

# Further sigma notation

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## Summary

Sigma notation is used to express many additions at once. Understanding what this notation is, how it works, and how to manipulate them is a valuable skill to learn for use in almost any area of mathematics.

## What is sigma notation?

If you want to add many things together, then it would be nice to have a quick way of writing this down! This is where **sigma notation** comes in. (USES!)

### Definition of sum and sigma notation

A **sum** is any addition of two or more real numbers. If  $a_k, a_{k+1}, \dots, a_n$  are real numbers (where  $k$  and  $n$  are some natural numbers with  $k \leq n$ ), then you can use **sigma notation** to write their sum as

$$a_k + a_{k+1} + \dots + a_n = \sum_{i=k}^n a_i$$

where the right hand side reads 'the sum from  $i = k$  to  $i = n$  of the elements  $a_i$ '. The symbol  $i$  is known as the **index** of the sum; the index of a sum can notionally be any letter.

### \* Examples

Here's some examples of sigma notation.

**i Example 1**

What is the value of  $\sum_{i=1}^{10} i$ ?

You can use the definition above to write this out as a sum and then calculate it:

$$\sum_{i=1}^{10} i = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = 55.$$

**i Example 2**

What is the value of  $\sum_{n=2}^5 n^2$ ?

Before tackling a problem using sigma notation, it can be best to read it out loud.

Here,

$$\sum_{n=2}^5 n^2 \text{ is 'the sum from } n = 2 \text{ to } n = 5 \text{ of } n^2\text{'}$$

This translates to

$$\sum_{n=2}^5 n^2 = 2^2 + 3^2 + 4^2 + 5^2$$

$$\text{and } 2^2 + 3^2 + 4^2 + 5^2 = 4 + 9 + 16 + 25 = 54.$$

**i Example 3**

What is the value of  $\sum_{n=1}^N n = S$ ?

In this case, you're being asked to find  $S = 1 + 2 + 3 + \dots + N$ . The following method is due to [Gauss](#), who came up with this answer during a maths lesson at school when he was seven (hinting at the genius to follow).

First of all, you can reorder  $S$  to write that  $S = N + (N - 1) + \dots + 2 + 1$ . Adding two lots of  $S$  together gives the following:

$$\begin{array}{rcccccccc} S & = & 1 & + & 2 & + & 3 & + & \dots & + & N \\ + S & = & N & + & (N-1) & + & (N-2) & + & \dots & + & 1 \\ \hline 2S & = & (N+1) & + & (N+1) & + & (N+1) & + & \dots & + & (N+1) \end{array}$$

Therefore,  $2S$  is  $N$  lots of  $(N + 1)$ ; you can write this as  $2S = N(N + 1)$ . Dividing both sides by 2 gives  $S = N(N + 1)/2$ .

## Writing sums using sigma notation

In this section, you will learn how to do the opposite of the above. That is, given a sequence of numbers, you will learn how to write their sum using sigma notation. The crux of this process is to recognise a pattern in the sequence of given numbers. It's best to learn this using examples.

### **i** Example 4

Write  $2 + 4 + 6 + 8 + 10 + 12$  using sigma notation.

You can tell that these are the first six multiples of 2; so you can list these elements as  $2n$  for  $n = 1$  up to  $n = 6$ . Therefore, you can write that

$$2 + 4 + 6 + 8 + 10 + 12 = \sum_{n=1}^6 2n.$$

### **i** Example 5

Write  $1 + 2 + 4 + 8 + 16 + 32$  using sigma notation.

These are the first 6 numbers in the sequence  $2^n$  for  $n = 0$  up to  $n = 5$ . Therefore, you can write

$$1 + 2 + 4 + 8 + 16 + 32 = \sum_{n=0}^5 2^n.$$

### **i** Example 6

Write  $-1 + 2 - 3 + 4 - 5$  using sigma notation.

For these types of sequences it's useful to keep in mind the sequence  $(-1)^n$ , which alternates between 1 and  $-1$ . Hence, you can write these elements as  $(-1)^n n$  for  $n = 1$  up to  $n = 5$ . Using sigma notation, it will look like this:

$$\sum_{n=1}^5 (-1)^n n.$$

## Properties

In this section you will learn about a few properties of sigma notation which means you'll have a toolkit to rearrange sums!

The first property you'll learn about sigma notation is *distributivity*. This property allows you to take constants from inside the sigma notation to outside the summation.

### Distributivity

Let  $a_k, a_{k+1}, \dots, a_n$  be a sequence of numbers (where  $k$  and  $n$  are integers with  $k \leq n$ ) and  $C$  be any constant. Then

$$\sum_{i=k}^n C a_i = C \sum_{i=k}^n a_i.$$

You can see this is true by writing the entire sum out, like this:

$$\begin{aligned} \sum_{i=k}^n C a_i &= C a_k + C a_{k+1} + C a_{k+2} + \dots + C a_n \\ &= C(a_k + a_{k+1} + a_{k+2} + \dots + a_n) \\ &= C \sum_{i=k}^n a_i \end{aligned}$$

### Example 7

What is the value of  $\sum_{n=2}^5 6n^2$ ?

Using distributivity,  $\sum_{n=2}^5 6n^2 = 6 \sum_{n=2}^5 n^2$ . From Example 2, you know that  $\sum_{n=2}^5 n^2 = 54$ . Therefore,  $6 \sum_{n=2}^5 n^2 = 6 \times 54 = 324$ .

Another property of sigma notation is something we'll call *combining sums*. This lets you write two sums in sigma notation as one sum.

### Combining sums

Let  $a_k, a_{k+1}, \dots, a_n$  and  $b_k, b_{k+1}, \dots, b_n$  be two sequences of numbers (where  $k$  and  $n$  are numbers with  $k \leq n$ ). Then

$$\sum_{i=k}^n a_i + \sum_{i=k}^n b_i = \sum_{i=k}^n (a_i + b_i)$$

and

$$\sum_{i=k}^n a_i - \sum_{i=k}^n b_i = \sum_{i=k}^n (a_i - b_i).$$

### Warning

Note in the property above that both sequences start at  $k$  and end at  $n$ .

Similar to the distributive property, you can show this is true by writing the entire sum out:

$$\begin{aligned}
\sum_{i=k}^n a_i + \sum_{i=k}^n b_i &= (a_k + a_{k+1} + \dots + a_n) + (b_k + b_{k+1} + \dots + b_n) \\
&= (a_k + b_k) + (a_{k+1} + b_{k+1}) + \dots + (a_n + b_n) \\
&= \sum_{i=k}^n (a_i + b_i).
\end{aligned}$$

In a similar way, you can show that  $\sum_{i=k}^n a_i - \sum_{i=k}^n b_i = \sum_{i=k}^n (a_i - b_i)$  is also true.

### **i** Example 8

Write  $\sum_{n=1}^6 2n + \sum_{n=0}^5 2^n$  as a single sum.

First, notice that the indices of these sequences are different, so before you can use the combining sums property you need to do a process called *reindexing*. Reindexing a sum in sigma notation means rewriting the same sum using different indicies. For your purposes, you can reindex  $\sum_{n=1}^6 2n$  as  $\sum_{n=0}^5 (2n + 2)$ . You can now use the combining sums property:

$$\sum_{n=1}^6 2n + \sum_{n=0}^5 2^n = \sum_{n=0}^5 (2n + 2) + \sum_{n=0}^5 2^n = \sum_{n=0}^5 (2n + 2^n + 2).$$

### **i** Note

In the above example, you could also reindex  $\sum_{n=0}^5 2^n$  instead of  $\sum_{n=1}^6 2n$ . Give it a go!

### **i** Splitting a sum

Let  $a_k, a_{k+1}, \dots, a_n$  and  $b_k, b_{k+1}, \dots, b_n$  be two sequences of numbers (where  $k$  and  $n$  are integers with  $k < n - 1$ ). Then for any integer  $m$  such that  $k \leq m < n$ ,

$$\sum_{i=k}^n a_i = \sum_{i=k}^m a_i + \sum_{i=m+1}^n a_i.$$

Again, similar to above you can show this by writing the entire sum out. This is left to you.

### Example 9

Write  $\sum_{i=10}^{100} i$  as the product of three sums.

There are a large number of ways to do this an example of which is:

$$\sum_{i=10}^{11} i + \sum_{i=12}^{98} i + \sum_{i=99}^{100} i.$$

Note that this example has not been particularly useful mathematically but is to illustrate the property in action.

This property is used more when working with infinite summations.

## Double sums

Sometimes, you'll want to multiply two sums together. This can be written succinctly using something called *double sums*.

### Double sums

Let  $a_k, a_{k+1}, \dots, a_n$  and  $b_t, b_{t+1}, \dots, b_m$  be two sequences of numbers (where  $k, n, t$ , and  $m$  are integers with  $k \leq n$  and  $t \leq m$ ). Then the **double sum**  $\sum_{i=k}^n \sum_{j=t}^m a_i b_j$  is defined as

$$\begin{aligned} \sum_{i=k}^n \sum_{j=t}^m a_i b_j &= a_k b_t + a_k b_{t+1} + \dots + a_k b_m + a_{k+1} b_t + a_{k+1} b_{t+1} \\ &\quad + \dots + a_{k+1} b_m + \dots + a_n b_m. \end{aligned}$$

### Tip

You might find it easier to remember the above by thinking of  $\sum_{i=k}^n \sum_{j=t}^m a_i b_j$  as  $a_1(\sum_{j=t}^m b_j) + a_2(\sum_{j=t}^m b_j) + \dots + a_n(\sum_{j=t}^m b_j)$ .

You will now see how this relates to multiplying two sums together. Suppose that  $a_k, a_{k+1}, \dots, a_n$  and  $b_t, b_{t+1}, \dots, b_m$  are like above, and consider the product  $(\sum_{i=k}^n a_i)(\sum_{j=t}^m b_j)$ . Writing it all out and performing the multiplication, you get

$$\begin{aligned}
\left(\sum_{i=k}^n a_i\right) \left(\sum_{j=t}^m b_j\right) &= (a_k + a_{k+1} + \dots + a_n)(b_t + b_{t+1} + \dots + b_m) \\
&= a_k b_t + a_k b_{t+1} + \dots + a_k b_m + a_{k+1} b_t + a_{k+1} b_{t+1} + \\
&\quad \dots + a_{k+1} b_m + a_{k+2} b_t + \dots + a_n b_m \\
&= \sum_{i=k}^n \sum_{j=t}^m a_i b_j
\end{aligned}$$

You can write this as a result:

**i** Double sums and products of two sums

Let  $a_k, a_{k+1}, \dots, a_n$  and  $b_t, b_{t+1}, \dots, b_m$  be two sequences of numbers (where  $k, n, t$ , and  $m$  are integers with  $k \leq n$  and  $t \leq m$ ). Then

$$\sum_{i=k}^n \sum_{j=t}^m a_i b_j = \left(\sum_{i=k}^n a_i\right) \left(\sum_{j=t}^m b_j\right).$$

**i Example 10**

Write  $(1 + 2 + 3 + 4 + 5 + 6)(2 + 4 + 6 + 8 + 10 + 12)$  as a double sum and as a product of two sums.

First, notice you can write out the above expression in the form  $(1)(2) + (1)(4) + \dots(1)(12) + (2)(2) + (2)(4) \dots(3)(2) + \dots(6)(12)$

From the definition above you may now rewrite the expression to the double sum

$$\sum_{i=1}^6 \sum_{j=1}^6 i * 2j$$

using the distributivity property this can be written as

$$2 \sum_{i=1}^6 \sum_{j=1}^6 ij$$

This can then be written using the product of two sums rule above to

$$2 \sum_{i=1}^6 i \sum_{j=1}^6 j$$

It is evident that the two sums are the same with different index variables this means that they can be combined to form

$$2 \sum_{k=1}^6 k^2$$

$k$  has been used to differentiate the new sum from the ones involving  $i$  and  $j$  before but as always the choice of index variable is relatively unimportant

### Quick check problems

1. What is the value of  $\sum_{i=2}^6 i$ .

Answer: The value of the above is: \_\_\_\_.

2. Given  $\sum_{j=1}^{100} i$  Identify the index of the sum.

Answer: The index is \_



3. You are given several statements below based on the properties of sums. Identify whether they are true or false.

(a) The sum  $3 + 6 + 9 + 12$  can be expressed as  $\sum_{i=0}^4 3i$  Answer: TRUE / FALSE.

(b) The sum  $-1 + 1 - 1 + 1$  can be expressed as  $\sum_{i=1}^4 -i$  Answer: TRUE / FALSE.

(c)  $\sum_{i=1}^{100} i = \sum_{i=0}^{101} i$  Answer: TRUE / FALSE.

(d)  $\sum_{i=1}^{100} 6i = 6 \sum_{i=0}^{100} i$  Answer: TRUE / FALSE.

(e)  $\sum_{i=1}^{100} 9i + \sum_{i=1}^{100} 3i = \sum_{i=1}^{100} 27i^2$  Answer: TRUE / FALSE.

(f)  $\sum_{i=1}^{100} 12i - \sum_{i=1}^{100} 4i = 8 \sum_{i=1}^{100} i$  Answer: TRUE / FALSE.

4. You are given several statements below based on the properties of sums. Identify whether they are true or false.

(a)  $\sum_{i=1}^{10} \sum_{j=2}^6 ij$  can be expressed as  $\left( \sum_{i=2}^6 i \right) \left( \sum_{j=1}^{10} j \right)$  Answer: TRUE / FALSE.

(b)  $\left( \sum_{i=1}^5 2i \right) \left( \sum_{j=5}^{10} 3j \right)$  can be expressed as  $6 \left( \sum_{i=1}^5 \sum_{j=5}^{10} ij \right)$  Answer: TRUE / FALSE.

(c) The sum  $(1+2+3+4+5+6)(-1-2)(3+6+9)$  can be expressed as  $\sum_{i=1}^6 \sum_{j=1}^2 \sum_{k=1}^3 -3ijk$   
Answer: TRUE / FALSE.

## Further reading