Rationalizing the denominator

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Summary

Rationalizing the denominator is a technique for simplifying fractions involving square roots in the denominator. This study guide also covers the topic of quadratic conjugates which are sometimes used to rationalize the denominator of a fraction.

Before reading this guide, it is recommended that you read Guide: Laws of indices and Guide: Expanding Brackets. The only irrational numbers you will see in this guide are are square roots, if you want to learn more about irraional numbers, have a look at the Guide: Number Theory.

What is rationalizing the denominator?

When you rationalize the denominator, you rewrite a fraction so that the denominator contains no square roots or other irrational numbers.

For example, in the fraction $\frac{3}{\sqrt{2}}$, the denominator contains a square root, which is irrational.

You want to rewrite the fraction so that it looks like this: $\frac{3\sqrt{2}}{2}$ where the numerator can be irrational, but the denominator is free of square roots and a rational number, in this case 2.

This process is helpful for doing operations like addition or subtraction on the fraction and can also be useful when trying to approximate such a fraction.

> **Note**
>
> When you rewrite fractions you might multiply the numerator and denominator by the same value k where k is any number. When you are doing this, you are multiplying the fraction by $\frac{k}{k}$ which is the same as multiplying the fraction by 1.
> You are therefore not changing the value of the fraction, you are rewriting it.

Expressions of the form $\frac{a}{b\sqrt{c}}$

For fractions like $\frac{a}{b\sqrt{c}}$, where a is any number and b and c are integers, you rationalize the denominator by multiplying both the numerator and denominator by the square root in the denominator: $\sqrt{c}$. This gives you:

$$\frac{a}{b\sqrt{c}} \cdot \frac{\sqrt{c}}{\sqrt{c}} = \frac{a\sqrt{c}}{b(\sqrt{c})^2}$$

Simplifying the denominator gives you

$$\frac{a\sqrt{c}}{bc}$$

The denominator is now rational and free of square roots as b and c are integers so bc is also an integer. Example 1 shows you how to rationalize the fraction you saw above: $\dfrac{3}{\sqrt{2}}$

Example 1

Simplify $\dfrac{3}{\sqrt{2}}$ by rationalizing the denominator:

To rationalize this, you multiply both the numerator and denominator by $\sqrt{2}$, giving you:

$$\frac{3}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{3\sqrt{2}}{(\sqrt{2})^2}$$

Simplifying gives you: $\dfrac{3\sqrt{2}}{2}$

Example 2

Simplify $\dfrac{3}{2\sqrt{5}}$ by rationalizing the denominator:

Multiply both the numerator and denominator by $\sqrt{5}$, giving you:

$$\frac{3}{2\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{3\sqrt{5}}{2(\sqrt{5})^2}$$

Simplifying gives you: $\dfrac{3\sqrt{5}}{10}$

Example 3

Simplify $\frac{5+\sqrt{2}}{7\sqrt{3}}$ by rationalizing the denominator:
Multiply both the numerator and denominator by $\sqrt{3}$, giving you:

$$\frac{5+\sqrt{2}}{7\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{(5+\sqrt{2})\sqrt{3}}{7(\sqrt{3})^2}$$

Expanding the brackets and simplifying then gives you:

$$\frac{5\sqrt{3}+\sqrt{2}\sqrt{3}}{7(3)} = \frac{5\sqrt{3}+\sqrt{6}}{21}$$

What is a quadratic conjugate?

Definition of a quadratic conjugate

Given an expression of the form $b + c\sqrt{d}$, where b, c and d are integers, the quadratic conjugate is the expression with the same terms but with the opposite sign in front of the term with a square root.
The quadratic conjugate of the expression $b + c\sqrt{d}$ is therefore $b - c\sqrt{d}$.

Example 4

What is the quadratic conjugate of $7 + 2\sqrt{3}$:

$$7 - 2\sqrt{3}$$

Example 5

What is the quadratic conjugate of $-2 - 3\sqrt{5}$:

$$-2 + 3\sqrt{5} = 3\sqrt{5} - 2$$

These conjugates are useful for eliminating square roots when rationalizing denominators of a particular form. Multiplying an expression by its quadratic conjugate eliminates the square root. You can see this when you multiply $b + c\sqrt{d}$ by its quadratic conjugate to get

$$(b + c\sqrt{d})(b - c\sqrt{d}) = b^2 - bc\sqrt{d} + bc\sqrt{d} - (c\sqrt{d})^2$$

Simplifying this gives you

$$b^2 - c^2d$$

This result is rational, free of square roots.

Expressions of the form $\frac{a}{b+c\sqrt{d}}$

Here a is any number and b, c and d are integers. In this case, rationalizing the denominator involves multiplying the numerator and the denominator by the quadratic conjugate of $b + c\sqrt{d}$, which would be $b - c\sqrt{d}$. As you saw above, the denominator will become $b^2 - c^2d$. The entire fraction therefore becomes:

$$\frac{a}{b + c\sqrt{d}} \cdot \frac{b - c\sqrt{d}}{b - c\sqrt{d}} = \frac{a(b - c\sqrt{d})}{b^2 - c^2d}$$

As you can see, the denominator is now $b^2 - c^2d$ and has been successfully rationalized.

Example 6

Simplify $\frac{2}{7 + 2\sqrt{5}}$ by rationalizing the denominator. Provide your answers in their simplest form and with a positive denominator:

The quadratic conjugate of the denominator is:

$$7 - 2\sqrt{5}$$

Multiplying the numerator and denominator by the quadratic conjugate gives you:

$$\frac{2}{7 + 2\sqrt{5}} \cdot \frac{7 - 2\sqrt{5}}{7 - 2\sqrt{5}} = \frac{2(7 - 2\sqrt{5})}{49 - 14\sqrt{5} + 14\sqrt{5} - (2\sqrt{5})^2}$$

Expanding the brackets and simplifying the denominator gives you:

$$\frac{14 - 4\sqrt{5}}{49 - 4(5)}$$

Further simplifying the denominator gives you:

$$\frac{14 - 4\sqrt{5}}{29}$$

Example 7

Simplify $\dfrac{3}{2+5\sqrt{3}}$ by rationalizing the denominator. Provide your answers in their simplest form and with a positive denominator:

The quadratic conjugate of the denominator is:

$$2 - 5\sqrt{3}$$

Multiplying the numerator and denominator by the quadratic conjugate gives you:

$$\frac{3}{2+5\sqrt{3}} \cdot \frac{2-5\sqrt{3}}{2-5\sqrt{3}} = \frac{3(2-5\sqrt{3})}{4-10\sqrt{3}+10\sqrt{3}-(5\sqrt{3})^2}$$

Expanding the brackets and simplifying the denominator gives you:

$$\frac{6-15\sqrt{3}}{4-25(3)}$$

Further simplifying the denominator gives you:

$$\frac{6-15\sqrt{3}}{-71}$$

Multiplying both the numerator and the denominator by -1 to get a positive denominator gives you:

$$\frac{15\sqrt{3}-6}{71}$$

Example 8

Simplify $\frac{2+\sqrt{7}}{5-\sqrt{3}}$ by rationalizing the denominator. Provide your answers in their simplest form and with a positive denominator:

The quadratic conjugate of the denominator is:

$$5+\sqrt{3}$$

Multiplying the numerator and denominator by the quadratic conjugate gives you:

$$\frac{2+\sqrt{7}}{5-\sqrt{3}}\cdot\frac{5+\sqrt{3}}{5+\sqrt{3}}=\frac{(2+\sqrt{7})(5+\sqrt{3})}{(5-\sqrt{3})(5+\sqrt{3})}$$

Expanding the brackets gives you:

$$\frac{2(5)+2\sqrt{3}+5\sqrt{7}+\sqrt{7}(\sqrt{3})}{25+5\sqrt{3}-5\sqrt{3}-(\sqrt{3})^2}$$

This simplifies to:

$$\frac{10+2\sqrt{3}+5\sqrt{7}+\sqrt{21}}{25-9}$$

Simplifying the denominator gives you:

$$\frac{10+2\sqrt{3}+5\sqrt{7}+\sqrt{21}}{16}$$

Quick check problems

Rationalize the denominator for each of the following expressions. Provide your answers in their simplest form and with a positive denominator.

1. $\frac{5}{\sqrt{2}}$
2. $\frac{7}{5\sqrt{3}}$
3. $\frac{7}{2+\sqrt{5}}$
4. $\frac{5\sqrt{2}}{7-2\sqrt{2}}$

Further reading

For more questions on the subject, please go to Questions: Rationalizing the denominator.

Version history

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