The Scalar Product

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Summary

Calulcating the scalar product, interpreting it geometrically and being careful of certain special cases are all important steps in working with vectors.

Before reading this guide, it is recommended that you read (Guide: Introduction to Vectors, Guide: Vector Addition and Scalar Multiplication and Guide: Sigma Notation).

What is the scalar product?

When working with vectors, you want to be able to combine them in order to solve vector-related problems. The **scalar product** (named as such since the result is a scalar), also known as the **dot product** (due to the symbol for the operation), gives you one way of doing this. The scalar product can tell you how much two vectors are aligned and can be used to find the angle between them.
It also has applications in applied mathematics and physics.

This guide will give both the algebraic and geometric definition of the scalar product, display certain properties of the operation and bring awareness to special cases that arise.

Algebraic definition of the scalar product

Let $\mathbf{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$, then the **scalar product**, denoted $\mathbf{u} \cdot \mathbf{v}$, is defined as follows:

$$\mathbf{u} \cdot \mathbf{v} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = u_1v_1 + u_2v_2 + \ldots + u_nv_n$$

This may also be written, using sigma notation, as

$$\mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^{n} u_iv_i$$

As the name suggests, the scalar product of two vectors is a scalar. It is important to be able to calculate the scalar product of vectors of all dimensions.

WARNING

Vectors must have the same dimension to have a scalar product.

Example 1

Take $\mathbf{u} = \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 2 \\ 7 \\ 2 \end{pmatrix}$, then the scalar product is

$$\begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 7 \\ 2 \end{pmatrix} = (3)(2) + (5)(7) + (1)(2) = 6 + 35 + 2 = 43$$

Example 2

Take $\mathbf{u} = \begin{pmatrix} 10 \\ 13 \\ 3 \\ 2 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 5 \\ 1 \\ 20 \\ 1 \end{pmatrix}$, then the scalar product is

$$\begin{pmatrix} 10 \\ 13 \\ 3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 1 \\ 20 \\ 1 \end{pmatrix} = (10)(5) + (13)(1) + (3)(20) + (2)(1) = 50 + 13 + 60 + 2 = 125$$

Example 3

Take $\mathbf{u} = \begin{pmatrix} -1 \\ 7 \\ 1 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$, then the scalar product is

$$\begin{pmatrix} -1 \\ 7 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} = (-1)(-1) + (7)(-1) + (1)(2) = 1 - 7 + 2 = -4$$

You can also take the scalar product of vectors in cartesian form.

Example 4

Take $\mathbf{u} = 4\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = \mathbf{i} - \mathbf{j} + 3\mathbf{k}$, then

$$\mathbf{u} \cdot \mathbf{v} = (4\mathbf{i} + 2\mathbf{j} + \mathbf{k}) \cdot (\mathbf{i} - \mathbf{j} + 3\mathbf{k}) = (4)(1) + (2)(-1) + (1)(3) = 5$$

Properties of the scalar product

Definition of Commutative, Distributive

An operation is **commutative** if it gives the same result whatever order the values are in. For example, $ab = ba$, so multiplication is commutative.
An operation is **distributive** (under addition) if $a(b + c) = (ab) + (ac)$.

The scalar product has some nice properties:

(a) The scalar product of a vector with itself is the square of its magnitude:

$$\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$$

(b) Multiplying a vector, $\mathbf{u}$ by a scalar, c, multiplies its scalar product with another vector, $\mathbf{v}$ by the same scalar, c:

$$(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v})$$

(c) The scalar product of a vector, $\mathbf{u}$ with the zero vector, $\mathbf{0}$ is 0:

$$\mathbf{u} \cdot \mathbf{0} = 0$$

(d) The scalar product is **commutative**:

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$$

(e) The scalar product is **distributive**:

$$\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$$

Example 5

Take $\mathbf{u} = \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 2 \\ 7 \\ 2 \end{pmatrix}$, as in example 1. Then you know

$$\mathbf{u} \cdot \mathbf{v} = 43$$

. Also,

$$\mathbf{v} \cdot \mathbf{u} = \begin{pmatrix} 2 \\ 7 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix} = (2)(3) + (7)(5) + (2)(1) = 6 + 35 + 2 = 43$$

So, $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$, that is, commutative.

Example 6

Take $\mathbf{u} = \begin{pmatrix} 14 \\ 2 \\ 100 \end{pmatrix}$, then

$$\mathbf{u} \cdot \mathbf{u} = 14^2 + 2^2 + 100^2 = 10200$$

and, the magnitude is

$$|\mathbf{u}| = \sqrt{14^2 + 2^2 + 100^2} = \sqrt{10200}$$

and so,

$$|\mathbf{u}|^2 = \mathbf{u} \cdot \mathbf{u}$$

Proof of commutativity

You've just seen one example showing that the scalar product is commutative. How about for all vectors? Let $\mathbf{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$, then

$$\mathbf{u} \cdot \mathbf{v} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n = v_1 u_1 + v_2 u_2 + \ldots + v_n u_n$$

since multiplication is commutative (that is $u_1 v_1 = v_1 u_1$). And,

$$\mathbf{v} \cdot \mathbf{u} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} = v_1 u_1 + v_2 u_2 + \ldots + v_n u_n = \mathbf{u} \cdot \mathbf{v}$$

Hence, the scalar product is commutative.

The Geometric Interpretation

Geometric defintion of the Scalar Product

The scalar product can also be defined as follows:

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos(\psi)$$

WARNING

In the above, the vectors cannot be **anti-parallel** or **parallel** and must be of non-zero length. The special cases in which they do not follow these conditions are dealt with below.

Having seen the geometric definition of the scalar product, you may now be wondering about where it comes from and its interpretation. You have the definitions, but how do these relate to the vectors you're taking the scalar product of? Take two arbitrary vectors $\mathbf{u}$ and $\mathbf{v}$. Place their base at the same point. Call this base point A. Consider the plane formed by the tip of $\mathbf{u}$ (at point B) and formed at the tip of $\mathbf{v}$ (at point C). The points A,B,C form a plane. Now, let a be the length of $\mathbf{u}$, b be the length of $\mathbf{v}$ and c be the length of $\mathbf{v} - \mathbf{u}$, like in the diagram below.

Using trigonometry, the height of the triangle in figure 1 is $h = b\sin(\psi)$. Looking at the diagram again, $l = b\cos(\psi)$ and

$$x = a - l = a - b\cos(\psi)$$

Using Pythagoras's theorem,

$$h^2 + x^2 = c^2$$

$$b^2 \sin^2(\psi) + (a - b\cos(\psi))^2 = c^2 = |\mathbf{v} - \mathbf{u}|^2$$

(the length of $\mathbf{v} - \mathbf{u}$). Expanding out the brackets on the left hand side obtains

$$b^2 - 2ab\cos(\psi) + a^2 = |\mathbf{v} - \mathbf{u}|^2$$

Using property (a) from above:

$$|\mathbf{v} - \mathbf{u}|^2 = (\mathbf{v} - \mathbf{u}) \cdot (\mathbf{v} - \mathbf{u})$$

Also, by distributivity and property (a),

$$(\mathbf{v} - \mathbf{u}) \cdot (\mathbf{v} - \mathbf{u}) = \mathbf{v} \cdot \mathbf{v} - 2\mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{u} = |\mathbf{v}|^2 - 2\mathbf{u} \cdot \mathbf{v} + |\mathbf{u}|^2$$

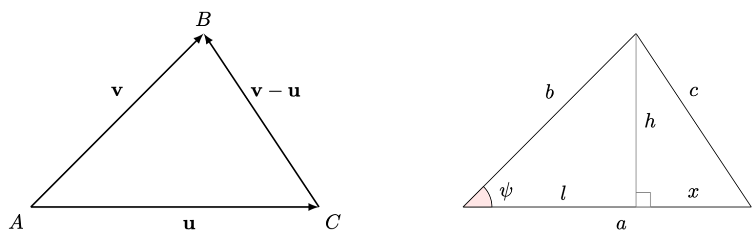

Figure 1: Geometric interpretation of the Scalar Product, showing the vectors v and u and their difference and a triangle with the corresponding lengths.

Recall from above that $|\mathbf{u}|^2 = a$ and $|\mathbf{v}|^2 = b$, then

$$b^2 - 2ab\cos(\psi) + a^2 = b^2 - 2\mathbf{u}\cdot\mathbf{v} + a^2$$

Cancelling the terms a^2 and b^2,

$$ab\cos(\psi) = \mathbf{u}\cdot\mathbf{v}$$

So,

$$\cos(\psi) = \frac{(\mathbf{u}\cdot\mathbf{v})}{(ab)} = \frac{(\mathbf{u}\cdot\mathbf{v})}{(|\mathbf{u}||\mathbf{v}|)}$$

So, in geometric terms, the above derivation tells you that the scalar product between two vectors is the product of their magnitudes multiplied by the angle between them.

Example 7

Take $\mathbf{u} = \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 2 \\ 7 \\ 2 \end{pmatrix}$, as in Example 1. It was calculated then that the scalar product is 43. So, you can use the above definition to calculate the angle, ψ between the two vectors.

$$\cos(\psi) = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \frac{43}{(\sqrt{3^2+5^2+1^2})(\sqrt{2^2+7^2+2^2})} = \frac{43}{(\sqrt{35})(\sqrt{57})} \approx 0.96$$

So, $cos(\psi) \approx 0.96$ and hence, $\psi = \cos^{-1}(0.96) \approx 15.7^\circ$

Example 8

Take $\mathbf{u} = 4\mathbf{i}+2\mathbf{j}+\mathbf{k}$ and $\mathbf{v} = \mathbf{i}-\mathbf{j}+3\mathbf{k}$, as in Example 4. It was calculated then that the scalar product between them is 5. So, the angle between them can be calculated as follows:

$$\cos(\psi) = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \frac{5}{(\sqrt{4^2+2^2+1^2})(\sqrt{1^2+(-1)^2+3^2})} = \frac{5}{(\sqrt{21})(\sqrt{11})} \approx 0.33$$

$$\psi \approx \cos^{-1}(0.33) \approx 71^\circ$$

! IMPORTANT

The scalar product can be a negative number! All this means is that the angle is between 90° or 270° exclusively, or one vector is in the opposite direction to the other.

Special Cases...

There are a few specific cases that you need to be careful with when dealing with the scalar product. These are:
(a) When $\mathbf{u}$ and $\mathbf{v}$ are parallel. In this case, the angle between the vectors is 0. So, the dot product is the magnitudes multiplied, that is

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}|$$

(b) When $\mathbf{u}$ and $\mathbf{v}$ are anti-parallel, then the dot product is negative the magnitudes multiplied, that is

$$\mathbf{u} \cdot \mathbf{v} = -|\mathbf{u}||\mathbf{v}|$$

(c) When either $\mathbf{u}$ or $\mathbf{v}$ is equal to 0 (or both are), the dot product is equal to 0, that is

$$\mathbf{u} \cdot \mathbf{0} = \mathbf{0} \cdot \mathbf{v} = 0$$

(d) Two vectors $\mathbf{u}$ and $\mathbf{v}$ are perpendicular if and only if their scalar product is equal to 0. This is because the angle between the two vectors is 90° and as $\cos(90) = 0$,

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos(90) = 0$$

.

The scalar product of perpendicular vectors

As mentioned in the above result, the scalar product of two perpendicular vectors is 0. Given two vectors, you can use this fact to test whether they are perpendicular or not.

Example 9

Take $\mathbf{u} = \begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 1 \\ 3 \\ -3 \end{pmatrix}$. Then,

$$\mathbf{u} \cdot \mathbf{v} = \begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \\ -3 \end{pmatrix} = (3)(1) + (3)(3) - (4)(3) = 0$$

So, $\mathbf{u}$ and $\mathbf{v}$ are indeed perpendicular.

Example 10

Let $\mathbf{u} = \begin{pmatrix} 2 \\ 1 \\ \lambda \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 5 \\ 6 \\ 1 \end{pmatrix}$. You can find the value of λ for which $\mathbf{u}$ and $\mathbf{v}$ are perpendicular as follows:

$$\mathbf{u} \cdot \mathbf{v} = \begin{pmatrix} 2 \\ 1 \\ \lambda \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 6 \\ 1 \end{pmatrix} = (2)(5) + (1)(6) + (\lambda)(1) = 10 + 6 + \lambda = 16 + \lambda$$

As $\mathbf{u}$ and $\mathbf{v}$ are perpendicular, the scalar product is 0. So,

$$16 + \lambda = 0$$

Hence, $\lambda = -16$

Quick check problems

1. Using the algebraic definition, what is the scalar product of the vectors $\mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 2 \\ 10 \\ 15 \end{pmatrix}$?
2. What is the scalar product of $\mathbf{u} \begin{pmatrix} 5 \\ 7 \\ -13 \end{pmatrix}$ with itself?
3. What is the scalar product of $\mathbf{v} = 3\mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\mathbf{u} = 4\mathbf{i} + 5\mathbf{j} + \mathbf{k}$?
4. Find the angle between the vectors $\mathbf{u} = \begin{pmatrix} 7 \\ -4 \\ 11 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 5 \\ 2 \\ 2 \end{pmatrix}$ to 2 significant figures.

[For more questions on the subject, please go to Questions: Scalar Product.]