Dealing With Powers and nth roots

Zoe Gemmell

Summary

Simplifying expressions involving powers and nth roots is a valuable skill to learn. In this guide you will learn a few manipulation techniques that will hopefully be useful going further into your maths career.

Before reading this guide, it is recommended that you read (Guide: Laws of indices). This guide uses · for multiplication. Power is another word for indice/index.

This guide will focus on expressions with bases and powers. You can recognise them from combinations of expressions in the following format:

$$base^{power}$$

What is a base?

A base is a number that is raised to a power. It can be any number you can think of; single digit, decimal numbers, a million. This is the number that gets multiplied by itself.

What is a power you might then ask? A power is any number that you can raise a base to, it dictates how many times the base is multiplied by itself.

> **Example 1**
>
> $$3^5 \text{ means } \underbrace{3 \cdot 3 \cdot 3 \cdot 3 \cdot 3}_{5\,times}$$
>
> You would say this out loud as 'three multiplied by itself five times', 'three to the fifth power' or 'three to the five'.

Working with powers - The Definitions

Before you have a look at the laws here are some definitions:

Definition of expression

A mathematical expression is a combination of symbols that has a finite length.

Definition of commutativity

Changing the order of the numbers involved in your expression does not change the answer.

Example 2

$2 + 3 = 5 = 3 + 2$

The order of the 2 and 3 can be swapped and the answer stays the same.

Definition of terminoligy in a fraction

The numerator of a fraction is everything on top of the line, the denominator is everything underneath the line:

$$\frac{numerator}{denominator}$$

Tip

Both addition and multiplication are commutative

Now onto the laws:

Law 7: Multiplication of variables with the same indices but different bases

When multiplying variables with the same powers you can expand and rewrite them into a nicer format:

$$a^r \cdot b^r = (ab)^r$$

Where the bases are multiplied together then raised to the rth power.

Example 3

$$\begin{aligned} 2^3 \cdot 3^3 &= \underbrace{(2 \cdot 2 \cdot 2)}_{3\,times} \cdot \underbrace{(3 \cdot 3 \cdot 3)}_{3\,times} \\ &= 2 \cdot 3 \cdot 2 \cdot 3 \cdot 2 \cdot 3 && (\text{ by commutativity}) \\ &= (2 \cdot 3)^3 \end{aligned}$$

Law 8: Division of variables with the same indices but different bases

When dividing variables with the same powers you can expand and rewrite them into a nicer format:

$$\frac{a^r}{b^r} = (\frac{a}{b})^r$$

Where the numerator base (a) is divided by the denominator base (b) then raised to the rth power.

Example 4

$$\begin{aligned} \frac{4^4}{5^4} &= \frac{\overbrace{4 \cdot 4 \cdot 4 \cdot 4}^{4\,times}}{\underbrace{5 \cdot 5 \cdot 5 \cdot 5}_{4\,times}} \\ &= \frac{4}{5} \cdot \frac{4}{5} \cdot \frac{4}{5} \cdot \frac{4}{5} && (\text{ by commutativity}) \\ &= (\frac{4}{5})^4 \end{aligned}$$

Law 9: fractional power, the nth root

When multiplying by bases with fractional powers (some sort of nth root) :

$$a^{1/n} \cdot b = \sqrt[n]{a \cdot b^n}$$

Where the aim of the manipulation is to put everything inside the square root sign, then to continue with the calculation.

Example 5

$$
\begin{aligned}
2^{1/3} \cdot 4 &= 2^{1/3} \cdot 4^{3/3} \\
&= \sqrt[3]{2 \cdot 4^3} \\
&= \sqrt[3]{2 \cdot (2 \cdot 2)^3} \\
&= \sqrt[3]{2^7} && (\,(2^2)^3 = 2^{2 \cdot 3} = 2^6) \\
&= 2^{7/3}
\end{aligned}
$$

Law 10: Multiplication under the root sign

When you have a term under a nth root sign which you can factorise (split up into different factors) this can be done and the nth root signs can be dealt with separately.

$$\sqrt[n]{a \cdot b} = \sqrt[n]{a} \cdot \sqrt[n]{b}$$

Where you are using the commutativity of multiplication to split up the nth root and deal with terms separately.

Example 6

$$
\begin{aligned}
\sqrt{12} &= \sqrt{4 \cdot 3} \\
&= \sqrt{4} \cdot \sqrt{3} \\
&= 2 \cdot \sqrt{3}
\end{aligned}
$$

Solving equations

Using the laws above you can solve equations

Tip

When dealing with equations, you may find yourself faced with the following form:

$$\frac{1}{a+\sqrt{b}}$$

In which case you will want to use the **conjugate** of the denominator to *creatively* multiply the fraction by 1. The conjugate of $a+\sqrt{b}$ is $a-\sqrt{b}$, to find this for any espression you change the sign in front of the root.

$$\begin{aligned}
\frac{1}{a+\sqrt{b}} \cdot 1 &= \frac{1}{a+\sqrt{b}} \cdot \frac{a-\sqrt{b}}{a-\sqrt{b}} \\
&= \frac{a-\sqrt{b}}{(a+\sqrt{b})(a-\sqrt{b})} \\
&= \frac{a-\sqrt{b}}{a^2+a\sqrt{b}-a\sqrt{b}-(\sqrt{b})^2} \\
&= \frac{a-\sqrt{b}}{a^2-b}
\end{aligned}$$

Now the fraction only has square roots in the numerator, this is better because it can often make expressions you are dealing with more concise.

Example 7

$$\begin{aligned}
2^{x+1} \cdot 3^x &= 72 \\
2(2 \cdot 3)^x &= 72 \\
(2 \cdot 3)^x &= 36 \\
6^x &= 36
\end{aligned}$$

From here you can count up powers of 6 to work out the value of x: $6^1 = 6$, $6^2 = 36$ so $x = 2$

Example 8

Here is a rather long example using multiple laws from this guide and the intro to indices guide.

$$\begin{aligned}
\sqrt{(6x)^3} &= 8\sqrt{27} \\
\sqrt{x^3}\cdot\sqrt{6^3} &= 8\sqrt{27} && \text{(Using Law 7 and Law 10)} \\
\sqrt{x^3}\cdot\sqrt{(2\cdot 3)^3} &= 8\sqrt{27} && \text{(Using Law 7)} \\
\sqrt{x^3}\cdot\sqrt{2^3}\sqrt{3^3} &= 8\sqrt{27} && \text{(Using Law 10)}
\end{aligned}$$

to simplify the appearance of the calculation, swap 2^3 for 8

$$\begin{aligned}
\sqrt{x^3}\cdot\sqrt{8}\sqrt{27} &= 8\sqrt{27} \\
\sqrt{x^3}\cdot\sqrt{8} &= 8 && \text{(cancel } \sqrt{27}\text{)} \\
\sqrt{x^3} &= \frac{8}{8^{1/2}} \\
\sqrt{x^3} &= \sqrt{8} && \text{(Using Law 2)}
\end{aligned}$$

Finally, switch back to 2^3

$$x^3 = 8 = 2^3$$

So x=2

Quick check problems

1. What is the singular of indices?
2. Solve $3x^3 \cdot 5 = 45$ for x
3. Determine whether the following calculations are correct:

(a) $\dfrac{3x^a}{y^{8b}} \cdot \dfrac{4z^a}{y^d} = \dfrac{12(xz)^a}{y^{d+8b}}$

(b) $\dfrac{4^2}{2^4} = 2$

(c) $(7^2)^0 = 1$

(d) $x^2 + x^3$ is greater than x for all values of x

For more questions on the subject, please go to Questions: Needs to be added