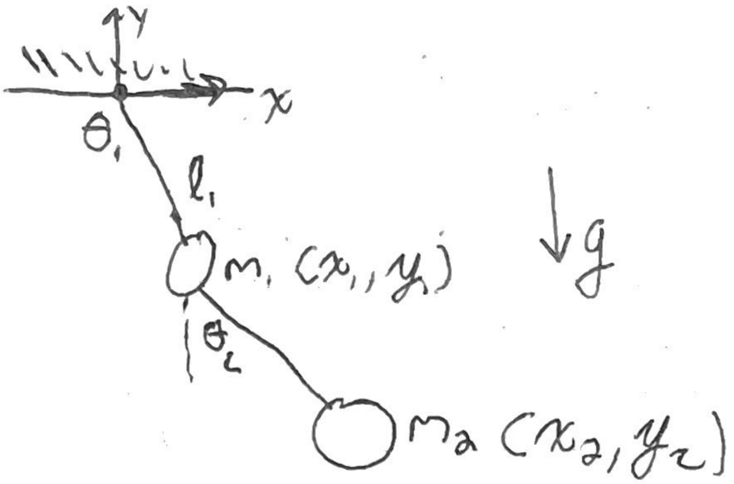
Derivation of Double Pendulum (2 DoF)

Thank you Good Vibrations with Free Ball



Assume

1). Point masses
2). Mass-less rigid rods
3). gravity is present

Kinematic Constraints

$$x_1 = l_1 \cdot \sin\theta_1 \quad (1) \qquad x_2 = l_1 \sin\theta_1 + l_2 \sin\theta_2 \quad (3)$$
$$y_1 = -l_1 \cos\theta_1 \quad (2) \qquad y_2 = -l_1 \cos\theta_1 - l_2 \cos\theta_2 \quad (4)$$

Velocities

$$\dot{x}_1 = \dot{\theta}_1 \cdot l_1 \cos\theta_1 \quad (5) \qquad \dot{x}_2 = \dot{\theta}_1 l_1 \cos\theta_1 + \dot{\theta}_2 l_2 \cos\theta_2 \quad (7)$$
$$\dot{y}_1 = +l_1 \dot{\theta} \sin\theta \quad (6) \qquad \dot{y}_2 = \dot{\theta}_1 l_1 \sin\theta_1 + \dot{\theta}_2 l_2 \sin\theta_2 \quad (8)$$

Potencial Energy

$$V = m_1 g \underset{(2)}{y_1} + m_2 g \underset{(4)}{y_2} \quad (9)$$

$$V = -(m_1 + m_2) g l_1 \cos\theta - m_2 g l_2 \cos\theta_2 \quad (10)$$

Kinetic Energy

$$T = \tfrac{1}{2} m_1 v_1^2 + \tfrac{1}{2} m_2 v_2^2 = \tfrac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \tfrac{1}{2} m_2 (\dot{x}_2 + \dot{y}_2)$$

$\longrightarrow$

Substitute equations

⑤–⑧ → ⑪ ~~into~~

$$T = \tfrac{1}{2}m_1\left[l_1^2\dot{\theta}_1^2\cancel{(\cos^2\theta_1+\sin^2\theta_1)}\right] + \tfrac{1}{2}m_2\left[(l_1\dot{\theta}_1\cos\theta_1 + l_2\dot{\theta}_2\cos\theta_2)^2 + (l_1\dot{\theta}_1\sin\theta_1 + l_2\dot{\theta}_2\sin\theta_2)^2\right]$$

$$= \tfrac{1}{2}m_1 l_1^2\dot{\theta}_1^2 + \tfrac{1}{2}m_2\left[l_1^2\dot{\theta}_1^2\cancel{(\cos^2\theta_1+\sin^2\theta_1)} + l^2\dot{\theta}_2\cancel{(\cos^2\theta+\sin^2\theta)} + 2l_1l_2\dot{\theta}_1\dot{\theta}_2\left[\underbrace{\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2}_{\cos(\theta_1-\theta_2)}\right]\right]$$

$$T = \tfrac{1}{2}m_1 l_1^2\dot{\theta}_1^2 + \tfrac{1}{2}m_2(l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2 + 2l_1l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2))$$

$$L = T - V = \tfrac{1}{2}m_1 l_1^2\dot{\theta}_1^2 + \tfrac{1}{2}m_2(l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2)) + (m_1+m_2)gl_1\cos\theta_1 + m_2gl_2\cos\theta_2$$

Lagranges Equation

$$\frac{d}{dt}\left(\frac{dL}{d\dot{q}_i}\right) - \frac{dL}{dq_i} = Q_i \quad (15)$$

$$\theta_1: \frac{dL}{d\dot{\theta}_1} = m_1l_1^2\dot{\theta}_1 + m_2l_1^2\dot{\theta}_1 + m_2l_1l_2\dot{\theta}_2\cos(\theta_1-\theta_2) \quad (16)$$

$$\frac{d}{dt}\left(\frac{dL}{d\dot{\theta}_1}\right) = (m_1+m_2)l_1^2\ddot{\theta}_1 + m_2l_1l_2\ddot{\theta}_2\cos(\theta_1-\theta_2) - m_2l_1l_2\dot{\theta}_2\sin(\theta_1-\theta_2)\cdot(\dot{\theta}_1-\dot{\theta}_2) \quad (17)$$

$$\frac{dL}{d\theta_1} = -m_2l_1l_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2) - (m_1+m_2)gl_1\sin\theta_1 \quad (18)$$

Computing the ~~Lagrangian~~ EQN (15)

~~m₁+m₂~~ Skipping Algebra ...

$$(m_1+m_2)l_1\ddot{\theta}_1 + m_2l_2\ddot{\theta}_2\cos(\theta_1-\theta_2) + m_2l_2\dot{\theta}_2^2\sin(\theta_1-\theta_2) + (m_1+m_2)g\sin\theta_1 = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i$$

Applying same proccess as Previous page, the resulting equation of motion is:

$$M_2 l_2 \ddot{\theta}_2 + M_2 l_1 \ddot{\theta}_1 \cos(\theta_1 + \theta_2) - M_2 l_1 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2)$$
$$+ M_2 g \sin\theta_2 = 0$$