

# Solving exponential equations

Ritwik Anand, Zheng Chen, Zoe Gemmell

## Summary

This guide applies the laws of indices to solve equations involving powers; a key skill in mathematics and many areas of science.

*It is highly recommended that you read [Guide: Laws of indices](#) before reading this guide. In addition, [Guide: Logarithms](#) is recommended but not required for some of the material in this guide.*

In [Guide: Laws of indices](#), you learned about the different ways that you can manipulate expressions involving powers (like  $x^3$  and  $2^{80}$ ) and  $n$ th roots (such as  $\sqrt[4]{x}$  and  $2^{1/80}$ ). In general mathematical life however, this knowledge will not be enough; you will need to apply these laws of indices to help you solve equations.

Solving equations involving indices is a key skill in areas whenever exponential growth or decay plays a role: economics, physics (particularly nuclear physics), chemistry, biology, and many more.

Before you get started however, it is worth reiterating the recommendation above:

### ! Important

**This guide assumes an excellent knowledge of the laws of indices. Please make sure that you have read [Guide: Laws of indices](#) before continuing.**

The numbering of the laws will follow the numbering in this guide and on [Factsheet: Laws of indices](#).

In addition, an initial understanding of logarithms will be required later in the guide. While this understanding is partially explained in this guide, you may want to familiarize yourself with [Guide: Logarithms](#) before attempting Examples 7, 8, 9, and 10.

## Discussion of techniques

In general, the golden rule of solving equations involving indices is the following:



Tip

**Make sure both sides are the in the same base before simplifying.** This is because if  $a$  is a positive number not equal to 1 and  $a^m = a^n$ , then you can immediately say that  $m = n$  and solve the equation from there.

However, you may be wondering; why on earth is this true? How can you ignore the base in this case? Or, more pertinently, *what if you can't write the equation in a form where both sides have the same base?*

What you can do in this case is isolate the variable on one side of the equation, and have a constant on the other (see [Guide: Rearranging equations](#); perhaps it looks like  $a^x = b$ . From there, you can take **logarithms** to base  $a$  of both sides.

Since for all  $a > 0$  with  $a \neq 1$  and all real numbers  $y$ :

$$a^{\log_a(y)} = y \quad \text{and} \quad \log_a(a^y) = y$$

it follows that

$$\log_a(a^x) = \log_a b$$

implies that

$$x = \log_a b$$

solving the equation! What this means is that logarithms undo exponentiation and exponentiation undoes logarithms.

In particular, this is why  $a^m = a^n$  implies that  $m = n$ . Taking logarithms to base  $a$  of both sides of  $a^m = a^n$  gives  $\log_a a^m = \log_a a^n$ ; using the above result then yields  $m = n$ .

However, please be aware of the following:



Warning

**You cannot take logarithms of a negative number!** So an expression like  $\log_a(-4)$  is **not** defined.

This is particularly pertinent in Example 9 below.

In this guide, logarithms are only used to undo exponentiation to solve equations; there will not be any applications of the laws of logarithms nor the change of base rule. (For more on these, see [Guide: Logarithms](#).)

# Examples

## Initial examples

### **i** Example 1

Solve  $x^{\frac{1}{2}} = 8$ .

You can start squaring both sides of the equation to get  $(x^{\frac{1}{2}})^2 = 8^2$ . Using Law 3,  $(x^{1/2})^2 = x^{2/2} = x$ ; so then you get the answer  $x = 64$ . To check if the answer is correct, you can substitute 64 back into the equation:  $64^{\frac{1}{2}} = \sqrt[2]{64} = 8$ .

### **i** Example 2

Solve  $x^{-2} = 9$ .

Using Law 5,  $x^{-2} = 9$  can be re-expressed as  $\frac{1}{x^2} = 9$ . Multiplying both sides by  $x^2$  gives you  $1 = 9x^2$ . Then dividing both sides by 9 gives you  $\frac{1}{9} = x^2$ . Remembering you can have positive and negative roots, you get  $x = \frac{1}{3}$  or  $x = -\frac{1}{3}$  as the two solutions to this equation.

### **i** Example 3

Solve  $3^{4x} = 27^{5-x}$ .

You can notice here that the two sides of the equation have different bases; so you need to write these in the same base. As  $27 = 3^3$ , the equation can be rewritten as:  $3^{4x} = (3^3)^{5-x}$ . Then using Law 3, you can write  $3^{4x} = 3^{15-3x}$ . Since the bases of both sides are equal, you can say that  $4x = 15 - 3x$ . Rearranging gives  $x = 15/7$ .

**i Example 4**

Solve  $x^{\frac{5}{3}} + 3x^{\frac{2}{3}} = 10x^{-\frac{1}{3}}$ .

If you look at all of the indices in the question, the denominator is 3 and that is a hint of what you need to multiply by. Multiply by  $x^{\frac{1}{3}}$  on both sides of the equation; using Law 1, this gives

$$\begin{aligned}x^{\frac{5}{3}} \cdot x^{\frac{1}{3}} + 3x^{\frac{2}{3}} \cdot x^{\frac{1}{3}} &= 10x^{-\frac{1}{3}} \cdot x^{\frac{1}{3}} \\x^{\frac{6}{3}} + 3x^{\frac{3}{3}} &= 10x^0\end{aligned}$$

Using Law 4, you can further simplify this expression to get  $x^2 + 3x = 10$ , and so  $x^2 + 3x - 10 = 0$ . This is a quadratic equation; factorizing this equation gives you  $(x+5)(x-2) = 0$ , then you can get  $x = -5$  and  $x = 2$  as the two possible solutions to this equation.

**i Example 5**

Solve  $2^{x+1} \cdot 3^x = 72$ .

Here, you will need to condense the left-hand side into a single base. Using Law 1, you can write that

$$2^{x+1} \cdot 3^x = 2(2^x \cdot 3^x) = 72.$$

Next, use Law 7 to combine  $2^x$  and  $3^x$  into a single base;

$$2(2^x \cdot 3^x) = 2((2 \cdot 3)^x) = 2 \cdot 6^x = 72.$$

Therefore,  $6^x = 36$ . Since  $36 = 6^2$ , it follows that  $x = 2$ . (Or indeed, you could have taken logarithms of both sides to base 6.)

### Example 6

Solve  $\sqrt{(6x)^3} = 8\sqrt{27}$ .

First of all, you can use Law 7 to write  $(6x)^3 = 6^3 \cdot x^3$ . Using Law 10, you can then write the left hand side of this equation as

$$\sqrt{(6x)^3} = \sqrt{6^3 \cdot x^3} = \sqrt{6^3} \cdot \sqrt{x^3}$$

You can now work on  $\sqrt{6^3}$ ; using Law 7 and Law 10, you can write

$$\sqrt{6^3} = \sqrt{2^3 \cdot 3^3} = \sqrt{8} \cdot \sqrt{27}.$$

Therefore, the initial equation becomes

$$\sqrt{(6x)^3} = \sqrt{8} \cdot \sqrt{27} \cdot \sqrt{x^3} = 8\sqrt{27}.$$

Dividing both sides by  $\sqrt{8}$  (thereby using Laws 2 and 6) and cancelling the  $\sqrt{27}$  gives

$$\sqrt{x^3} = \frac{8}{\sqrt{8}} = \frac{8^1}{8^{1/2}} = 8^{1/2} = \sqrt{8}.$$

Squaring both sides to undo the square root gives  $x^3 = 8$ , and so  $x = 2$ .

## Examples involving logarithms

### Example 7

Solve  $e^{3x} = 12$ .

Taking the logarithm of both sides to base  $e$  gives you  $\log_e(e^{3x}) = \log_e(12)$ . Using the definition of logarithms, you can express the equation as  $3x = \ln(12)$ . Rearranging the equation gives you  $x = \frac{\ln(12)}{3}$  and this is your final answer.

### Tip

Although  $e$  here was treated as some constant, it is actually a very important constant called *Euler's number*.

In addition, there is a special name for  $\log_e(x)$ ; this is the **natural logarithm** of  $x$ , often written  $\ln(x)$ .

To see more about Euler's number  $e$  and natural logarithms, please read [Guide: Logarithms](#).

### **i** Example 8

Given the equation  $6^x = 3^{x+1}$ , solve for  $x$ .

First of all, you can notice that as 6 is not a power of 3 (or vice versa) getting these in the same base is difficult. What you can do instead is use the laws of indices to get an expression of the form  $a^x = b$  and then take logarithms. Using Law 1, you can write that

$$6^x = 3^x \cdot 3^1 = 3(3^x)$$

Dividing both sides by  $3^x$  and using Law 8 gives

$$\left(\frac{6^x}{3^x}\right) = \left(\frac{6}{3}\right)^x = 3$$

and so you are left with  $2^x = 3$ . Taking logarithms of both sides to base 2 gives  $x = \log_2(3)$ ; and this is your final answer!

### **i** Example 9

Given the equation  $5^{2x} + 7(5^x) - 30 = 0$ , you are asked to solve for  $x$ .

Start by letting  $y = 5^x$ . Then, you can rewrite the equation given as  $(5^x)^2 + 7(5^x) - 30 = 0$  using Law 3, which is the same as writing

$$y^2 + 7y - 30 = 0.$$

Recognizing that this is a quadratic equation (see [Guide: Introduction to quadratic equations](#) for more), you can use the quadratic formula or otherwise to show that

$$y = -10 \quad \text{or} \quad y = 3.$$

As  $y = 5^x$ , this means that  $5^x = -10$  or  $5^x = 3$ . By taking the logarithm of both sides to base 5, you can show that

$$x = \log_5(-10) \quad \text{or} \quad x = \log_5(3).$$

Remember from above that the logarithm of a negative number is not defined, so  $x = \log_5(3)$  is the only viable solution.

The final example uses many of the laws of indices!

### **i** Example 10

Solve  $3^{3x} = 5^{x-4}$ .

This example is similar to Example 8, only with a few extra steps. Once again, as 3 is not a power of 5 or vice versa, the strategy is to write the equation in the form  $a^x = b$  and then take logarithms of both sides.

First of all, use Law 1 to write  $5^{x-4} = 5^x \cdot 5^{-4}$  and Law 3 to write  $3^{3x} = (3^3)^x$ . Since  $3^3 = 27$ , the equation becomes

$$27^x = 5^x \cdot 5^{-4}$$

Dividing both sides by  $5^x$  gives

$$\frac{27^x}{5^x} = 5^{-4}$$

Using Law 8 and Law 5, you can write

$$\left(\frac{27}{5}\right)^x = \frac{1}{5^4} = \frac{1}{625}$$

Taking logarithms to base  $27/5$  on both sides gives  $x = \log_{27/5}(1/625)$ , which is your final answer.

## Quick check problems

1. Solve  $3x^3 \cdot 5 = 405$  for  $x$ .
2. Solve  $(9x - 1)^{1/3} = 4$  for  $x$ .
3. Solve  $x^{3/2} = \frac{125}{27}$  for  $x$ .

## Further reading

For more questions on the subject, please go to [Questions: Solving exponential equations](#).

## Version history

v1.0: initial version created 08/23 by Ritwik Anand, Zheng Chen, and Zoë Gemmell as part of a University of St Andrews STEP project.

- v1.1: edited 04/24 by TC.