Introduction to rearranging equations

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Summary

Being able to rearrange equations is a crucial skill in mathematics as it can help us to simplify problems and better understand the relationships between variables. This will guide introduce the concept and take you through how to undo common operations like addition, multiplication, and powers.

Introduction

Mathematical formulas are used in all walks of life. From converting between temperature scales, to working out volumes of spheres, these are important mathematical expressions often linking two or more variables. Being able to manipulate these mathematical expressions could allow you to extract information and solve for specific variables. This concept of **rearranging an equation** is vital in handling mathematical formulas, solving all sorts of equations (linear, quadratic, simultaneous, vector), and much much more. It is a crucial tool in almost any academic subject that handles numbers or data.

This guide will introduce you to rearranging equations by stipulating the golden rule of rearranging equations, demonstrate how certain pairs of mathematical operations 'undo' each other, and explain strategies and techniques for rearranging even the most complicated looking formulas.

The golden rule of rearranging equations

The **golden rule of rearranging equations** is the following.

> **!** Important
>
> **You must treat both sides of an equation exactly the same.**

This means if you add $5xy$ to one side an equation, you must add $5xy$ to the other side of the equation. If you multiply one side of an equation by 189273, you must multiply the other side of the equation by 189273. If you divide both sides of an equation by $2lp$, then... well, you get the picture.

The reason for this is balance. Suppose that $x = 2$. If you wanted to multiply the left hand side by 2 but not the right hand side, you would end up with $2x = 2$. But as you can see here, this has changed the assumption that $x = 2$, as in this case x is now equal to 1! So to preserve equality in the expression, you must follow the golden rule of rearranging equations.

Undoing operations

Suppose you are given an expression like

$$5x^3y^3 + \frac{6z}{w^4} = 4abc^2$$

How would you go about rearranging this equation to something of the form $x = ...$? This process is known as **rearranging for** x or **making** x **the subject of the equation**. You can notice that in the above equation, x has been raised to the power of 3, it has been multiplied by both 5 and y^3, and it has been added to $6z/w^4$. To make x the subject of the equation, you will need to 'undo' the operations on x until only a single instance of x remains.

The premise of rearranging equations is to 'undo' what operations have been applied to a particular variable to express that variable on its own. This guide will focus on the following three pairs of operations:

- addition and subtraction
- multiplication and division
- powers of n and nth roots, where n is some number not equal to 0 or 1.

Of these

- subtraction 'undoes' addition and addition 'undoes' subtraction
- division 'undoes' multiplication and multiplication 'undoes' division
- taking the nth root 'undoes' raising to the power of n, and raising to the power of n 'undoes' taking the nth root.

The *order* of undoing these operations is often critical. Addition, subtraction, multiplication and division involve **two numbers**, but raising to the power of n and taking nth roots involves **one number**. These properties influence the order in which you should undo operations and therefore rearrange the equation. You should keep these properties in mind as you study the content in the guide.

Addition and subtraction

The reason why you can undo addition by subtraction (and subtraction by addition) is because anything added to 0 is itself. It is best to see this behaviour in an example: Example 1 uses subtraction to rearrange an equation in x.

Example 1

Given the equation $x + a = b$ you can make x the subject of the equation by subtracting the variable a from both sides of the equation (remembering the golden rule!)
This will give you

$$x + a - a = b - a$$

Since $a - a = 0$, the equation becomes

$$x + 0 = b - a$$

Since $x + 0 = x$ for any number x, you can write that

$$x = b - a$$

and since x is isolated on the left hand side, you have finished.

Example 1 is written in lots of detail to explain what is going on. In practice, you will not need to be this thorough in your working.

Example 2

If you want to make x the subject of the equation $x - 8 = ab$ you can add 8 to both sides.
Doing this gives you

$$x - 8 + 8 = ab + 8$$

which simplifies to

$$x = ab + 8$$

Tip

Going forward, you may not even have to write the middle step of Example 2; you can start with $x - 8 = ab$ and go straight to $x = ab + 8$. This is an example of natural progression in mathematics; once a concept has been mastered, then you can use it without specifying details in future.

Multiplication and division

The reason why you can undo multiplication with division (and division with multiplication) is that any number multiplied by 1 is itself. However, the following health warning applies:

Important

Most of the time, you can undo multiplication in an equation by division. The only time you are not able to do this is if you divide by 0; which is strictly forbidden in mathematics and carries a heavy punishment. The expression $0/0$ is **undefined and should not be written**.

The example below uses division to 'undo' the multiplication in the equation in x. As with Example 1, this example uses more detail than is necessary to illustrate the concepts involved.

Example 3

Given the equation $8x = 9$, you can make x the subject by dividing both sides by 8 (which is not equal to 0). Doing this gives

$$\frac{8x}{8} = \frac{9}{8}$$

This can then be simplified. As $\frac{8}{8}$ is equal to 1, the expression simplifies to

$$1 \cdot x = \frac{8}{9}$$

which, as $1 \cdot x = 1$ for all numbers x, can be simplified to

$$x = \frac{8}{9}$$

and since x is on its own on the left-hand side, you are finished.

Similarly, you can undo division in an equation by using multiplication. Since no equation will ever be divided by 0 to begin with, you can always undo division.

This next example combines what you have learned so far.

Example 4

How do you make x the subject of the equation $\frac{3}{4}x + a = 6$?
In fact, you have a little bit of flexibility in this question. One way you can do it is isolate a multiple of x on one side by subtracting by a first. Doing this gives

$$\frac{3}{4}x = 6 - a$$

You know that you can undo the division of x by 4 by multiplying both sides of the equation by 4. Here, that you have to multiply **all** of the right hand side by 4; this includes both terms 6 and $-a$. This is an important step; as if you don't do this, you are violating the golden rule in not treating both sides of the equation exactly the same! To make sure you do not fall into this trap, it can be useful to use brackets to ensure everything on the right-hand side is multiplied by 4. Anyway, the next step in the rearranging is

$$3x = 4(6 - a)$$

Now, dividing both sides by 3 (making sure to divide *everything* on the right-hand side by 3) gives

$$x = \frac{4(6 - a)}{3}$$

and you are done.

There are many ways to do this question. You could do the division by 3 first, then the subtraction by $a/3$, then the multiplication by 4. Here, the order of operations doesn't matter so much, but isolating a multiple of the chosen variable on one side is often the best way to go about rearranging equations.

This forms the basis of the general strategy for rearranging any equation:

Tip

When rearranging an equation for a variable, try to get exactly one term involving that variable on one side of the equation; then do what you need to do to isolate that variable.

It is also worth restating:

> **!** Important
>
> When multiplying/dividing one side of an equation by a number a, make sure to multiply/divide **everything** on that side of an equation by a.

This is a common health warning in rearranging equations, as you will see throughout the rest of the guide.

Powers and roots

Undoing powers of n using nth roots

How would you undo a power such as $x^2 = 9$, $y^3 = 64$ or even $x^n = a$ (where $n \geq 2$ is a whole number)?

The idea is to use roots. However, there are some considerations to be made when taking the nth root of an equation.

Take the first of these examples, which is $x^2 = 9$. Here, you are trying to rearrange for x, where x is some number than when squared gives 9. You would immediately think that $x = 3$, and you'd be right; but x could *also* be -3. How do you account for this? The idea is to use the square root symbol $\sqrt{}$, but $x = \sqrt{9}$ can't be two solutions at once. To get around this, you could define the square root symbol to **always** mean the *positive* square root of a number, and write $\pm$ to mean 'plus or minus'. Therefore, if $x^2 = 9$, then $x = \pm\sqrt{9} = \pm 3$ to rearrange for x.

> **⚠** Warning
>
> The square root of a negative number is **not** a real number. If you want to rearrange an equation like $x^2 = -4$ for x, then you will need to use **complex numbers**. See [Guide: Introduction to complex numbers] for more.
> Complex numbers are not considered in the rest of this guide.

So, what about $y^3 = 64$? Here there is only one number that when cubed is equal to 64; this number is 4. This is because $(-4)^3 = -64$. So in fact, you are able to take the *cube root* $\sqrt[3]{}$ of any number and not have to worry about signs; if $y^3 = 64$ then $y = \sqrt[3]{64} = 4$. You can contrast this to the case of the square roots!

Now, what about $x^n = a$? If n is even, then x^n must be a square number, and so must be positive. In addition, if n is even then $(-x)^n = x^n$, and so you must also take positive and negative roots.

Undoing powers using nth roots

Suppose that $x^n = a$.

- If n is even and $a \geq 0$, then
$$x = \pm \sqrt[n]{a}$$
where $\sqrt[n]{a}$ is the positive nth root of a.
- If n is odd, then
$$x = \sqrt[n]{a}$$
where $\sqrt[n]{a}$ is the nth root of a.

Thankfully, the situation of undoing nth roots using powers of n is not as complex (no pun intended!). Since there is no ambiguity about the sign of an nth root, you can raise to the power of n to undo the root in any case:

Undoing nth roots using powers of n

Suppose that $\sqrt[n]{x} = a$. Then $x = a^n$.

When rearranging equations involving powers of n and nth roots, you should *always* aim to isolate your variable on one side before undoing the power of n or the nth root. This is because:

Important

In general,
$$a^n + b^n \neq (a+b)^n \quad \text{and} \quad \sqrt[n]{a} + \sqrt[n]{b} \neq \sqrt[n]{a+b}$$

Example 5

You are given the equation $2z = 4x^2 + 5y$. To make x the subject of this equation, you should isolate $4x^2$ on one side of the equation like so:

$$4x^2 = 5y - 2z$$

Now, to apply the square root, you need to have x^2 on its own. So divide by 4 to get

$$x^2 = \frac{5y - 2z}{4}$$

Now you can then take the square root of both sides to get

$$x = \pm\sqrt{\frac{5y - 2z}{4}}$$

and this is your answer.

For roots in equations, you can use inverse operation of taking the equation to a power.

Example 6

If you want to make x the subject of the equation $\sqrt[5]{\frac{x^3}{2}} = b - 1$, you need to undo the fifth root around $x^3/2$. This can be done by raising both sides to the power of 5. Following the health warning above, you must remember to raise **everything** on the right hand side to the power of 5. Doing this gives

$$\frac{x^3}{2} = (b-1)^5$$

You cannot take cube roots yet however, because of that 2. Multiplying 2 from both sides gives

$$x^3 = 2(b-1)^5.$$

Now you can cube root to get

$$x = \sqrt[3]{2(b-1)^5}$$

and you are done.

Finally, let's return to the initial example at the start of the guide.

Example 7

Let's take

$$5x^3y^3 + \frac{6z}{w^4} = 4abc^2$$

How do you rearrange for x?
First of all, you need to isolate the term involving x on one side of the equation. You can subtract $6z/w^4$ from both sides of the equation to get

$$5x^3y^3 = 4abc^2 - \frac{6z}{w^4}$$

You now need to get a power of x on its own by dividing through by 5 and y^3. You can either do these separately or together. So dividing both sides of the equation by $5y^3$

gives

$$x^3 = \frac{4abc^2}{5y^3} - \frac{6z}{5w^4y^3}$$

Now, taking the cube root of both sides gives

$$x = \sqrt[3]{\frac{4abc^2}{5y^3} - \frac{6z}{5w^4y^3}}$$

and this is your answer.

Quick check problems

1. What is the subject of the following equation $x^3 - \frac{x}{7} - 16 = y$?
2. You are given five statements below. Decide whether they are true or false.

(a) Rearranging $x^2 - 4 = 16$ gives $x = \pm 2$.

(b) Division undoes multiplication by any number.

(c) The equation $m^2 + 4m = 7l + m$ is equivalent to $m + 7l = m^2 + 4m$.

(d) The equation $p = \sqrt{q + 4r^2}$ has q as the subject of the equation.

(e) The equation $\sqrt{a^2 + 4b + 2c} = c^2$ is equivalent to $c^4 - 2c = a^2 + 4b$.

Further reading

For more about why powers of n and nth roots undo each other, see Guide: Laws of indices..

For more questions on this topic, please go to Questions: Introduction to rearranging equations.

Version history and licensing

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