Calculus With Algebraic Notes

Math Notes

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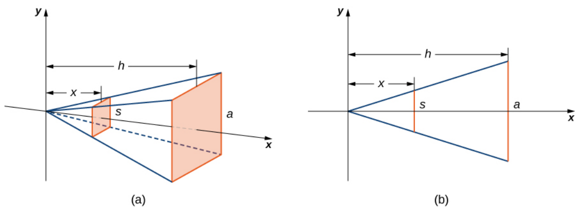
Contents

1 Volume

1.1 Volume: Given cross-sectional shapes

The below picture is given as an example. This object's base is defined by it's cross-sections which are squares who's length is defined by a graph's height.



The idea with these problems is that we have to find the area function, once we find the area function we can apply that to our integral with the bounds given. The simple formula is:

$$V = \int_a^b A(x)dx \tag{1}$$

1.2 Rotational Volume: Washer Method

Volume by washer is the easlist to visualize, circular disks are formed by the areas from heights of the two functions. We know the area function is defined by

$$A = \pi r^2$$
$$A = \pi (f(x))^2$$

Now let's consider, $g(x)$, where $g(x) \leq f(x)$, on the interval $[a, b]$.

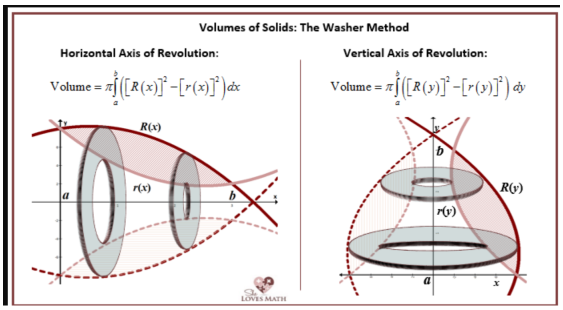
$$A = \pi f(x)^2 - \pi g(x)^2 = \pi (f(x)^2 - g(x)^2)$$

Now let's put into integral form:

$$V = \pi \int_a^b f(x)^2 - g(x)^2 dx \tag{2}$$

Now if we revolved around another axis, we need to find the distance from the functions to that axis, if ϕ is the axis which is less then f, g:

$$V = \pi \int_a^b (f(x) - \phi)^2 - (g(x) - \phi)^2 dx \tag{3}$$



1.3 Rotational Volume: Shell Method

Volume by shell is another method of gathering a rotational volume when you are presented with a different axis of rotation. While the washer method is much more easier for many problems, when rotating around the y-axis with a function that can't be easily set in terms of y, volume by shell is a better way to go.

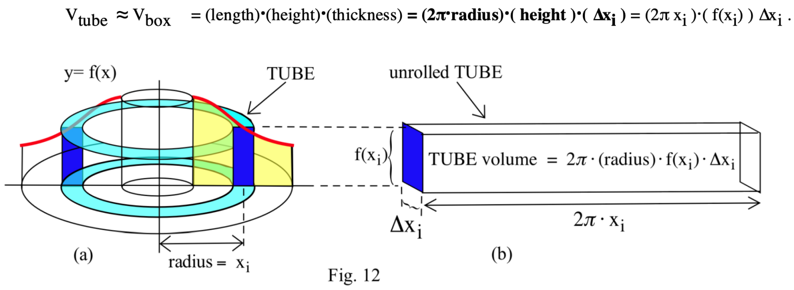


Fig. 12

We can now write the formula for volume now:

$$V = lwh \tag{4}$$

$$V = 2\pi x_i f(x)\Delta x \tag{5}$$

Now the integral form, given $f = f(x)$, and $g = g(x)$, where $f \geq g$, on the interval $[a, b]$

$$V = 2\pi \int_a^b x(f - g)dx \tag{6}$$

2 Application of Integration

There are many applications of integration. The area between curves, average values, and volumes are ones that are covered in the past section. These are more involved application problems.

2.1 Work

Work is defined by a force applied over a distance.

$$W = Fd \tag{7}$$

Now we can define force in two different ways depending on the given units. If the problem is given weight in terms of pounds or Newtons, then it becomes a simple problem.

$$W = \int_a^b w(x)(d)dx \tag{8}$$

Where $w(x)$ is the weight per slice, and d is distance traveled.
Now if the given units are in terms of density and volume. We need a new formula:

$$W = g\delta \int_a^b V(d)dx \tag{9}$$

Consider this problem : a trough is shaped by the graph $y = x^6$. The cross sections are horizontal slices that run vertically up the trough. The trough is 4 meters tall and 5 meters wide. Water is filled in the trough up to 3m. The density of water is 1000 kg per m^3. If the water is being pumped from the top, much work is being done? Since these are slices are horizontal, we need the function in terms of y.

$$y = x^6 \tag{10}$$

$$x = \pm(y)^{1/6} \rightarrow x = 2y^{1/6} \tag{11}$$

This is the width of our trough as we run up the height, so $\Delta h = dy$. Our length of the trough is 5 meters so our volume is:

$$V = 5(2y^{1/6})\Delta h \rightarrow V = 10y^{1/6}\Delta h \tag{12}$$

Force is equal to mass times accleration, which means that we need to apply $m = \delta V$. Then we apply our acceleration, which is gNow we can do our distance. 4 is the max height and it will travel by a height of y, so the distance travel is $4 - y$. That means our work formula is

$$W = Fd = \delta V(4 - y)g = g\delta(10y^{1/6})(4 - y)\Delta h \tag{13}$$

$$W = 10\delta g \int_0^3 (y^{1/6})(4 - y)dy \tag{14}$$

2.2 Center of Mass

Center of mass can be very interesting and involved application of integrals.

To find the center of mass alongside a line, we need to refer to Archimedes' Law of Lever: where a rod will be balanced, if $m_1 d_1 = m_2 d_2$, m is the mass alongside the rod, and d is the distance between the $\bar{x}$ and the mass, where that is the center posiition.

We can write d in terms of the $\bar{x}$ and the position of the mass.

$$\begin{aligned} d_1 &= \bar{x} - x_1 \\ d_2 &= x_2 - \bar{x} \\ m_1(\bar{x} - x_1) &= m_2(x_2 - \bar{x}) \\ m_1\bar{x} - m_1 x_1 &= m_2 x_2 - m_2\bar{x} \\ m_1\bar{x} + m_2\bar{x} &= m_1 x_1 + m_2 x_2 \\ \bar{x} &= \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} \end{aligned}$$

We can now try applying the above formula to the y-axis. Because, we're dealing with a plane, moments of can exist within alongside the y, or x-axis. These moments are where the object has the tendency of rotating around. We can calculate these momenets by taking the sum of the total of the point masses.

1. Moment of x-axis is equal to

$$M_x = \sum_{i=0}^{n} m_i y_i \tag{15}$$

2. Moment of y-axis is equal to

$$M_y = \sum_{i=0}^{n} m_i x_i \tag{16}$$

The center of mass is defined by a ordered pair: $(\bar{x}, \bar{y})$. Where the each is defined by it's respective moments, where $m_t = \sum m_i$

$$C(\bar{x}, \bar{y}) \rightarrow \bar{x} = \frac{M_x}{m_t}, \bar{y} = \frac{M_y}{m_t} \tag{17}$$

As you can see, the center of mass is an average of the all the mass points inside this point defined plane.

We can now take this from dealing with point masses to an actual plane, defined by curves. Let's say we have two curves, $f(x), g(x)$, where $f(x) \geq g(x)$ on the interval of $[a, b]$. This plane, just to make it general, has different density at different points on it. The mass at some point is equal to the desntiy function times it's area, where $\delta(x)$ is the density function.

$$M_i = \delta(x)(f(x) - g(x)) \tag{18}$$

Given that we are working with an interval, the total mass is defined by:

$$m_t = \int_a^b \delta(x)(f(x) - g(x))dx \tag{19}$$

Now we have our total mass, we need to find our momements. Let's go back to what we where given in our moment equations in equation 1.

$$M_x = my$$

We know that, in a thin slice, that the average value is going to be the center point, so we can rewrite y in terms of that average.

$$M_x = m(\frac{f(x) + g(x)}{2})$$

We can now plug that back into our integration formula,

$$M_x = \int_a^b \delta(x)(\frac{f(x) + g(x)}{2})(f(x) - g(x))dx \tag{20}$$

$$M_x = \frac{1}{2}\int_a^b \delta(x)(f(x)^2 - g(x)^2)dx \tag{21}$$

Now for the moment of y,

$$M_y = mx \tag{22}$$

$$My = \int_a^b x\delta(x)(f(x) - g(x))dx \tag{23}$$

The three final equations we have for the center of mass problems are:

$$m = \int \delta(x)(f(x) - g(x))dx \tag{24}$$

$$\bar{x} = \frac{M_y}{m} = \frac{1}{m}\int_a^b x\delta(x)(f(x) - g(x))dx \tag{25}$$

$$\bar{y} = \frac{M_x}{m} = \frac{1}{2m}\int_a^b \delta(x)(f(x)^2 - g(x)^2)dx \tag{26}$$

2.3 Hydrostatic Force

Hydrostatic force is the force an object feels when it's submerged by a liquid. It's defined by the liquid's density, surface area of the object, and the acceleration felt by gravity. The force changes since different parts of the object is submerged at different heights.

$$F = \delta dgA \tag{27}$$

Do note, that this formula only works with horizontal strips as vertical plates will have different pressures at different depths.

Let's do an example problem . Consider a triangle who's base is 6 meters long and it's right on the water's surface. The triangle's height is 4 meters. The force is going to be applied down on the triangle, so if we use a vertical x-axis, where $x = 0$ is the surface of the water, and $x = 4$ is the depth at which the tip of the triangle is, we can defined our horizontal strips has Δx. The black strip in the image is a good repersentation of what we are going for.

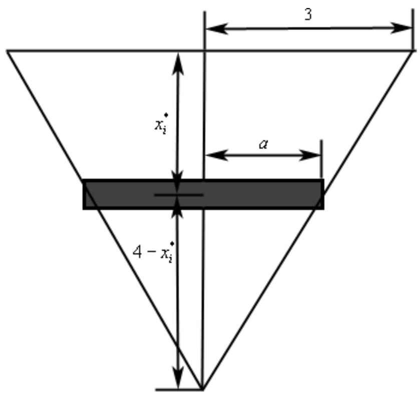


Figure 1: from: tutorial.math.lamar.edu/Classes/CalcII/HydrostaticPressure.aspx

Now for our area function, we can find the width of the triangle at different x distances, by using similar triangles.

$$\frac{3}{4} = \frac{a}{4 - x_i}$$
$$2a = \frac{3(4 - x_i)}{2}$$

Now since this is going to add all the strips together, we don't need to do our area of a triangle equation.

$$A = 2a\Delta x \tag{28}$$

$$F = \delta dgA \rightarrow F = \delta g \int_0^4 2a(x)dx \rightarrow F = \delta g \int_0^4 \frac{3x(4 - x)}{2} dx \tag{29}$$

2.4 Kinetic Energy

Kinetic energy is defined by this formula:

$$KE = \frac{1}{2}mv^2 \tag{30}$$

This is a simple equation for point masses, but if we had a bar rotating around some axis, then we need to consider that different parts of that bar will have differen KE. We need to setup a integral to calculate the total KE that rotational object has. Let's consider a bar rotating around the y-axis of a coordinate system. It rotates around that y-axis about 2 revolutions per second. The bar has $\delta = \frac{5kg}{m}$, and is 3 meters wide. We know that mass of a section of the bar is going to be:

$$m_i = \delta \Delta x \tag{31}$$

Now we need to find out the speed of each part of the object. The object makes two revolutions per second. That means the distances traveled is going $2 * 2\pi r d$. This is because, the circle formed by rotating is going to equalthe distance of the circluar path times the distance from the axis. Now writing it in terms of our x, the velocity is equal to:

$$v_i = 2(2\pi r x_i) \tag{32}$$

Now we can write in terms of the integral:

$$KE = \frac{1}{2}mv^2 \rightarrow KE = \frac{\delta}{2}\int_0^3 (4\pi(3)x)^2 dx \tag{33}$$

$$KE = 72\delta\pi^2 \int_0^3 x^2 dx \tag{34}$$

3 Vectors