Proof: Scalar product

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Explanations as to why properties of the scalar product are true.

*Before reading this proof sheet, it is recommended that you read* [*Guide: The scalar product*](../studyguides/scalarproduct.qmd)*. In addition, reading* [*Guide: Introduction to vectors*](../studyguides/introductiontovectors.qmd) *and* [*Guide: Vector addition and scalar multiplication*](../studyguides/addandsm.qmd) *is essential, and reading either* [*Guide: Trigonometry (degrees)*](../studyguides/trigonometry-degrees.qmd) *or* [*Guide: Trigonometry (radians)*](../studyguides/trigonometry-radians.qmd) *is useful.*

The starting point of this proof sheet is the algebraic definition of the scalar product:

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| Reminder of algebraic definition of the scalar product |
| Let and be two vectors. The **scalar product** of and , written as , is given by |

From here, the proof sheet will start with the proof of properties (1) to (5), which can be done using the algebraic definition of the scalar product. Then, the equivalence of the two definitions of scalar product is shown. Once this is done, it is safe to use the geometric definition of the scalar product in showing properties (6) and (7).

This peculiar structure is necessary to ensure that no un-proved statements are used before they are known! This guide uses column notation for vectors; this is purely for space reasons.

# Proof of properties (1) – (5)

### Proof of property (1)

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| Property (1) |
| For all vectors and : |

Suppose that , . By the algebraic definition of scalar product Since , , and , you can write You can recognize this final term as and so as required.

### Proof of property (2)

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| Property (2) |
| The scalar product of any vector with the zero vector is 0, so: |

Suppose that and you know that . By the algebraic definition of scalar product as required.

### Proof of property (3)

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| Property (3) |
| For all vectors , it follows that: |

Suppose that , , and . You know from [Guide: Vector addition and scalar multiplication](../studyguides/addandsm.qmd) that

So from the algebraic definition of the scalar product: Expanding the brackets and rearranging gives You can recognize these final two terms as and respectively. so as required.

### Proof of property (4)

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| Property (4) |
| If are vectors and (pronounced ‘lambda’) is a scalar, then |

Suppose that and are vectors and that is a scalar. You know from [Guide: Vector addition and scalar multiplication](../studyguides/addandsm.qmd) that

By the algebraic definition of scalar product Factorizing the right hand side by a common factor of gives You can recognize this final term as and so You can recognize the term in brackets as , and so as required.

### Proof of property (5)

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| Property (5) |
| The scalar product of a vector with itself is the square of its magnitude: |

Using the algebraic definition for the scalar product, you can work out that for a general vector :

# Proof of equivalence of algebraic and geometric definitions

The main goal in this proof is to show that as defined above is **also** equal to .

In order to prove the equivalence of these definitions, you will need properties (1), (3) and (5) from [Guide: The scalar product](../studyguides/scalarproduct.qmd).

Place the starts of the two vectors and at the same point. Call this base point . Notice that the angle of and at the point is the smallest angle between them; call this angle .

Consider the plane formed by the end of (at point ) and formed at the tip of (at point ). The points form a plane. Now, let be the length of , be the length of and be the length of .

The points therefore form a triangle with side lengths . Drop a perpendicular from to the line ; this perpendicular line has length , and splits the line into lengths , where the line of length is from the point to the intersection of the perpendicular.

All of this information is shown in [Figure 1](#fig-4).

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| Figure 1: Geometric interpretation of the scalar oroduct, showing the vectors , , and their difference and a triangle with the corresponding lengths. |

Using trigonometry, the height of the triangle in [Figure 1](#fig-4) is . Looking at the diagram again, and

Using Pythagoras’s theorem,

and so

(the length of . Expanding out the brackets on the left hand side obtains

Using property (5) from above:

Using property (1), property (3) and property (5),

Remember from above that and , then

Cancelling the terms and gives

and so as required.

# Proof of properties (6) and (7)

### Proof of property (6)

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| Property (6) |
| If two vectors and are **parallel** (so is a scalar multiple of by a positive scalar; see [Guide: Vector addition and scalar multiplication](../studyguides/addandsm.qmd)), then  Similarly, if and are **anti-parallel** (so is a scalar multiple of by a negative scalar; see [Guide: Vector addition and scalar multiplication](../studyguides/addandsm.qmd)), then |

Suppose that and are parallel; so they point in the same direction. This means that the smallest angle between and is . Therefore, as (see [Guide: Trigonometry (degrees)](../studyguides/trigonometry-degrees.qmd) or [Guide: Trigonometry (radians)](../studyguides/trigonometry-radians.qmd)), it follows from the geometric definition of the scalar product that

Now suppose that and are anti-parallel; so they point in completely opposite directions. This means that the smallest angle between and is degrees or radians. Since the cosine of this value is (see [Guide: Trigonometry (degrees)](../studyguides/trigonometry-degrees.qmd) or [Guide: Trigonometry (radians)](../studyguides/trigonometry-radians.qmd)), it follows from the geometric definition of the scalar product that

### Proof of property (7)

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| Property (6) |
| If two non-zero vectors and are perpendicular, then their scalar product is equal to . On the other hand, if the scalar product of two non-zero vectors and is equal to , then and are perpendicular. |

Suppose that and are perpendicular; so the smallest angle between them is degrees or radians. The cosine of a right angle is (see [Guide: Trigonometry (degrees)](../studyguides/trigonometry-degrees.qmd) or [Guide: Trigonometry (radians)](../studyguides/trigonometry-radians.qmd)). So using the geometric definition of the scalar product gives

Now suppose that . It then follows from the geometric definition of the scalar product that

Since both and are non-zero, neither of their magnitudes are . So , where is the smallest angle between and . Since , the only value of in this range such that is radians (so ). Therefore, and are perpendicular.

# Further reading

[Click this link to go back to Guide: The scalar product.](../studyguides/scalarproduct.qmd)

[For questions on this topic, please go to Questions: The scalar product.](../questions/qs-scalarproduct.qmd)

## Version history

v1.0: created in 05/24 by tdhc, based on work of Isabella Lewis as part of a University of St Andrews STEP project.

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