Further sigma notation

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Summary

Sigma notation is used to express many additions at once. Understanding what this notation is, how it works, and how to manipulate them is a valuable skill to learn for use in almost any area of mathematics.

What is sigma notation?

If you want to add many things together, then it would be nice to have a quick way of writing this down! This is where **sigma notation** comes in. (USES!)

> **Definition of sum and sigma notation**
>
> A **sum** is any addition of two or more real numbers. If $a_k, a_{k+1}, \dots, a_n$ are real numbers (where k and n are some natural numbers with $k \leq N$), then you can use **sigma notation** to write their sum as
>
> $$a_k + a_{k+1} + \dots + a_N = \sum_{i=k}^{N} a_i$$
>
> where the right hand side reads 'the sum from $i = k$ to $i = n$ of the elements a_i'. The symbol i is known as the **index** of the sum; the index of a sum can notionally be any letter.

Examples

Here's some examples of sigma notation.

Example 1

What is the value of $\sum_{i=1}^{10} i$?

You can use the definition above to write this out as a sum and then calculate it:

$$\sum_{i=1}^{10} i = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = 55.$$

Example 2

What is the value of $\sum_{n=2}^{5} n^2$?

Before tackling a problem using sigma notation, it can be best to read it out loud. Here,

$$\sum_{n=2}^{5} n^2 \text{ is 'the sum from } n = 2 \text{ to } n = 5 \text{ of } n^2\text{'.}$$

This translates to

$$\sum_{n=2}^{5} n^2 = 2^2 + 3^2 + 4^2 + 5^2$$

and $2^2 + 3^2 + 4^2 + 5^2 = 4 + 9 + 16 + 25 = 54$.

Example 3

What is the value of $\sum_{n=1}^{N} n = S$?

In this case, you're being asked to find $S = 1+2+3+...+N$. The following method is due to Gauss, who came up with this answer during a maths lesson at school when he was seven (hinting at the genius to follow).

First of all, you can reorder S to write that $S = N + (N-1) + ... + 2 + 1$. Adding two lots of S together gives the following:

$$\begin{array}{cccccccccccc}
 & S & = & 1 & + & 2 & + & 3 & + & ... & + & N \\
+ & S & = & N & + & (N-1) & + & (N-2) & + & ... & + & 1 \\
\hline
 & 2S & = & (N+1) & + & (N+1) & + & (N+1) & + & ... & + & (N+1)
\end{array}$$

Therefore, $2S$ is N lots of $(N+1)$; you can write this as $2S = N(N+1)$. Dividing both sides by 2 gives $S = N(N+1)/2$.

Writing sums using sigma notation

In this section, you will learn how to do the opposite of the above. That is, given a sequence of numbers, you will learn how to write their sum using sigma notation. The crux of this process is to recognise a pattern in the sequence of given numbers. It's best to learn this using examples.

Example 4

Write $2+4+6+8+10+12$ using sigma notation.
You can tell that these are the first six multiples of 2; so you can list these elements as $2n$ for $n=1$ up to $n=6$. Therefore, you can write that

$$2+4+6+8+10+12=\sum_{n=1}^{6} 2n.$$

Example 5

Write $1+2+4+8+16+32$ using sigma notation.
These are the first 6 numbers in the sequence 2^n for $n=0$ up to $n=5$. Therefore, you can write

$$1+2+4+8+16+32=\sum_{n=0}^{5} 2^n.$$

Example 6

Write $-1+2-3+4-5$ using sigma notation.
For these types of sequences it's useful to keep in mind the sequence $(-1)^n$, which alternates between 1 and -1. Hence, you can write these elements as $(-1)^n n$ for $n=1$ up to $n=5$. Using sigma notation, it will look like this:

$$\sum_{n=1}^{5}(-1)^n n.$$

Properties

In this section you will learn about a few properties of sigma notation which means you'll have a toolkit to rearrange sums!

The first property you'll learn about sigma notation is *distribuitivity*. This property allows you to take constants from inside the sigma notation to outside the summation.

Distribuitivity

Let $a_k, a_{k+1}, \ldots, a_n$ be a sequence of numbers (where k and n are integers with $k \leq n$) and C be any constant. Then

$$\sum_{i=k}^{n} Ca_i = C \sum_{i=k}^{n} a_i.$$

You can see this is true by writing the entire sum out, like this:

$$\begin{aligned} \textstyle\sum_{i=k}^{n} Ca_i &= Ca_k + Ca_{k+1} + Ca_{k+2} + \ldots + Ca_n \\ &= C(a_k + a_{k+1} + a_{k+2} + \ldots + a_n) \\ &= C \textstyle\sum_{i=k}^{n} a_k \end{aligned}$$

Example 7

What is the value of $\sum_{n=2}^{5} 6n^2$?

Using distributivity, $\sum_{n=2}^{5} 6n^2 = 6 \sum_{n=2}^{5} n^2$. From Example 2, you know that $\sum_{n=2}^{5} n^2 = 54$. Therefore, $6 \sum_{n=2}^{5} n^2 = 6 \times 54 = 324$.

Another property of sigma notation is something we'll call *combining sums*. This lets you write two sums in sigma notation as one sum.

Combining sums

Let $a_k, a_{k+1}, \ldots, a_n$ and $b_k, b_{k+1}, \ldots, b_n$ be two sequences of numbers (where k and n are numbers with $k \leq n$). Then

$$\sum_{i=k}^{n} a_i + \sum_{i=k}^{n} b_i = \sum_{i=k}^{n} (a_i + b_i)$$

and

$$\sum_{i=k}^{n} a_i - \sum_{i=k}^{n} b_i = \sum_{i=k}^{n} (a_i - b_i).$$

Warning

Note in the property above that both sequences start at k and end at n.

Similar to the distributive property, you can show this is true by writing the entire sum out:

$$\begin{aligned}\sum_{i=k}^{n} a_i + \sum_{i=k}^{n} b_i &= (a_k + a_{k+1} + \ldots + a_n) + (b_k + b_{k+1} + \ldots + b_n) \\ &= (a_k + b_k) + (a_{k+1} + b_{k+1}) + \ldots + (a_n + b_n) \\ &= \sum_{i=k}^{n} (a_k + b_k).\end{aligned}$$

In a similar way, you can show that $\sum_{i=k}^{n} a_i - \sum_{i=k}^{n} b_i = \sum_{i=k}^{n} (a_i - b_i)$ is also true.

Example 8

Write $\sum_{n=1}^{6} 2n + \sum_{n=0}^{5} 2^n$ as a single sum.
First, notice that the indices of these sequences are different, so before you can use the combining sums property you need to do a process called *reindexing*. Reindexing a sum in sigma notation means rewriting the same sum using different indicies. For your purposes, you can reindex $\sum_{n=1}^{6} 2n$ as $\sum_{n=0}^{5} (2n+2)$. You can now use the combining sums property:

$$\sum_{n=1}^{6} 2n + \sum_{n=0}^{5} 2^n = \sum_{n=0}^{5} (2n+2) + \sum_{n=0}^{5} 2^n = \sum_{n=0}^{5} (2n + 2^n + 2).$$

Note

In the above example, you could also reindex $\sum_{n=0}^{5} 2^n$ instead of $\sum_{n=1}^{6} 2n$. Give it a go!

Splitting a sum

Let $a_k, a_{k+1}, \ldots, a_n$ and $b_k, b_{k+1}, \ldots, b_n$ be two sequences of numbers (where k and n are integers with $k < n - 1$). Then for any integer m such that $k \leq m < n$,

$$\sum_{i=k}^{n} a_i = \sum_{i=k}^{m} a_i + \sum_{i=m+1}^{n} a_i.$$

Again, similar to above you can show this by writing the entire sum out. This is left to you.

Example 9

Write $\sum_{i=10}^{100} i$ as the product of three sums.
There are a large number of ways to do this an example of which is:

$$\sum_{i=10}^{11} i + \sum_{i=12}^{98} i + \sum_{i=99}^{100} i.$$

Note that this example has not been particularly useful mathematically but is to illustrate the property in action.
This property is used more when working with infinite summations.

Double sums

Sometimes, you'll want to multiply two sums together. This can be written succinctly using something called *double sums*.

Double sums

Let $a_k, a_{k+1}, \dots, a_n$ and $b_t, b_{t+1}, \dots, b_m$ be two sequences of numbers (where k, n, t, and m are integers with $k \leq n$ and $t \leq m$). Then the **double sum** $\sum_{i=k}^{n} \sum_{j=t}^{m} a_i b_j$ is defined as

$$\begin{aligned} \sum_{i=k}^{n} \sum_{j=t}^{m} a_i b_j &= a_k b_t + a_k b_{t+1} + \dots + a_k b_m + a_{k+1} b_t + a_{k+1} b_{t+1} \\ &\quad + \dots + a_{k+1} b_m + \dots + a_n b_m. \end{aligned}$$

Tip

You might find it easier to remember the above by thinking of $\sum_{i=k}^{n} \sum_{j=t}^{m} a_i b_j$ as $a_1(\sum_{j=t}^{m} b_j) + a_2(\sum_{j=t}^{m} b_j) + \dots + a_n(\sum_{j=t}^{m} b_j)$.

You will now see how this relates to multiplying two sums together. Suppose that $a_k, a_{k+1}, \dots, a_n$ and $b_t, b_{t+1}, \dots, b_m$ are like above, and consider the product $(\sum_{i=k}^{n} a_i)(\sum_{j=t}^{m} b_j)$. Writing it all out and performing the multiplication, you get

$$
\begin{aligned}
\left(\sum_{i=k}^{n} a_i\right)\left(\sum_{j=t}^{m} b_j\right) &= (a_k + a_{k+1} + \dots + a_n)(b_t + b_{t+1} + \dots + b_m) \\
&= a_k b_t + a_k b_{t+1} + \dots + a_k b_m + a_{k+1} b_t + a_{k+1} b_{t+1} + \\
&\qquad \dots + a_{k+1} b_m + a_{k+2} b_t + \dots + a_n b_m \\
&= \sum_{i=k}^{n} \sum_{j=t}^{m} a_i b_j
\end{aligned}
$$

You can write this as a result:

Double sums and products of two sums

Let $a_k, a_{k+1}, \dots, a_n$ and $b_t, b_{t+1}, \dots, b_m$ be two sequences of numbers (where $k, n, t,$ and m are integers with $k \leq n$ and $t \leq m$). Then

$$\sum_{i=k}^{n} \sum_{j=t}^{m} a_i b_j = \left(\sum_{i=k}^{n} a_i\right)\left(\sum_{j=t}^{m} b_j\right).$$

Example 10

Write $(1+2+3+4+5+6)(2+4+6+8+10+12)$ as a double sum and as a product of two sums.

First, notice you can write out the above expression in the form $(1)(2)+(1)(4)+\ldots(1)(12)+(2)(2)+(2)(4)\ldots(3)(2)+\ldots(6)(12)$

From the definition above you may now rewrite the expression to the double sum

$$\sum_{i=1}^{6}\sum_{j=1}^{6} i*2j$$

using the distrubitivity property this can be written as

$$2\sum_{i=1}^{6}\sum_{j=1}^{6} ij$$

This can then be written using the product of two sums rule above to

$$2\sum_{i=1}^{6} i\sum_{j=1}^{6} j$$

It is evident that the two sums are the same with different index variables this means that they can be combined to form

$$2\sum_{k=1}^{6} k^2$$

k has been used to differentiate the new sum from the ones involving i and j before but as always the choice of index variable is relatively unimportant

Quick check problems

1. What is the value of $\sum_{i=2}^{6} i$.

Answer: The value of the above is: ___.

2. Given $\sum_{j=1}^{100} i$ Identify the index of the sum.

Answer: The index is _

3. You are given several statements below based on the properties of sums. Identify whether they are true or false.

(a) The sum $3+6+9+12$ can be expressed as $\sum_{i=0}^{4} 3i$ Answer: TRUE / FALSE.

(b) The sum $-1+1-1+1$ can be expressed as $\sum_{i=1}^{4} -i$ Answer: TRUE / FALSE.

(c) $\sum_{i=1}^{100} i = \sum_{i=0}^{101} i$ Answer: TRUE / FALSE.

(d) $\sum_{i=1}^{100} 6i = 6\sum_{i=0}^{100} i$ Answer: TRUE / FALSE.

(e) $\sum_{i=1}^{100} 9i + \sum_{i=1}^{100} 3i = \sum_{i=1}^{100} 27i^2$ Answer: TRUE / FALSE.

(f) $\sum_{i=1}^{100} 12i - \sum_{i=1}^{100} 4i = 8\sum_{i=1}^{100} i$ Answer: TRUE / FALSE.

4. You are given several statements below based on the properties of sums. Identify whether they are true or false.

(a) $\sum_{i=1}^{10}\sum_{j=2}^{6} ij$ can be expressed as $\left(\sum_{i=2}^{6} i\right)\left(\sum_{j=1}^{10} j\right)$ Answer: TRUE / FALSE.

(b) $\left(\sum_{i=1}^{5} 2i\right)\left(\sum_{j=5}^{10} 3j\right)$ can be expressed as $6\left(\sum_{i=1}^{5}\sum_{j=5}^{10} ij\right)$\$ Answer: TRUE / FALSE.

(c) The sum $(1+2+3+4+5+6)(-1-2)(3+6+9)$ can be expressed as $\sum_{i=1}^{6}\sum_{j=1}^{2}\sum_{k=1}^{3} -3ijk$\$

Answer: TRUE / FALSE.

Further reading