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Design & Analysis of Algorithms

1. Asympotic Notations

Asympotic Notations are the mathematical notations used to describe the running time of an algorithm when the input tends towards a particular value or a limiting value.

There are mainly three asympotic notations :-

i) Big-O notations
ii) Omega (Ω) notations
iii) theta (Θ) notations

I. Big-O notations

It represents the upper bound of the running time of an algorithm.

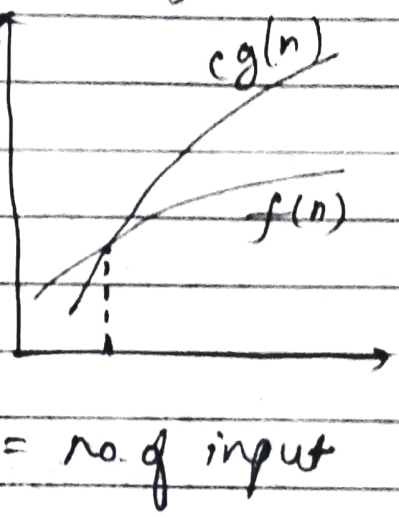
$O(g(n)) = \{ f(n) :$ there exist positive constants c & n_0 such that $0 \leq f(n) \leq cg(n)$ for all $n \geq n_0 \}$

where $c > 0$ and $n \geq n_0$.

- This notations is known as the upper bound of the algorithm, or a worst case of an algorithm.

Ex. $f(n) = 3\log n + 100$
$g(n) = \log n$

$3\log n + 100 <= c * \log(n)$
$c = 150$ & $n > 2$ (undefined at $n = 1$)

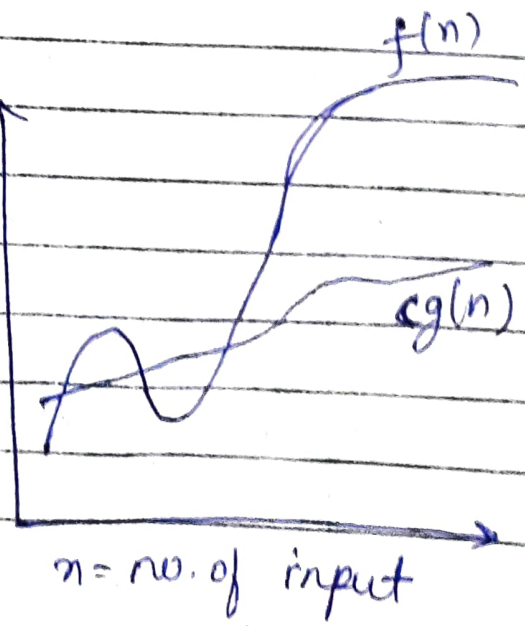


II. Omega (Ω) Notation

- Omega notations represents the lower bound of the running time of an algorithm.

$\Omega(g(n)) = \{f(n)$: there exist positive constant c & n_0 such that $0 \leq c\,g(n) \leq f(n)$ for all n, $n \geq n_0\}$

- This notation is known as the lower bound of the algorithm, or a best case of an algorithm.



Example:- $f(n) = 3n + 2$

$c \cdot g(n) \leq f(n)$ $\quad c$ = constant $\quad g(n) = n$

$c \cdot n \leq 3n + 2$

$cn - 3n \leq 2$

$n(c-3) \leq 2 \Rightarrow n \leq \frac{2}{c-3}$

if we assume $c = 4$, then $n_0 = 2$

$c = 4, \; n_0 = 2$

III Theta (Θ) notations

- Theta notations encloses the function from above & below since it represent the upper & the lower bound of the running time of an algorithm.

$\Theta(g(n)) = \{ f(n) :$ there exist positive constant $c1$, $c2$ & n_0 such that

$$0 \leq c1 * g(n) \leq f(n) \leq c2 * g(n)$$

for all $n > n_0$.

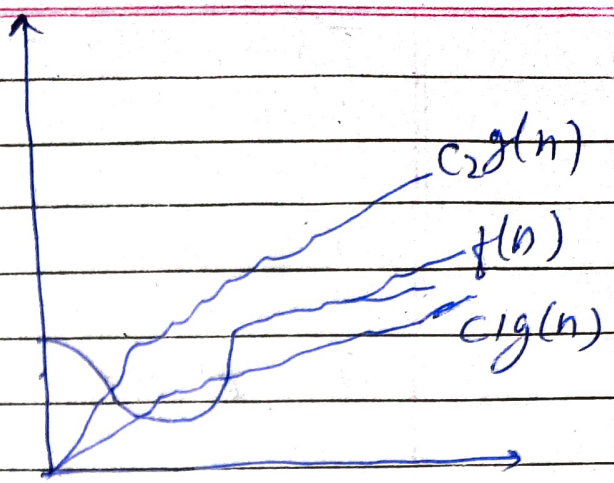
- This is known as tight Bounds of an algorithm or a average case of algorithm.
- Example

$f(n) = 5n^3 + 16n^2 + 3n + 8$

$= 5n^3 \leq 5n^3 + 16n^2 + 3n + 18 \leq (5+16+3+8)\, n^3$

$= 5n^3 \leq f(n) \leq 32n^3$

$C_1 = 5, \quad C_2 = 32, \quad n_0 = 1$



$f(n) \longleftrightarrow \Theta(n^3)$

Ans 2:- $i = 1, 2, 4, 8, 16, \ldots\ldots$ k^{th} terms $\ldots\ldots n$

$G_n = ar^{n-1}$

$G_n = 1(2)^{k-1}$

$n = 2^{k-1}$

$\log_2 n = (k-1)\log_2 2$

$k = \log_2 n + 1$

$O(n) = \log n$

Ans 3:- $T(n) = 3T(n-1)$

$T(n-1) = 3T(n-2)$

$T(n) = 3 \times 3T(n-2)$

$T(n-2) = 3T(n-3)$

$T(n) = 3 \times 3 \times 3T(n-3)$

$T(n) = 3^3 T(n-3)$

$T(n-3) = 3T(n-4)$

$T(n) = 3^3 \times 3T(n-4)$

$T(n) = 3^4 \times T(n-4)$

⋮

General form

$T(n) = 3^i T(n-i) - I \qquad T(0) = 1$

$T(n-i) = T(0)$

$n - i = 0 \qquad \boxed{n = i}$

Put $n = i$ in Eq I

$T(n) = 3^n T(n-n)$

$T(n) = 3^n T(0) \qquad (T(0) = 1 \{Given\})$

$T(n) = 3^n$

$T(n) = O(3^n)$

Any-4 $T(n) = \{2T(n-1) - 1 \quad \text{if } n > 0 \text{ ; otherwise } 1\}$

$T(n) = 2T(n-1) - 1$

$T(n) = 2 \times (2T(n-2) - 1) - 1$

$T(n) = 2^2 T(n-2) - 2 - 1$

$T(n) = 2^2 (2T(n-3) - 1) - 2 - 1$

⋮

$T(n) = 2^i (T(n-i) - (2^{i-1} + 2^{i-2} + 2^{i-3} + \cdots + 1)$

$T(n-i) = T(0)$

$n - i = 0$

$n = i$

Time complexity = $O(1)$

Ans 5 & 11 :-

No. of step (k)	i	j
0	0	1
1	1	2
2	3	3
3	6	4
4	10	5
5	15	6
6	21	7
⋮	⋮	⋮
k step	n	⋮

$T(n) = O(k)$

$i = 0, 1, 3, 6, 10, \ldots\ldots\ldots n$

$Sn = 1 + 3 + 6 + 10 + 15 + \ldots\ldots\ldots + n$

$Sn = 1 + 3 + 6 + 10 + \ldots\ldots\ldots + (n-1) + n$

$-(+) \cdot (-)$

$0 = 1 + 2 + 3 + 4 + 5 + \ldots\ldots\ldots + n$

$n = 1 + 2 + 3 + 4 \ldots\ldots\ldots$ k-step

$$n = \frac{k}{2}[2(1) + (k-1)1]$$

$$2n = k[2 + k - 1]$$

$$2n = k^2 + k \Rightarrow 2n = (k + 1/2)^2 - (1/2)^2$$

$$2n + (1/2)^2 = (k + 1/2)^2$$

$$k + 1/2 = \sqrt{2n + (1/2)^2}$$

$$k = \sqrt{2n + (1/2)^2} - 1/2$$

$$T(n) = T(k)$$

$$T(n) = T\left(\sqrt{2n + (1/2)^2} - 1/2\right)$$

$$T(n) = O(\sqrt{n})$$

Ans. 6

```
void function(int n) {
    int i, count = 0;
    for( i=1; i*i<=n; i++)
        count++;
}
```

Ans :-6 → Since, i is moving from 1 to $\sqrt{n}$ with linear growth so,

$$\boxed{T(n) = O(\sqrt{n})}$$

Ans-7:-

```
void function (int n)
{
    int i, j, k, count = 0;
for ( i=n/2 ; i<=n; i++)
  for(j=1 ; j<=n ; j=j*2)
    for( k=1; k<=n ; k=k*2)
        count++;
}
```

⇒ $O(n \log n \log n)$

⇒ $O(n(\log)^2)$

8.

```
function( int n)                    →T(n)
{  if (n==1) return;
for(i=1 to n)                       →n
  { for (j=1 to n)                  →n
    {
      printf ("*"); }
    }
      function (n-1); }             →T(n-1)
```

$T(n) = T(n-1) + n^2$

$T(n) = T(n-2) + n^2 + (n-1)^2$

$T(n) = T(n-3) + n^2 + (n-1)^2 + (n-2)^2$

⋮

General Term

$$T(n) = T(n-i) + n^2 + (n-1)^2 + (n-2)^2 + \cdots + (n-i)^2$$

$$T(n-i) = T(1)$$

$$n = i+1 \Rightarrow \boxed{n-1 = i}$$

$$T(n) = T(n-(n-1)) + n^2 + (n-1)^2 + (n-2)^2 + \cdots + (n-(n-1))^2$$

$$T(n) = T(1) + n^2 + (n-1)^2 + (n-2)^2 + \cdots + 1^2$$

$$T(n) = 1 + 1^2 + 2^2 + 3^2 + \cdots + n^2$$

$$T(n) = \frac{n(n+1)(2n+1)}{6}$$

$$\boxed{T(n) = O(n^3)}$$

9. Time complexity of

```
void function (int n)
{
    for(i=1; i<n; )
        for(j=1; j<=n; j=j+i)
            print("#")
    }
}
```

⇒ $O(n\sqrt{n})$

10. If $c > 1$, then the exponential c^n for outgrows any term, so that ans is

$$n^k \text{ is } O(c^n)$$

12. $T(n) = T(n-1) + T(n-2) + c$

$$T(n-2) \leq T(n-1)$$

$$T(n) = 2T(n-1) + c$$

$T(n-1) = 2T(n-2) + c$

$$T(n) = 2(2T(n-2) + c) + c$$

$$T(n) = 2^2 T(n-2) + 2c + c$$

$T(n-2) = 2T(n-3) + c$

$$T(n) = 2^2(2T(n-3) + c) + 2c + c$$

$$T(n) = 2^3 T(n-3) + 2^2 c + 2c + c$$

⋮

General Term

$$T(n) = 2^i T(n-i) + (2^0 + 2^1 + 2^2 + \cdots + 2^{i-1})c$$

$$n - i = 0$$

$$\boxed{n = i}$$

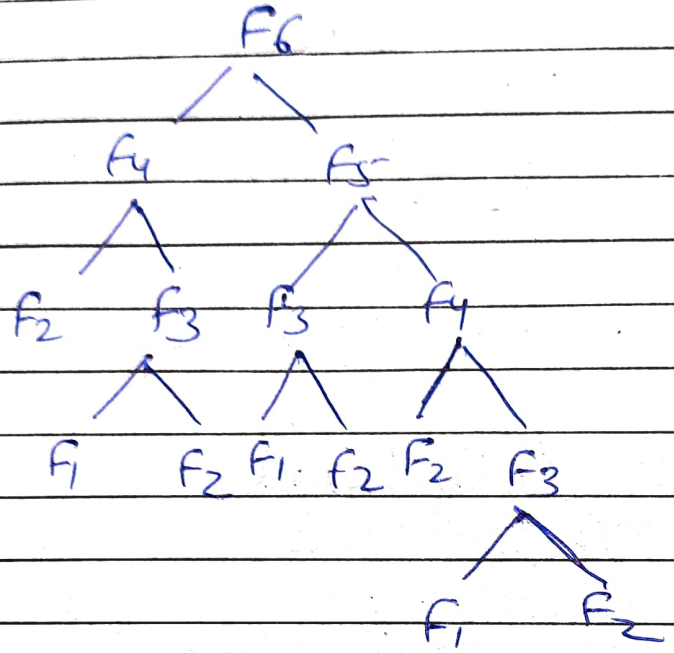
$$T(n) = 2^n T(0) + (2^0 + 2^1 + 2^2 + \cdots + 2^{n-1})c$$

$$T(n) = 2^n(1) + \frac{2^0(2^{n-1} - 1)}{2-1} c$$

$T(n) = 2^n(1+c) + c$

$T(n) = O(2^n)$

fib(6)



The maximum depth is proportional to the N, hence the space complexity of fiboracci recursvice is O(n)

13. i) $n \log n$

```
void fun() {
int i, j;

for (i=1; i<=n; i++)
{
    for (j=0; j<=n; j=j*2)
        printf("#");
printf("\n");
}
}
```

ii) n^3

```
void fun(int n)
{
  int i, j, k;
  for(i=0; i<=n; i++)
  {
    for(j=0; j<=n; j++)
    {
      for(k=0; k<=n; k++)
        printf("#");
    }
  }
}
```

iii) $\log(\log n)$

```
void Sieve of Eratosthenes(int n)
{
  bool prime[n+1];
  memset(prime, true, sizeof(prime));
  for(int p=2; p*p<=n; p++)
  {
    if(prime[p]==true)
    {
      for(int i=p*p; i<=n; i+=p)
        prime[i] = false;
    }
  }
}
```

```
for (int p=2; p<=n; p++)
    if (prime[p])
        cout << p << endl;
}
```

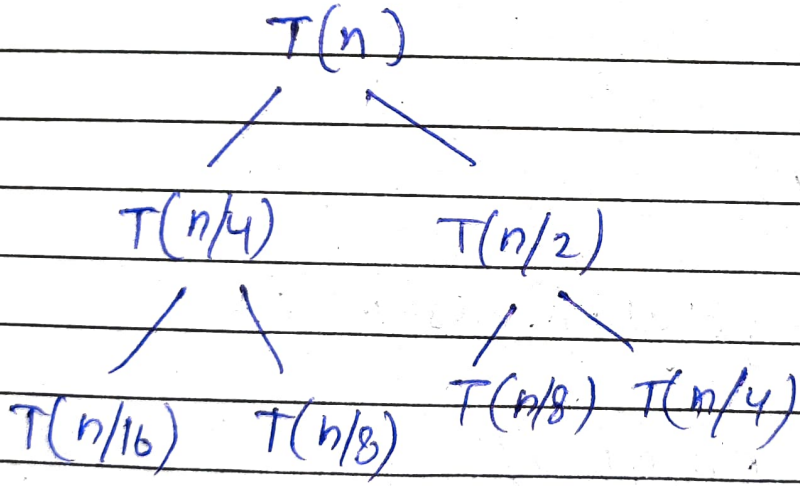
14.

$$T(n) = T(n/4) + T(n/2) + cn^2 \qquad T(1) = c$$

$n = n/2$

$$T(n/2) = T(n/8) + T(n/4) + c(n^2/4)$$

$$T(n) = T(n/4) + 2T(n/16) + c(n^2/16 + n^2/4 + n^2)$$



$$T(n) = c\left[n^2 + \frac{5n^2}{16} + \frac{25n^2}{256} + \cdots\right]$$

$$T(n) = n^2 c\left[1 + \frac{5}{16} + \frac{5^2}{16^2} + \cdots\right]$$

$$\boxed{T(n) = O(n^2)}$$

15. for $i=1$, the inner loop is executed n times
for $i=2$, the inner loop is executed $n/2$ times.
for $i=3$, the inner loop is executed $n/3$ times.
for $i=4$, the inner loop is executed $n/4$ times
⋮
for $i=n$, the inner loop is executed n/n times.

$$\text{Total time} = n + n/2 + n/3 + \cdots\cdots + n/n$$
$$= n(1 + 1/2 + 1/3 + \cdots\cdots + 1/n)$$
$$= n \log n$$

$$\boxed{T(n) = O(n \log n)}$$

16. $O(\log(\log n))$

18.(a) $100, \log\log n, \log n, \sqrt{n}, n, n\log n, n^2, 2^n, 2^{2n}, 4^n, n!$

(b) $1, \log(\log(n)), \sqrt{\log n}, \log n, \log(2n), \log(n!), 2\log(n), n, 2n, 4n, n\log(n), n^2, 2(2^n), n!$

(c) $96, \log_8 n, \log_2 n, \log(n!), 5n, n\log_6 n, n\log_2 n, 8n^2, 7n^3, 8^{2n}, n!$

19. LINEAR_SEARCH (A, key)

```
comp ← 0 , f ← 0
for i = 1 to A.length
    comp ← comp + 1
    if A[i] == key
        print "Element founded"
        f = 1
if f == 0
    print "Element not found"
print comp
```

20. INSERTION_SORT (A)

```
for j = 2 to A.length
    key = A[j]
    i = j - 1
    while i > 0 and A[i] > key
        A[i+1] = A[i]
        i = i - 1
    A[i+1] = key
```

Recursive Method of insertion Sort

```
INSERTION_SORT (A, n)
    if n ≤ 1
        return
```

INSERTION_SORT (A, n-1)

key = [n-1];

j = n-2 :

while j ≥ 0 & A[j] > key

A[j+1] = A[j]

j = j-1

A[j+1] = key

Insertion sort considers one input element per iteration & produces a partial solution without considering future elements that why it is called online sorting

Other sorting algorithm that have been discussed in lectures are :-

- Bubble sorting
- Selection Sorting
- Quick Sort
- Merge Sort
- Heap sort
- Counting Sort

Q1.

	Best case	Average case	Worst case
Bubble Sort	$\Omega(N)$	$\Theta(N^2)$	$O(N^2)$
Selection Sort	$\Omega(N^2)$	$\Theta(N^2)$	$O(N^2)$
Insertion Sort	$\Omega(N)$	$\Theta(N^2)$	$O(N^2)$
Merge Sort	$\Omega(N \log N)$	$\Theta(N \log N)$	$O(N \log N)$
Heap Sort	$\Omega(N \log N)$	$\Theta(N \log N)$	$O(N \log N)$
Quick Sort	$\Omega(N \log N)$	$\Theta(N \log N)$	$O(N^2)$
Counting Sort	$\Omega(N+K)$	$\Theta(N+K)$	$O(N+K)$

Q2.

	In Place	Stable	Online
Bubble Sort	Yes	yes	yes
Insertion Sort	yes	yes	yes
Selection Sort	Yes	No	yes
Merge Sort	No	yes	yes
Quick Sort	yes	No	yes
Heap Sort	Yes	NO	yes
Count Sort	No	yes	yes

Q3. Linear Search

```
LINEAR_SEARCH (A, key)
    found ← 0
for i=1 to N
    if A[i] == key
        found ← 1
        print "Element found"
        break
```

if found == 0
print "Element Not found"

Time complexity – O(n)
Space Complexity – O(1)

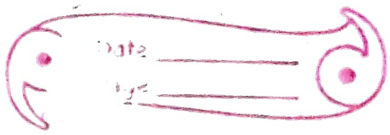
Binary Search (Iterative)

```
BINARY_SEARCH (A, beg, end, key)
  while beg ≤ end
        mid = beg + (end - beg)/2
    if mid == key
        return mid
    if A[mid] < key
        beg = mid + 1
    if A[mid] > key
        end = mid - 1

  return -1
```

Time complexity – $O(\log_2 n)$
Space complexity – O(1)

Binary Search (Recursion)

```
BINARY_SEARCH (A, beg, end, key)

  if end ≥ beg
      mid = (beg + end)/2
```



if A[mid] == item
    return mid+1
else if A[mid] < item
    return BINARY_SEARCH (A, mid+1, end, key)

else,
    return BINARY_SEARCH (A, beg, mid-1, end)

return -1

Time complexity — O(logn)
Space complexity — O(1)

24. T(n) = T(n/2) + c