Introduction to complex numbers

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Summary

Complex numbers are an extremely important concept in mathematics, as you can express a solution to any algebraic equation using complex numbers. This guide introduces the idea of a complex number in the form $z = a + bi$, as well as the concepts of real and imaginary parts, complex conjugates, and Argand diagrams.

What is a complex number?

Real numbers, which consist of all possible decimal numbers, can do lots of things very well. For instance, you can measure all possible lengths using real numbers, which you cannot do using rational numbers a/b with a, b whole numbers and b non-zero. You can also solve any equation $ax = b$ for x, with a, b non-zero real numbers.

But real numbers cannot do everything. Consider the equation

$$x^2 = -1.$$

Since the square of any real number is greater than or equal to 0, it follows that this equation has **no** solutions which are real numbers. So you will have to look elsewhere for the solution to such an equation; perhaps defining a new kind of number. Setting i to be a number such that $i^2 = -1$ solves this issue; this i is known as the **imaginary unit**.

It turns out that introducing this imaginary unit i is enough to express solutions to **all** equations involving an x^2 term, known as **quadratic equations**; for more on these, see Guide: Introduction to quadratic equations. By using this imaginary unit as the basis for a new number system containing the real numbers, you can solve all possible quadratic equations. This number system is known as the **complex numbers**.

Complex numbers appear almost everywhere in mathematics. Aside from being used to solve any quadratics, they have important applications in the theory of numbers more generally, in higher forms of algebra, the study of functions, the theory of electromagnetism, solving differential equations, and even measuring the motion of tornadoes!

This guide will focus on introducing the concept of a complex number, with the exact definition of complex numbers together with some initial examples of complex numbers. Then, the concepts of real and imaginary parts of complex numbers are explored, the complex conjugate is defined, and the pictorial representation of a complex number via Argand diagrams is explained.

Initial definitions of complex numbers

i Definition of the imaginary unit

i Definition of a complex number

Any number $z = a + bi$ where a and b are real numbers is called a **complex number**. Sometimes the complex number can be written as $z = a + ib$. This particularly happens when b is a square root such as $\sqrt{5}$, or the output of a function such as $\sin(x)$.

i **Example 1**

You are given the quadratic equation $2x^2 + 4x - 8 = 0$. The variable of the quadratic equation is x, and the coefficients are $a = 2, b = 4, c = -8$.

There are some special kinds of complex numbers:

- If $b = 0$, then $z = a$ is known as **real number**. So every real number is also a complex number.
- If $a = 0$, then $z = bi$ is known as a **purely imaginary** number.

Purely imaginary numbers are used to express solutions to the equation $x^2 = -a$, where $a > 0$ is a positive real number.

i Solutions to $x^2 = -a$

Let $a > 0$ be a positive real number. Then the solutions to $x^2 = -a$ are given by

$$x = i\sqrt{a} \qquad \text{and } x = -i\sqrt{a}.$$

i **Example 2**

Real and imaginary parts

Complex conjugate

Argand diagrams

Here, $a \neq 0$ as if it was, then the equation would no longer be a quadratic equation!

The general shape of a quadratic equation is known as a **parabola**. A figure of two parabolas is given in Figure 1; the left hand graph is if $a > 0$, whereas the right hand graph is when $a < 0$.

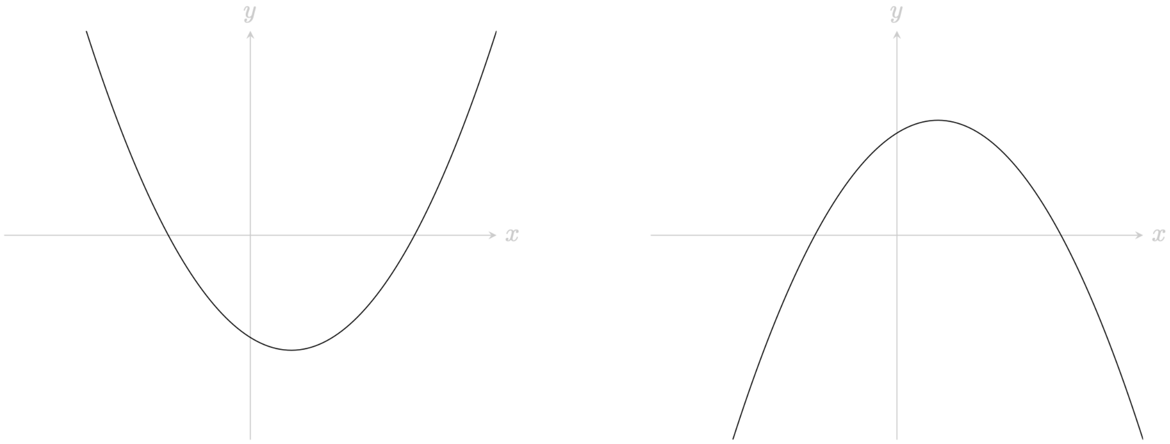

Figure 1: A pair of parabolas. (left) A graph of a quadratic $ax^2 + bx + c$ where $a > 0$. (right) A graph of a quadratic $ax^2 + bx + c$ where $a < 0$.

It is very important to be able to identify the variable in a quadratic equation, as well as the coefficients a, b, c. In the course of your mathematical study, it may be that the variable of a quadratic equation is not only letters like x, y, z, but squares or cubes like x^2 and y^3, or even functions like e^x, $\cos(x)$, and $\sin(y)$.

> **Example 1**
>
> You are given the quadratic equation $2x^2 + 4x - 8 = 0$. The variable of the quadratic equation is x, and the coefficients are $a = 2, b = 4, c = -8$.

Example 2

Here, the equation $y^4 - 10y^2 + 25 = 0$ may look like a quartic equation, but it is actually a quadratic equation. Using the laws of indices, you can rewrite the equation as $(y^2)^2 - 10y^2 + 25 = 0$. Therefore, the variable of the quadratic equation is y^2, and the coefficients are $a = 1$, $b = -10$, $c = 25$.

Example 3

You are given the equation $-e^{2x} + 4e^x - 5 = 0$. Using the laws of indices, you can rewrite the equation as $-(e^x)^2 + 4e^x - 5 = 0$. The variable of the quadratic equation is e^x, and the coefficients are $a = -1$, $b = 4$, $c = -5$. This is not the only solution to the coefficients; since the right-hand side is equal to 0, you can multiply the equation through by -1 to get $(e^x)^2 - 4e^x + 5 = 0$, which gives $a = 1$, $b = -4$ and $c = 5$. Both solutions are equally valid.

Example 4

You are given the equation $t + 1 = \frac{4}{t-3}$. This really is a quadratic equation! You can multiply both sides by $x - 3$ to get

$$(t + 1)(t + 3) = 4$$

You can then expand the brackets to get

$$t^2 + t + 3t + 3 = 4$$

and so $t^2 + 4t + 3 = 4$. Finally, you are able to subtract 4 from both sides to get $t^2 + 4t - 1 = 0$. It follows that the variable of the quadratic equation is t, and the coefficients are $a = 1$, $b = 4$, $c = -1$.

Solving a quadratic equation

To solve the quadratic equation, you could use one of three methods:

- You could **factorise** the quadratic equation $ax^2 + bx + c = 0$ into linear equations $(mx + n)(px + q)$, then work out the roots when each of these linear equations is zero. See (Guide: Factorisation) for more.
- You could **complete the square** in order to reduce the quadratic equation $ax^2 + bx +$

$c = 0$ into the form $(x + b/2a)^2 = d$, and then solve from there (not forgetting the negative root). See (Guide: Completing the square) for more.

- You could **use the quadratic formula**; for a quadratic equation $ax^2 + bx + c = 0$, the two roots to the quadratic equation are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Each method is equally valid, but some may involve more work than others. It is up to you to decide which method is best for each quadratic you encounter; but it is thoroughly advised that if you are not sure which method is best, then the quadratic formula is the one to choose. See Guide: Using the quadratic formula for more.

The discriminant

What the roots of the quadratic formula look like are determined by the term $b^2 - 4ac$; this term has a special name.

> **i The discriminant**
>
> The term $D = b^2 - 4ac$ is known as the **discriminant** of the quadratic equation $ax^2 + bx + c = 0$.

There are then three separate cases for solutions to quadratic equations.

- If $D = b^2 - 4ac$ is positive, then $\sqrt{D}$ is a real number and the two roots of the quadratic equation $ax^2 + bx + c = 0$ are

$$x = \frac{-b + \sqrt{D}}{2a} \qquad \text{and} \qquad x = \frac{-b - \sqrt{D}}{2a}$$

  These two roots are both real numbers and distinct from each other. You can observe this behaviour on a graph in Figure 2; the parabola crosses the x-axis in two places.

- If $D = b^2 - 4ac = 0$ is zero, then $\sqrt{D} = 0$. In this case, the two roots of the quadratic equation $ax^2 + bx + c = 0$ are

$$x = \frac{-b}{2a} \qquad \text{and} \qquad x = \frac{-b}{2a}$$

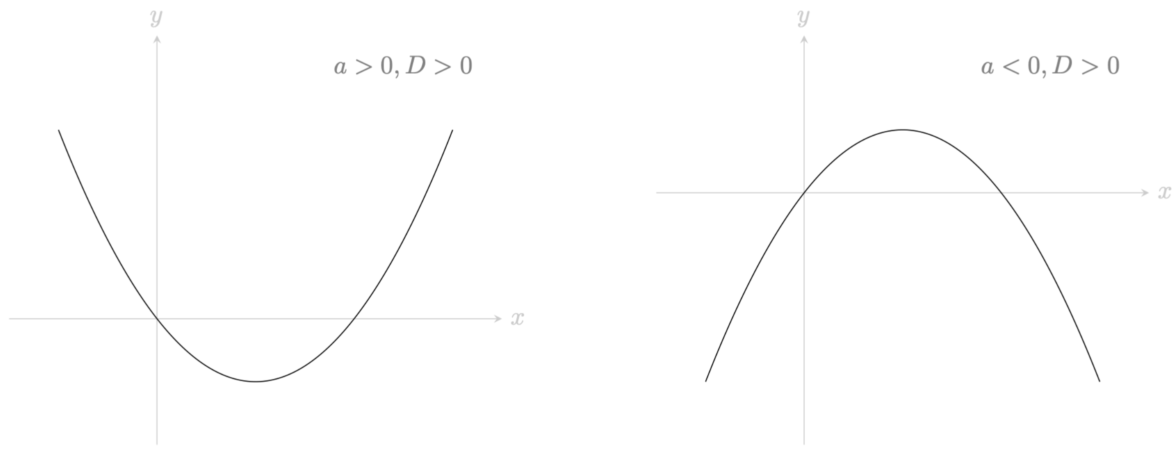

Figure 2: A pair of parabolas. (left) A graph of a quadratic $ax^2 + bx + c$ where $a > 0$ and $D > 0$. (right) A graph of a quadratic $ax^2 + bx + c$ where $a < 0$ and $D > 0$.

These two roots are given by the same real number. To be sure that you express both roots, you can write '$x = -b/2a$ twice'. You can observe this behaviour on a graph in Figure 3; the parabola touches the x-axis in exactly one place.

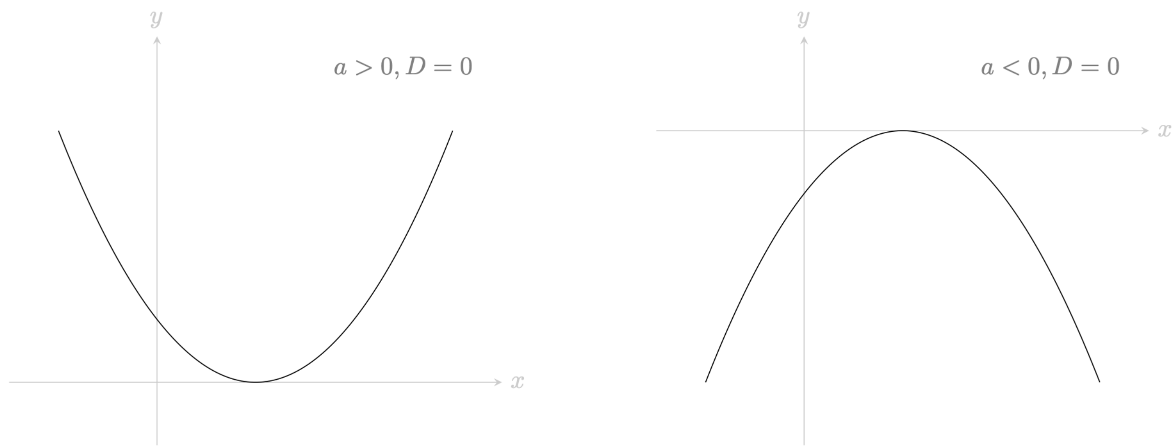

Figure 3: A pair of parabolas. (left) A graph of a quadratic $ax^2 + bx + c$ where $a > 0$ and $D = 0$. (right) A graph of a quadratic $ax^2 + bx + c$ where $a < 0$ and $D = 0$.

- If $D = b^2 - 4ac$ is negative, then $\sqrt{D}$ is not a real number. In this case, the two roots of the quadratic equation are **complex numbers**. You can express the two roots of the quadratic equation by

$$x = \frac{-b + i\sqrt{-D}}{2a} \qquad \text{and} \qquad x = \frac{-b - i\sqrt{-D}}{2a}$$

where i is the imaginary unit (so $i^2 = -1$; see (Guide: Introduction to complex numbers)). In a graph, the parabola does not cross the x-axis at all; this indicates that there are no real solutions to this quadratic equation. See Figure 4 for a picture.

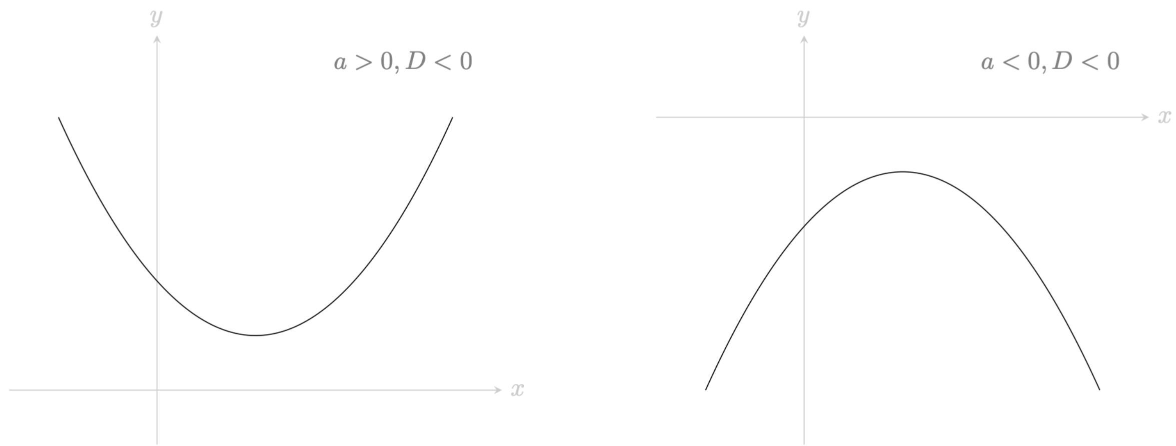

Figure 4: A pair of parabolas. (left) A graph of a quadratic $ax^2 + bx + c$ where $a > 0$ and $D > 0$. (right) A graph of a quadratic $ax^2 + bx + c$ where $a < 0$ and $D < 0$.

Warning

You can use the discriminant to check how many roots a quadratic equation has in the variable given to you. However, this is at most a maximum number of solutions. Conditions on that variable may also reduce the number of valid solutions, particularly if you have real valued functions. For instance, since $e^x > 0$ for all real number x, there are no solutions in x if you find $e^x = -1$.

Here's some examples of using the discriminant and known properties of functions to rule out solutions.

Example 5

In Example 1, you identified the coefficients of $2x^2+4x-8 = 0$ as $a = 2, b = 4, c = -8$. Using these, you can work out the value of the discriminant $D = b^2 - 4ac$ as

$$D = (4)^2 - 4(2)(-8) = 16 + 64 = 80.$$

Since $D = 80$, you can say that this quadratic equation has two distinct real roots in $x = r_1$ and $x = r_2$.

Example 6

In Example 2, you identified the coefficients of $y^4 - 10y^2 + 25 = 0$ as $a = 1, b = -10, c = 25$, and the variable as y^2. Using these, you can work out the value of the discriminant $D = b^2 - 4ac$ as

$$D = (-10)^2 - 4(1)(25) = 100 - 100 = 0.$$

Since $D = 0$, you can say that this quadratic equation has at most one real root r in terms of y^2.
Whether or not the equation itself has real solutions in y depends on whether r is positive or negative! You cannot take the square root of a negative number, so if r is negative the equation has no real solutions. If r is positive, then the equation has two real roots in y; that is, $y = \pm\sqrt{r}$.

Example 7

In Example 3, you identified the coefficients of $-e^{2x} + 4e^x - 5 = 0$ as $a = -1, b = 4, c = -5$, and the variable as e^x. Using these, you can work out the value of the discriminant $D = b^2 - 4ac$ as

$$D = (4)^2 - 4(-1)(-5) = 16 - 20 = -4.$$

Since $D = -4$, you can say that this quadratic equation has complex roots.
This equation therefore has no real solutions in x. This is because e^x is real for any real x; if e^x is complex, it follows that x cannot be real.

Example 8

In Example 4, you rearranged the equation $t + 1 = \frac{4}{t-3}$ to $t^2 + 4t - 1 = 0$, and therefore identified the coefficients as $a = 1, b = 4, c = -1$, and the variable as t. Using these, you can work out the value of the discriminant $D = b^2 - 4ac$ as

$$D = (4)^2 - 4(1)(-1) = 16 + 4 = 20.$$

Since $D = 20$, you can say that this quadratic equation has two distinct real roots in t.

Quick check problems

1. What is the discriminant of the quadratic equation $x^2 - x - 1 = 0$?
2. You are given the quadratic equation $4h^2 - h + 101 = 0$. Identify the variable, and the coefficients a, b, c.
3. You are given three statements below. Decide whether they are true or false.

(a) The quadratic equation $m^2 + 4m + 4 = 0$ has two distinct real roots.

(b) The quadratic equation $m^2 - 4m - 4 = 0$ has exactly one real root.

(c) The quadratic equation $4m^2 + 4m + 4 = 0$ has no real roots.

Further reading

For more questions on the subject, please go to Questions: Introduction to complex numbers.

[For how to add, subtract, multiply, and divide complex numbers, please go to Guide: Arithmetic on complex numbers.]

Version history

v1.0: initial version created 09/24 by tdhc.