Introduction to Logarithms

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Summary

Using logarithms to condense large numbers is a fundamental skill when working in Mathematics. Understanding the laws of logarithms, working with the natural logarithm and learning how to change the base of a logarithm are key in mastering their functionality.

Before reading this guide, it is recommended that you read the guides on 'Laws of indices', as well as 'Dealing with powers and nth roots'.

What are Logarithms?

In the same way that subtraction 'undoes' addition, and division 'undoes' multiplication, you can think of the logarithm as the function that 'undoes' exponentiation. Logarithms are useful because they essentially let you work backwards through a calculation by undoing exponential effects. Furthermore, logarithms allow us to express large numbers in a convenient way, as well as model various phenomena as they occur in a logarithmic fashion. An example would be a hot object cooling down; this decays logarithmically.

In this guide, you will learn the definition of a logarithm, the laws concerning logarithms, what the natural logarithm is and finally, how to change the base of a logarithm.

Definition of a logarithm

In its simplest form, a logarithm answers "How many of one number multiply together to make another number." This is what motivates the following definition:

$$a^x = b$$

or

$$\log_a(b) = x$$

where $a > 1$, and $b > 0$.
x is known as the *exponent*, a is the *base* and b is the *argument*. You would read the equation above as "logarithm of b to the base a is equal to x."

Important

Note that the logarithm function can only evaluate **real, positive numbers**. Therefore, you can't compute the logarithm of any number that is lesser than or equal to 0.

Example 1

Suppose you are given the equation $\log_{10}(1000) = x$, and you need to find x. You can rewrite the equation as $10^x = 1000$, and from here, you can see that $x = 3$.

Example 2

Now consider the equation $\log_x(8) = 3$. Rewriting this, you can see that this is equal to $x^3 = 8$. Therefore $x = 2$.

Generally, if the base is not specified, it is assumed that the base is 10. For example, $\log(a) = b$ means that the logarithm of a to the base 10 is equal to b.

Example 3

Consider the equation from Example 1. This can be rewritten as $\log(1000) = x$, and the solution for x is once again $x = 3$

That being said, it is good practice to always specify the base of the logarithm you're using.

Following from the definition, you can make two key observations.

Inverse of a logarithm

$$\log_b(b^k) = k$$

The logarithm of an exponential where the base is equal to the base of the logarithm is equal to the exponent.

Inverse of an exponent

$$b^{log_b(k)} = k$$

Raising a logarithm of a number to its own base is equal to the number itself.

The results above can be proved using the definition of the logarithm.

Laws of Logarithms

There are several rules of logarithms that allows us to expand, condense and solve logarithmic equations. For this section, assume that $a > 1$.

Law 1: Product Rule

$$\log_a(M \cdot N) = \log_a(M) + \log_a(N)$$

The logarithm of a product is the sum of the logarithms of the factors.

Example 4

1. $\log_a(100) = \log_a(10 \cdot 10) = \log_a(10) + \log_a(10)$
2. $\log_a(27) = \log_a(3 \cdot 9) = \log_a(3) + \log_a(9)$

Law 2: Quotient Rule

$$\log_a\left(\frac{M}{N}\right) = \log_a(M) - \log_a(N)$$

The logarithm of a ratio of two numbers is equal to the logarithm of the **numerator** minus the logarithm of the **denominator**

Example 5

1. $\log_a\left(\frac{5}{4}\right) = \log_a(5) - \log_a(4)$
2. $\log_a(27) = \log_a\left(\frac{54}{2}\right) = \log_a(54) - \log_a(2)$

Law 3: Power Rule

$$\log_a(M^k) = k \cdot \log_a(M)$$

The logarithm of an exponential number is the exponent multiplied with the logarithm of the base.

Example 6

1. $\log_a(4) = \log_a(2^2) = 2 \cdot \log_a(2)$
2. $\log_a(27) = \log_a(3^3) = 3 \cdot \log_a(3)$

Law 4: Zero Rule

$$\log_a(1) = 0$$

The logarithm of 1 to any base is **always** equal to 0.
This is true as

$$a^0 = 1$$

using the definition of the logarithm.

Example 7

1. $\log_4(1) = 0$
2. $\log_{10}(1) = 0$

Law 5: Identity Rule

$$\log_a(a) = 1$$

The logarithm of the argument, where the argument is equal to the base of the logarithm, is **always** equal to 1.
This is true as

$$a^1 = a$$

using the definition of the logarithm.

Example 8

1. $\log_4(4) = 1$
2. $\log_{10}(10) = 1$

The Natural Logarithm

To learn about the natural logarithm, you first need to know of **Euler's number**.

> **i** Euler's Number
>
> **Euler's number**, denoted *e*, is an irrational mathematical constant that is approximately equal to 2.71828.

Euler's number appears naturally in various phenomena involving exponential growth and decay.

Now, the natural logarithm is a mathematical function used to study problems specifically involving exponential growth and decay. Unlike the logarithms we encountered above, this logarithm always has *e* as its base.

> **i** The Natural Logarithm
>
> The **natural logarithm**, often denoted $\ln(x)$, is $\log_e(x)$.

The Natural Logarithm shares the same domain as the logarithms above, namely that it only evaluates **positive, real** numbers. All of the laws from above apply as well.

> **i** **Example 9**
>
> Suppose you're given the equation $\ln(x) = 4$, and you've been asked to find x. Using the definiton of a natural logarithm, you can show that $\log_e(x) = 4$. Then, using the definition of a logarithm, you can conclude that $x = e^4$.

The Natural Logarithm is quite useful as it's the **inverse** of the exponential function with base *e*. A graph of the natural logarithm, as well as its corresponding exponential curve is shown in Figure 1.

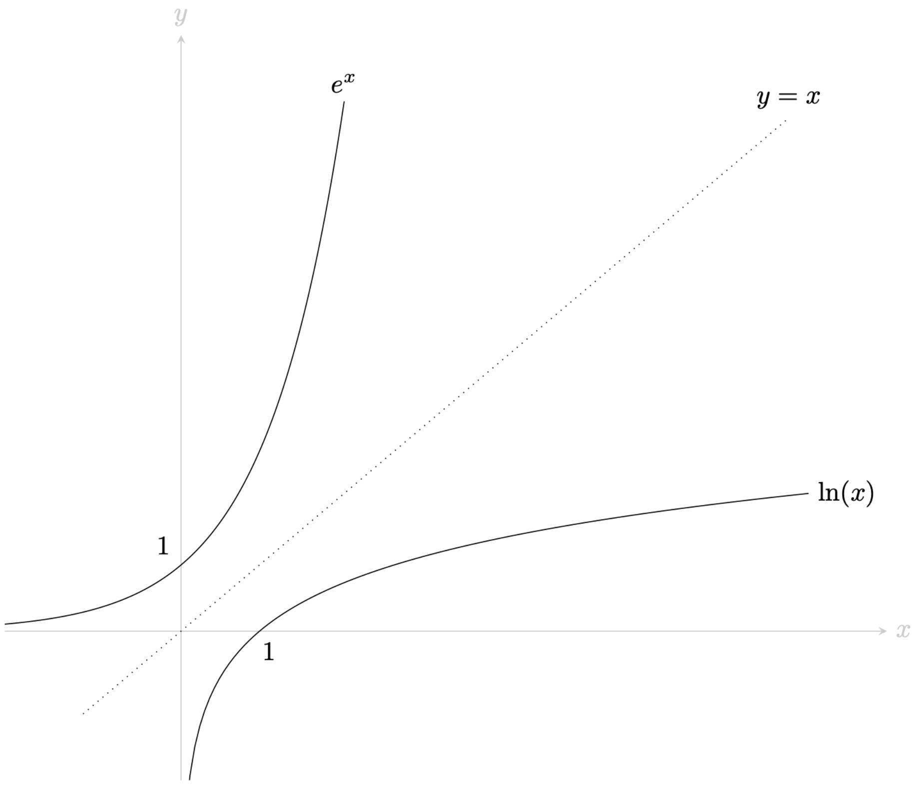

Figure 1: Figure 1: A graph that shows the natural logarithm $\ln(x)$, along with the e^x curve. Each curve is a reflection of the other in the line $y = x$, making them inverses of each other. Note that $\ln(x)$ only takes positive x values, and is undefined for $x = 0$.

Example 10

Let's say that you've been given the equation $e^x = 5$, and you need to solve for x. Taking the natural log on both sides of the equation gives us

$$\ln(e^x) = \ln(5)$$

Now, using Law 3, you can write the equation as

$$x\ln(e) = \ln(5)$$

From here, dividing both sides by $\ln(e)$ results in

$$x = \frac{\ln(5)}{\ln(e)}$$

However, you know from the definition of the logarithm that $\ln(e) = \log_e(e)$. Finally, $\log_e(e) = 1$ by the Inverse of a Logarithm rule.
Therefore, $x = \ln(5)$.

Changing Bases of Logarithms

Changing the base of a logarithm can be done for a variety of reasons. A few common reasons might be because;

- **Familiarity**: Sometimes, changing the base of a logarithm can make calculations more familiar. For example, working with base 10 logarithms is common in many areas of engineering, physics and chemistry, as they allow for simpler calculations by working with powers of 10.
- **Solving equations**: Changing the base of a logarithm can sometimes help make a calculation simpler. You could change the base in order to work in a base that suits the problem at hand better. An example would be wanting to work with the natural logarithm when working on problems involving Euler's number.
- **Using Calculators**: Some calculators may only allow logarithms of base 10 or the natural logarithm to be computed. For this reason, it's important to know how to write out logarithms of any base into one that's required for use.

Change of base rule

Changing the base of any logarithm can be done by using the following formula

$$\log_b(a) = \frac{\log_x(a)}{\log_x(b)}$$

where

- a is the argument.
- b is the original base, and $b > 0.$.
- x is the new base, and $x > 1$.

Example 11

You are given $\log_2(e^5)$ and asked to change the base to e.
Using the change of base rule,

$$\log_2(e^5) = \frac{\ln(e^5)}{\ln(2)}$$

Using the Inverse of a Logarithm rule, you can work out that $\ln(e^5) = 5$. Therefore, the equation above simplifies to

$$\log_2(e^5) = \frac{5}{\ln(2)}$$

Example 12

Given the equation $3^x = 7^{x+1}$, to solve for x;
First, you can take the logarithm of both sides of the equation with base $a > 1$,

$$\log_a(3^x) = \log_a(7^{x+1})$$

Next, using Law 3, you can bring the powers of the arguments outside of the logarithm as follows,

$$x \log_a(3) = (x+1) \log_a(7)$$

After that, expand the right side of the equation to obtain

$$x \log_a(3) = x \log_a(7) + \log_a(7)$$

Subtracting $x \log_a(7)$ from both sides of the equation gives you

$$x \log_a(3) - x \log_a(7) = \log_a(7)$$

From which you can pull a factor of x out on the left, giving you

$$x \cdot (\log_a(3) - \log_a(7)) = \log_a(7)$$

Finally, dividing both sides by $(\log_a(3) - \log_a(7))$, you conclude that

$$x = \frac{\log_a(7)}{\log_a(3) - \log_a(7)}$$

Example 13

Given the equation $5^{2x} + 7(5^x) - 30 = 0$, you are asked to solve for x.
Start by letting $y = 5^x$. Then, you can rewrite the equation given as $(5^x)^2 + 7(5^x) - 30 = 0$ using laws of indices, which is the same as writing

$$y^2 + 7y - 30 = 0$$

Recognizing that this is a quadratic equation, you can use the quadratic formula, or otherwise, to show that

$$y = -10 \quad \text{or} \quad y = 3$$

As $y = 5^x$, this means that $5^x = -10$ or $5^x = 3$. By taking the logarithm of both sides, you can show that

$$x \log_a(5) = \log_a(-10) \quad \text{or} \quad x \log_a(5) = 3$$

Recall that the logarithm of a negative number is not defined, so $x \log_a(5) = 3$ is the only viable solution. Therefore, you can conclude that

$$x = \frac{3}{\log_a(5)}$$

Quick check problems

1. What is the value of the logarithm of $\log_{10}(10000)$?
2. What is the value $\log_3(27)$?
3. Determine whether the following can be evaluated:

(a) $\log_{14}(135)$?

(b) $\log_3(-9)$?

(c) $\log_{1234}(12.34)$?

4. Let $a > 1$. Rewrite $\log_a\left(\frac{2x}{3y^2}\right)$ as $m \log_a(2x) + n \log_a(3y^2)$ where m and n are constants. What are the values of m and n?