Calculus
With
Algebraic Notes

Math Notes

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Contents

1 Vectors

Multivariable functions are functions that depend on many different independent variables. Multivariable calculus mainly deals with moving the concepts learned in single variable to multivariable. We are going to be dealing with multiple functions that have several independent variables. The first step is to figure out how we are going to deal with the new N-D space. The best way to deal with that is using vectors.

1.1 Simple Vector Notations

We can use simple vectors to notate 3D points and graphs. This would be a point:

$$P_1 = (x, y, z)$$

Now a vector repersenting a position of this point would be called a position vector:

$$\vec{p} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Vectors have a defined magnitude and direction. The magnitude is computed and notated like this:

$$||\vec{p}|| = \sqrt{\sum_{i=0}^{n} p_i^2}$$

Where n is the number of components the vector has. Now is we take the magnitude and divide the vector components by it's mangitude, we turned the vector into an unit vector.

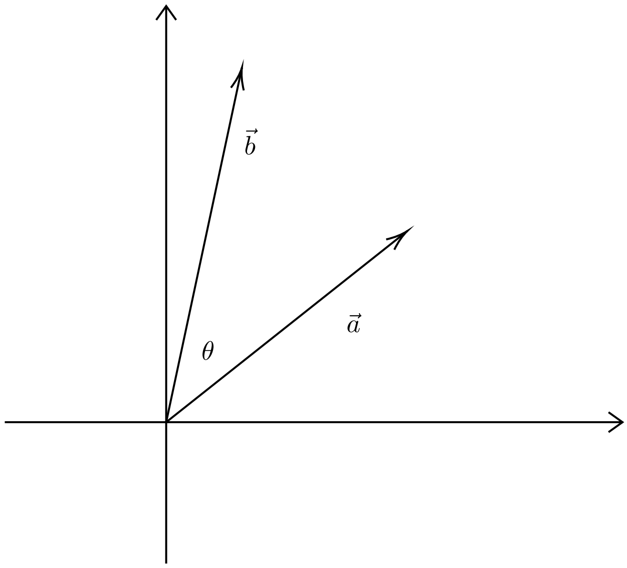
$$\hat{p} = \frac{\vec{p}}{||\vec{p}||}$$

This retains it's direction but has a mangitude of one. Now this is useful for making some computation easier. Vectors can be written in standard form, with their respect components, $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$.

$$\vec{p} = a\hat{i} + b\hat{j} + c\hat{k}$$

1.2 Dot Product

The dot product is called the scaler multiplication of two vectors in $\mathbb{R}^n$. Meaning that the dot product can be used in any space if the two vectors share the same amount of components. The dot product takes two vectors and returns a scalar product. That product can be use to determine if the two vectors are parallel or perpendicular.



Now the dot product can be computed in multiple ways:

$$\vec{P} \cdot \vec{B} = \sum_{i=0}^{n} (\vec{P}_i)(\vec{B}_i)$$
$$\vec{P} \cdot \vec{B} = (||\vec{P}|| \cdot ||\vec{B}||) cos(\theta)$$

These two formulas can be used together to determine if the dot product is parallel or perpendicular. Consider if the dot product is zero. Solving for θ would result in a value of $\frac{\pi}{2}$. This would mean that the two vectors are perpendicular to each other. The parallel case is harder. A vector facing in the opposite direction can still be parallel. So that means that angle between the two vectors can be π or 0. Reversing the equation will result in:

$$\theta = cos^{-1}(\frac{\vec{P} \cdot \vec{B}}{||\vec{P}|| \cdot ||\vec{B}||})$$

Now, another way to test if the vectors are parallel is using the unit vector, $\hat{v}$. Now a unit vector is a vector that retains it's direction, but its magnitude would be 1.

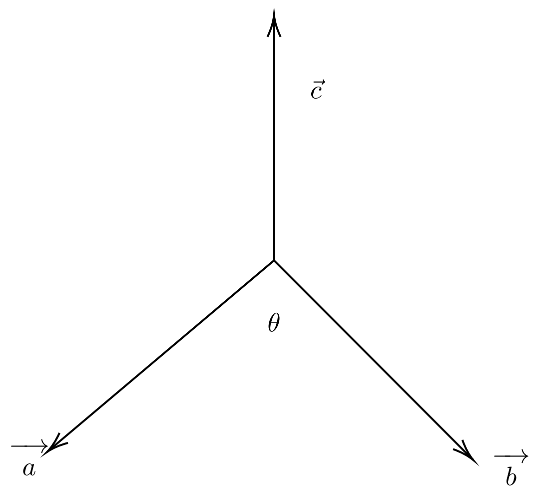
$$\hat{v} = \frac{\vec{v}}{||\vec{v}||}$$

Now, a vector is parallel only if this the unit vector $\hat{a}$ is a multiple of $\hat{v}$.

$$\hat{a} = \pm c \cdot \hat{v}$$

1.2.1 Cross Product

The cross product produces a $\mathbb{R}^3$ vector from two $\mathbb{R}^3$ vectors. The product of this a vector in the same space that of the original space. This also can be use to determine the perpendicular vector between the two vector factors.



Now to compute the cross product, there are two major methods. The first one is called method of co-factors, but considering that cross products only require $\mathbb{R}^3$, the second method would be better.

$$\vec{a} \times \vec{b} = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{bmatrix} \begin{matrix} \hat{i} & \hat{j} \\ a_x & a_y \\ b_x & b_y \end{matrix}$$

$$\vec{a} \times \vec{b} = \begin{bmatrix} (a_y b_z - a_z b_y)\hat{i} \\ (a_z b_x - a_x b_z)\hat{j} \\ (a_y b_x - a_x b_y)\hat{k} \end{bmatrix}$$

The idea is to draw diagonal lines between the components. This will result in the equation we see above. As mentioned above, using the cross product can produce a perpendicular vector. But the cross product can be use to determine if the vectors are parallel. If the cross product returns a zero vector, $\vec{0}$, then the two vectors used will be parallel to each others.

2 Equation of Lines

Equations of lines are a simple introduction to how we are going to deal with functions that exist in three dimensional space. Typically, given a set of points, you can take those points and develop a position vector for each of those points. Using those position vectors, you can determine a direction vector.

2.1 Forms of Equation

The first equation is called the vector function equation.

$$\vec{r}(t) = \vec{p_1} + (t)\vec{q}$$

The second equation is called the parametric form:

$$\begin{aligned} x &= v_x t \\ y &= v_y t \\ z &= v_z t \end{aligned}$$

We get this form by solving the first vector equation for their individual components. By setting these equations together we get what is called the systematic equations:

$$x^{-1}(v_x) = y^{-1}(v_y) = z^{-1}(v_z)$$

2.2 Example Problem: Create an equation of the line

Given, $p_1(2, -4, 1)$ and $p_2(0, 4, -10)$, find the equation of line that passes through those points.

$$\vec{p_1} = \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix}$$

$$\vec{p_2} = \begin{pmatrix} 0 \\ 4 \\ -10 \end{pmatrix}$$

Using those position vectors, we can form a direction vector.

$$\vec{q} = \vec{p_2} - \vec{p_1}$$

$$\vec{q} = \begin{pmatrix} -2 \\ 8 \\ -11 \end{pmatrix}$$

Now that we have a vector, we take the original position vector. Now scale it by a parameter t.

$$\vec{r} = \vec{p_1} + t\vec{q}$$
$$\vec{r} = \begin{pmatrix} 2 - 2t \\ -4 + 8t \\ 1 - 11t \end{pmatrix}$$

2.3 Example problem: Testing if a line is perpendicular or parallel

Given, $p_1(2,0,9)$ and $p_2(-4,1,-5)$, is the line through those points parallel, perpendicular, or neither to $\vec{r_i}(t) = \begin{pmatrix} 5 \\ 1 - 9t \\ -8 - 4t \end{pmatrix}$.
First we need an equation of the line through the first two points.

$$\vec{d} = \vec{p_2} - \vec{p_1} = \begin{pmatrix} -6 \\ 1 \\ -14 \end{pmatrix}$$

The equation of the line is

$$\vec{r} = \vec{p_1} + \vec{d}t$$
$$\vec{r} = \begin{pmatrix} 2 - 6t \\ t \\ 9 - 14t \end{pmatrix}$$

Now we can compare it to the original equation of the line. The position vector parallel to the original equation is just the coefficents of the t. But since those coefficents are not multiples of the coefficents of the new equation, the two lines are not parallel. To compare if they are perpendicular or not, we have to compute the dot product. The dot product will result in a non zero value, meaning that the two lines are not perpendicular to each other.

2.4 Example problem: Testing if a line intersects another line

Determine if these two lines intersect each other. Given line $\vec{r}_1$ and line $\vec{r}_2$.

$$\vec{r}_1 = \begin{pmatrix} 8 + t \\ 5 + 6t \\ 4 - 2t \end{pmatrix}$$

$$\vec{r}_2 = \begin{pmatrix} -7 + 12t \\ 3 - t \\ 14 + 8t \end{pmatrix}$$

We will need to use the parametric equations of lines to solve this problem. First we have to consider that $\vec{r}_1(t) \neq \vec{r}_2(t)$ for the same t. A note about parametric equations should be noted later. So we instead say $\vec{r}_1(t_1) = \vec{r}_2(t_2)$. Now we solve that system of equation.

$$\begin{aligned} 8 + t_1 &= -7 + 12t_2 \\ 5 + 6t_1 &= 3 - t_2 \\ 4 - 2t_1 &= 14 + 8t_2 \end{aligned}$$