Proof: the quadratic formula

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Summary

An explanation as to why the quadratic formula is true.

Before reading this proof sheet, it is recommended that you read [Guide: Completing the square].

Proof of the quadratic formula

Remember from Guide: Using the quadratic formula that the **quadratic formula** is used to find roots of any quadratic equation:

> **i** The quadratic formula
>
> Let $ax^2+bx+c = 0$ be a quadratic equation (where $a \neq 0$). The roots to this quadratic equation are given by
> $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
> where one of the roots is given by the term $(-b + \sqrt{b^2 - 4ac})/2a$ and the other given by the term $(-b - \sqrt{b^2 - 4ac})/2a$.

In order to prove that these really are the solutions to the quadratic, you can **complete the square** on $ax^2 + bx + c$ using the fact that $a \neq 0$. See (Guide: Completing the square) for why this works. ::: {.callout-note appearance="simple"}

Proof of the quadratic formula

First of all, as $a \neq 0$ you can divide $ax^2 + bx + c = 0$ through by a to get

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Taking the c/a term over to the other side gives

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

Completing the square gives
$$\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} = -\frac{c}{a}$$

You can rearrange to get
$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} = \frac{b^2 - 4ac}{4a^2}$$

Now the result is starting to come together. Taking square roots of both sides (not forgetting that it could be positive or negative) gives
$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

and rearranging gives
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

as required.

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Further reading

Guide: Using the quadratic formula

Questions: Using the quadratic formula

Version history

v1.0: created in 04/24 by tdhc.