Credit Default Modeling: Structural Modeling with Empirical Default Probabilities

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December 10, 2013

Abstract

This is a technical document to describe the structural credit default model as implemented in partial fulfillment of the requirements for the course Application Programming for Financial Engineering (IEOR 4500).

Here, we merely state the implementation details for the model and will go through the rationale during our presentation.

1 Introduction

We begin with the Black & Cox based first-passage time model [1] to get default probabilities assuming the asset evolves as Geometric Brownian Motion, then alter these probabilities based on empirical default frequencies and finally used the risk-neutral default probabilities to derive a closed-form expression for the par CDS spread.

2 The Initial Default Model

The firm's assets V are modeled as composed of Equity E and Debt D. Equity volatility can be observed but the Asset volatility needs to be inferred as [2]

$$\sigma_V = \sigma_E \frac{E}{E+D} \qquad \text{where } V = E + D$$

Default happens when the value of the assets touches a barrier level which is close to the level of debt, but not known with certainty. This barrier level is assumed to be stochastic and modeled as GBM with a zero drift and volatility σ_B which is a nonlinear function of the equity volatility [3]. We express σ_B as

$$\sigma_B = 0.05 + \min\{0.3, \sigma_E\}$$

Hence, the diffusion can be viewed as asset price evolving as a Geometric Brownian Motion beginning at $V_0 = 1$ as

$$V_t = \mathrm{e}^{\mu t + \sigma B_t} \qquad \text{where } \sigma = \sqrt{\sigma_V^2 + \sigma_B^2}$$

we need to compute the probability of hitting a barrier H

$$H = k \cdot \frac{E}{D+E}$$

The value of the adjustment factor, $k \in (0.8, 1)$ is a constant for each firm at any point of time and is a non-linear function of its interest coverage[4].

For Brownian motion starting at 0 with drift μ^* and variance σ^2, the probability of hitting a barrier $H^* < 0$ in time T is given by [5]

$$\mathbf{P}^T_{\text{default}} = \Phi\left(\frac{H^* - \mu^* T}{\sigma\sqrt{T}}\right) + \mathrm{e}^{\frac{2\mu H^*}{\sigma^2}} \Phi\left(\frac{H^* + \mu^* T}{\sigma\sqrt{T}}\right)$$

Hence, for Geometric Brownian motion with drift μ and volatility σ with a barrier at H, we substitute

$$\mu^* = \mu - \frac{\sigma^2}{2}, \qquad H^* = \log H$$

in the above equation to get the default probability for the firm over time T.

3 Empirical Mapping

The method above will give a default probability for the firm, which we call P_D.

This severely underestimates the probability of default when the barrier is far away and overestimates the default probabilities for firms very close to the barrier levels.

To remedy this, we map $P_D \to P_D^P$, the empirical default probability using a KMV style approach (to be described in the presentation) [6].

For the purposes of this project, we use the proprietary $P_D \to P_D^P$ developed by Hanoz Kalwachwala [7] at Barker Investment Management, Singapore.

From this real empirical default probability, we obtain the empirical risk-neutral default probability of default, P_D^Q as [8]

$$PD^Q = \Phi\left(\Phi^{-1}\left(P_D^P\right) + \frac{\mu - r}{\sigma}\sqrt{T}\right)$$

Using estimates of the firm asset growth μ and process volatility σ, one can derive the entire term-structure of default probabilities.

4 Credit Default Swaps

The survival probability $P_{\text{surv}} = 1 - P_{\text{default}}$. Using this survival probability, in continuous time, we equate the contingent and fee legs of the CDS contract to obtain the par CDS spread c

$$\underbrace{c\int_0^T \mathrm{e}^{-rs} P_{\text{surv}}\, ds}_{\text{fee leg}} = \underbrace{(1-R)\int e^{-rs}\, dP_{\text{surv}}}_{\text{contingent leg}}$$

where R is the recovery rate [2].

Hence,

$$c = \frac{(1-R)\int e^{-rs}\, dP_{\text{surv}}}{\int_0^T \mathrm{e}^{-rs} P_{\text{surv}}\, ds}$$

The J.P. Morgan CDS model gives the equivalent expression in discrete time as [9]

$$c = \frac{(1-R)\sum_{i=1}^n DF_i\left(P_{\text{surv},i-1} - P_{\text{surv},i}\right)}{\sum_{i=1}^n DF_i\left(P_{\text{surv},i-1} + P_{\text{surv},i}\right)\cdot \frac{\Delta_i}{2}}$$

Assuming the simple GBM model of asset evolution, $P_{\text{default}} = P_D$ and we can obtain closed form solutions for the above integrals [2] to obtain the value of c.

$$c = \frac{r(1-R)G(t)}{1 - P_{\text{surv}}(t)\mathrm{e}^{-rt} - G(t)}$$

where [10]

$$G(t) = \left(\frac{1}{H}\right)^{(\gamma+Z)} \Phi\left(\frac{\log H}{\sigma\sqrt{T}} - Z\sigma\sqrt{T}\right) + \left(\frac{1}{H}\right)^{(\gamma-Z)} \Phi\left(\frac{\log H}{\sigma\sqrt{T}} + Z\sigma\sqrt{T}\right)$$

and

$$\gamma = \frac{1}{2} - \frac{\mu}{\sigma^2}, \quad Z = \sqrt{\gamma^2 + \frac{2r}{\sigma^2}}$$

Since we have the value of P_D^Q, we obtain the value of σ_{implied} such that

$$P_{D,GBM}\left(\sigma_{\text{implied}}\right) = P_D^Q$$

and plug this σ_{implied} in the above formula for c.

The value of c is in close agreement to the value obtained using the computationally more complex numerical J.P. Morgan CDS formula.

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