as well.

CHAPTER **21**

INDUCTANCE

Let's consider a solenoid that is connected to a circuit. Remember, the solenoid will create a induced current if there is change in the magnetic flux. If we passed a varying current through the solenoid, we will create an induced emf inside the very same circuit that the solenoid was in. By Lenz's Law, this induced emf will opposes the same change in current that caused it. We call this **self-inductance**. There is also another type of inductance called mutual inductance but that is not important for upcoming discussions. We note that the inductance is equal to,

$$\mathrm{L} = \frac{N\Phi}{i}$$

We note that if i, changes over time, then so does the magnetic flux. The equation can be rewritten,

$$\mathrm{L}i = N\Phi$$
$$L\frac{\mathrm{di}}{\mathrm{dt}} = N\frac{\mathrm{d}\Phi}{\mathrm{dt}}$$

By Faraday's Law,

$$\mathcal{E} = -N(\frac{\mathrm{d}\Phi}{\mathrm{dt}}) = -L\frac{\mathrm{di}}{\mathrm{dt}}$$

Which is the self-induced emf of the solenoid in a circuit of varying current.

21.1 Inductors

As noted above, the Lenz's Law would mean that the solenoid would have an emf that would oppose the very change in current that creates it. This means the the inductor acts like a stop sign, preventing the varying current from changing to quickly. We further note the behavior of inductors in a circuit. We know that from electrostatics, that a conservative electric field would have,

$$\oint \vec{\mathrm{E}} \cdot \mathrm{d}\vec{\ell} = 0$$

but the inductor will have induced electric field that isn't conservative, equal to the emf created by the inductor.

$$\oint \vec{\mathrm{E}}_n \cdot \mathrm{d}\vec{\ell} = -L\frac{\mathrm{di}}{\mathrm{dt}}$$

But the integral only differs from the regular electric field around the inductor, so we can change the integral from some point a to point b around the inductor. Since we consider the inductor to be an ideal inductor,

where inside a conductor the total electric field is going to equal zero, we know that the induced electric field must cancel out with the regular electric field on the surface (otherwise there wouldn't be any force moving the current through the solenoid). We state that $\vec{\mathrm{E}}_n = -\vec{\mathrm{E}}$, and solving the loop integral will yield,

$$\oint \vec{\mathrm{E}}_n \cdot \mathrm{d}\vec{\ell} = \int_a^b -\vec{\mathrm{E}} \cdot \mathrm{d}\vec{\ell} = -L\frac{\mathrm{di}}{\mathrm{dt}}$$

This just confirms that the voltage potential between a and b, which is also the voltage potential between the terminals of the inductor, is equal to it's inductance times the change in the current.

21.1.1 Computing Inductance

Let's say we have a toroidal solenoid. We want to compute it's inductance, sort of like how we wanted to compute the capacitance of a capacitor. The solenoid is typically the shape we use for the inductor (not toroidal), but the process would be similar. First we consider the magnetic flux of the inductor,

$$\Phi = \vec{\mathrm{B}} \cdot \vec{\mathrm{A}} = \frac{\mu_0 NIA}{2\pi r}$$

Now we just simply find the inductance of the solenoid,

$$\mathrm{L} = \frac{N\Phi}{i} = \frac{\mu_0 N^2 A}{2\pi r}$$

The inductance is solely dependent on the geometry of the object.

21.1.2 Energy Stored In Inductors

Inductors store energy like capacitors. We first consider the a increasing current going through an inductor. Let's consider the power,

$$\mathrm{P} = VI$$

We already know that power is a change in energy, a transfer between the external source that is carrying in the current into the inductor.

$$\int_0^t \mathrm{P}\mathrm{dt} = \int V_{ab} I \ \mathrm{dt}$$

We know that $V = L\frac{\mathrm{dI}}{\mathrm{dt}}$, solving the integral on the right-hand side,

$$\int_0^i \mathrm{L}I \ \mathrm{dI} = \frac{1}{2}\mathrm{L}i^2$$

were i is the final current. This energy is stored in the magnetic field of the coil. Like how the electric field stores the energy of the capacitor. We can solve the previous equation for a toroidal solenoid, and find the density per unit volume,

$$u_b = \frac{1}{2}\mu \frac{N^2 I^2}{(2\pi r)^2}$$

which can be expressed in terms of a magnetic field component,

$$u_b = \frac{B^2}{2\mu_0}$$

CHAPTER **22**

CIRCUITS

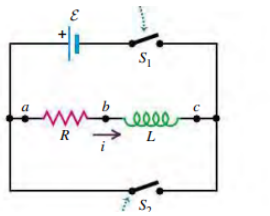
Each circuit combination has different behavior, though they are connected pretty closely. This is due to how the behavior of some components interact with other components. Mostly how these components change the behavior of currents. Some circuits would need a good review of spring motion, but not really required.

22.1 Kirchhoff's Junction and Loop Rules

22.2 RC Circuits

22.3 RL Circuits

Let's look at the circuit below to develop ideas of how the RL circuit works. We already know that the inductor will help prevent a rapid change in current. So when the switch changes from open to begin, the current can't change from it's zero value to it's final value in a instant. If this happened then the change in current through the inductor would be infinite, and the voltage through the inductor would be infinite as well.



RL Circuit

We first start with the loop rule for voltages,

$$\mathcal{E} - iR - L\frac{\mathrm{di}}{\mathrm{dt}} = 0$$

Solve for the change in current,

$$\frac{\mathrm{di}}{\mathrm{dt}} = \frac{\mathcal{E}}{L} - \frac{R}{L}i$$

Now we can start considering the position of the switches. We know at the start, $i = 0$, so there wouldn't be any voltage through the resistor. But that doesn't mean the inductor's voltage is zero as well. Remember the voltage through the inductor depends on the change in the current through the inductor. As you can see from the equation,

$$\frac{\mathrm{di}}{\mathrm{dt}}_{initial} = \frac{\mathcal{E}}{L}$$

From the second part of the equation, we know when the current increases, the resistance over inductance term will increase as well. So the rate of increasing current is actually going to decrease. Meaning that the current will reach a steady state value later in time. Which in turn means that the change in current will be zero later in time as well.

$$\frac{\mathrm{di}}{\mathrm{dt}} = \frac{\mathcal{E}}{L} - \frac{R}{L}i = 0$$
$$I = \frac{\mathcal{E}}{R}$$

We gathered the final current I. This result makes sense. It doesn't depend on the inductor at all because the change in current is zero. Meaning the voltage across the inductor would be zero as well.

22.3.1 Current Growth: Closing the Switch

If we solve the differential equation, from time $t_0 = 0$, where we closed the switch,

$$i = \frac{\mathcal{E}}{R}(1 - e^{-(R/L)t})$$

We note that the time constant is $\tau = L/R$

22.3.2 Current Decay: Opening the Switch

We note that when the switch is closed after a long time, we get our final current value. The second we open the switch, we redefine that time to be $t_0 = 0$, the moment we open the switch. The current can't change instantly, so we know that at t_0 the current would be the I_{final}.

$$i = I_{final}e^{-Rt/L}$$

22.3.3 Voltage Growth and Decay

It would make sense that the voltage growth and decay would also follow the same time scale. Since the voltage growth across the resistor is equal to,

$$V_r = iR = \frac{\mathcal{E}}{R}(1 - e^{-t/\tau})R = \mathcal{E}(1 - e^{-t/\tau})$$

and decay would be equal to

$$V_r = iR = I_{final}e^{-Rt/L}R = V_{final}e^{-t/\tau}$$

The voltage decay through the inductor is defined to be,

$$V_L = L\frac{\mathrm{di}}{\mathrm{dt}} = L\frac{\mathcal{E}}{L}e^{-t/\tau} = \mathcal{E}e^{-t/\tau}$$

As you can see the inductor's voltage is actually decreasing from the moment we closed the switch.

$$V_L = L\frac{\mathrm{di}}{\mathrm{dt}} = -V_{final}e^{-t/\tau}$$

The voltage is still decreasing from when the switch was closed for a long time to it being open. A trend that should be noticed is that the V_final will typically equal the $\mathcal{E}$. But do note this is for a single resistor and inductor configuration, you have to consider the other remaining resistors and inductors in a multi-RL component circuit.

22.4 LC-Circuits

So LC circuits behavior differs greatly from the RL and RC circuits before. We first consider what happens when the capacitor discharges. The capacitor discharges but it can't do instantly so the current starts from 0 to some final value. This means that if there is an inductor connected, the that inductor will experience a change in current, creating an induced emf. This induced emf will be in the opposite direction of the current being discharged from the capacitor. Meaning the capacitor will actually be reversed in polarity because a current is charing it in the reverse direction. And the process happens again, the capacitor and inductor will oscillate back and forth between each other. We call this undamped electrical oscillation because there is no loss in energy.

22.4.1 Equation of Oscillation

Solving the loop equation gives us a second order linear differential equation, which is in terms of charge (because $i = \frac{\mathrm{dq}}{\mathrm{dt}}$). We note this equation,

$$q = Q\cos(\omega t + \phi)$$

where ω is the angular frequency of the oscillation, and ϕ is the phase angle that depends on the initial conditions. Note that the angular frequency is terms of rads/second. It's not regular frequency which is cycles per second.

$$\omega = \frac{1}{\sqrt{LC}} \rightarrow f = \frac{1}{2\pi\sqrt{LC}}$$

We can take it's derivative to find the current oscillation,

$$i = -\omega Q \sin(\omega t + \phi)$$

The phase angle value can be determine by the initial condition. If at $t = 0$, the capacitor has it's maximum charge (it's fully charged), then we know that the current has to be zero. So the phase angle at this point would be zero. If the charge is zero, the capacitor is uncharged, a non-zero current value, then the phase angle is $\pm\frac{\pi}{2}$.

22.4.2 Energy of Oscillation

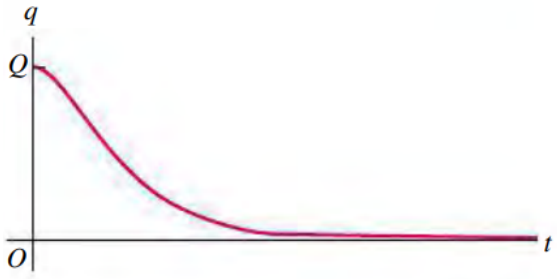
Since there is no lose in energy, the oscillation will happen forever. What happens is the energy gets transfered between the capacitor and inductor. We know that the maximum energy is going also be equal to energy in the capacitor when it's fully charged. It then will be equal to the sum of the energy of the inductor and the capacitor with dependence on the charge at specific times.

$$\frac{1}{2}Li^2 + \frac{q^2}{2C} = \frac{Q^2}{2C}$$

22.5 LRC Circuit

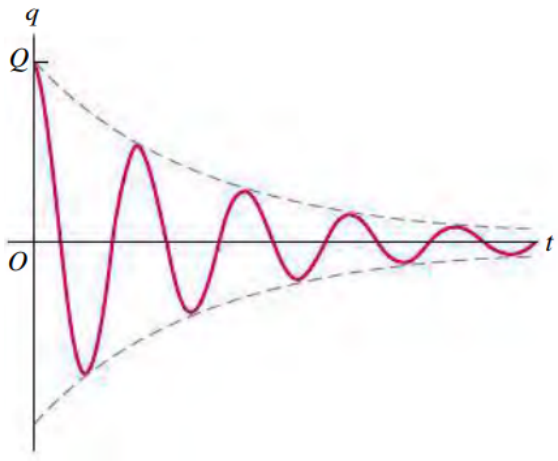
LRC circuits take into account of every circuit component we have. Like LC circuits, we have an electric oscillation but unlike that oscillation we have a damping factor. The resistor in the circuit will cause the energy loss. There are two different types of damping when we talk about the LRC circuit.

First is the **over-damping**, when this occurs (large values of R), the oscillation doesn't occur at all. It's just an exponential decay.



Over-damped

The second is the **under-damping**, where you start seeing the regular exponential decay oscillation.

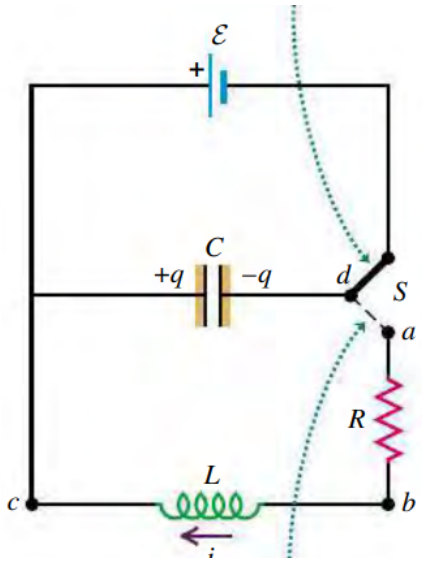


Under-damping

There is a point were you can find the critical damp value. At this point, you start to see no oscillation at all and it marks the border between the oscillation and no oscillation.

22.5.1 Equation of Current

By using the loop rule we can develop the equations of current and charge for the LRC circuits.



LRC

First we consider the switch to first be closed in position d. This allows the capacitor to gain the max charge of the emf. We note that the charge value is equal to $Q = C\mathcal{E}$, when after this happens we open the switch to close it at position a. This disconnects the emf source, and now we have a circuit that is just the inductor,

capacitor, and resistor. By the loop rule,

$$-iR - L\frac{\mathrm{di}}{\mathrm{dt}} - \frac{q}{C} = 0$$

whose, solution in terms of the charge is,

$$q = Ae^{-t/2\tau}\cos(\sqrt{\frac{1}{LC} + \frac{1}{4\tau^2}}t + \phi)$$

A and the phase angle are determine by initial conditions. For example, if the capacitor is at max charge at time $t = 0$, then $A = Q_{max}$ and the phase angle would be zero. The angular frequency of the under-damped oscillation,

$$\omega = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

When the square root terms cancel each other, or when $R = \sqrt{4L/C}$, the critical damping occurs, and you will see no more oscillation.

22.6 AC Circuits

AC circuits are circuits that have alternating currents. The emf source for example a rotating coil of wire in a magnetic field. But either way, this will produce a current that is cycles. We call this source a **alternator**. For example, we note this,

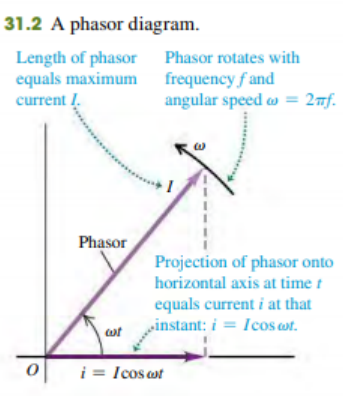
$$v = V\cos(\omega t)$$

As a voltage at some time, where V is the maximum voltage. The current similarly is described as

$$i = I\cos(\omega t)$$

22.6.1 Phasor Diagrams

We use rotating vector diagrams to represent the varying voltages and current. The diagram shows the instantaneous value (the small v and i which varies with respect to time) as a projection onto a horizontal axis of a vector with a length equal to the maximum amplitude of the quantity. The vector would rotate counterclockwise with a speed equal to the ω (angular frequency).



Phasor Diagrams

22.6.2 Measuring AC

We can use diodes to measure the current in an ac circuit. Diode is a device that conducts better in one direction than in the other, i.e one side has zero resistance while the other side has infinite resistance. We find the rectified average current I_{rav} to be defined as

$$I_{rav} = \frac{2}{\pi} I$$

where we consider the constant factor as an average value of the cosine or sine.

22.6.3 Root-Mean-Square Values

We can use the root-mean-square values of current and voltage. We start by squaring the current formula,

$$i^2 = I^2 \cos^2(\omega t)$$

$$i^2 = I^2 \frac{1}{2}(1 + \cos(2\omega t)) = \frac{1}{2} I^2 + \frac{1}{2} I^2 \cos(2\omega t)$$

The average value of cosine with a period of 2ω is zero because half the time it's positive and the other half it's negative. So we get the root-mean-square value as

$$I_{rms} = I \frac{\sqrt{2}}{2}$$

This hold rule for voltage as well,

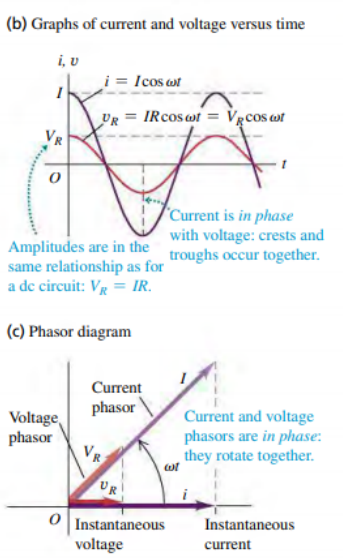
$$V_{rms} = V \frac{\sqrt{2}}{2}$$

22.6.4 Resistors in AC Circuits

Since we have a varying current, we can describe the voltage across the resistor as,

$$v_r = iR = (IR)\cos(\omega t)$$

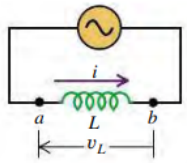
If you note the phasor diagram, the voltage-resistor phasor is just proportional to the instantaneous current phasor. That means they share the same frequency and are in phase of each other. The current and voltage of the resistor are parallel at very instant.



Resistor Phasor

22.6.5 Inductors in AC Circuits

We setup a simple circuit with one inductor and an ac source.



Inductor Circuit

There is no total resistance but there is a voltage across the inductor. The induced emf is usually given by Faraday's Law. But note that this isn't the case here. Because if the current is in the positive direction (counterclockwise) and increasing (from a to b), then the change in current is also positive meaning the induced emf would be to the left opposing the increasing clockwise current. That means point a will have a higher potential then point b.

$$v_L = L\frac{\mathrm{di}}{\mathrm{dt}} = L\frac{\mathrm{d}}{\mathrm{dt}}(I\cos(\omega t)) = -I\omega L\sin(\omega t)$$

We note that the current and voltage phasors are out of phase by a quarter cycle (sine waves are out of phase to cosine waves by $\pi/2$). We usually describe the voltage being out of phase to the current, so we note that we can write the voltage at one point with respect to another is

$$v = V\cos(\omega t + \phi)$$

when the current is some source defined by

$$i = I\cos(\omega t)$$

The phase angle describes how much the voltage is out of phase to the current. For a pure resistor that phase angle is 0, while pure inductors will have a phase angle of 90 degrees. We note the inductive reactance of the inductor as

$$X_L = \omega L$$

Which the units are ohms, this inductive reactance is just a description of the self-induced emf that opposes any change in the current through the inductor. This value is directly proportional to the frequency of the circuit. Larger values of X_L will produce smaller values of I, consider that $V_L = IX_L$. This means a high frequency voltage applied across will produce a small current, while the lower voltages would produce will give rise to a larger current.

22.6.6 Capacitor in AC Circuit

The derivation is similar to the previous ones, so I'm just going to give the resulting formula.

$$v_c = \frac{I}{\omega C}\sin(\omega t)$$

The peaks of the capacitor voltage occurs at max and min of the $q = \frac{I}{\omega}\sin(\omega t)$. It lags behind the current phase by an angle of $\phi = -\frac{\pi}{2}$. We can rewrite the equation

$$v_c = \frac{I}{\omega C}\cos(\omega t - \frac{\pi}{2})$$

Like before, we defined the capacitive reactance of the capacitor as

$$X_c = \frac{1}{\omega C}$$

The relationship of the capacitive reactance and current is the reverse of the inductor. Higher capacitance will mean a lower reactance, allowing higher frequency currents to pass through, while the reverse is true as well. Lower frequency currents will pass through with lower capacitance.

22.6.7 Summary of AC Circuit Components

TABLE 31.1 Circuit Elements with Alternating Current

Circuit Element	Amplitude Relationship	Circuit Quantity	Phase of v
Resistor	$V_R = IR$	R	In phase with i
Inductor	$V_L = IX_L$	$X_L = \omega L$	Leads i by 90°
Capacitor	$V_C = IX_C$	$X_C = 1/\omega C$	Lags i by 90°

Table of Circuit Elements

CHAPTER **23**

MECHANICAL WAVES

Motion that exhibits back and forth motion is called periodic motion or oscillations. There are different types of waves. But before we go over that we will do a simple review of oscillations. The simplest spring will exhibit this type of motion. We will consider a spring on a frictionless horizontal surface. It's will be attached to a simple mass, such that we don't have to consider the gravitational force.

We define our origin to be at the point were the position of the spring is at equilibrium. It's neither stretched out or compressed. Now if $x > 0$, where x is the displacement from the equilibrium point, then we say that the spring is stretched. That means we would have the spring come back to the equilibrium point with some force, given by Hook's Law.

$$\vec{\mathbf{F}} = -k\vec{\mathbf{x}}$$

Once it reaches it's equilibrium point, the motion will drag it behind the equilibrium point, and since the displacement is non-zero but smaller then the initial displacement, it will have some acceleration. This is acceleration will point back to the equilibrium point, and then the displacement would be forward from the origin point. This is an example of oscillations, and of course we can simplify it down into an equation that gives us this motion.

Since we are dealing with waves, we can showcase this motion with the following formula,

$$x(t) = A\cos(\omega t + \phi)$$

The later sections will go more in-depth with the types of waves we will be dealing with.

23.1 Waves

First we should understand what a wave is. A wave in simple terms is a way to describe changes in a medium (this implies changes from it's equilibrium state) from one region of the system to another. Meaning that is something that disturbs the system as a whole. An important thing is that waves carry energy from one point to another, not matter.

23.2 Types of Mechanical Waves

There are different types of waves, the first type is the simplest to understand. The mechanical waves are waves that travel through some medium. For example, sound waves travel through the air. In this case, the air around us is the medium. There are two different types of mechanical waves, and other waves in this category can be formed from a combination of these types.

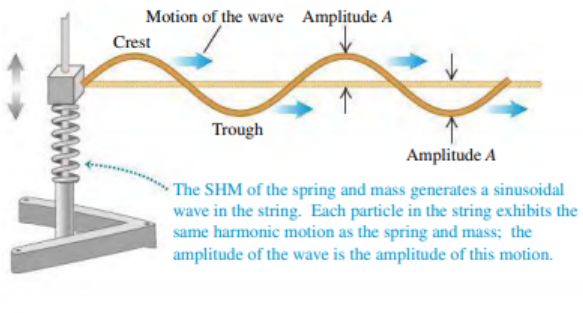
A transverse wave are waves that are produced by displacements in the medium that are perpendicular to the direction of travel of the wave. The up/down motion on a string is an example of a transverse wave. Since the wave travels forward, but the displacement from the equilibrium point is the up/down motion, which is perpendicular to how the wave is traveling through the spring, is why it's considered to be transverse.

A longitudinal wave are waves that are produced by displacements in the medium that are parallel to the direction of travel of the wave. Like try pushing a flat plate to compress some water in a tank, the direction of the displacement is the same as the direction of wave that travels through the water.

An important thing to note is that medium doesn't actually move. The wave moves energy, not matter, so the particles in the wave. The **wave speed** is the name given to the speed of which the wave propagates through the medium.

23.3 Periodic Transverse Waves

The string example will be used to showcase periodic transverse waves.



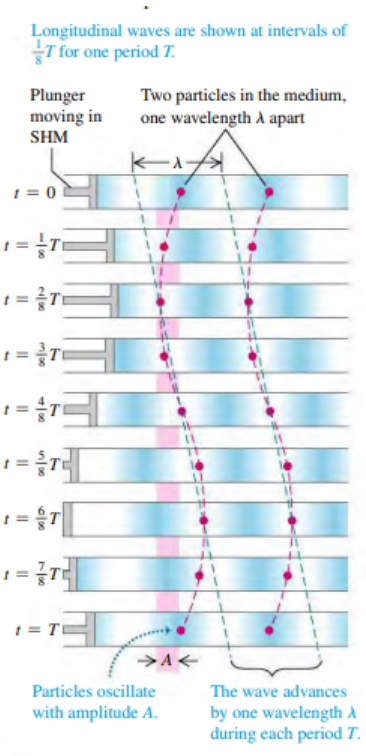
Simple Spring

Let's say it has an amplitude A, frequency f, the angular frequency $\omega = 2\pi f$, and the period of $T = 1/f = 2\pi/\omega$. We can use these pieces of information to form a sinusoidal function that can represent the periodic wave. A single wave pulse would just have one peak, but a wave is made up peaks and valleys, so a wave is sum or a continuous series of pulses. When this wave moves through the medium, or in this case the string, every particle in that medium undergoes a simple harmonic motion with the same frequency we applied to the spring at one end.

NOTE!, that the motion of the string is actually just going up and down as the wave passes through. The medium is disturbed by the wave, causing it to undergo a simple harmonic motion, while the wave travels alongside the length of the string.

23.4 Periodic Longitudinal Waves

The principles above for transverse waves apply to longitudinal waves, but notice the difference in the waves.



Simple Spring

Transverse waves have disturbances perpendicular to the motion of the wave, but here the wave is actually traveling alongside each wave. The differing time marks makes it hard to see, but the vertical direction is trying to show how the the wavelength changes with respect to time. The are areas of compression and rarefaction (reduced density). These regions gives us the marks that we can use to determine the wavelength. As time advances, the compression and rarefaction regions move as well.

23.5 Wave Speed

Since the wave is periodic, the shape of the pattern is repeating, so we can measure one repetition as the wavelength, λ. Then the wave speed would be give by the phrase, how many cycles per second (f, Hz) times the length of one cycle (wavelength),

$$v = \lambda f$$

There is an inverse relationship between the wavelength and frequency. As the frequency increases you have more cycles per second of course. That means each "cycle" is much smaller then the previous frequency. That means the length of the repetition is actually smaller. So the wavelength decreases with respect to frequency for a constant wave speed. The wave speed remains constant for all frequencies of said waves, and the wave speed is said to be determined by the properties of the material of the medium.

23.6 Mathematical Description of a Wave

As in the introduction, the wave formula I introduced was a simple 1D formula that just looked at a single particle with respect to time. But let's try finding a more complex and general description of a wave. If we consider a string whose length will be the x-axis, we know that the waves on this string will be transverse. The particles at position x will be displaced by some y with respect to the wave. So we know that y will have a dependence on where we are looking at (x) and what moment at time we are looking at (t). So $y = y(x, t)$ will be a multi variable function. For longitudinal waves, the principle holds true but it's important to note, that the displacement is parallel to the x-axis.

First let's considered what's happening to particles at different points at differing points in time. We know that every point in the medium will oscillate in some form of a SHM with the same amplitude and frequency. But the oscillations are not in step with each other. For example, let's note a point on the string as B. B's at max y at $t = 0$, then returns to $y = 0$ after some $t = \frac{2}{8}T$. This happens to all the other points as well, but the exact timing lags behind. This means, let's say point C, which is further down from B, hit's max at $t = \frac{4}{8}T$ and returns at $t = \frac{6}{8}T$. Each particles' motion differs by some fraction of a single cycle. We note these differences as **phase differences**.

Let's work backwards for now, consider the case $y(0, t)$, where we are looking at the particle at very start of the string. We know that $\omega = 2\pi f$, so our equation becomes

$$y(x = 0, t) = A\cos(\omega t) = A\cos(2\pi f t)$$

Remember we are developing a wave equation that tracks the wave disturbances, not the particle's motions. We can use the equation to showcase the motion of the particles but we have to think about the properties of a periodic wave to actually develop it. Let's work backward. Let's say at time $t = 0$ and $x = 0$, the particle is at a max displacement of $y = A$, the wave will travel from $x = 0$ to some point x. The time for it to travel x-distance would be given by $\frac{x}{v}$, where v is the wave speed. We know since the motion lags, at time t, the motion at position x would be the same as the motion at position $x = 0$ at a earlier point in time $t = t - \frac{x}{v}$. So we can replace the original t in our equation with the expression we got for the earlier time.

$$y(x, t) = A\cos[\omega(\frac{x}{v} - t)]$$

We can write the equation down into more usable forms. First we consider,

$$y(x, t) = A\cos[2\pi\omega(\frac{x}{\lambda} - \frac{t}{T})]$$

Then we let $k = \frac{2\pi}{\lambda}$, called the wave number,

$$y(x, t) = A\cos[kx - \omega t]$$

Now let's consider a wave function that is moving backwards or in the negative x direction. We know from the previous statement, that the motion at position x at time t would equal the motion at position $x = 0$ at an earlier time. This means that the looking forward in time, the motion would remain the same as it did in it's past. In reverse, looking back in time, the motion would remain the as in the future. We can summarize that statement by saying, at a time from $t + \frac{x}{v}$, at position x, it would remain the same position $x = 0$ at time $t = 0$.

$$y(x, t) = A\cos[kx + \omega t]$$

We note that the quantity that is inside the cosine, $(kx + \omega t)$ is called the phase, and it's always measured in radians. The phase can be use to determine what is going on in the sinusoidal cycle. Meaning that you can use to determine for any values of x and t what is going to occur. For example, if it was a max amp at y, the phase could be any value of $n\pi$, where n is either 0 or any even integer.

23.6.1 Wave Functions

To summarize this section, the wave traveling in a positive direction would be,

$$y(x,t) = A\cos[kx - \omega t]$$

the wave traveling in a negative direction would be,

$$y(x,t) = A\cos[kx + \omega t]$$

23.6.2 Phase Velocity

The wave speed is the speed which the wave travels, or in our case if we where following the wave, how fast we would have to move alongside a point in the given phase. Consider the phase,

$$kx - \omega t = 0$$

This statement for a forward moving wave is true because the wave speed is constant. Meaning if we took the derivative of displacement (velocity),

$$\frac{\mathrm{dx}}{\mathrm{dt}} = \frac{\omega}{k} = v$$

We can get an expression of velocity for the wave.

23.6.3 Graph of the Wave Function

We can develop the graph of the wave function by looking at 2D snapshots of it. Consider the snapshot when $t = 0$. This means we are going to look at a graph of displacement vs position of the particles on the length of the string. Or in simpler terms, the shape of the string at $t = 0$. Of course, this means that you will be getting the wavelength from the graph instead of the period. In reverse, when you set the position equal to zero, you are looking at the displacement of a particle at position $x = 0$ over some function of time. This means you will be able to get the period of wave from this snapshot.

23.7 Particle Velocity and Acceleration

We can find the particle's transverse velocity from the wave equation. Remember that a transverse wave will be do to a displacement from the equilibrium perpendicular to the direction of travel of the said wave. So the individual particles at any moment t, will be oscillating up and down. The velocity we are measuring is that said motion.

We start with the wave equation, and we take the partial derivative of the said equation with respect to time. Note, I'm using the simplification of the partial derivative operator.

$$\partial_t y(x,t) = \partial_t(A\cos(kx-\omega t)) = -\omega A\cos(kx-\omega t) = \omega A\sin(kx-\omega t)$$

$$v_y = \omega A\sin(kx-\omega t)$$

Note that v_y is use to note the difference from the wave speed v. Now for the transverse acceleration,

$$a_y = \frac{\partial y}{\partial \mathrm{t}}2 = -\omega^2 A\sin(kx-\omega t)$$

23.7.1 Slope and Curvature

The partial with respect time can give us information of the velocity and acceleration of the wave. But the partial with respect to x can us information of the geometry of the wave. We can use it find the slope of the string and curvature of the string,

$$\frac{\partial y}{\partial \mathrm{x}} = -kA\sin(kx-\omega t)$$
$$\frac{\partial y^2}{\partial \mathrm{x}^2} = -k^2 A\cos(kx-\omega t)$$

23.8 Wave Equation

One of the most important equations is the ratio between the second partial derivatives of the wave function.

$$\frac{\partial y^2}{\partial \mathrm{x}^2}/\frac{\partial y^2}{\partial \mathrm{t}^2} = \omega^2/k^2 = v^2$$

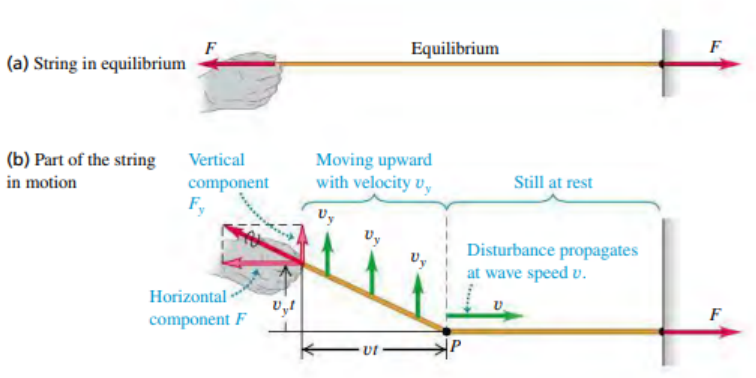
$$\frac{\partial y^2}{\partial \mathrm{x}^2} = \frac{1}{v^2}\frac{\partial y}{\partial \mathrm{t}}2$$

This is the 1D wave equation, that states that any disturbance can propagate as a wave along the x-axis with the wave speed v. The wave equation doesn't necessarily state that sinusoidal waves need be the only waves that fit the equation.

23.9 Speed of a Transverse Wave

We are going to try to determine the speed of a transverse wave on a string. The speed of a transverse wave on a string are due to the tension in the string and the mass per unit length (or linear mass density). The increased tension also increases the restoring forces that make sure the string returns back to it's original state. This would also increase the wave speed as well. Well increasing the mass per unit length would slow down the wave speed as it would feel heavier.

Let's consider a string that is perfectly flexible, in it's equilibrium position. The tension on the string is F. The linear mass density on the string is μ. At time $t = 0$, there is a constant upward force on end of the string. The other end is held down by the tension.



The moving parts of the string, move up with a constant transverse velocity. We note the impulse-momentum theorem, the total change in transverse component of momentum of the moving string is equal to the impulse. Meaning from $t = 0$ to time t, the force applied F_y, is equal to the transverse momentum of the string. This should make sense as we are considering waves that move with a constant wave speed.

$$F_y t = m v_y$$

We note that the point of where the string is moving and not moving moves also with a constant velocity. We note this point as P. As P moves, the amount of mass increases, meaning that there has to be an increase in momentum. The force doesn't necessarily change, but the mass does, so the momentum increases. We note that P moves at the wave speed. We note that the total force at the start of the string, where our wave originates, has the horizontal components of F and F_y. this is because we have no motion directed alongside the string (remember this is a transverse wave, if we did it would be a combination between a transverse and longitudinal wave).

Note the geometry of the diagram, we can use similar triangles to derivative an expression for wave speed.

$$\frac{F_y}{F} = \frac{v_y t}{vt}$$
$$F_y = F\frac{v_y}{v}$$

We know the impulse is equal the transverse force applied over time t,

$$F_y t = F\frac{v_y}{v} t$$

We set this expression equal to the momentum,

$$m_i v_y = F\frac{v_y}{v}t$$

Now m_i is the total mass of the segment of length, and since we know the length is going to equal the distance that the point P travels. Since it originates at the origin and it travels with a wave speed v. So the total distance is equal to vt, and the total mass would be the linear mass density times the distance, $\mu * vt$.

$$\begin{aligned}\mu v t v_y &= F\frac{v_y}{v}t \\ v &= \sqrt{F/\mu}\end{aligned}$$

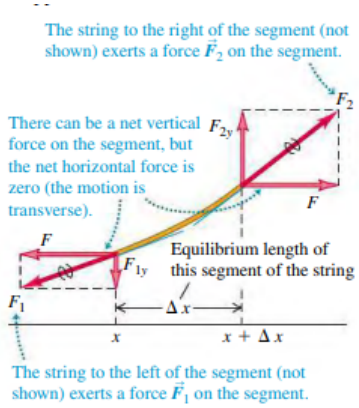
23.9.1 The Speed of Mechanical Waves

Later on, it can be seen that all mechanical waves have some expression of this form,

$$v = \sqrt{\frac{\text{Restoring force returning the system to equilibrium}}{\text{Inertia resisting the return to equilibrium}}}$$

In case of the string, the inertia comes from the mass density or mass of the string. The tension is part of the restoring force of the string.

23.9.2 Second Method: Newton's Second Law



We can use Newton's second law to find the exact same result. First we we consider small mass segments of the string in it's equilibrium position, $m_i = \mu\Delta x$. The small force segments can be broken down into their respective force components. The x-component would have equal magnitude to F, but add up to zero because we are dealing with transverse motion.We note that the ratio of the forces in the y direction vs the horizontal forces is equal to the slope at those points.

$$\frac{F_{1y}}{F} = -\frac{\partial y}{\partial \mathrm{x}}_x$$
$$\frac{F_{2y}}{F} = \frac{\partial y}{\partial \mathrm{x}}_{x+\Delta x}$$

Solve for the individual F_{iy}, then add them up,

$$F_y = F_{1y} + F_{2y} = F\left[\frac{\partial y}{\partial \mathrm{x}}_{x+\Delta x} - \frac{\partial y}{\partial \mathrm{x}}_x\right]$$

We then use Newton's Second Law,

$$F\left[\frac{\partial y}{\partial \mathrm{x}}_{x+\Delta x} - \frac{\partial y}{\partial \mathrm{x}}_x\right] = \mu\Delta x\frac{\partial y^2}{\partial \mathrm{t}^2}$$
$$\frac{\mu}{F}\frac{\partial y^2}{\partial \mathrm{t}^2} = \frac{\left[\frac{\partial y}{\partial \mathrm{x}}_{x+\Delta x} - \frac{\partial y}{\partial \mathrm{x}}_x\right]}{\Delta x}$$
$$\frac{\partial y^2}{\partial \mathrm{x}^2} = \frac{\mu}{F}\frac{\partial y^2}{\partial \mathrm{t}^2}$$

As you can see we get the wave equation again, meaning that $v = \sqrt{F/\mu}$. The same result we got last time.

23.10 Energy in Wave Motion

Every wave motion has energy alongside it. To create a wave, we needed to apply a force. That said point were the applied force is moves, so we do work on the system. As the wave moves, the medium exerts a force on one region of space to another. This how a wave moves energy from one region of space to another. For example, one a string, lets denote a point on it called A. Let's say that this wave moves left to right from point A. To the left of A, we have a force alongside the string, the restorative force, that points from right to left. We note that F_y/F_x is equal to the negative slope of string at that point. As noted before, the F_x will always equal horizontal F, and add up to zero. So the slope is actually F_y/F, which is equal to the partial of wave function $y(x,t)$ with respect to x.

$$F_y(x,t) = -F\frac{\partial y}{\partial \mathrm{x}}$$

When point a moves in the y-direction, the force F_y does work on this point (trying to bring it back to equilibrium), and transfers energy to the right of A. We defined power to be $\vec{\mathrm{F}}\vec{\mathrm{v}}$, so we can try doing this with the wave function. The power is from the work of moving the point back to equilibrium times it's transverse velocity.

$$P(x,t) = F_y(x,t)v_y(x,t) = -F\frac{\partial y}{\partial \mathrm{x}}\frac{\partial y}{\partial \mathrm{t}}$$

$$P(x,t) = Fk\omega A^2 \sin^2\left[kx - \omega t\right]$$

We note that we already found that $v^2 = F/\mu$, so we can finally express the power function like this,

$$P(x,t) = \sqrt{\mu F}\omega^2 A^2 \sin^2\left[kx - \omega t\right]$$

Of course, we can find the max power since sine has a max value of 1, and over time, any whole number of cycles yields an average value for sine of 1/2.

$$P_max = \sqrt{\mu F}\omega^2 A^2$$

$$P_a v = \frac{1}{2}\sqrt{\mu F}\omega^2 A^2$$

23.10.1 Wave Intensity

Waves on a string are one-dimensional, while the other types of waves can carry energy in multiple directions. We defined **intensity**, I, to be the time average rate at which energy is transported by the wave, per unit area, across a surface perpendicular to the direction of propagation. The units are usually measured in watts per square meter. For example, if a wave propagates through 3D space, equally in all directions, then we know the surface is going to be a sphere. The I is going to depend on the distance from the source, and $I \propto \frac{1}{r^2}$. The intensity is defined to be in this case,

$$I_1 = \frac{P}{4\pi r_1^2}$$

Now if you had another sphere, with a different radius, r_2, with no energy absorbed between the two spheres, then the power must be the same for both of them. So the ratio between intensity would just be the ratio between the radi.

$$\frac{I_1}{I_2} = \frac{r_2^2}{r_1^2}$$

23.11 Wave Interferences, Superposition, and Boundary Conditions

Interference is when two or more waves overlap with each other. It occurs in the same region of space in the medium. This is where the boundary of the medium comes in to play. If there is a fixed end in the string, what happens when you send a wave across that said string? Well you get a wave reflected back to you, kind of like Newton's Third Law. This is actually because the string's wave exerts a upward force on that said fixed boundary, which the boundary sends a downward on force on the string. This means the wave is actually now flipped, the pulse could have been upwards but now the reflected force on the string now is making the pulse go downwards. The pulse that started the wave is called the **incident** pulse, and the reflected pulse moves in the opposite direction of that said pulse. When the boundary is free, the opposite happens. Since there is nothing to absorb the impact of the restoring force, there is nothing to invert the wave, so the wave moves back in the opposite direction but upwards as well.

23.11.1 Superposition

You can take two wave functions and add them together, basically because the additive property of the wave function comes from the fact that you can have sums of linear wave functions.

$$y(x,t) = y_1(x,t) + y_2(x,t)$$

If both y_1 and y_2 individually satisfies the wave equation, then their sum will satisfy the wave equation. But this only occurs in systems that have linear wave functions, and their wave equation is not linear. By this metric, you can't use the principle of superposition.

23.12 Standing Waves on a String

Consider a boundary condition such that both tends of the string are fixed. What happens is that when a wave passes through, the wave is reflected on both sides. This causes the wave to look like it's oscillating in place. We call this a **standing wave**. Standing waves have things called **nodes** and **anti-nodes**. Nodes are places were the standing wave remains at zero, while anti-nodes oscillate between min and max amps.

The principle of standing waves can be described by **interference**. Since the waves are reflected at both ends, you got a wave traveling to the right and to the left. Places were they meet add up, and if those places you get a zero value from the sum, those places are called **nodes**. This is called **destructive interference**. The places that the waves add up to their respective maximum are called **anti-nodes**. That is called **constructive interference**.

23.12.1 Standing Wave Equation

We can get the standing wave equation by adding two wave functions with equal amps. We consider y_1 to be the incident wave, and y_2 to be the reflected wave. The y_1 is traveling to the left, while y_2 would be traveling to the right. At $x = 0$, y_1 traveled from the right to the left, and is reflected. That means y_2 will be starting from $x = 0$, and will travel to the right from the left.

$$y_1(x,t) = -A\cos(kx + \omega t)$$
$$y_2(x,t) = A\cos(kx + \omega t)$$

The sign represents what is happening at $x = 0$. Remember the y_1 is being reflected, so that means it's amp is at it's min. At this point y_1 and y_2 are at a phase difference of $\pi rads$. We add them up,

$$y(x,t) = 2A\sin(kx)\sin(\omega t)$$

You can find positions were the nodes occur by finding where the x-sine factor is zero.

$$x = 0, \frac{\pi}{k}, \frac{2\pi}{k}, \frac{3\pi}{k}$$

Or know that $k = 2\pi/\lambda$

$$x = 0, \frac{\lambda}{2}, \frac{2\lambda}{2}, \frac{3\lambda}{2}$$

23.12.2 Normal Modes of Strings

A standing wave that has some length L, has to be such that the length is some integer multiple of $\frac{\lambda}{2}$.

$$L = n\frac{\lambda}{2}$$

You can solve for frequency as well, and this is called the fundamental frequency,

$$f_n = n\frac{v}{2L}$$

These are also called harmonics. The **normal mode** of a oscillating system is a motion that is where all particles of the said system move sinusoidally with the same frequency. You can find the fundamental frequency of the first using the restoring force / inertia from,

$$f_1 = \frac{1}{2L}\sqrt{\frac{F}{\mu}}$$